Sharp large deviations for Gaussian quadratic forms

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Outline

Strong law and central limit theorem

Let (X_n) be a sequence of iid random variables and set

$$S_n = \sum_{k=1}^{n} X_k.$$

If $X_n \in \mathbb{L}^2(\mathbb{R})$ with $\mathbb{E}[X_n] = m$ and $\text{Var}(X_n) = \sigma^2$, we have

$$\frac{S_n}{n} \longrightarrow m \qquad \text{a.s.}$$

$$\frac{S_n - nm}{\sqrt{n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2)$$

Idea. Require more on (X_n) to obtain sharp results on tails

The Gaussian sample

Assume that $X_n \sim \mathcal{N}(0, \sigma^2)$ so that $S_n \sim \mathcal{N}(0, \sigma^2 n)$. For all $c > 0$,

$$\mathbb{P}(S_n \geqslant nc) \sim \frac{\sigma}{c\sqrt{2\pi n}} \exp\Big(-\frac{c^2 n}{2\sigma^2}\Big).$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \geqslant nc) = -\frac{c^2}{2\sigma^2}.$$

Question. Is this limit true in the non Gaussian case ?

Outline

Let L be the log-Laplace of (X_n). The Fenchel-Legendre dual of L is

$$I(c) = \sup_{t \in \mathbb{R}} \{ct - L(t)\}.$$

Theorem (Cramer-Chernov)

The sequence (S_n/n) satisfies an **LDP** *with rate function I*

- **Upper bound:** *for any closed set $F \subset \mathbb{R}$*

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}\Big(\frac{S_n}{n} \in F\Big) \leqslant -\inf_F I,$$

- **Lower bound:** *for any open set $G \subset \mathbb{R}$*

$$\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}\Big(\frac{S_n}{n} \in G\Big) \geqslant -\inf_G I.$$

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On the Cramer-Chernov theorem

The rate function I is convex with $I(m) = 0$. Therefore, for all $c > m$,

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(S_n \geqslant nc) = -I(c).$$

- **Gaussian:** If $X_n \sim \mathcal{N}(0, \sigma^2)$ with $\sigma^2 > 0$,

$$I(c) = \frac{c^2}{2\sigma^2}.$$

- **Exponential:** If $X_n \sim \mathcal{E}(\lambda)$ with $\lambda > 0$,

$$I(c) = \begin{cases} \lambda c - 1 - \log(\lambda c) & \text{if } \ c > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Outline

On the Bahadur-Rao theorem

Theorem (Bahadur-Rao)

Assume that L is finite on all $\mathbb{R}$ and that the law of (X_n) is absolutely continuous. Then, for all $c > m$,

$$\mathbb{P}(S_n \geqslant nc) = \frac{\exp(-nI(c))}{\sigma_c t_c \sqrt{2\pi n}} \Big[1 + o(1)\Big]$$

where t_c is given by $L'(t_c) = c$ and $\sigma_c^2 = L''(t_c)$.

Remark. The core of the proof is the Berry-Esséen theorem.

Theorem (Bahadur-Rao)

(S_n/n) satisfies an **SLDP** *associated with L. For all $c > m$, it exists $(d_{c,k})$ such that for any $p \geqslant 1$ and n large enough*

$$\mathbb{P}(S_n \geqslant nc) = \frac{\exp(-nI(c))}{\sigma_c t_c \sqrt{2\pi n}} \left[1 + \sum_{k=1}^{p} \frac{d_{c,k}}{n^k} + \mathcal{O}\left(\frac{1}{n^{p+1}}\right) \right].$$

Remark. The coefficients $(d_{c,k})$ may be explicitly given as functions of the derivatives $l_k = L^{(k)}(t_c)$. For example,

$$d_{c,1} = \frac{1}{\sigma_c^2} \left(\frac{l_4}{8\sigma_c^2} - \frac{5 l_3^2}{24\sigma_c^4} - \frac{l_3}{2 t_c \sigma_c^2} - \frac{1}{t_c^2} \right).$$

