Combinatorial aspect of Artin L functions

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15/12/2015

Lignes directrices

Introduction

The purpose of this talk is to study the factorisation of Dedekind ζ functions as product of Artin L-functions.
If $\zeta_K(s) = L_1(s) \cdots L_n(s)$, the absolute value of the discriminant of K the product of the conductor of the L_i will be equal to $|\mathrm{Disc}(K)|$, so they will normally be smaller.
Since the cost of computing a L function is proportional to the squareroot of the conductor, this will speed up the computation.

Exercises

Exercise

Let $M \in M_n(K)$ be a matrix, then the following equality of formal power series in $K[[T]]$ holds :

$$\log(\det(1 - TM) = -\sum_{n\geq 1} \mathrm{Tr}(M^n) T^n/n$$

Exercise

Let $\sigma \in S_n$ be a permutation and set $M_\sigma = (\delta_{i,\sigma(j)})$ in $\mathrm{GL}_n(K)$. The map $\sigma \mapsto M_\sigma$ is called the natural representation of S_n, and furthermore $\mathrm{Tr}(M_\sigma)$ is the number of fixed points of σ.

Let K be a number field of degree n and F its Galois closure and set $G = \mathrm{Gal}(F/\mathbb{Q})$ then G acts transitively on the n embedding of K in F.
This allows to identify G as a conjugacy class of a transitive subgroup of $\mathfrak{S}_n$. The map $\mathfrak{S}_n \to GL_n(\mathbb{Q})$ restrict to a representation of G called the natural representation.

Hasse-Weil zeta function of a ring

Let R be a finitely generated ring.

Theorem (Nullstellensatz for $\mathbb{Z}$)

if M is a maximal ideal of R, then the quotient M/R is finite.

We note $\mathcal{N}(M)$ the cardinal of the quotient M/R.

Definition

The zeta function of R is defined by the formal Dirichlet series

$$\zeta_R(s) = \prod_M \frac{1}{1 - \mathcal{N}(M)^{-s}}$$

where the product run over all maximal ideals of R.

Examples

1. $\zeta_{\mathbb{Z}} = \zeta$, the Riemann ζ-function.
2. More generally, if K is a number field and $\mathbb{Z}_K$ its ring of integers, then $\zeta_{\mathbb{Z}_K} = \zeta_K$, the Dedekind ζ-function of K.

Euler product

Let p be a prime number then

$$\zeta_{R/pR}(s) = \prod_{M, p \in M} \frac{1}{1 - \mathcal{N}(M)^{-s}}$$

where the product run over all maximal ideals of R containing p. It follows that ζ_R can be written as an ordinary Euler product

$$\zeta_R(s) = \prod_p \zeta_{R/pR}(s) \quad .$$

where the ring R/pR are finitely generated $\mathbb{F}_p$ algebras.

Zeta function of an $\mathbb{F}_p$ algebra

A finitely generated $\mathbb{F}_p$ algebra A is isomorphic to $\mathbb{F}_p[X_1, \cdots, X_n]/I$ for some ideal I. We set $V(K) = \{(x_1, \cdots, x_n) \in K^n | P(x_1, \cdots, x_n) = 0 \forall P \in I\}$.
We define the uppercase Z function of A as $Z(p^{-s}) = \zeta(s)$.

Exercise

$$Z_A(T) = \exp(\sum_{n \geq 1} |V(F_{p^n})| T^n / n)$$

Theorem

Lefchetz trace formula Assuming that A is good, then $|V(F_{p^n})| = \sum_{i \geq 0} (-1)^i \mathrm{Tr}(\phi^{n*} | H^i)$ *where the* H^i *are cohomology group for a Weil cohomology.*

Lefchetz fixed point formula

Let P be a squarefree polynomial over $\mathbb{C}$ and $V = \{\alpha_i | 1 \geq i \geq n\}$ the complex roots. Topologically, this is just n points. The homology of V is $H_0(V, \mathbb{Q}) = \mathbb{Q}^n$, $H_i(V, \mathbb{Q}) = 0$ if $i > 0$, and the points (α_i) induce a basis B of $H_0(V, \mathbb{Q})$. An homeomorphism S induces a permutation σ of $(\alpha_i)_{i=1}^n$. The matrix of S_* in the basis B is the matrix M_σ, whose trace is the number of fixed points of σ hence of S.
This is a special case of the Lefchetz fixed point formula but the only case we will need.

