

# On the asymptotic behavior of bifurcating autoregressive processes

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# Outline

## 1 Introduction

- Bifurcating autoregressive processes
- The state of the art

## 2 Least squares estimation

- Martingale assumptions
- Our least squares estimators

## 3 Main results

- A keystone lemma
- Strong laws of large numbers
- Central limit theorems

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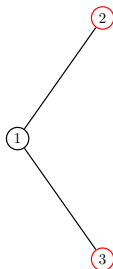
## 3 Main results

- A keystone lemma
- Strong laws of large numbers
- Central limit theorems

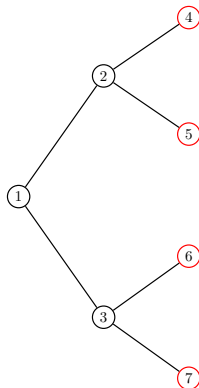
# Cell division

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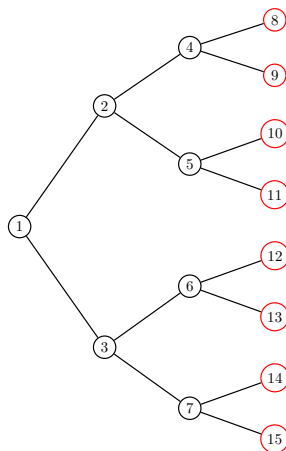
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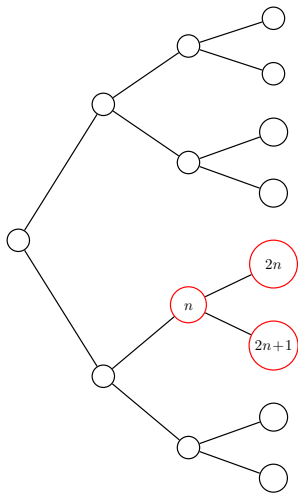
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## Ancestors and offsprings



The offsprings of cell  $n$  are labelled

- $2n$ ,
- $2n + 1$ .

For each cell  $n$ ,  $X_n$  can be

- the diameter,
- the age,...



# Bifurcating AutoRegressive processes

Consider the **BAR**( $p$ ) process given, for all  $n \geq 2^{p-1}$ , by

$$\begin{cases} X_{2n} &= a + \sum_{k=1}^p b_k X_{\lfloor \frac{n}{2^{k-1}} \rfloor} + \varepsilon_{2n}, \\ X_{2n+1} &= c + \sum_{k=1}^p d_k X_{\lfloor \frac{n}{2^{k-1}} \rfloor} + \varepsilon_{2n+1} \end{cases}$$

where  $\lfloor x \rfloor$  stands for the integer part of  $x$  and  $(\varepsilon_{2n}, \varepsilon_{2n+1})$  is the driven noise of the process.

## Goal

Statistical inference on the unknown parameters  $(a, b, c, d)$ .

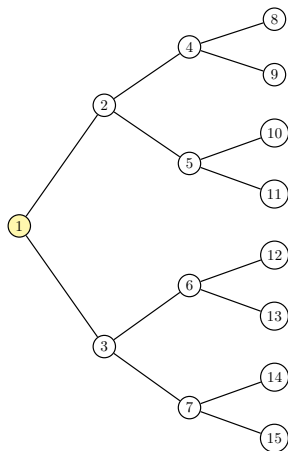
# Bifurcating AutoRegressive processes

For simplicity, we focus on the **BAR(1)** process given, for all  $n \geq 1$ , by

$$\begin{cases} X_{2n} &= a + b X_n + \varepsilon_{2n}, \\ X_{2n+1} &= c + d X_n + \varepsilon_{2n+1} \end{cases}$$

where  $0 < \max(|b|, |d|) < 1$  and  $|a| + |c| \neq 0$ .