Outline

Gaussian quadratic forms

Let (X_n) be a centered stationary real Gaussian process with spectral density $g \in \mathbb{L}^\infty(\mathbb{T})$

$$\mathbb{E}[X_n X_k] = \frac{1}{2\pi} \int_{\mathbb{T}} \exp(i(n-k)x) g(x)\, dx.$$

We are interested in the behavior of

$$\boldsymbol{W_n = \frac{1}{n} X^{(n)t} M_n X^{(n)}}$$

where (M_n) is a sequence of Hermitian matrices of order n and

$$X^{(n)} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}.$$

Toeplitz and Cochran

Let $T_n(g)$ be the covariance matrix of $X^{(n)}$. By the Cochran theorem,

$$W_n = \frac{1}{n} \sum_{k=1}^{n} \lambda_k^n Z_k^n$$

- $\lambda_1^n, \ldots, \lambda_n^n$ are the eigenvalues of $T_n(g)^{1/2} M_n T_n(g)^{1/2}$,
- $Z_1^n, \ldots, Z_n^n$ are iid with $\chi^2(1)$ distribution.

The normalized cumulant generating function of W_n is

$$L_n(t) = \frac{1}{n} \log \mathbb{E}\Big[\exp(ntW_n) \Big] = -\frac{1}{2n} \sum_{k=1}^{n} \log(1 - 2t\lambda_k^n)$$

as soon as $t \in \Delta_n = \left\{ t \in \mathbb{R} \;/\; 2t\lambda_k^n < 1 \right\}$.

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Outline

LDP assumption. There exists $\varphi \in \mathbb{L}^\infty(\mathbb{T})$ not identically zero such that, if $m_\varphi = \text{essinf } \varphi$ and $M_\varphi = \text{esssup } \varphi$,

(H_1)
$$m_\varphi \leqslant \lambda_k^n \leqslant M_\varphi$$

and, for all $h \in \mathbb{C}([m_\varphi, M_\varphi])$,

(H_1)
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} h(\lambda_k^n) = \frac{1}{2\pi} \int_{\mathbb{T}} h(\varphi(x))dx.$$

Under (H_1), the asymptotic cumulant generating function is

$$L(t) = -\frac{1}{4\pi} \int_{\mathbb{T}} \log(1 - 2t\varphi(x))dx$$

where $t \in \Delta = \{t \in \mathbb{R} \;/\; 2\max(m_\varphi t, M_\varphi t) < 1\}$.

Large deviation principle

The Fenchel-Legendre dual of L is

$$I(c) = \sup_{t \in \mathbb{R}} \left\{ ct + \frac{1}{4\pi} \int_{\mathbb{T}} \log(1 - 2t\varphi(x))dx \right\}.$$

Theorem (Bercu-Gamboa-Lavielle)

*If (H_1) holds, the sequence (W_n) satisfies an **LDP** with rate function I. In particular, for all $c > \mu$*

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(W_n \geqslant c) = -I(c).$$

with $\mu = \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{T}} \varphi(x)dx.$

Outline

Sharp large deviation results

SLDP assumption. There exists H such that, for all $t \in \Delta$

$$(H_2) \qquad L_n(t) = L(t) + \frac{1}{n}H(t) + o\Big(\frac{1}{n}\Big)$$

where the remainder is uniform in t.

Theorem (Bercu-Gamboa-Lavielle)

Assume that (H_1) and (H_2) hold. Then, for all $c > \mu$

$$\mathbb{P}(W_n \geqslant c) = \frac{\exp(-nI(c) + H(t_c))}{\sigma_c t_c \sqrt{2\pi n}} \Big[1 + o(1)\Big]$$

where t_c is given by $L'(t_c) = c$ and $\sigma_c^2 = L''(t_c)$.

