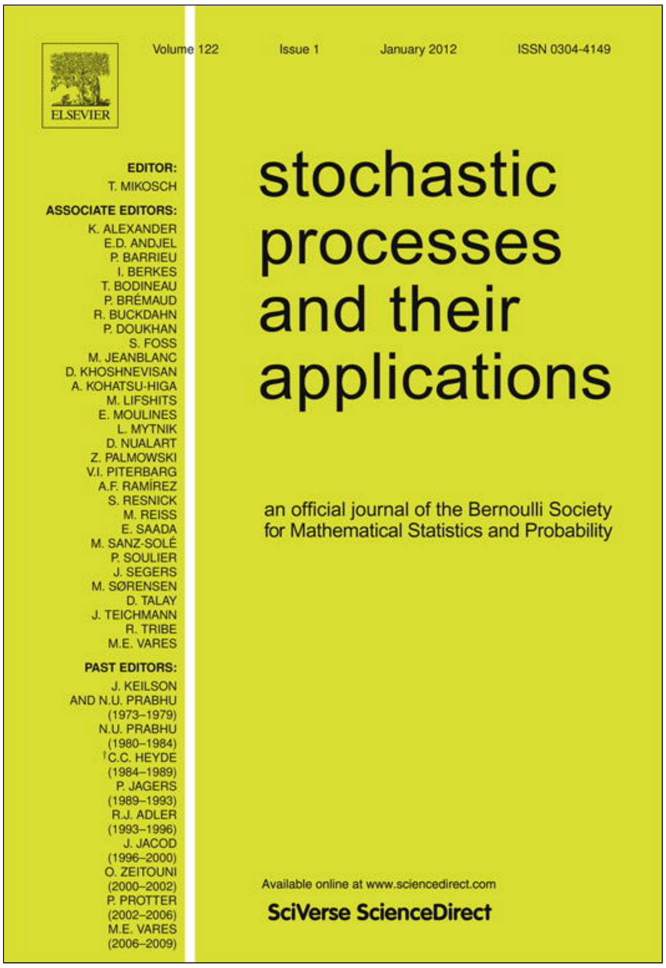
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Fluctuations of interacting Markov chain Monte Carlo methods

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Abstract

We present a multivariate central limit theorem for a general class of interacting Markov chain Monte Carlo algorithms used to solve nonlinear measure-valued equations. These algorithms generate stochastic processes which belong to the class of nonlinear Markov chains interacting with their empirical occupation measures. We develop an original theoretical analysis based on resolvent operators and semigroup techniques to analyze the fluctuations of their occupation measures around their limiting values.

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Keywords: Multivariate central limit theorems; Random fields; Martingale limit theorems; Self-interacting Markov chains; Markov chain Monte Carlo algorithms

1. Introduction

1.1. Nonlinear measure-valued equations

Let $(S^{(l)}, \mathcal{S}^{(l)})_{l \geq 0}$ be a sequence of measurable spaces. For any $l \geq 0$ we denote by $\mathcal{P}(S^{(l)})$ the set of probability measures on $S^{(l)}$. Suppose we have a sequence of probability measures

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$\pi^{(l)} \in \mathcal{P}(S^{(l)})$ such that, for any $l \geq 1$, $\pi^{(l)}$ satisfies the following nonlinear measure-valued equation

$$\Phi^{(l)}(\pi^{(l-1)}) = \pi^{(l)} \tag{1.1}$$

for some mappings $\Phi^{(l)} : \mathcal{P}(S^{(l-1)}) \to \mathcal{P}(S^{(l)})$. We will also use the convention $\Phi^{(0)}(\pi^{(-1)}) = \pi^{(0)}$.

In numerous scenarios, the probability measures $(\pi^{(l)})_{l\geq 0}$ need to be approximated numerically. Interacting particle methods have been previously proposed to approximate these probability distributions [10,8]. However they suffer from several limitations detailed in [7,9]. To bypass some of these limitations, an alternative class of algorithms known as interacting Markov chain Monte Carlo (i-MCMC) methods has been recently introduced in [7,9]. The main objective of this article is to present a multivariate Central Limit Theorem (CLT) for these methods. This extends significantly our previous result established in [5] which only applies to a restricted class of i-MCMC algorithms.

Before describing two general class of models where i-MCMC methods can be used, we introduce the notation adopted in this paper.

1.2. Notation and conventions

We denote respectively by $\mathcal{M}(E)$, $\mathcal{M}_0(E)$, $\mathcal{P}(E)$, and $\mathcal{B}(E)$, the set of all finite signed measures on some measurable space $(E, \mathcal{E})$ equipped with some σ-field $\mathcal{E}$, the convex subset of measures with null mass, the subset of all probability measures, and finally the Banach space of all bounded and measurable functions f on E equipped with the uniform norm $\|f\| = \sup_{x\in E} |f(x)|$ and the Borel σ-field associated to the supremum norm. We also denote by $\mathcal{B}_1(E) \subset \mathcal{B}(E)$ the unit ball of functions $f \in \mathcal{B}(E)$ with $\|f\| \leq 1$, and by $\mathrm{Osc}_1(E)$, the convex set of $\mathcal{E}$-measurable functions f with oscillations less than one; that is,

$$\mathrm{osc}(f) = \sup\{|f(x) - f(y)|; x, y \in E\} \leq 1.$$

We let $\mu(f) = \int \mu(dx) f(x)$ be the Lebesgue integral of a function $f \in \mathcal{B}(E)$ with respect to a measure $\mu \in \mathcal{M}(E)$. We slightly abuse the notation, and sometimes we denote by $\mu(A) = \mu(1_A)$ the measure of a measurable subset $A \in \mathcal{E}$.

We recall that a bounded integral operator M from a measurable space $(E, \mathcal{E})$ into an auxiliary measurable space $(F, \mathcal{F})$ is an operator $f \mapsto M(f)$ from $\mathcal{B}(F)$ into $\mathcal{B}(E)$ so that the functions

$$M(f)(x) = \int_F M(x, dy) f(y) \in \mathbb{R}$$

are $\mathcal{E}$-measurable and bounded for any $f \in \mathcal{B}(F)$. By Fubini's theorem, we recall that a bounded integral operator M from a measurable space $(E, \mathcal{E})$ into an auxiliary measurable space $(F, \mathcal{F})$ also generates a dual operator $\mu \mapsto \mu M$ from $\mathcal{M}(E)$ into $\mathcal{M}(F)$ defined by $(\mu M)(f) := \mu(M(f))$.

We denote by $\|M\| := \sup_{f\in\mathcal{B}_1(E)} \|M(f)\|$ the norm of the operator $f \mapsto M(f)$ and we equip the Banach space $\mathcal{M}(E)$ with the corresponding total variation norm $\|\mu\| = \sup_{f\in\mathcal{B}_1(E)} |\mu(f)|$. We let $\beta(M)$ be the Dobrushin coefficient of a bounded integral operator M defined by the following formula

$$\beta(M) := \sup\{\mathrm{osc}(M(f));\ f \in \mathrm{Osc}_1(F)\}.$$

When M has a constant mass, that is $M(1)(x) = M(1)(y)$ for any $(x, y) \in E^2$, the operator $\mu \mapsto \mu M$ maps $\mathcal{M}_0(E)$ into $\mathcal{M}_0(F)$ and $\beta(M)$ coincides with the norm of this operator.

We also denote by $(M^k)_{k\geq 0}$ the semigroup associated to M given by the recursive formulae $M^k(x,dz) = \int M^{k-1}(x,dy)M(y,dz)$, for $k \geq 1$ and $M^0 = Id$ the identity transition.

For any sequence of finite signed measures $(\mu_i)_{i\geq 0}$ defined on some collection of measurable spaces $(E_i, \mathcal{E}_i)_{i\geq 0}$, we define $(\mu_i \otimes \mu_j)(dx_i, dx_j) = \mu_i(dx_i)\mu_j(dx_j)$, $\mu_i{}^{\otimes 2}(dx_i, dx'_i) = (\mu_i \otimes \mu_i)(dx_i, dx'_i)$ and

$$\otimes_{i\leq k\leq j}\, \mu_k\ (dx_i, dx_{i+1}, \ldots, dx_j) = \prod_{k=i}^{j} \mu_k(dx_k).$$

For two bounded measurable functions f and g defined respectively on $(E, \mathcal{E})$ and $(F, \mathcal{F})$ we define $(f \otimes g)(x, x') = f(x)g(x')$.

We equip the set of distribution flows $\mathcal{M}(E)^{\mathbb{N}}$ with the uniform total variation distance defined by

$$\forall \eta = (\eta_n)_{n\geq 0}, \quad \mu = (\mu_n)_{n\geq 0} \in \mathcal{M}(E)^{\mathbb{N}} \qquad \|\eta - \mu\| := \sup_{n\geq 0} \|\eta_n - \mu_n\|.$$

We extend a given bounded integral operator $\mu \in \mathcal{M}(E) \mapsto \mu M \in \mathcal{M}(F)$ into an mapping

$$\eta = (\eta_n)_{n\geq 0} \in \mathcal{M}(E)^{\mathbb{N}} \mapsto \eta D = (\eta_n M)_{n\geq 0} \in \mathcal{M}(F)^{\mathbb{N}}.$$

Sometimes, we abuse the notation and denote the constant distribution flow by ν instead of $(\nu)_{n\geq 0}$ equal to a given measure $\nu \in \mathcal{P}(E)$.