# Binary tree, generations, subtrees



- Generation 0 is  $\mathbb{G}_0 = \{1\}$ .
- Generation 1 is  $\mathbb{G}_1 = \{2, 3\}$ .
- Generation 2 is  $\mathbb{G}_2 = \{4, 5, 6, 7\}$ .
- Generation  $n$  is

$$\mathbb{G}_n = \{2^n, 2^n + 1, \dots, 2^{n+1} - 1\}.$$

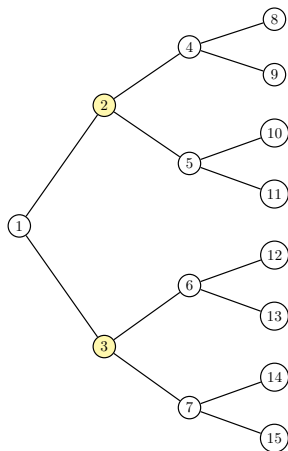
- Subtree up to time  $n$  is

$$\mathbb{T}_n = \bigcup_{k=0}^n \mathbb{G}_k.$$

- Cardinalities are  $|\mathbb{G}_n| = 2^n$  and

$$|\mathbb{T}_n| = 2^{n+1} - 1.$$

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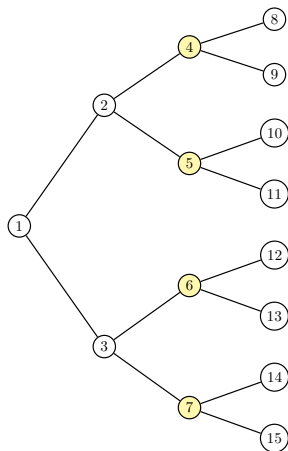
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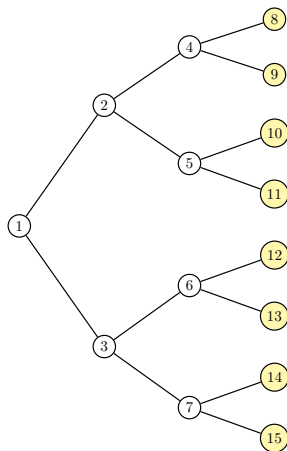
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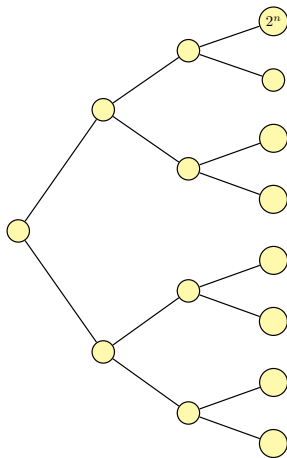
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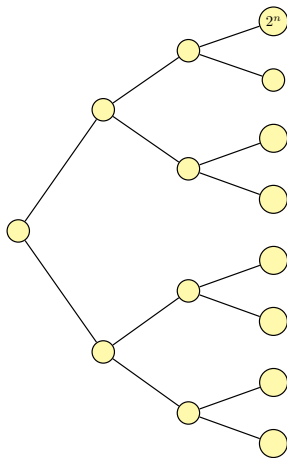
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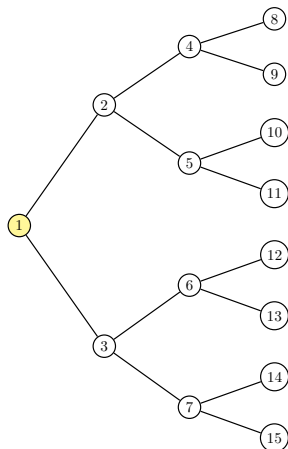
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# Filtration



• At time 0,  $\mathcal{F}_0 = \sigma\{X_1\}$ .

• At time 1,  $\mathcal{F}_1 = \sigma\{X_1, X_2, X_3\}$ .