Sharp large deviation results

SLDP assumption. For $p \geqslant 1$, there exists $H \in \mathcal{C}^{2p+3}(\mathbb{R})$ such that, for all $t \in \Delta$ and for any $0 \leqslant k \leqslant 2p+3$

$$(H_2(p)) \qquad L_n^{(k)}(t) = L^{(k)}(t) + \frac{1}{n} H^{(k)}(t) + \mathcal{O}\left(\frac{1}{n^{p+2}}\right)$$

where the remainder is uniform in t.

Remark. Assumption $(H_2(p))$ is not really restrictive. It is fulfilled in many statistical applications.

Theorem (Bercu-Gamboa-Lavielle)

For $p \geqslant 1$, assume that (H_1) and $(H_2(p))$ hold. Then, (W_n) satisfies an **SLDP** *of order p associated with L and H. For all $c > \mu$, it exists $(d_{c,k})$ such that for n large enough*

$$\mathbb{P}(W_n \geqslant c)$$

$$= \frac{\exp(-nI(c) + H(t_c))}{\sigma_c t_c \sqrt{2\pi n}} \left[1 + \sum_{k=1}^{p} \frac{d_{c,k}}{n^k} + \mathcal{O}\left(\frac{1}{n^{p+1}} \right) \right].$$

Remark. The coefficients $(d_{c,k})$ may be given as functions of the derivatives $l_k = L^{(k)}(t_c)$ and $h_k = H^{(k)}(t_c)$. For example,

$$d_{c,1} = \frac{1}{\sigma_c^2} \left(-\frac{h_2}{2} - \frac{h_1^2}{2} + \frac{l_4}{8\sigma_c^2} + \frac{l_3 h_1}{2\sigma_c^2} - \frac{5 l_3^2}{24 \sigma_c^4} + \frac{h_1}{t_c} - \frac{l_3}{2 t_c \sigma_c^2} - \frac{1}{t_c^2} \right).$$

Outline

Sum of squares

Our first statistical application is on the sum of squares

$$W_n = \frac{1}{n}\sum_{k=1}^{n} X_k^2.$$

We recall that $g \in \mathbb{L}^\infty(\mathbb{T})$ is the spectral density of the centered stationary real Gaussian process (X_n).

Theorem (Bryc-Dembo)

The sequence (W_n) satisfies an **LDP** *with rate function I which is the Fenchel-Legendre dual of*

$$L(t) = -\frac{1}{4\pi}\int_{\mathbb{T}} \log(1 - 2tg(x))dx.$$

Sum of squares

We shall often make use of the integral

$$\Delta(f) = \frac{1}{4\pi^2} \iint_{\mathbb{T}} \left(\frac{\log f(x) - \log f(y)}{\sin((x-y)/2)} \right)^2 dxdy.$$

Theorem (Bercu-Gamboa-Lavielle)

Assume that $g > 0$ on $\mathbb{T}$ and g admits an analytic extension on the annulus $A_r = \{z \in \mathbb{C} \ / \ r < |z| < r^{-1}\}$ with $0 < r < 1$. Then, (W_n) satisfies an **SLDP** *associated with L and H where*

$$H(t) = \frac{1}{2}\Delta(1 - 2tg).$$

Outline

Neyman-Pearson

Let $g_0, g_1 \in \mathbb{L}^1(\mathbb{T})$ be two spectral densities. We wish to test

$$H_0 : \langle\langle\, g = g_0 \,\rangle\rangle \qquad \textbf{against} \qquad H_1 : \langle\langle\, g = g_1 \,\rangle\rangle.$$

For this simple hypothesis, the most powerful test is based on

$$W_n = \frac{1}{n} X^{(n)t} \Big[T_n^{-1}(g_0) - T_n^{-1}(g_1) \Big] X^{(n)}.$$

LRT assumption.

$$(H_3) \qquad \log g_0 \in \mathbb{L}^1(\mathbb{T}) \qquad \text{and} \qquad \frac{g_0}{g_1} \in \mathbb{L}^\infty(\mathbb{T})$$

Theorem (Bercu-Gamboa-Lavielle)

