Exponential inequality for autoregressive processes in adaptive tracking

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Dedicated to Professor H. F. Chen on the occasion of his seventieth birthday

Outline

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   - Bennett's inequality
   - Bernstein's inequality
2. Exponential inequalities for martingales
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   - Freedman's inequality
   - De la Peña's inequality
3. Application to adaptive tracking
4. Two open problems

Hoeffding's inequality

Let (X_n) be a sequence of **independent** random variables and

$$S_n = \sum_{k=1}^{n} X_k.$$

Theorem (Hoeffding's inequality)

Assume that for each $1 \leqslant k \leqslant n$, $a_k \leqslant X_k \leqslant b_k$ a.s. for some constants $a_k < b_k$. Then, for all $x \geqslant 0$,

$$\mathbb{P}(|S_n - \mathbb{E}[S_n]| \geqslant x) \leqslant 2 \exp\Big(-\frac{2x^2}{\sum_{k=1}^{n}(b_k - a_k)^2}\Big).$$

Bennett's inequality

Theorem (Bennett's inequality)

Assume that (X_n) is square integrable and for each $1 \leqslant k \leqslant n$, $X_k \leqslant c$ a.s. for some constant $c > 0$. Then, for all $x \geqslant 0$,

$$\mathbb{P}(S_n - \mathbb{E}[S_n] \geqslant x) \leqslant \exp\Big(-\frac{V_n}{c^2} h\Big(\frac{xc}{V_n}\Big)\Big)$$

where V_n is the variance of S_n and

$$h(x) = (1+x)\log(1+x) - x.$$

Bernstein's inequality

For all $x \geqslant 0$,

$$h(x) \geqslant \frac{3x^2}{2(3+x)}.$$

Theorem (Bernstein's inequality)

Assume that (X_n) is square integrable and for each $1 \leqslant k \leqslant n$, $X_k \leqslant c$ a.s. for some constant $c > 0$. Then, for all $x \geqslant 0$,

$$\mathbb{P}(S_n - \mathbb{E}[S_n] \geqslant x) \leqslant \exp\Big(-\frac{x^2}{2(V_n + xc/3)}\Big).$$

Outline

Azuma-Hoeffding's inequality

Let (M_n) be a square integrable martingale adapted to $\mathbb{F} = (\mathcal{F}_n)$ with $M_0 = 0$. The **predictable** and the **total** quadratic variations of (M_n) are given by

$$< M >_n = \sum_{k=1}^{n} \mathbb{E}[\Delta M_k^2 | \mathcal{F}_{k-1}], \qquad [M]_n = \sum_{k=1}^{n} \Delta M_k^2$$

where $\Delta M_n = M_n - M_{n-1}$.

Theorem (Azuma-Hoeffding's inequality)

Assume that for each $1 \leqslant k \leqslant n$, $a_k \leqslant \Delta M_k \leqslant b_k$ a.s. for some constants $a_k < b_k$. Then, for all $x \geqslant 0$,

$$\mathbb{P}(|M_n| \geqslant x) \leqslant 2 \exp\left(- \frac{2x^2}{\sum_{k=1}^{n} (b_k - a_k)^2} \right).$$

Freedman's inequality

Theorem (Freedman's inequality)

Assume that for each $1 \leqslant k \leqslant n$, $\Delta M_k \leqslant c$ a.s. for some constant $c > 0$. Then, for all $x, y > 0$,

$$\mathbb{P}(M_n \geqslant x, <M>_n \leqslant y) \leqslant \exp\left(-\frac{x^2}{2(y+cx)}\right).$$

Theorem

Freedman's inequality also holds under the Bernstein moment condition: for all $n \geqslant 1$, $p \geqslant 2$ and for some constant $c > 0$,

$$\sum_{k=1}^{n} \mathbb{E}[|\Delta M_k|^p | \mathcal{F}_{k-1}] \leqslant \frac{p!}{2} c^{p-2} <M>_n \qquad a.s.$$

De la Peña's inequality

Definition. We shall say that (M_n) is **conditionally symmetric** if, for all $n \geqslant 1$, $\mathcal{L}(\Delta M_n | \mathcal{F}_{n-1})$ is symmetric.

Theorem (De la Peña's inequality)

If (M_n) is conditionally symmetric, then for all $x, y > 0$,

$$\mathbb{P}(M_n \geqslant x, [M]_n \leqslant y) \leqslant \exp\left(-\frac{x^2}{2y}\right).$$

Self-normalized martingales

Theorem (De la Peña's inequality)

If (M_n) is conditionally symmetric, then for all $x, y > 0$,

$$\mathbb{P}\Big(\frac{M_n}{[M]_n} \geqslant x\Big) \leqslant \sqrt{\mathbb{E}\Big[\exp\Big(-\frac{x^2}{2}[M]_n\Big)\Big]},$$

$$\mathbb{P}\Big(\frac{M_n}{[M]_n} \geqslant x, [M]_n \geqslant \frac{1}{y}\Big) \leqslant \exp\Big(-\frac{x^2}{2y}\Big).$$

Goal. Normalize by $<M>_n$ instead of $[M]_n$.