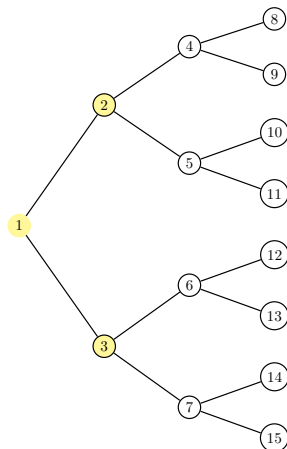
• At time 2,

$$\mathcal{F}_2 = \sigma\{X_1, X_2, X_3, X_4, X_5, X_6, X_7\}.$$

• At time  $n$ ,

$$\mathcal{F}_n = \sigma\{X_k \text{ with } k \in \mathbb{T}_n\}.$$

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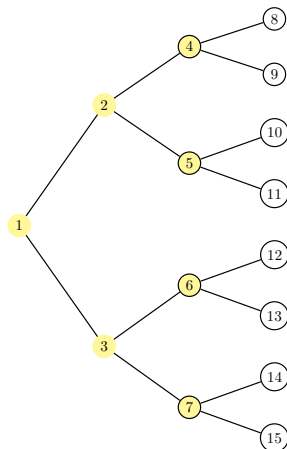
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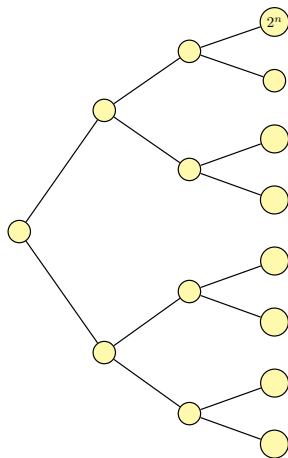
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# Previous works I

## Stationary processes

- Cowan, Staudte, Biometrics, 1986 : Introduction, motivation in Statistical Biology.
- Huggins, Annals of Statistic, 1996 : Maximum likelihood estimation for large trees.
- Huggins, Basawa, Journal of Applied Probability, 1999 and Australian Journal of Statistics, 2000 : Maximum likelihood estimation for large trees and higher order.
- Basawa, Zhou, Journal of Applied Probability, 2004 : exponential BAR processes, and Journal of Time Series Analysis, 2005 : CLT for BAR processes.

# Previous works II

## Non stationary processes

- Guyon, Annals of Applied Probability, 2007 : Least square estimation for Gaussian BAR processes via a Markov chain approach.
- Delmas, Marsalle, 2008, 2009 : The same as Guyon and possibility for a cell to die.
- Our work, EJP 2009 : Least square estimation for martingale difference  $\text{BAR}(p)$  processes by use of a martingale approach.

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# Martingale assumptions

(H.1)  $\forall n \geq 0$  and  $\forall k \in \mathbb{G}_{n+1}$ ,

$$\mathbb{E}[\varepsilon_k | \mathcal{F}_n] = 0 \quad \text{and} \quad \mathbb{E}[\varepsilon_k^2 | \mathcal{F}_n] = \sigma^2 > 0 \quad \text{a.s.}$$

(H.2)  $\forall n \geq 0$  and  $\forall k \neq \ell \in \mathbb{G}_{n+1}$ ,

- if  $[k/2] \neq [\ell/2]$ , then  $\varepsilon_k$  and  $\varepsilon_\ell$  are independent given  $\mathcal{F}_n$
- if  $[k/2] = [\ell/2]$ , then for  $\rho < \sigma^2$

$$\mathbb{E}[\varepsilon_k \varepsilon_\ell | \mathcal{F}_n] = \rho \quad \text{a.s.}$$

(H.3)

$$\sup_{n \geq 0} \sup_{k \in \mathbb{G}_{n+1}} \mathbb{E}[\varepsilon_k^4 | \mathcal{F}_n] < \infty \quad \text{a.s.}$$



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# Estimation of the parameter

We estimate  $\theta = (a, b, c, d)^t$  by the

## Least squares estimator

$$\hat{\theta}_n = \begin{pmatrix} \hat{a}_n \\ \hat{b}_n \\ \hat{c}_n \\ \hat{d}_n \end{pmatrix} = (\mathbf{I}_2 \otimes S_{n-1}^{-1}) \sum_{k \in \mathbb{T}_{n-1}} \begin{pmatrix} X_{2k} \\ X_k X_{2k} \\ X_{2k+1} \\ X_k X_{2k+1} \end{pmatrix}$$

where

$$S_n = \sum_{k \in \mathbb{T}_n} \begin{pmatrix} 1 & X_k \\ X_k & X_k^2 \end{pmatrix}.$$

# A martingale identity

$$(\hat{\theta}_n - \theta) = (I_2 \otimes S_{n-1}^{-1}) \sum_{k \in \mathbb{T}_{n-1}} \begin{pmatrix} \varepsilon_{2k} \\ X_k \varepsilon_{2k} \\ \varepsilon_{2k+1} \\ X_k \varepsilon_{2k+1} \end{pmatrix}.$$