- *If (H_3) holds, then under H_0, (W_n) satisfies an* **LDP** *with rate function I which is the Fenchel-Legendre dual of*

$$L(t) = -\frac{1}{4\pi}\int_{\mathbb{T}} \log\Big(\frac{(1-2t)g_1(x) + 2tg_0(x)}{g_1(x)}\Big)dx.$$

- *Assume that $g_0, g_1 > 0$ on $\mathbb{T}$ and g_0, g_1 admit an analytic extensions on the annulus $A_r = \{z \in \mathbb{C} \ / \ r < |z| < r^{-1}\}$ with $0 < r < 1$. Then, under H_0, (W_n) satisfies an* **SLDP** *associated with L and H where*

$$H(t) = \frac{1}{2}\Big(\Delta(g_1) - \Delta((1-2t)g_1 + 2tg_0)\Big).$$

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$$H(t) = \frac{1}{2}\Big(\Delta(g_1) - \Delta((1-2t)g_1 + 2tg_0)\Big).$$

One can explicitly calculate L and H. For example, consider the autoregressive process of order p. For all $x \in \mathbb{T}$

$$g(x) = \frac{\sigma^2}{|A(e^{ix})|^2}$$

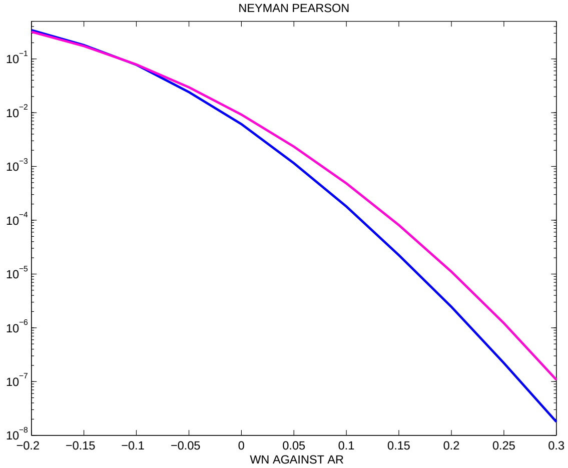
where A is a polynomial given by

$$A(e^{ix}) = \prod_{j=1}^{p}(1 - a_j e^{ix})$$

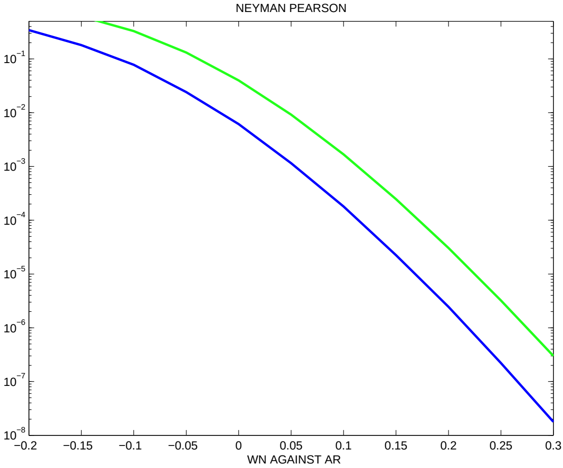
with $|a_j| < 1$. Then, we have $\displaystyle\frac{1}{2\pi}\int_{\mathbb{T}} \log g(x)dx = \log \sigma^2$ and