For any $\mathbb{R}^d$-valued function $f = (f^i)_{1\leq i\leq d} \in \mathcal{B}(F)^d$, any integral operator M from E into F, and any $\mu \in \mathcal{M}(F)$, we will slightly abuse the notation, and we write $M(f)$ and $\mu(f)$ the $\mathbb{R}^d$-valued function and the point in $\mathbb{R}^d$ given respectively by

$$M(f) := (M(f^1), \ldots, M(f^d)) \quad \text{and} \quad \mu(f) := (\mu(f^1), \ldots, \mu(f^d)).$$

We also simplify the notation and sometimes we write

$$M[(f^1 - M(f^1))(f^2 - M(f^2))](x)$$

instead of

$$M[(f^1 - M(f^1)(x))(f^2 - M(f^2)(x))](x) = M(f^1 f^2)(x) - M(f^1)(x)M(f^2)(x).$$

Unless otherwise stated, we denote by $c(k)$, $k \in \mathbb{N}$, a constant whose value may vary from line to line but only depends on the parameter k. Finally, we shall use $\sum_\emptyset = 0$ and $\prod_\emptyset = 1$.

1.3. Examples

We give here two classes of models where i-MCMC methods can be used.

Feynman–Kac models. In this context, we have

$$\forall l \geq 0\ \forall(\mu, f) \in (\mathcal{P}(S^{(l)}) \times \mathcal{B}(S^{(l+1)})) \quad \Phi^{(l+1)}(\mu)(f) := \mu(G_l L_{l+1}(f))/\mu(G_l) \quad (1.2)$$

where L_{l+1} is a Markov transition kernel from $S^{(l)}$ into $S^{(l+1)}$ and $G_l : S^{(l)} \to \mathbb{R}^+$. In this situation, the solution of the measure-valued equation (1.1) is given by

$$\pi^{(l)}(f) = \gamma^{(l)}(f)/\gamma^{(l)}(1) \quad \text{with } \gamma^{(l)}(f) := \mathbb{E}\left(f(X_l) \prod_{0\leq k<l} G_k(X_k)\right) \qquad (1.3)$$

where $(X_l)_{l\geq 0}$ is a Markov chain taking values in the state spaces $(S^{(l)})_{l\geq 0}$, with initial distribution $\pi^{(0)}$ and Markov transitions $(L_l)_{l\geq 1}$. These Feynman–Kac models arise in a large number of applications including nonlinear filtering, Bayesian statistics and physics; see [10,8]. Note that these models are quite flexible. For instance, the reference Markov chain may represent the paths from the origin up to the current time l of an auxiliary Markov chain $(X'_l)_{l\geq 0}$ taking values in some state spaces $(S'_l)_{l\geq 0}$ with initial distribution $\pi^{(0)}$ and Markov transitions $(L'_l)_{l\geq 1}$; that is, we have

$$X_l := (X'_0, \ldots, X'_l) \in S^{(l)} := (S'_0 \times \cdots \times S'_l) \tag{1.4}$$

and consequently

$$L_l(x_{l-1}, dy_l) = \delta_{x_{l-1}}(dy_{l-1}) L'_l(y'_{l-1}, dy'_l). \tag{1.5}$$

When $G_l(x_l) = G'_l(x'_l)$, that is the potential function only depends on the terminal value of the path $x_l = (x'_0, \ldots, x'_l)$, the measures $\pi^{(l)}$ correspond to the path space measures given for $l \geq 1$ by

$$\pi^{(l)}(dx_l) \propto \left\{ \prod_{0\leq k<l} G'_k(x'_k) \right\} \pi^{(0)}(dx'_0) \prod_{1\leq k\leq l} L'_k(x'_{k-1}, dx'_k) \tag{1.6}$$

where '$\propto$' means 'proportional to'. □

Interacting annealing models. These models were recently introduced in [4] and can be reinterpreted as a special case of (1.1). In this scenario, we have $S^{(l)} = S$ and a pre-determined sequence of probability distributions $(\pi^{(l)})_{l\geq 0}$ of the form

$$\pi^{(l)}(dx) = \frac{\exp(-\beta_l V(x))\lambda(dx)}{\lambda(\exp(-\beta_l V))}$$

where λ is a reference measure, $(\beta_l)_{l\geq 0}$ is an increasing positive sequence and $\lambda(\exp(-\beta_l V)) < \infty$. Based on this sequence, we build a sequence of mappings $(\Phi^{(l)})_{l\geq 0}$ satisfying (1.1) as follows. We introduce $\epsilon \in [0, 1)$ and two sequences of Markov kernels $(K_l)_{l\geq 0}$ and $(L_l)_{l\geq 0}$ where both K_l and L_l admit $\pi^{(l)}$ as an invariant measure. We then set

$$\forall l \geq 0\, \forall(\mu, f) \in (\mathcal{P}(S^{(l)}) \times \mathcal{B}(S^{(l+1)})) \quad \Phi^{(l+1)}(\mu)(f) := \Psi_l(\mu) L_{l+1} K_{\epsilon,l+1}(f) \tag{1.7}$$

where $\Psi_l : \mu \in \mathcal{P}(S) \mapsto \Psi_l(\mu) \in \mathcal{P}(S)$ is defined by

$$\Psi_l(\mu)(dx) := \frac{G_l(x)\mu(dx)}{\mu(G_l)} \tag{1.8}$$

for $G_l(x) = \exp(-(\beta_{l+1} - \beta_l)V(x))$ and

$$K_{\epsilon,l} := (1-\epsilon) \sum_{k\geq 0} \epsilon^k K_l^k.$$

It is easy to check that (1.1) is satisfied for the mappings (1.7). □

1.4. Interacting Markov chain Monte Carlo methods

We introduce a sequence of 'initial' probability measures $(\nu^{(l)})_{l\geq 0}$ on $(S^{(l)})_{l\geq 0}$. We also introduce a Markov transition $M^{(0)}$ from $S^{(0)}$ into itself, and a collection of Markov transitions

$M_\mu^{(l)}$ from $S^{(l)}$ into itself, indexed by the parameter $l \geq 0$, where $\mu \in \mathcal{P}(S^{(l-1)})$. We further assume that the invariant measure of each operator $M_\mu^{(l)}$ is given by $\Phi^{(l)}(\mu)$; that is we have

$$\forall l \geq 0\ \forall \mu \in \mathcal{P}(S^{(l-1)}) \quad \Phi^{(l)}(\mu) = \Phi^{(l)}(\mu) M_\mu^{(l)}.$$

For $l = 0$, we use the convention $M_\mu^{(0)} = M^{(0)}$ and $\Phi^{(0)}(\mu) = \pi^{(0)}$.

We now have all the elements to define the i-MCMC algorithm. The algorithm generates a sequence of processes $(X^{(l)})_{l \geq 0}$ where $X^{(k)} := (X_n^{(k)})_{n \geq 0}$ is the process at level k whose associated occupation measure at iteration n of the algorithm is denoted by

$$\eta_n^{(k)} := \frac{1}{n+1} \sum_{p=0}^{n} \delta_{X_p^{(k)}}.$$

At level $k = 0$, $X^{(0)}$ is a Markov chain on $S^{(0)}$ with $X_0^{(0)} \sim \nu^{(0)}$ and Markov transitions $M^{(0)}$; that is

$$\mathbb{P}(X_{n+1}^{(0)} \in dx \mid X_n^{(0)}) = M^{(0)}(X_n^{(0)}, dx).$$

At level $k \geq 1$, given a realization of the chain $X^{(k-1)}$, the k-th level chain $X^{(k)}$ is an inhomogeneous Markov chain with $X_0^{(k)} \sim \nu^{(k)}$ and Markov transitions $M_{\eta_n^{(k-1)}}^{(k)}$ at iteration n depending on the current occupation measure $\eta_n^{(k-1)}$ of the chain at level $(k-1)$; that is

$$\mathbb{P}(X_{n+1}^{(k)} \in dx \mid X^{(k-1)}, X_n^{(k)}) = M_{\eta_n^{(k-1)}}^{(k)}(X_n^{(k)}, dx). \tag{1.9}$$

The rationale behind this is that the k-th level chain $X_n^{(k)}$ behaves asymptotically as a homogeneous Markov chain with transition kernel $M_{\pi^{(k-1)}}^{(k)}$ of invariant probability measure $\pi^{(k)}$ as long as $\eta_n^{(k-1)}$ is a 'good' approximation of $\pi^{(k-1)}$.

These i-MCMC algorithms can be interpreted as non-standard adaptive MCMC schemes [2,3,14] where the parameters to be adapted are probability measures instead of finite-dimensional parameters. Algorithms relying on similar principles were first proposed in [1] and independently in [4]. Related algorithms where we also have a sequence of nested MCMC-like chains 'feeding' each other have also recently appeared in statistics [11] and physics [12].

We now give examples of such Markov kernels for the Feynman–Kac and interacting annealing models described in Section 1.3.