Consequently, if  $\Sigma_n = I_2 \otimes S_n$ , we have the martingale identity

$$\hat{\theta}_n - \theta = \Sigma_{n-1}^{-1} M_n$$

where  $M_n$  is the martingale given by

$$M_n = \sum_{k \in \mathbb{T}_{n-1}} \begin{pmatrix} \varepsilon_{2k} \\ X_k \varepsilon_{2k} \\ \varepsilon_{2k+1} \\ X_k \varepsilon_{2k+1} \end{pmatrix}.$$

# Estimation of the conditional variance and covariance

We estimate the conditional variance  $\sigma^2$  and covariance  $\rho$  by

## Conditional variance estimator

$$\hat{\sigma}_n^2 = \frac{1}{2|\mathbb{T}_{n-1}|} \sum_{k \in \mathbb{T}_{n-1}} (\hat{\varepsilon}_{2k}^2 + \hat{\varepsilon}_{2k+1}^2)$$

## Conditional covariance estimator

$$\hat{\rho}_n = \frac{1}{|\mathbb{T}_{n-1}|} \sum_{k \in \mathbb{T}_{n-1}} \hat{\varepsilon}_{2k} \hat{\varepsilon}_{2k+1}$$

where for all  $k \in \mathbb{G}_n$

$$\begin{cases} \hat{\varepsilon}_{2k} = X_{2k} - \hat{a}_n - \hat{b}_n X_k, \\ \hat{\varepsilon}_{2k+1} = X_{2k+1} - \hat{c}_n - \hat{d}_n X_k. \end{cases}$$

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# Strong laws for the noise

If  $(\varepsilon_n)$  satisfies (H.1), (H.2) and (H.3), we have

$$\lim_{n \rightarrow +\infty} \frac{1}{|\mathbb{T}_n|} \sum_{k \in \mathbb{T}_n} \varepsilon_k = 0 \quad \text{a.s.}$$

$$\lim_{n \rightarrow +\infty} \frac{1}{|\mathbb{T}_n|} \sum_{k \in \mathbb{T}_n} \varepsilon_k^2 = \sigma^2 \quad \text{a.s.}$$

$$\lim_{n \rightarrow +\infty} \frac{1}{|\mathbb{T}_{n-1}|} \sum_{k \in \mathbb{T}_{n-1}} \varepsilon_{2k} \varepsilon_{2k+1} = \rho \quad \text{a.s.}$$

# Some notations

First of all, let

$$\begin{aligned}\bar{a} &= \frac{a+c}{2}, & \bar{b} &= \frac{b+d}{2}, \\ \overline{ab} &= \frac{ab+cd}{2}, & \overline{a^2} &= \frac{a^2+c^2}{2}, & \overline{b^2} &= \frac{b^2+d^2}{2}.\end{aligned}$$

In addition, denote by  $\Gamma$  and  $L$  the two symmetric matrices

$$\Gamma = \begin{pmatrix} \sigma^2 & \rho \\ \rho & \sigma^2 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} 1 & \lambda \\ \lambda & \ell \end{pmatrix}$$

where

$$\lambda = \frac{\bar{a}}{1 - \bar{b}} \quad \text{and} \quad \ell = \frac{\overline{a^2} + \sigma^2 + 2\lambda\overline{ab}}{1 - \overline{b^2}}.$$



# A keystone lemma

## Lemma

If  $(\varepsilon_n)$  satisfies (H.1), (H.2) and (H.3), we have

$$\lim_{n \rightarrow \infty} \frac{S_n}{|\mathbb{T}_n|} = L \quad a.s.$$

## Limiting matrix

The limiting matrix  $L$  is positive definite.