Self-normalized martingales

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Self-normalized martingales

Definition. We shall say that (M_n) is **conditionally Gaussian** if for all $n \geqslant 1$,

$$\mathcal{L}(\Delta M_n | \mathcal{F}_{n-1}) = \mathcal{N}(0, \Delta{<}M{>}_n)$$

where $\Delta{<}M{>}_n = {<}M{>}_n - {<}M{>}_{n-1}$.

Theorem (Bercu)

If (M_n) is conditionally Gaussian, then for all $x > 0$,

$$\mathbb{P}\Big(\frac{M_n}{{<}M{>}_n} \geqslant x\Big) \leqslant \inf_{p>1} \left(\mathbb{E}\Big[\exp\Big(-(p-1)\frac{x^2}{2}{<}M{>}_n\Big)\Big]\right)^{1/p}.$$

Outline

Autoregressive process

Consider the autoregressive process given, for all $n \geqslant 0$, by

$$X_{n+1} = \theta X_n + U_n + \varepsilon_{n+1}$$

- $X_n \longrightarrow$ the system output,
- $U_n \longrightarrow$ the adaptive control that can be chosen,
- $\varepsilon_n \longrightarrow$ the driven noise.

We assume that the noise (ε_n) is iid with $\mathcal{N}(0, \sigma^2)$ distribution.
Our goal is to

- Estimate the unknown parameter θ,
- Control the dynamic of the process (X_n).

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Estimation

We estimate the parameter θ by the **least squares** estimator

$$\widehat{\theta}_n = \frac{\sum_{k=1}^n X_{k-1}(X_k - U_{k-1})}{\sum_{k=1}^n X_{k-1}^2}.$$

$$\boldsymbol{\widehat{\theta}_n - \theta = \sigma^2 \frac{M_n}{<M>_n}}$$

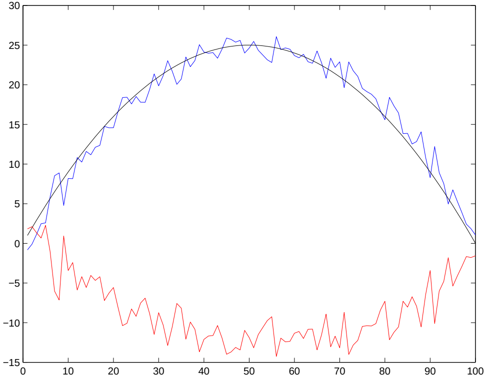
where

$$M_n = \sum_{k=1}^n X_{k-1}\varepsilon_k \qquad \text{and} \qquad <M>_n = \sigma^2 \sum_{k=1}^n X_{k-1}^2.$$

Adaptive Control

The role of U_n is to force X_n to track, step by step, a given reference trajectory (x_n). We use the **adaptive control**

$$U_n = x_{n+1} - \widehat{\theta}_n X_n.$$



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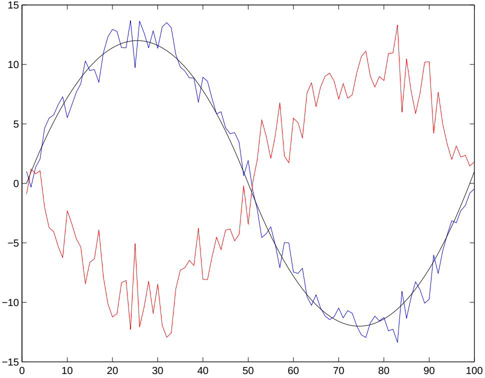

Exponential Inequality

Theorem (Bercu)

For all $n \geqslant 1$ and $x > 0$, we have

$$\mathbb{P}(|\widehat{\theta}_n - \theta| \geqslant x) \leqslant 2\exp\Big(-\frac{nx^2}{2(1+y_x)}\Big)$$

where y_x is the unique positive solution of the equation

$$(1+y)\log(1+y) - y = x^2.$$

Remark. For all $0 < x < 1/2$, $y_x < 2x$ so that

$$\mathbb{P}(|\widehat{\theta}_n - \theta| \geqslant x) \leqslant 2\exp\Big(-\frac{nx^2}{2(1+2x)}\Big).$$

Outline

Large Deviations

Consider the stable autoregressive process without control. Let

$$a = \frac{\theta - \sqrt{\theta^2 + 8}}{4} \qquad \text{and} \qquad b = \frac{\theta + \sqrt{\theta^2 + 8}}{4}.$$

Theorem (Bercu-Gamboa-Rouault)

The sequence $(\widehat{\theta}_n)$ satisfies an **LDP** *with rate function*

$$I(x) = \begin{cases} \frac{1}{2} \log \left(\frac{1 + \theta^2 - 2\theta x}{1 - x^2} \right) & \textit{if } \ x \in [a, b], \\ \log |\theta - 2x| & \textit{otherwise.} \end{cases}$$

In particular, for all $x > \theta$, $\displaystyle \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\widehat{\theta}_n \geqslant x) = -I(x).$

Goal. Investigate the **LDP** for $(\widehat{\theta}_n)$ in adpative tracking.

Moderate Deviations

Goal. Let (a_n) be a sequence of positive numbers increasing to infinity such that $a_n = o(n)$. Denote

$$\boldsymbol{V_n = \sqrt{\frac{n}{a_n}}(\widehat{\theta}_n - \theta).}$$

One can conjecture that (V_n) satisfies an **LDP** with speed a_n and rate function $I(x) = x^2/2$.