Feynman–Kac models. Assume we are working on path spaces $S^{(l)} := (S^{(l-1)} \times S'_l)$ (cf. (1.4)–(1.6)), we can select for $M_\mu^{(l)}$ a Metropolis–Hastings kernel of independent proposal distribution $(\mu \otimes L'_l)$ and target distribution $\Phi^{(l)}(\mu)$. More precisely, using the fact that

$$\Phi^{(l)}(\mu)(d(y_{l-1}, y'_l)) \propto \mu(dy_{l-1}) G'_{l-1}(y'_{l-1}) L'_l(y'_{l-1}, dy'_l)$$

the independent Metropolis–Hastings kernel $M_\mu^{(l)}$ using $\mu(dy_{l-1}) L'_l(y'_{l-1}, dy'_l)$ as a proposal distribution is given by

$$M_\mu^{(l)}(x_l, dy_l) = \mu(dy_{l-1}) L'_l(y'_{l-1}, dy'_l) \left(1 \wedge \frac{G'_{l-1}(y'_{l-1})}{G'_{l-1}(x'_{l-1})}\right)$$

$$+ \left(1 - \mu \left(1 \wedge \frac{G'_{l-1}(y'_{l-1})}{G'_{l-1}(x'_{l-1})}\right)\right) \delta_{x_l}(dy_l) \tag{1.10}$$

where we recall that $x_l = (x_{l-2}, x'_{l-1}, x'_l) = (x_{l-1}, x'_l) \in S^{(l)} = (S^{(l-1)} \times S'_l)$ and $y_l = (y_{l-1}, y'_l) \in S^{(l)} = (S^{(l-1)} \times S'_l)$. □

Interacting annealing models. In this case, we can select

$$M^{(l)}_\mu(x, dy) = \epsilon K_l(x, dy) + (1 - \epsilon) \Psi_{l-1}(\mu) L_l(dy). \tag{1.11}$$

One can easily check that $M^{(l)}_\mu$ admits $\Phi^{(l)}(\mu)$ as invariant probability measure. □

For sufficiently regular models, we proved in [7,9] that the occupation measures $\eta^{(l)}_n$ converge to the solution $\pi^{(l)}$ of Eq. (1.1), in the sense that $\lim_{n\to\infty} \eta^{(l)}_n(f) = \pi^{(l)}(f)$ almost surely for $f \in \mathcal{B}(S^{(l)})$. The articles [5,9] also provide a collection of non asymptotic $\mathbb{L}_r$-mean error estimates and exponential deviations inequalities. The fluctuation analysis of $\eta^{(l)}_n$ around the limiting measure $\pi^{(l)}$ has been initiated in [5] in the special case where $M^{(l)}_\mu(x_l, .) = \Phi^{(l)}(\mu)$. In this 'simpler' situation, the l-th level chain $X^{(l)} = (X^{(l)}_n)$ is given $X^{(l-1)}$ a collection of conditionally independent random variables with $X^{(l)}_0 \sim \nu^{(l)}$ and $X^{(l)}_n \sim \Phi^{(l)}(\eta^{(l-1)}_{n-1})$ for $n \geq 1$.

1.5. Contribution and organization of the paper

The present article studies the fluctuations of the occupation measures $(\eta^{(l)}_n)_{l\geq 0}$ associated to the class of i-MCMC algorithms towards their limiting values $(\pi^{(l)})_{l\geq 0}$. Briefly speaking, our analysis proceeds as follows. First, we study weighted sequences of local random fields $V^{(l)}_n$ which are related to the fluctuations of the occupation measures $\eta^{(l)}_p$ around their local invariant measures $\Phi^{(l)}(\eta^{(l-1)}_{p-1})$ for $p \leq n$. We show that these random fields $(V^{(l)}_n)_{l\geq 0}$ converge in law, as n tends to infinity and in the sense of finite dimensional distributions, to a sequence of independent and centered Gaussian fields $(V^{(l)})_{l\geq 0}$ with covariance functions defined in terms of the resolvent operator associated to the Markov transition $M^{(l)}_{\pi^{(l-1)}}$ and its invariant probability measure $\pi^{(l)}$. Finally, we deduce the fluctuations of $\eta^{(l)}_n$ around their limiting values $\pi^{(l)}$ by a simple application of the continuous mapping theorem (or the multivariate δ-method) applied to a first order decomposition of the error $\sqrt{n}[\eta^{(l)}_n - \pi^{(l)}]$ in terms of the random fields $(V^{(k)}_n)_{0\leq k\leq l}$.

The rest of the paper is organized as follows. The main result of the article is presented in full detail in Section 2. The regularity conditions are summarized in Section 2.1. In Section 2.2 we state a multivariate CLT in terms of the semigroup associated with a first order expansion of the mappings $\Phi^{(l)}$ appearing in (1.1). Section 3 addresses the fluctuation analysis of an abstract class of time inhomogeneous Markov chains. In Section 3.2, we present a preliminary resolvent analysis to estimate the regularity properties of resolvent operators and invariant measure type mappings. In Section 3.3, we apply these results to study the local fluctuations of a class of weighted occupation measures associated to self-interacting chains. Section 4 addresses the fluctuation analysis of local interaction random fields associated with i-MCMC algorithms. The proof of the main theorem presented in Section 2.2 is a direct consequence of a fluctuation theorem for local interaction random fields, and it is given at the end of Section 4.1. Finally, we establish in Section 5 that the regularity conditions discussed in Section 2.1 are also valid for a path space extension of i-MCMC algorithms.

2. Statement of some results

2.1. Regularity conditions

Our first regularity condition is a first order weak regularity condition on the mappings $\Phi^{(l)}$ governing the measure-valued equation (1.1). We assume that, for any $l \geq 0$, the mappings $\Phi^{(l+1)} : \mathcal{P}(S^{(l)}) \to \mathcal{P}(S^{(l+1)})$ satisfy the following first order local decomposition

$$[\Phi^{(l+1)}(\mu) - \Phi^{(l+1)}(\eta)] = (\mu - \eta)D_{l+1} + \Xi_l(\mu, \eta) \tag{2.1}$$

where $D_{l+1} : \mathcal{B}(S^{(l+1)}) \to \mathcal{B}(S^{(l)})$ is a bounded integral operator that may depend on the measure η and $\Xi_l(\mu, \eta)$ is a remainder signed measure on $S^{(l+1)}$ indexed by the set of probability measures $\mu, \eta \in \mathcal{P}(S^{(l)})$. We further require that

$$|\Xi_l(\mu, \eta)(f)| \leq \int |(\mu - \eta)^{\otimes 2}(g)| \Xi_l(f, dg) \tag{2.2}$$

for some integral operator Ξ_l from $\mathcal{B}(S^{(l+1)})$ into the set $\mathcal{T}_2(S^{(l)})$ of all tensor product functions $g = \sum_{i \in I} a_i (h_i^1 \otimes h_i^2)$, with $I \subset \mathbb{N}$, $(h_i^1, h_i^2)_{i \in I} \in (\mathcal{B}(S^{(l)})^2)^I$, and a sequence of numbers $(a_i)_{i \in I} \in \mathbb{R}^I$ such that

$$|g| := \sum_{i \in I} |a_i| \, \|h_i^1\| \, \|h_i^2\| < \infty \quad \text{and} \quad \chi_l := \sup_{f \in \mathcal{B}_1(S^{(l+1)})} \int |g| \Xi_l(f, dg) < \infty. \tag{2.3}$$

Our second set of regularity conditions are for the Markov kernels $M_\mu^{(l)}$. We assume these kernels satisfy the following two regularity conditions

$$m_l(n_l) := \sup_{\mu \in \mathcal{P}(S^{(l-1)})} \beta((M_\mu^{(l)})^{n_l}) < 1 \tag{2.4}$$

and

$$\|[M_\mu^{(l)} - M_\nu^{(l)}](f)\| \leq \int |[\mu - \nu](g)| \Gamma_{l,\mu}(f, dg) \tag{2.5}$$

for some collection of bounded integral operators $\Gamma_{l,\mu}$ from $\mathcal{B}(S^{(l)})$ into $\mathcal{B}(S^{(l-1)})$ and indexed by the set of measures $\mu \in \mathcal{P}(S^{(l-1)})$ with

$$\sup_{\mu \in \mathcal{P}(S^{(l-1)})} \int \Gamma_{l,\mu}(f, dg) \|g\| \leq \Lambda_l \|f\| \quad \text{and} \quad \Lambda_l < \infty.$$

We end this section with some comments on this set of conditions.