$$\Delta(g) = -\sum_{j=1}^{p}\sum_{k=1}^{p} \log(1 - a_j \overline{a}_k).$$

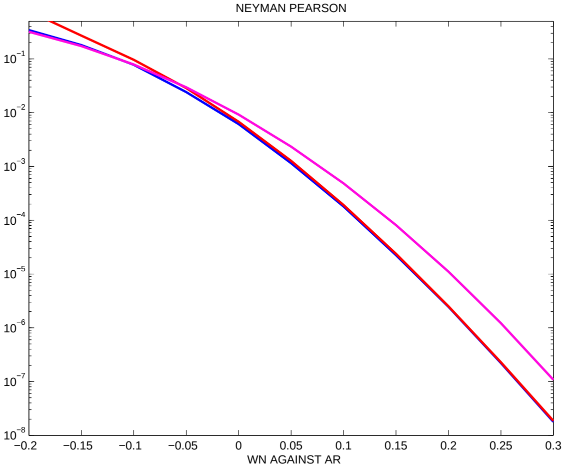
Neyman-Pearson



Neyman-Pearson



Neyman-Pearson



Outline

Stable autoregressive process

Consider the stable autoregressive process

$$X_{n+1} = \theta X_n + \varepsilon_{n+1}, \qquad |\theta| < 1$$

where (ε_n) is iid $\mathcal{N}(0, \sigma^2)$, $\sigma^2 > 0$. If X_0 is independent of (ε_n) with $\mathcal{N}(0, \sigma^2/(1-\theta^2))$ distribution, (X_n) is a centered stationary Gaussian process. For all $x \in \mathbb{T}$

$$g(x) = \frac{\sigma^2}{1 + \theta^2 - 2\theta \cos x}.$$

Let $\widehat{\theta}_n$ be the **least squares** estimator of the parameter θ

$$\widehat{\theta}_n = \frac{\displaystyle\sum_{k=1}^{n} X_k X_{k-1}}{\displaystyle\sum_{k=1}^{n} X_{k-1}^2}.$$

We have $\widehat{\theta}_n \longrightarrow \theta$ a.s. and $\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1-\theta^2)$. One can also estimate θ by the **Yule-Walker** estimator

$$\widetilde{\theta}_n = \frac{\displaystyle\sum_{k=1}^{n} X_k X_{k-1}}{\displaystyle\sum_{k=0}^{n} X_k^2}.$$

$$a = \frac{\theta - \sqrt{\theta^2 + 8}}{4} \qquad \text{and} \qquad b = \frac{\theta + \sqrt{\theta^2 + 8}}{4}.$$

Theorem (Bercu-Gamboa-Rouault)

- $(\widehat{\theta}_n)$ *satisfies an* **LDP** *with rate function*

$$J(x) = \begin{cases} \frac{1}{2} \log \left(\frac{1 + \theta^2 - 2\theta x}{1 - x^2} \right) & \text{if } x \in [a, b], \\ \log |\theta - 2x| & \text{otherwise.} \end{cases}$$

- $(\widetilde{\theta}_n)$ *satisfies an* **LDP** *with rate function*

$$I(x) = \begin{cases} \frac{1}{2} \log \left(\frac{1 + \theta^2 - 2\theta x}{1 - x^2} \right) & \text{if } x \in]-1, 1[, \\ +\infty & \text{otherwise.} \end{cases}$$

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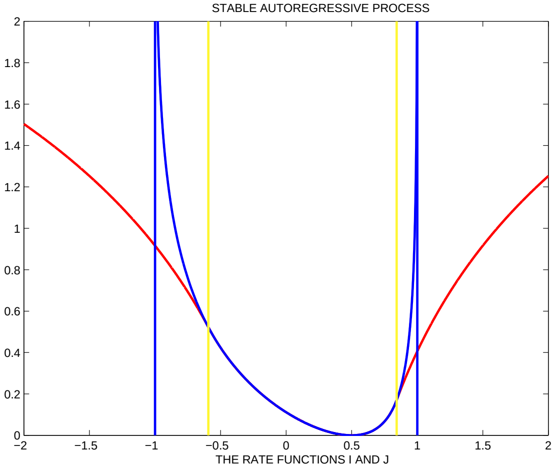
Least squares and Yule-Walker



Yule-Walker

Theorem (Bercu-Gamboa-Lavielle)