# Strong law for the parameter

## Theorem

Assume that  $(\varepsilon_n)$  satisfies (H.1), (H.2) and (H.3). Then,  $\hat{\theta}_n$  **converges a.s. to  $\theta$**  with the rate of convergence

$$\| \hat{\theta}_n - \theta \|^2 = \mathcal{O} \left( \frac{\log |\mathbb{T}_{n-1}|}{|\mathbb{T}_{n-1}|} \right) \quad \text{a.s.}$$

In addition, if  $\Lambda = \mathbb{I}_2 \otimes L$ , we have the **quadratic strong law**

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |\mathbb{T}_{k-1}| (\hat{\theta}_k - \theta)^t \Lambda (\hat{\theta}_k - \theta) = 4\sigma^2 \quad \text{a.s.}$$

## Tools for the strong laws

The strong laws are related to the random variable

$$V_n = (\hat{\theta}_n - \theta)^t \Sigma_{n-1} (\hat{\theta}_n - \theta)$$

where  $\Sigma_n = I_2 \otimes S_n$ .

# Tools for the strong laws

We have the main decomposition

$$V_{n+1} + \mathcal{A}_n = V_1 + \mathcal{B}_{n+1} + \mathcal{W}_{n+1},$$

where, if  $\Delta M_{n+1} = M_{n+1} - M_n$ ,

$$\mathcal{A}_n = \sum_{k=2}^n M_k^t (\Sigma_{k-1}^{-1} - \Sigma_k^{-1}) M_k,$$

$$\mathcal{B}_{n+1} = 2 \sum_{k=2}^n M_k^t \Sigma_k^{-1} \Delta M_{k+1},$$

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# Strong law for the conditional variance

Denote

$$\sigma_n^2 = \frac{1}{2|\mathbb{T}_{n-1}|} \sum_{k \in \mathbb{T}_{n-1}} (\varepsilon_{2k}^2 + \varepsilon_{2k+1}^2).$$

## Theorem

Assume that  $(\varepsilon_n)$  satisfies (H.1), (H.2) and (H.3). Then,  $\hat{\sigma}_n^2$  converges a.s. to  $\sigma^2$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k \in \mathbb{T}_{n-1}} (\hat{\varepsilon}_{2k} - \varepsilon_{2k})^2 + (\hat{\varepsilon}_{2k+1} - \varepsilon_{2k+1})^2 = 4\sigma^2 \quad \text{a.s.}$$

$$\lim_{n \rightarrow \infty} \frac{|\mathbb{T}_n|}{n} (\hat{\sigma}_n^2 - \sigma_n^2) = 4\sigma^2 \quad \text{a.s.}$$

# Strong law for the conditional covariance

Denote

$$\rho_n = \frac{1}{|\mathbb{T}_{n-1}|} \sum_{k \in \mathbb{T}_{n-1}} \varepsilon_{2k} \varepsilon_{2k+1}.$$

## Theorem

Assume that  $(\varepsilon_n)$  satisfies (H.1), (H.2) and (H.3). Then,  $\hat{\rho}_n$  converges a.s. to  $\rho$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k \in \mathbb{T}_{n-1}} (\hat{\varepsilon}_{2k} - \varepsilon_{2k})(\hat{\varepsilon}_{2k+1} - \varepsilon_{2k+1}) = 2\rho \quad \text{a.s.}$$

$$\lim_{n \rightarrow \infty} \frac{|\mathbb{T}_n|}{n} (\hat{\rho}_n - \rho_n) = 4\rho \quad \text{a.s.}$$



# More assumptions

(H.4)  $\forall n \geq 0$  and  $\forall k \in \mathbb{G}_{n+1}$ ,

$$\mathbb{E}[\varepsilon_k^4 | \mathcal{F}_n] = \tau^4 \quad \text{a.s.}$$

Moreover,  $\forall k \neq \ell \in \mathbb{G}_{n+1}$  with  $[k/2] = [\ell/2]$  and for  $\nu^2 < \tau^4$

$$\mathbb{E}[\varepsilon_k^2 \varepsilon_\ell^2 | \mathcal{F}_n] = \nu^2 \quad \text{a.s.}$$

(H.5)

$$\sup_{n \geq 0} \sup_{k \in \mathbb{G}_{n+1}} \mathbb{E}[\varepsilon_k^8 | \mathcal{F}_n] < \infty \quad \text{a.s.}$$

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# Central limit theorems

## Theorem

If  $(\varepsilon_n)$  satisfies (H.1) to (H.5), we have

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# Central limit theorems

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!!!! Many thanks for your attention !!!!