The regularity condition (2.1)–(2.2) is a first order refinement of a Lipschitz type condition we used in [7,9] to derive a series of $\mathbb{L}_p$-mean error bounds and exponential inequalities. This condition has been introduced in [5] for studying the fluctuations of the simple i-MCMC algorithm corresponding to $M_\mu^{(l)}(x, .) = \Phi^{(l)}(\mu)$. The regularity condition (2.4) is an ergodicity condition on the Markov kernel $M_\mu^{(l)}$. Finally the regularity condition (2.5) is a local Lipschitz type continuity condition on the kernel $M_\mu^{(l)}$. This condition is less stringent than the one used in [9] where it is assumed that (2.5) holds for some operators $\Gamma_{l,\mu} = \Gamma_l$ that do not depend on μ. Therefore, most of the asymptotic results presented in [9] do not apply in the present context. Nevertheless, it can be checked that the inductive proof of the $\mathbb{L}_p$-mean error bounds presented in Theorem 5.2 in [5] hold true under the weaker condition (2.5); thus, for every $l \geq 0$ and any function $f \in \mathcal{B}(S^{(l)})$, we know that $\eta_n^{(l)}(f)$ converges almost surely to $\pi^{(l)}(f)$ as $n \to \infty$. The main advantage of the set of conditions presented here is that it is stable under a state space enlargement (see Section 5), so that the asymptotic analysis of such algorithms, including the multivariate CLT presented in the next section, applies directly without further work to i-MCMC algorithms on path spaces.

We illustrate these regularity conditions for the models discussed in Section 1.3. We further assume in the rest of this section that $(G_l)_{l\geq 0}$ is a collection of (0, 1]-valued potential functions on some state space $(S^{(l)})_{l\geq 0}$ such that

$$\forall l \geq 0 \quad \inf_{S^{(l)}} G_l > 0. \tag{2.6}$$

Feynman–Kac models. To establish (2.1)–(2.2), we observe that the mapping Ψ_l defined in (1.8) can be rewritten in terms of a nonlinear transport equation

$$\Psi_l(\mu)(dy) = (\mu \mathcal{S}_{l,\mu})(dy) := \int \mu(dx) \mathcal{S}_{l,\mu}(x, dy)$$

where

$$\mathcal{S}_{l,\mu}(x, dy) = G_l(x)\delta_x(dy) + (1 - G_l(x))\,\Psi_l(\mu)(dy).$$

Using the decomposition

$$\begin{aligned}\Psi_l(\mu) - \Psi_l(\eta) = (\mu - \eta)\mathcal{S}_{l,\eta} + \mu(\mathcal{S}_{l,\mu} - \mathcal{S}_{l,\eta}) \Rightarrow \Psi_l(\mu) - \Psi_l(\eta) \\ = \frac{1}{\mu(G_l)}(\mu - \eta)\mathcal{S}_{l,\eta}\end{aligned} \tag{2.7}$$

we prove the first order decomposition

$$\Psi_l(\mu) - \Psi_l(\eta) = (\mu - \eta)D_l' + \Xi_{l-1}'(\mu, \eta)$$

with the integral operators D_l' defined for any $f \in \mathcal{B}(S^{(l)})$ by $D_l'(f) := (\eta(G_l))^{-1}\mathcal{S}_{l,\eta}(f)$, and the remainder measures

$$\Xi_{l-1}'(\mu, \eta)(f) := \left[\frac{1}{\mu(G_l)} - \frac{1}{\eta(G_l)}\right](\mu - \eta)\mathcal{S}_{l,\eta}(f).$$

Using the fact that

$$\left[\frac{1}{\mu(G_l)} - \frac{1}{\eta(G_l)}\right] = \frac{(\eta - \mu)(G_l)}{\mu(G_l)\eta(G_l)}$$

and

$$(\mu - \eta)\mathcal{S}_{l,\eta}(f) = (\mu - \eta)(G_l f) - (\mu - \eta)(G_l)\,\Psi_l(\eta)(f)$$

we obtain

$$\begin{aligned}|\Xi_{l-1}'(\mu, \eta)(f)| &\leq \frac{1}{\inf G_l^2}[|(\mu - \eta)^{\otimes 2}(G_l \otimes (G_l f))| + \|f\|\,|(\mu - \eta)^{\otimes 2}(G_l \otimes G_l)|] \\ &:= \int |(\mu - \eta)^{\otimes 2}(g)|\Xi_{l-1}'(f, dg)\end{aligned}$$

with the integral operator

$$\Xi'_{l-1}(f, dg) = \frac{1}{\inf G_l^2}(\delta_{G_l \otimes (G_l f)}(dg) + \|f\| \delta_{G_l \otimes G_l}(dg)).$$

We check that the mappings (1.2) satisfy (2.1) using the fact that

$$\Phi^{(l+1)}(\mu) - \Phi^{(l+1)}(\eta) = (\Psi_l(\mu) - \Psi_l(\eta))L_{l+1} = (\mu - \eta)D_{l+1} + \Xi_l(\mu, \eta) \tag{2.8}$$

with the first order operator $D_{l+1} = D'_l L_{l+1}$ and the remainder measure $\Xi_l(\mu, \eta) = \Xi'_{l-1}(\mu, \eta)L_{l+1}$. The remainder measure satisfies (2.2) for

$$\Xi_l(f, dg) = \frac{1}{\inf G_l^2}(\delta_{G_l \otimes (G_l L_{l+1}(f))}(dg) + \|L_{l+1}(f)\| \delta_{G_l \otimes G_l}(dg)). \tag{2.9}$$

We mention that in this case the parameter χ_l defined in (2.3) is such that

$$\chi_l \leq 2 \sup G_l^2 / \inf G_l^2.$$

Assume we are working on path spaces $S^{(l)} := (S^{(l-1)} \times S'_l)$ where $G_l(x_l) = G'_l(x'_l)$ and $(S'_l)_{l \geq 0}$ are finite spaces. If we use for $M_\mu^{(l)}(x, dy)$ the independent Metropolis–Hastings kernel (1.10), (2.4) is satisfied as $\|G'_l\| \leq 1$; e.g., [13, Theorem 2.1]. Additionally, we have

$$\begin{aligned}
&\left| \int (M_\mu^{(l)}(x_l, dy_l) - M_\nu^{(l)}(x_l, dy_l)) f(y_l) \right| \\
&\quad \leq \left| \int (\mu - \nu)(dy_{l-1}) L'_l(y'_{l-1}, dy'_l) \left(1 \wedge \frac{G'_{l-1}(y'_{l-1})}{G'_{l-1}(x'_{l-1})} \right) f(y_l) \right| \\
&\qquad + \|f\| \left| \int (\mu - \nu)(dy_{l-1}) \left(1 \wedge \frac{G'_{l-1}(y'_{l-1})}{G'_{l-1}(x'_{l-1})} \right) \right|.
\end{aligned}$$

So (2.5) is satisfied for

$$\Gamma_{l,\mu}(f, dg) = \sum_{x'_{l-1} \in S'_{l-1}} \delta_{L'_l(f)\left(1 \wedge \frac{G'_{l-1}}{G'_{l-1}(x'_{l-1})}\right)}(g) + \|f\| \delta_{1 \wedge \frac{G'_{l-1}}{G'_{l-1}(x'_{l-1})}}(g)$$

where

$$\begin{aligned}
\int \Gamma_{l,\mu}(f, dg) \|g\| &= \sum_{x'_{l-1} \in S'_{l-1}} \left\| L'_l(f) \left(1 \wedge \frac{G'_{l-1}}{G'_{l-1}(x'_{l-1})} \right) \right\| + \|f\| \left\| 1 \wedge \frac{G'_{l-1}}{G'_{l-1}(x'_{l-1})} \right\| \\
&\leq 2 \left(\sum_{x'_{l-1} \in S'_{l-1}} \left\| 1 \wedge \frac{G'_{l-1}}{G'_{l-1}(x'_{l-1})} \right\| \right) \|f\|. \quad \square
\end{aligned}$$

Interacting annealing models. To establish (2.1)–(2.2), we can proceed similarly to Feynman–Kac models. It is sufficient to substitute $L_{l+1}K_{\epsilon,l+1}$ to L_{l+1} in (2.8)–(2.9). In this context, the Markov transitions given in (1.11) are such that

$$M_\eta^{(l)}(x, dy) \geq (1 - \epsilon) \Psi_{l-1}(\mu) L_l(dy) \Longrightarrow \beta(M_\eta^{(l)}) \leq \epsilon$$

so (2.4) is satisfied with $n_l = 1$ and $m_l(1) \leq \epsilon$. Moreover, we have

$$[M_\mu^{(l)} - M_\nu^{(l)}](f) = (1-\epsilon)[\Psi_{l-1}(\mu) - \Psi_{l-1}(\nu)]L_l(f).$$

Using the decomposition (2.7) one proves that

$$\|[M_\mu^{(l)} - M_\nu^{(l)}](f)\| \leq \frac{1-\epsilon}{\inf G_{l-1}} |(\mu-\nu)(\mathcal{S}_{l-1,\mu} L_l(f))|.$$

Hence condition (2.5) is satisfied with $\Gamma_{l,\mu}(f, dg) = \frac{1-\epsilon}{\inf G_{l-1}} \delta_{\mathcal{S}_{l-1,\mu} L_l(f)}(dg)$ and $\Lambda_l \leq (1-\epsilon)/\inf G_{l-1}$. □

2.2. *A multivariate central limit theorem*

To describe precisely the fluctuations of the empirical measures $\eta_n^{(l)}$ around their limiting value $\pi^{(l)}$, we need a few additional notations. We denote by $D_{k,l}$ with $0 \leq k \leq l$ the semigroup associated with the bounded integral operators D_k introduced in (2.1). More formally, we have

$$\forall 1 \leq k \leq l \quad D_{k,l} = D_k D_{k+1} \ldots D_l.$$

For $k > l$, we use the convention $D_{k,l} = Id$, the identity operator.