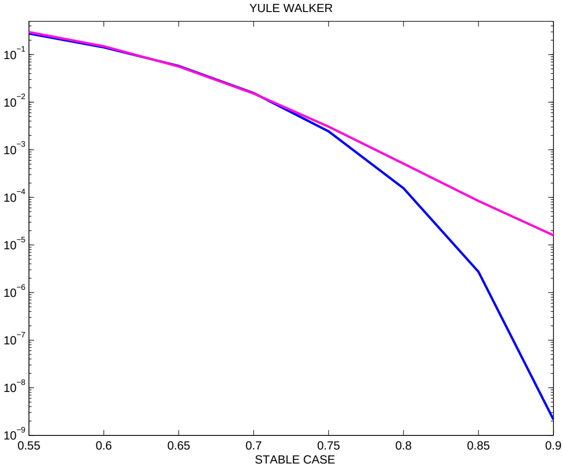
The sequence $(\widetilde{\theta}_n)$ satisfies an **SLDP**. *For all $c \in \mathbb{R}$ with $c > \theta$ and $|c| < 1$, it exists a sequence $(d_{c,k})$ such that for any $p \geqslant 1$ and n large enough*

$$\mathbb{P}(\widetilde{\theta}_n \geqslant c) = \frac{\exp(-nI(c) + H(c))}{\sigma_c t_c \sqrt{2\pi n}} \left[1 + \sum_{k=1}^{p} \frac{d_{c,k}}{n^k} + \mathcal{O}\left(\frac{1}{n^{p+1}}\right) \right]$$

$$t_c = \frac{c(1+\theta^2) - \theta(1-c^2)}{1-c^2}, \qquad \sigma_c^2 = \frac{1-c^2}{(1+\theta^2-2\theta c)^2},$$

$$H(c) = -\frac{1}{2} \log\left(\frac{(1-c\theta)^4}{(1-\theta)^2(1+\theta^2-2\theta c)(1-c^2)^2} \right).$$

Yule-Walker


YULE WALKER
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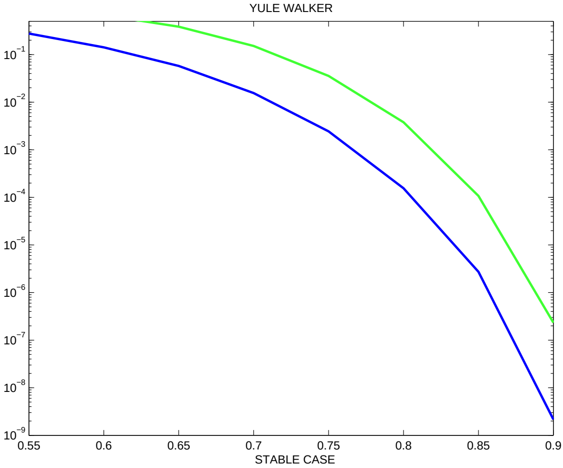
STABLE CASE

Yule-Walker


YULE WALKER
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STABLE CASE

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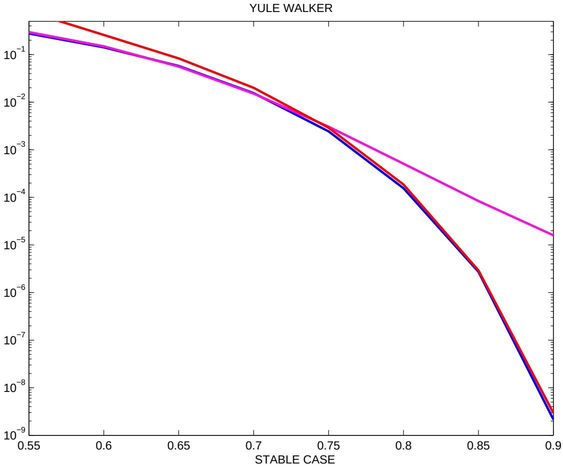
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STABLE CASE

Explosive autoregressive process

Consider the explosive autoregressive process

$$X_{n+1} = \theta X_n + \varepsilon_{n+1}, \qquad |\theta| > 1$$

where (ε_n) is iid $\mathcal{N}(0, \sigma^2)$, $\sigma^2 > 0$. The **Yule-Walker** estimator satisfies $\widetilde{\theta}_n \longrightarrow 1/\theta$ a.s. together with

$$|\theta|^n \Big(\widetilde{\theta}_n - \frac{1}{\theta} \Big) \xrightarrow{\mathcal{L}} \frac{(\theta^2 - 1)}{\theta^2} \, \mathcal{C}$$

where $\mathcal{C}$ stands for the Cauchy distribution.