Artin L functions

Let K be a number field and P be an irreducible polynomial over $\mathbb{Z}$ such that $K = \mathbb{Q}[X]/(P)$, Then for all p but a finite number $\mathcal{O}_K/p\mathcal{O}_K \cong \mathbb{F}_p[X]/(P)$. If V is as above then $|V(F_{p^n})| = \mathrm{Tr}(\phi^{n*}|H^0)$. where ϕ is the dual of the Frobenius operator.
So $Z_V(T) = \sum_{n\geq 1} \mathrm{Tr}(\phi^{n*}|H^0)T^n/n$ which give
$Z_V(T) = 1/\det(1 - T\phi^*|H^0)$

To factor Z_V, the idea is to decompose H^0 as a direct sum of subspaces (E_i) that are stable under ϕ. Indeed if $H^0 = \bigoplus_i E_i$ then

$$\mathrm{Tr}(\phi^{n*}|H^0) = \sum_i \mathrm{Tr}(\phi^{n*}|E_i)$$

and $Z_V(T) = \prod_i 1/\det(1 - T\phi^*|E_i)$.
Note that H^0 is mostly independent of p, only the Frobenius action is. If we choose the E_i to be stable by all the Frobenius (hence the whole Galois group), we get a factorisation of ζ_K. Such subspace are naturally identified to representation of the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ where F is the Galois closure of K. So if ρ is such a representation, $L_{\rho,p} = \sum_{n\geq 1} \mathrm{Tr}(\rho(\phi^{n*}|H^0))T^n/n$ which give $L_{\rho,p}(T) = 1/\det(1 - T\rho(\phi^*|H^0))$. And globally $L_\rho(s) = \prod_p 1/\det(1 - p^{-s}\rho(\phi_p))$ where ϕ_p is a Frobenius $\left(\frac{\mathfrak{p}}{K/\mathbb{Q}}\right)$ for any ideal $\mathfrak{p}$ of $\mathcal{O}_K$ above p.

If F/K is an extension of number fields, we define $L_\rho(s) = \prod_{\mathfrak{p}} L_{\rho,\mathfrak{p}}$
where if $\mathfrak{p}$ is not ramified, $L_{\rho,\mathfrak{p}} = 1/\det(1-\mathcal{N}(\mathfrak{p})^{-s}\rho(\phi_{\mathfrak{p}}))$. where $\phi_{\mathfrak{p}}$ is a Frobenius $\left(\frac{\mathfrak{p}}{K/\mathbb{Q}}\right)$ for any ideal $\mathcal{P}$ of $\mathcal{O}_F$ above $\mathfrak{p}$. and if $\mathfrak{p}$ is ramified, and I be the inertia subgroup of $\mathfrak{p}$ and D the composition subgroup. Let ϕ be an automorphism such that $\phi(x) = x^{\mathcal{N}(\mathfrak{p})} \pmod{\mathcal{P}}$. ϕ is unique modulo an element of I. Let W the subset of V of elements that are fixed by $\rho(I)$,
$L_{\rho,\mathfrak{p}} = 1/\det(1-\mathcal{N}(\mathfrak{p})^{-s}\rho|W(\phi))$.

We define the degree of an Artin L-function as the product $\dim \rho \deg K$. We will say that an Artin L function is irreducible if we cannot write it as a product of two non-constant Artin L functions.
Artin L function associated to irreducible representations are not in general irreducible if the base field is not $\mathbb{Q}$.
Two Artin L functions associated to different representations can be equal.

It follows that $\zeta_F = L_\rho$ where ρ is the adjunct representation of G. Since ρ is a direct sum of irreducible representation we have the factorisation : $\zeta_L = \prod_{\rho \text{ irred}} L_\rho^{\dim \rho}$.

Links with Hecke L-functions

Let L/K an abelian extension, then it can be described by class field theory parameters $(\mathfrak{m}, C)$ such that by Artin reciprocity $\mathcal{C}\ell_{\mathfrak{m}}(K)/C \cong \mathrm{Gal}(L/K)$. This isomorphism links a character χ of $\mathcal{C}\ell_{\mathfrak{m}}(K)/C$ with an irreducible representation ρ of $G = \mathrm{Gal}(L/K)$ such that $L_{\chi} = L_{\rho}$.

Theorem

Hecke Artin L-functions associated to non-trivial representations of degree 1 *admit an holomorphic continuation to the whole complex plane, and can be completed to a function* Λ *which satisfies* $\Lambda(1-s) = \epsilon\overline{\Lambda}(\overline{s})$.

(Artin L-functions of trivial representations are Dedekind ζ functions).

Theorem

Brauer Artin L-functions admits a meromorphic continuation to the whole complex plane and can be completed to a function Λ *which satisfies* $\Lambda(1-s) = \epsilon\overline{\Lambda}(\overline{s})$.

Conjecture

Artin Artin L-functions associated to non-trivial irreducible representation are holomorphic on the whole complex plane.

This is proven for all supersolvable groups. This is also true for A_4 but not for $\hat{A}_4 = SL_2(\mathbb{F}_3)$.

If K is a number field of degree n, let F be its Galois closure and $G = \mathrm{Gal}(F/K)$. The action of G on the n embedding of K in F define a monomorphism from G to S_n. The natural representation of S_n leads to a n dimension representation ρ of G and furthermore $\zeta_K = L_\rho$. Note that the trivial representation appears in ρ, so $\zeta_K(s) = \zeta(s)L_{\rho'}$.