Using this notation, the multivariate CLT describing the fluctuations of the i-MCMC algorithm around the solution of the Eq. (1.1) is stated as follows. We remind the reader that the integral operator D_{l+1} may depend on the measure $\pi^{(l)}$.

Theorem 2.1. *For every $k \geq 0$, the sequence of random fields $(U_n^{(k)})_{n\geq 0}$ on $\mathcal{B}(S^{(k)})$ defined below*

$$U_n^{(k)} := \sqrt{n+1}[\eta_n^{(k)} - \pi^{(k)}]$$

converges in law, as n tends to infinity and in the sense of finite dimensional distributions, to a sequence of Gaussian random fields $U^{(k)}$ on $\mathcal{B}(S^{(k)})$ given by the following formula

$$U^{(k)} := \sum_{0\leq l\leq k} \frac{\sqrt{(2l)!}}{l!} V^{(k-l)} D_{(k-l)+1,k}. \tag{2.10}$$

Here $(V^{(l)})_{l\geq 0}$ stands for a collection of independent and centered Gaussian fields with a variance function given by

$$\mathbb{E}(V^{(l)}(f)^2) = \pi^{(l)}[(f-\pi^{(l)}(f))^2] + 2\sum_{n\geq 1} \pi^{(l)}[(f-\pi^{(l)}(f))(M^{(l)}_{\pi^{(l-1)}})^n(f-\pi^{(l)}(f))]. \tag{2.11}$$

In the special case where $M_\mu^{(l)}(x,.) = \Phi^{(l)}(\mu)$ for all $l \geq 1$, that is $\mathbb{E}(V^{(l)}(f)^2) = \pi^{(l)}[(f-\pi^{(l)}(f))^2]$, the result corresponds to the one obtained previously in [5]. This special class of i-MCMC algorithms behaves as a sequence of independent random variables with distributions $\Phi^{(l)}(\eta_n^{(l-1)})$ given by the local invariant measures of MCMC chains with transition kernels $M^{(l)}_{\eta_n^{(l-1)}}(x,\cdot)$. In the more general case considered here, the additional terms on the right hand side of (2.11) reflects the fluctuations of these MCMC algorithms around their limiting invariant probability measures.

Finally we note that, if it was possible to sample exactly from $\pi^{(k-1)}$, then we would have $U^{(k)} = V^{(k)}$. However, we need to approximate $\pi^{(k-1)}$ using an MCMC kernel which itself relies on an MCMC approximation of $\pi^{(k-2)}$ and so on. The price to pay for these additional approximations appears clearly in (2.10).

A toy example. Consider a Feynman–Kac model where $S^{(l)} := (S^{(l-1)} \times S)$ with $S = \{1, 2\}$ and

$$G'_l(1) = p^{(\beta_{l+1}-\beta_l)}, \qquad G'_l(2) = q^{(\beta_{l+1}-\beta_l)}$$

for $p = 1 - q > 0$, $(\beta_l)_{l\geq 0}$ is an increasing positive sequence and

$$L'_{l+1} = \begin{pmatrix} 1 - \pi_{l+1}(2) & \pi_{l+1}(2) \\ \pi_{l+1}(1) & 1 - \pi_{l+1}(1) \end{pmatrix}$$

where

$$\pi_{l+1}(1) = 1 - \pi_{l+1}(2) = \frac{p^{\beta_{l+1}}}{p^{\beta_{l+1}} + q^{\beta_{l+1}}}.$$

We have $\pi_{l+1}L'_{l+1} = \pi_{l+1}$ with $\pi_{l+1} = (\pi_{l+1}(1)\ \pi_{l+1}(2))$ and it is easy to check that the resulting Feynmac–Kac probability measure $\pi^{(l+1)}$ on S^{l+2} admits as a marginal distribution π_{l+1} at time $l+1$. For the sake of illustration, we derive the expression of all the terms appearing in the variance of $U^{(k)}(f)$ in Eq. (2.10) of Theorem 2.1.

In this scenario, we have established in Section 2.1 that $D_{l+1} = D'_l L_{l+1}$ with $D'_l = S_{l,\pi^{(l)}}/\pi^{(l)}(G_l) = S_{l,\pi^{(l)}}/\pi_l(G'_l)$ where, recalling the notation $x_l = (x'_0, \ldots, x'_l)$, we obtain

$$S_{l,\pi^{(l)}}(x_l, y_l) = G'_l(x'_l)\delta_{x_l}(y_l) + (1 - G'_l(x'_l))\frac{\pi^{(l)}(y_l)G'_l(y'_l)}{\pi_l(G'_l)},$$
$$L_{l+1}(x_l, y_{l+1}) = \delta_{x_l}(y_l)L'_{l+1}(y'_l, y'_{l+1})$$

so

$$D_{l+1}(x_l, y_{l+1}) = \frac{1}{\pi_l(G'_l)}\{G'_l(x'_l)\delta_{x_l}(y_l)L'_{l+1}(y'_l, y'_{l+1}) + (1 - G'_l(x'_l))\pi^{(l+1)}(y_{l+1})\}.$$

Finally the Markov transition kernel $M^{(l)}_{\pi^{(l)}}(x_l, y_l)$ satisfies

$$M^{(l)}_{\pi^{(l)}}(x_l, y_l) = \pi^{(l)}(y_{l-1})L'_l(y'_{l-1}, y'_l)\left(1 \wedge \frac{G'_{l-1}(y'_{l-1})}{G'_{l-1}(x'_{l-1})}\right) + \left(1 - \pi^{(l)}\left(1 \wedge \frac{G'_{l-1}(y'_{l-1})}{G'_{l-1}(x'_{l-1})}\right)\right)\delta_{x_l}(y_l).$$

Clearly even in this toy example, the expression of the variance of $U^{(k)}(f)$ is unfortunately analytically intractable. It is additionally only possible to compute numerically this variance for very small values of k as the semigroup $D_{1,k}$ and the transition matrix $M^{(k)}_{\pi^{(k)}}$ would have to be computed for a number of values increasing exponentially fast with k. $\square$

Since the original version of this paper [6], the authors have become aware of a recent paper of Yves Atchadé [4] which analyzes the local fluctuations of some related algorithms; namely the importance-resampling MCMC algorithm which is an interacting annealing model as described in Section 1.3 and a version of the equi-energy sampler [11]. In [4], the author provides a CLT associated to some random measures $\pi_2^{(k)}$ which converge almost surely to $\pi_2^{(k)}$ as n tends to infinity. We provide here a multivariate CLT valid for any l (and not only $l = 2$). The variance expression we obtain depends explicitly on the first order semigroup $D_{k,l}$ and the fluctuations of the i-MCMC algorithm on lower indexed levels.

3. On the fluctuations of time inhomogeneous Markov chains

To establish the proof of our main result (Theorem 2.1), it is necessary to first provide some general results about the fluctuations of time inhomogeneous Markov chains with Markov transitions that may depend on some predictable flow of distributions on some possibly different state space.

3.1. Description of the model

We consider a collection of Markov transitions M_η on some measurable space $(S, \mathcal{S})$ indexed by the set of probability measures $\eta \in \mathcal{P}(S')$, on some possibly different measurable space $(S', \mathcal{S}')$. We further assume that there exists an integer $n_0 \geq 0$ such that

$$m(n_0) := \sup_{\eta \in \mathcal{P}(S')} \beta(M_\eta^{n_0}) < 1 \quad \text{and we set } p(n_0) := 2n_0/(1 - m(n_0)). \tag{3.1}$$

We also assume that for any pair of measures $(\eta, \mu) \in \mathcal{P}(S')^2$ we have

$$\|[M_\mu - M_\eta](f)\| \leq \int |[\mu - \eta](g)| \Gamma_\mu(f, dg) \tag{3.2}$$

for some collection of bounded integral operator Γ_μ from $\mathcal{B}(S)$ into $\mathcal{B}(S')$, indexed by the set of measures $\mu \in \mathcal{P}(S')$ with

$$\sup_{\mu \in \mathcal{P}(S')} \int \Gamma_\mu(f, dg) \|g\| \leq \Lambda \|f\| \quad \text{for some finite constant } \Lambda < \infty.$$

We consider an increasing sequence of σ-fields $(\mathcal{F}_n)_{n\geq 0}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We let η_n be a $\mathcal{P}(S')$-valued random process adapted to the filtration $\mathcal{F}_n$ (i.e. each probability distribution η_n is $\mathcal{F}_n$-measurable). We further assume that $\mathcal{F}_n$ contains the σ-field generated by the random states X_p from the origin $p = 0$ up to the current time horizon $p = n$ of an S-valued non homogeneous Markov chain X_n with a prescribed initial distribution $\nu \in \mathcal{P}(S)$, and some transitions defined by

$$\forall n \geq 0 \quad \mathbb{P}(X_{n+1} \in dx \mid \mathcal{F}_n) = M_{\eta_n}(X_n, dx). \tag{3.3}$$

For example, $\mathcal{F}_n$ could be $\sigma((X'_p, X_p), 0 \leq p \leq n)$, i.e. the canonical sigma field associated with the $(S' \times S)$-valued process $(X'_n, X_n)_{n\geq 0}$, and $\eta_n = \frac{1}{(n+1)} \sum_{p=0}^n \delta_{X'_p}$ is the flow of occupation measures of $(X'_n)_{n\geq 0}$. In this context, (3.3) reflects the fact that, given $(\eta_n)_{n\geq 0}$, the process $(X_n)_{n\geq 0}$ is a Markov chain with random Markov transitions defined in terms of the occupation measures $(\eta_n)_{n\geq 0}$.