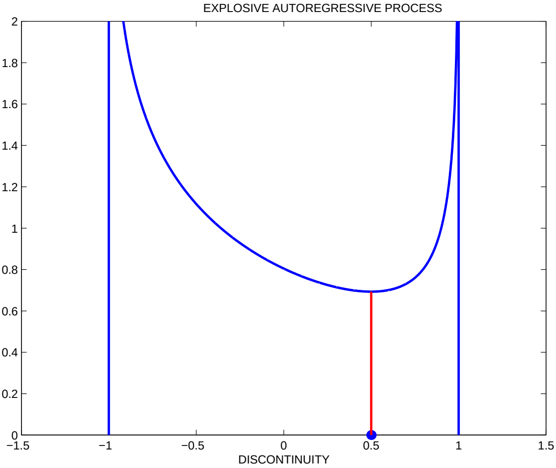
Explosive autoregressive process

Theorem (Bercu)

The sequence $(\widetilde{\theta}_n)$ *satisfies an* **LDP** *with rate function*

$$I(x) = \begin{cases} \frac{1}{2}\log\left(\frac{1+\theta^2-2\theta x}{1-x^2}\right) & \text{if } x \in]-1,1[, \ x \neq 1/\theta, \\ 0 & \text{if } x = 1/\theta, \\ +\infty & \text{otherwise.} \end{cases}$$

Discontinuity



Yule-Walker

Theorem (Bercu)

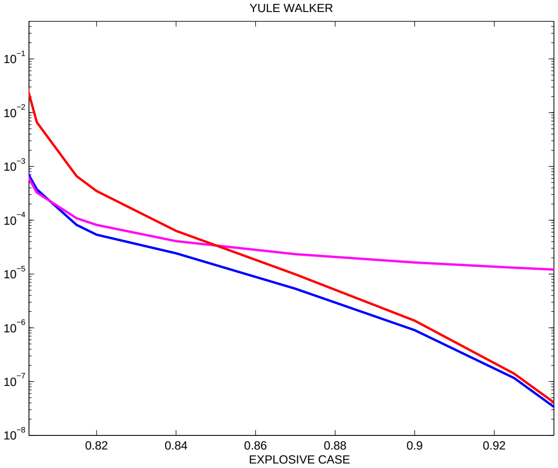
The sequence $(\widetilde{\theta}_n)$ *satisfies an* **SLDP**. *For all* $c \in \mathbb{R}$ *with* $c > 1/\theta$ *and* $|c| < 1$, *it exists a sequence* $(d_{c,k})$ *such that for any* $p \geqslant 1$ *and* n *large enough*

$$\mathbb{P}(\widetilde{\theta}_n \geqslant c) = \frac{\exp(-nI(c) + H(c))}{\sigma_c t_c \sqrt{2\pi n}} \left[1 + \sum_{k=1}^{p} \frac{d_{c,k}}{n^k} + \mathcal{O}\left(\frac{1}{n^{p+1}}\right) \right]$$

$$t_c = \frac{(\theta c - 1)(\theta - c)}{1 - c^2}, \qquad \sigma_c^2 = \frac{1 - c^2}{(1 + \theta^2 - 2\theta c)^2},$$

$$H(c) = -\frac{1}{2} \log \left(\frac{(\theta c - 1)^2}{(1 + \theta^2 - 2\theta c)(1 - c^2)} \right).$$

Yule-Walker


YULE WALKER
10^−1
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EXPLOSIVE CASE