We further assume that the variations of $(\eta_n)_n$ are controlled by some sequence of random variables $\tau(n)$ in the sense that

$$\forall n \geq 0 \quad \|\eta_n - \eta_{n-1}\| \leq \tau(n), \quad \text{and we set } \overline{\tau}(n) := \sum_{0 \leq p \leq n} \tau(p). \tag{3.4}$$

For $n = 0$ we use the convention $\eta_{-1} = 0$, the null measure on S'.

3.2. Regularity properties of resolvent operators

The main simplification of conditions (3.1) comes from the fact that M_η has a unique invariant measure

$$\Phi(\eta)M_\eta = \Phi(\eta) \in \mathcal{P}(S).$$

In addition, the so-called resolvent operators

$$P_\eta : f \in \mathcal{B}(S) \to P_\eta(f) := \sum_{n\geq 0}[M_\eta^n - \Phi(\eta)](f) \in \mathcal{B}(S) \tag{3.5}$$

are well defined absolutely convergent series that satisfy the Poisson equation given by

$$\begin{cases}(M_\eta - Id)P_\eta = (\Phi(\eta) - Id)\\ \Phi(\eta)P_\eta = 0.\end{cases}$$

Resolvent operators are classical tools for the asymptotic analysis of time inhomogeneous Markov chains. In our context the Markov chain interacts with a flow a probability measures. To analyze the situation where this flow converges to some limiting measure, it is convenient to study the regularity properties of the resolvent operators P_η as well as the ones of the invariant measure mapping $\Phi(\eta)$ associated with M_η.

Proposition 3.1. *Under the regularity conditions (3.1) and (3.2), we have*

$$\sup_{\eta\in\mathcal{P}(S')} \|P_\eta\| \leq p(n_0). \tag{3.6}$$

In addition, for any $f \in \mathcal{B}(S)$ and any $(\mu, \eta) \in \mathcal{P}(S')$ we have the following Lipschitz type inequalities

$$|[\Phi(\eta) - \Phi(\mu)](f)| \leq \int |[\eta - \mu](g)|\, \Upsilon_\mu(f, dg) \tag{3.7}$$

and

$$\|[P_\eta - P_\mu](f)\| \leq \int |[\eta - \mu](g)|\, \Upsilon'_\mu(f, dg) \tag{3.8}$$

where $(\Upsilon_\mu, \Upsilon'_\mu)$ is a pair of bounded integral operators from $\mathcal{B}(S)$ into $\mathcal{B}(S')$ indexed by the set of measures $\mu \in \mathcal{P}(S')$ such that

$$\int \|g\|\, \Upsilon_\mu(f, dg) \leq p(n_0)\Lambda\|f\|$$

and

$$\int \|g\|\, \Upsilon'_\mu(f, dg) \leq p(n_0)(1 + p(n_0))\Lambda\|f\|.$$

Proof. The first result (3.6) is proved in [9]. For completeness, it is sketched here. We use the fact that

$$P_\eta(f)(x) = \sum_{n\geq 0} \int [M_\eta^n(f)(x) - M_\eta^n(f)(y)]\,\Phi(\eta)(dy)$$

to check that

$$\|P_\eta(f)\| \leq \sum_{n\geq 0} \mathrm{osc}(M_\eta^n(f))$$

and

$$\|P_\eta(f)\| \leq \left[\sum_{n\geq 0} \beta(M_\eta^n)\right] \mathrm{osc}(f) \Rightarrow \|P_\eta\| \leq 2\sum_{p\geq 1}\sum_{r=0}^{n_0-1} \beta(M_\eta^{(p-1)n_0+r})$$
$$\leq \frac{2n_0}{1-\beta(M_\eta^{n_0})}.$$

The result (3.6) follows straightforwardly. The proof of (3.7) is based on the following decomposition

$$[\Phi(\eta)-\Phi(\mu)](f) = \{[\Phi(\eta)-\Phi(\mu)]M_\mu + \Phi(\eta)[M_\eta - M_\mu]\}(f_\mu)$$

with $f_\mu := (f - \Phi(\mu)(f))$. Under our regularity conditions on the integral operators M_μ, we find that

$$|[\Phi(\eta)-\Phi(\mu)](f)| \leq |[\Phi(\eta)-\Phi(\mu)]M_\mu(f_\mu)| + \|[M_\eta - M_\mu](f_\mu)\|$$
$$\leq |[\Phi(\eta)-\Phi(\mu)]M_\mu(f_\mu)| + \int |[\mu-\eta](g)|\Gamma_\mu(f_\mu, dg).$$

This recursion readily implies (3.7) with the integral operator given by

$$\Upsilon_\mu(f, dg) := \sum_{n\geq 0} \Gamma_\mu(M_\mu^n(f_\mu), dg).$$

Finally we observe that

$$\int \|g\|\,\Upsilon_\mu(f, dg) \leq \sum_{n\geq 0}\int \|g\|\Gamma_\mu(M_\mu^n(f_\mu), dg) \leq \Lambda \sum_{n\geq 0} \|M_\mu^n(f_\mu)\|.$$

Arguing as above, we conclude that

$$\int \|g\|\,\Upsilon_\mu(f, dg) \leq \Lambda \sum_{n\geq 0} \mathrm{osc}(M_\mu^n(f)) \leq p(n_0)\Lambda\|f\|.$$

This ends the proof of (3.7). The proof of (3.8) follows the same type of arguments. We observe that

$$P_\eta - P_\mu = P_\mu(M_\eta - M_\mu)P_\eta + [\Phi(\mu)-\Phi(\eta)]P_\eta.$$

To check this formula, we first use the fact that $M_\mu P_\mu = P_\mu M_\mu$ to prove that

$$P_\mu(M_\mu - Id) = (M_\mu - Id)P_\mu = (\Phi(\mu) - Id).$$

This yields

$$P_\mu(M_\mu - Id)P_\eta = (\Phi(\mu) - Id)P_\eta.$$

Using the Poisson equation and the fact that $P_\mu(1) = 0$ we also have the decomposition

$$P_\mu(M_\eta - Id)P_\eta = P_\mu(\Phi(\eta) - Id) = -P_\mu.$$

Combining these two formulae, we conclude that

$$P_\mu(M_\eta - M_\mu)P_\eta = [P_\eta - P_\mu] - [\Phi(\mu) - \Phi(\eta)]P_\eta.$$

This ends the proof of the decomposition given above. It is now easily checked that

$$\begin{aligned}\|[P_\mu - P_\eta](f)\| &\le \|(M_\mu - M_\eta)P_\mu(f)\| + \int |[\eta-\mu](g)|\,\Upsilon_\mu(P_\mu(f), dg)\\ &\le \int |[\mu-\eta](g)|\{\Gamma_\mu(P_\mu(f), dg) + \Upsilon_\mu(P_\mu(f), dg)\}.\end{aligned}$$

The end of the proof follows the same type of arguments as before. This ends the proof of the proposition. □

3.3. Local fluctuations of weighted occupation measures

This section is concerned with the fluctuation analysis of the occupation measures of the time inhomogeneous Markov chain introduced in (3.3). In Section 4, we shall use these results to analyze the fluctuations of i-MCMC algorithms. The fluctuation analysis of this type of models is related to the fluctuations of weighted occupation measures with respect to some weight array type functions.

Definition 3.2. We let $\mathcal{W}$ be the set of non negative and non increasing weight array functions $w = (w_n(p))_{0\le p\le n,\ 0\le n}$, satisfying the following conditions

$$\exists m \ge 1 \quad \text{such that } \sum_{n\ge 0} w_n^m(0) < \infty$$

with

$$\forall \epsilon \in [0,1] \quad \varpi(\epsilon) := \lim_{n\to\infty} \sum_{0\le p\le \lfloor \epsilon n\rfloor} w_n^2(p) < \infty$$

and some scaling function ϖ such that $\lim_{(\epsilon_0,\epsilon_1)\to(0+,1-)}(\varpi(\epsilon_0), \varpi(\epsilon_1)) = (0,1)$.

We observe that the traditional and constant fluctuation rate sequences $w_n(p) = 1/\sqrt{n}$ belong to $\mathcal{W}$, with the identity function $\varpi(\epsilon) = \epsilon$. In our setup it is necessary to introduce more general sequences; see proof of Theorem 2.1 in Section 4.

Definition 3.3. We associate to a given weight array function $w \in \mathcal{W}$ the mapping

$$W : \eta \in \mathcal{M}(S)^{\mathbb{N}} \mapsto W(\eta) = (W_n(\eta))_{n\ge 0} \in \mathcal{M}(S)^{\mathbb{N}}$$

defined for any flow of measures $\eta = (\eta_n)_{n\ge 0} \in \mathcal{P}(S)$, and any $n \ge 0$, by the weighted measures

$$W_n(\eta) = \sum_{0\le p\le n} w_n(p)\eta_p.$$

The next proposition presents a pivotal decomposition formula of the weighted occupation measures in terms of a martingale on a fixed time horizon with a negligible remainder bias term.

Proposition 3.4. *We consider the flow of random measures* $\zeta := (\zeta_n)_{n\in\mathbb{N}} \in \mathcal{M}(S)^{\mathbb{N}}$ *defined for any* $n \ge 0$ *by the following formula*

$$\forall n \ge 0 \quad \zeta_n = [\delta_{X_n} - \Phi(\eta_{n-1})].$$

For $n = 0$, we use the convention $\Phi(\eta_{-1}) = \nu$ so that $\zeta_0 = [\delta_{X_0} - \nu]$. For any weight array function $w \in \mathcal{W}$, the weighted measures $W_n(\zeta)$ satisfy the following decomposition

$$W_n(\zeta)(f) = \sum_{0 \le p \le n} w_n(p) \Delta \mathbb{M}_{p+1}(f) + \mathbb{L}_n(f) \tag{3.9}$$

with the martingale increments

$$\Delta \mathbb{M}_{p+1}(f) = \mathbb{M}_{p+1}(f) - \mathbb{M}_p(f) := (P_{\eta_{p-1}}(f)(X_{p+1}) - M_{\eta_p} P_{\eta_{p-1}}(f)(X_p)) \tag{3.10}$$

and a remainder signed measure $\mathbb{L}_n$ such that

$$\|\mathbb{L}_n\| \le w_n(0)(1 + p(n_0)\overline{\tau}(n))(2 + p(n_0))\Lambda.$$

Proof. We let $P_{\eta_{n-1}}$ be the integral operator solution of the Poisson equation associated with the Markov transition $M_{\eta_{n-1}}$ with an invariant measure $\Phi(\eta_{n-1})$. By construction, we have

$$\zeta_n(f) = [f(X_n) - \Phi(\eta_{n-1})(f)] = P_{\eta_{n-1}}(f)(X_n) - M_{\eta_{n-1}}(P_{\eta_{n-1}}(f))(X_n).$$

For $n = 0$, we use the convention $P_{\eta_{-1}} = Id$ and $M_{\eta_{-1}} = \nu$. The proof of (3.9) is based on the following decomposition

$$\zeta_n(f) = A_n(f) + B_n(f) + C_n(f) + \Delta \mathbb{M}_{n+1}(f)$$

with the random processes $A_n(f)$, $B_n(f)$ and $C_n(f)$ defined below

$$\begin{aligned}
A_n(f) &:= [P_{\eta_n} - P_{\eta_{n-1}}](f)(X_{n+1}) \\
B_n(f) &:= [P_{\eta_{n-1}}(f)(X_n) - P_{\eta_n}(f)(X_{n+1})] \\
C_n(f) &:= [M_{\eta_n} - M_{\eta_{n-1}}] P_{\eta_{n-1}}(f)(X_n).
\end{aligned}$$

Using the Lipschitz inequality (3.8) presented in Proposition 3.1, we prove that

$$\begin{aligned}
|A_n(f)| &\le \|[P_{\eta_n} - P_{\eta_{n-1}}](f)\| \\
&\le \int |[\eta_n - \eta_{n-1}](g)| \Upsilon'_{\eta_n}(f, dg) \le \tau(n) p(n_0)(1 + p(n_0))\Lambda \|f\|.
\end{aligned}$$

In addition, using the Lipschitz regularity condition (3.2), we also obtain

$$\begin{aligned}
|C_n(f)| &\le \|[M_{\eta_n} - M_{\eta_{n-1}}](P_{\eta_{n-1}}(f))\| \\
&\le \left\| \int |[\eta_n - \eta_{n-1}](g)| \Gamma_{\eta_n}(P_{\eta_{n-1}}(f), dg) \right\| \\
&\le \tau(n)\Lambda \|P_{\eta_{n-1}}\| \|f\| \le \tau(n)\Lambda p(n_0) \|f\|.
\end{aligned}$$

By definition of the weighted measure $W_n(\zeta)$, we have

$$\begin{aligned}
W_n(\zeta) &:= \sum_{0 \le p \le n} w_n(p) \zeta_p(f) \\
&= \sum_{0 \le p \le n} w_n(p) \Delta \mathbb{M}_{p+1}(f) + \sum_{0 \le p \le n} w_n(p)(A_p(f) + B_p(f) + C_p(f)).
\end{aligned} \tag{3.11}$$

From previous calculations, we have

$$\left| \sum_{0 \le p \le n} w_n(p)(A_p(f) + C_p(f)) \right| \le w_n(0)\overline{\tau}(n)\Lambda p(n_0)(2 + p(n_0))\|f\|.$$

Finally, we use the following decomposition

$$\sum_{0\le p\le n} w_n(p)B_p(f) = \sum_{0\le p\le n} [w_n(p)P_{\eta_{p-1}}(f)(X_p) - w_n(p+1)P_{\eta_p}(f)(X_{p+1})] + \sum_{0\le p\le n} [w_n(p+1) - w_n(p)]P_{\eta_p}(f)(X_{p+1})$$

with the convention $w_n(n+1) = 0$. This implies that

$$\left|\sum_{0\le p\le n} w_n(p)B_p(f)\right| \le 2w_n(0)\|f\| + p(n_0)\|f\| \sum_{0\le p\le n} [w_n(p) - w_n(p+1)] = (2 + p(n_0))\|f\|w_n(0).$$

The end of the proof is now a direct consequence of formula (3.11). □

Now, we are in position to state and to prove the main result of this section.

Theorem 3.5. *Assume there exist a measure η and some $m \ge 1$ such that*

$$\forall f \in \mathcal{B}_1(S') \quad \mathbb{E}(|\eta_n(f) - \eta(f)|^m) \le \epsilon_m(n) \quad \textit{with} \sum_{n\ge 0} \epsilon_m(n) < \infty.$$

We let $V_n := W_n(\zeta)$ be the sequence of random fields on $\mathcal{B}(S)$ associated with a given weight array function $w \in \mathcal{W}$ and defined in (3.9). We suppose that $w \in \mathcal{W}$ is chosen so that $w_n(0)\overline{\tau}(n)$ tends to 0 as $n \to \infty$. In this situation, V_n converges in law as $n \to \infty$ to a Gaussian random field V on $\mathcal{B}(S)$ such that

$$\forall (f, g) \in \mathcal{B}(S)^2 \quad \mathbb{E}(V(f)V(g)) = \Phi(\eta)[C_\eta(f, g)]$$

with the local covariance function

$$C_\eta(f, g) := M_\eta[(P_\eta(f) - M_\eta P_\eta(f))(P_\eta(g) - M_\eta P_\eta(g))].$$

Proof. Using Proposition 3.4, it is clearly sufficient to prove that the random fields

$$W_n'(\zeta) := \sum_{0\le p\le n} w_n(p)\Delta\mathbb{M}_{p+1} \tag{3.12}$$

converge in law to the Gaussian random field V as $n \to \infty$. To use the Lindeberg CLT for triangular arrays of $\mathbb{R}^d$-valued random variables, we let $f = (f^i)_{1\le i\le d} \in \mathcal{B}(S)^d$ be a collection of d-valued functions and we consider the $\mathbb{R}^d$-valued random variables $W_n'(\zeta)(f) = (W_n'(\zeta)(f^i))_{1\le i\le d}$. We further denote by $\mathcal{F}_p$ the σ-field generated by the random variables X_q for any $q \le p$. By construction, for any functions f and $g \in \mathcal{B}(S)$ and for every $0 \le p \le n$ we find that

$$\mathbb{E}(w_n(p)\Delta\mathbb{M}_{p+1}(f) \mid \mathcal{F}_p) = 0$$
$$\mathbb{E}(w_n(p)^2\Delta\mathbb{M}_{p+1}(f)\Delta\mathbb{M}_{p+1}(g) \mid \mathcal{F}_p) = w_n(p)^2 C_p'(f, g)(X_p)$$

with the local covariance function

$$C'_p(f,g) := M_{\eta_p}[(P_{\eta_{p-1}}(f) - M_{\eta_p}P_{\eta_{p-1}}(f))(P_{\eta_{p-1}}(g) - M_{\eta_p}P_{\eta_{p-1}}(g))].$$