Outline

Consider the stable Ornstein-Uhlenbeck process

$$dX_t = \theta X_t dt + dW_t, \qquad \theta < 0$$

where (W_t) is a standard Brownian motion. We establish similar **SLDP** for **the energy**

$$S_T = \int_0^T X_t^2 \, dt,$$

the **maximum likelihood** estimator of θ

$$\widehat{\theta}_T = \frac{\int_0^T X_t \, dX_t}{\int_0^T X_t^2 \, dt} = \frac{X_T^2 - T}{2 \int_0^T X_t^2 \, dt},$$

and the **log-likelihood ratio** given, for $\theta_0, \theta_1 < 0$, by

$$W_T = (\theta_0 - \theta_1) \int_0^T X_t \, dX_t - \frac{1}{2}(\theta_0^2 - \theta_1^2) \int_0^T X_t^2 \, dt.$$

Stable Ornstein-Uhlenbeck process

Theorem (Bryc-Dembo)

The sequence (S_T/T) *satisfies an* **LDP** *with rate function*

$$J(x) = \begin{cases} \dfrac{(2\theta c + 1)^2}{8c} & \text{if } c > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem (Florens-Pham)

The sequence $(\widehat{\theta}_T)$ *satisfies an* **LDP** *with rate function*

$$I(x) = \begin{cases} -\dfrac{(c - \theta)^2}{4c} & \text{if } c < \theta/3, \\ 2c - \theta & \text{otherwise.} \end{cases}$$

Stable Ornstein-Uhlenbeck process

Theorem (Bercu-Rouault)

The sequence $(\widehat{\theta}_T)$ satisfies an **SLDP**. *For all $\theta < c < \theta/3$, it exists a sequence $(d_{c,k})$ such that, for any $p \geqslant 1$ and T large enough*

$$\mathbb{P}(\widehat{\theta}_T \geqslant c) = \frac{\exp(-TI(c) + H(c))}{\sigma_c t_c \sqrt{2\pi T}} \left[1 + \sum_{k=1}^{p} \frac{d_{c,k}}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right]$$

$$t_c = \frac{c^2 - \theta^2}{2c}, \qquad H(c) = -\frac{1}{2}\log\left(\frac{(c+\theta)(3c-\theta)}{4c^2}\right),$$

$\sigma_c^2 = -1/2c$. *Similar expansion holds for $c > \theta/3$ with $c \neq 0$.*

Stable Ornstein-Uhlenbeck process

Theorem (Bercu-Rouault)

- *For $c = 0$, it exists a sequence (b_k) such that, for any $p \geqslant 1$ and T large enough*

$$\mathbb{P}(\widehat{\theta}_T \geqslant 0) = \frac{\exp(\theta T)}{\sqrt{\pi T}\sqrt{-\theta}}\left[1 + \sum_{k=1}^{p} \frac{b_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right].$$

- *For $c = \theta/3$, it exists a sequence (d_k) such that, for any $p \geqslant 1$ and T large enough*

$$\mathbb{P}(\widehat{\theta}_T \geqslant c) = \frac{\exp(-TI(c))}{4\pi T^{1/4}\tau_\theta}\left[1 + \sum_{k=1}^{2p} \frac{d_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right)\right]$$

where $\tau_\theta = (-\theta/3)^{1/4}/\Gamma(1/4)$.

Stable Ornstein-Uhlenbeck process

Theorem (Bercu-Rouault)

- *For* $c = 0$*, it exists a sequence* (b_k) *such that, for any* $p \geqslant 1$ *and* T *large enough*

$$\mathbb{P}(\widehat{\theta}_T \geqslant 0) = \frac{\exp(\theta T)}{\sqrt{\pi T}\sqrt{-\theta}}\left[1 + \sum_{k=1}^{p} \frac{b_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right].$$

- *For* $c = \theta/3$*, it exists a sequence* (d_k) *such that, for any* $p \geqslant 1$ *and* T *large enough*

$$\mathbb{P}(\widehat{\theta}_T \geqslant c) = \frac{\exp(-TI(c))}{4\pi T^{1/4}\tau_\theta}\left[1 + \sum_{k=1}^{2p} \frac{d_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right)\right]$$

where $\tau_\theta = (-\theta/3)^{1/4}/\Gamma(1/4)$*.*