Using Proposition 3.1, after some tedious but elementary calculations we find that

$$\|C'_p(f,g) - C_\eta(f,g)\| \le c(\eta)\left\{\int |[\eta_{p-1} - \eta](h)|\,\Upsilon^1_\eta((f,g),dh) + \int |[\eta_p - \eta](h)|\,\Upsilon^2_\eta((f,g),dh)\right\}$$

with a pair of bounded integral operators Υ^i_η, $i = 1, 2$, from $\mathcal{B}(S)^2$ into $\mathcal{B}(S')$ such that

$$\int \|h\|\,\Upsilon^i_\eta((f,g),dh) \le c(\eta)\|f\|\,\|g\|.$$

In the above displayed formula, $c(\eta) < \infty$ stands for a finite constant whose value only depends on the measure η. Under our assumptions, the following almost sure convergence result readily follows

$$\lim_{p\to\infty} \|C'_p(f,g) - C_\eta(f,g)\| = 0. \tag{3.13}$$

On the other hand, using Proposition 3.4, for any function $h \in \mathcal{B}(S)$ we have the decomposition

$$\sum_{0\le p\le n} w_n^2(p)(h(X_p) - \Phi(\eta_{p-1})(h)) = \sum_{0\le p\le n} w_n^2(p)\Delta\mathbb{M}_{p+1}(h) + \mathbb{L}'_n(h)$$

with the martingale increments $\Delta\mathbb{M}_{p+1}(h)$ given in (3.10), and a remainder signed measure $\mathbb{L}'_n$ such that

$$\|\mathbb{L}'_n\| \le w_n^2(0)(1 + p(n_0)\overline{\tau}(n))(2 + p(n_0))\Lambda.$$

Using a Burkholder–Davis–Gundy type inequality for martingales, we find that for any $m \ge 1$

$$\mathbb{E}\left(\left|\sum_{0\le p\le n} w_n^2(p)\Delta\mathbb{M}_{p+1}(h)\right|^m\right)^{\frac{1}{m}} \le a(m)\left(\sum_{0\le p\le n} w_n(p)^4\right)^{\frac{1}{2}} \operatorname{osc}(h)$$

for some constants $a(m)$ whose values only depend on the parameter m. Thus, we find that

$$\mathbb{E}\left(\left|\sum_{0\le p\le n} w_n^2(p)\Delta\mathbb{M}_{p+1}(h)\right|^m\right)^{\frac{1}{m}} \le a(m)w_n(0)\left(\sum_{0\le p\le n} w_n(p)^2\right)^{\frac{1}{2}} \operatorname{osc}(h).$$

Under our assumptions on the weight functions w, if we take $h = C_\eta(f,g)$ then by (3.13) we obtain the following almost sure convergence result

$$\lim_{n\to\infty} \sum_{p=0}^n w_n(p)^2 C_\eta(f,g)(X_p) = \lim_{n\to\infty} \sum_{p=0}^n w_n(p)^2\,\Phi(\eta_{p-1})(C_\eta(f,g)).$$

We now combine the regularity property (3.7) with the generalized Minkowski inequality to prove that for any function $h \in \mathcal{B}(S)$ and any $m \ge 1$

$$\mathbb{E}\left(\left|\sum_{p=0}^n w_n(p)^2[\Phi(\eta_{p-1})(h) - \Phi(\eta)(h)]\right|^m\right)^{\frac{1}{m}}$$

$$\leq \sum_{p=0}^{n} w_n(p)^2 \int \mathbb{E}(|[\eta_{p-1} - \eta](g)|^m)^{\frac{1}{m}}\, \Upsilon_\mu(h, dg)$$
$$\leq (p(n_0)\Lambda\|h\|) \left(w_n(0)^2 + \sum_{p=1}^{n} w_n(p)^2 \epsilon_m(p-1) \right).$$

This readily implies that

$$\mathbb{E}\left(\left| \sum_{p=0}^{n} w_n(p)^2 [\Phi(\eta_{p-1})(h) - \Phi(\eta)(h)] \right|^m \right)^{\frac{1}{m}} \leq w_n(0)^2 (p(n_0)\Lambda\|h\|) \times \left(1 + \sum_{p\geq 0} \epsilon_m(p)\right).$$

If we choose $h = C_\eta(f, g)$, this yields the following almost sure convergence result

$$= \lim_{n\to\infty} \sum_{p=0}^{n} w_n(p)^2\, \Phi(\eta_{p-1})(C_\eta(f, g)) = \Phi(\eta)(C_\eta(f, g)).$$

To summarize, we have proved the following series of almost sure convergence results

$$\begin{aligned}\lim_{n\to\infty} \sum_{p=0}^{n} w_n(p)^2 C'_p(f, g)(X_p) &= \lim_{n\to\infty} \sum_{p=0}^{n} w_n(p)^2 C_\eta(f, g)(X_p) \\ &= \lim_{n\to\infty} \sum_{p=0}^{n} w_n(p)^2\, \Phi(\eta_{p-1})(C_\eta(f, g)) \\ &= \Phi(\eta)(C_\eta(f, g)).\end{aligned}$$

Therefore, we also have the almost sure convergence result

$$\lim_{n\to\infty} \sum_{p=0}^{n} w_n(p)^2 \mathbb{E}(\Delta\mathbb{M}_{p+1}(f)\Delta\mathbb{M}_{p+1}(g) \mid \mathcal{F}_p) = \Phi(\eta)(C_\eta(f, g)).$$

Since we have $\vee_{0\leq p\leq n} w_n(p) = w_n(0) \to 0$, as $n \to \infty$, the Lindeberg condition is satisfied and we conclude that the sequence of random fields $W'_n(\zeta)$ defined in (3.12) converges in law to the Gaussian random field V as $n \to \infty$. This ends the proof of the theorem. □

We end this section with an alternative and simpler representation of the covariance function of the random field V presented in Theorem 3.5. We have

$$C_\eta(f, f)(x) = \int M_\eta(x, dy)[P_\eta(f)(y) - M_\eta(P_\eta(f))(x)]^2.$$

Using the decomposition

$$\begin{aligned}P_\eta(f)(y) - M_\eta(P_\eta(f))(x) &= [P_\eta(f)(y) - P_\eta(f)(x)] + [P_\eta(f)(x) - M_\eta(P_\eta(f))(x)] \\ &= [P_\eta(f)(y) - P_\eta(f)(x)] + [f(x) - \Phi(\eta)(f)]\end{aligned}$$

and the fact that

$$\begin{aligned}\int M_\eta(x, dy)[P_\eta(f)(y) - P_\eta(f)(x)] &= [M_\eta(P_\eta(f))(x) - P_\eta(f)(x)] \\ &= -[f(x) - \Phi(\eta)(f)]\end{aligned}$$

we prove the formula

$$C_\eta(f,f)(x) = \int M_\eta(x,dy)[P_\eta(f)(y) - P_\eta(f)(x)]^2 - [f(x) - \Phi(\eta)(f)]^2.$$

On the other hand, recalling that $\Phi(\eta) = \Phi(\eta)M_\eta$ and using again the Poisson equation we also have

$$\int \Phi(\eta)(dx)M_\eta(x,dy)[P_\eta(f)(y) - P_\eta(f)(x)]^2 = 2\Phi(\eta)[P_\eta(f)(f - \Phi(\eta)(f))]$$

and

$$2\Phi(\eta)[P_\eta(f)(f - \Phi(\eta)(f))] = 2\Phi(\eta)[(f - \Phi(\eta)(f))^2] + 2\sum_{n\geq 1} \Phi(\eta)[M_\eta^n(f - \Phi(\eta)(f))(f - \Phi(\eta)(f))].$$

Hence we have proved the following proposition.

Proposition 3.6. *The limiting covariance function presented in Theorem 3.5 is alternatively defined for any function $f \in \mathcal{B}(S)$ by the following formula*

$$\Phi(\eta)[C_\eta(f,f)] = \Phi(\eta)[(f - \Phi(\eta)(f))^2] + 2\sum_{n\geq 1} \Phi(\eta)[(f - \Phi(\eta)(f))M_\eta^n(f - \Phi(\eta)(f))].$$

4. A fluctuation theorem for local interaction fields

4.1. Introduction

This section presents the fluctuation analysis of a class of weighted random fields associated to i-MCMC algorithms. Following the local fluctuation analysis for time inhomogeneous Markov chains presented in Section 3.3, we introduce the following weighted random fields.

Definition 4.1. We consider the flow of random measures

$$\forall l \geq 0\ \forall n \geq 0 \quad \delta_n^{(l)} := \left[\delta_{X_n^{(l)}} - \Phi^{(l)}(\eta_{n-1}^{(l-1)})\right].$$

For $n = 0$, we use the convention $\Phi^{(l)}(\eta_{-1}) = \nu^{(l)}$ so that $\delta_0^{(l)} = [\delta_{X_0^{(l)}} - \nu^{(l)}]$. We associate to a sequence of weight array functions $(w^{(l)})_{l\geq 0} \in \mathcal{W}^{\mathbb{N}}$ the following flow of random fields $(W_n^{(l)}(\delta^{(l)}))_{l\geq 0}$ on the sets of functions $(\mathcal{B}(S^{(l)}))_{l\geq 0}$

$$\forall l \geq 0\ \forall n \geq 0 \quad W_n^{(l)}(\delta^{(l)}) := \sum_{0\leq p\leq n} w_n^{(l)}(p)\delta_p^{(l)}.$$

We observe that the regularity conditions (2.4) and (2.5) ensure that the collection of Markov operators $M_\eta^{(l)}$ and their invariant measures $\Phi^{(l)}(\eta)$ satisfy the regularity conditions (3.1) and (3.2) introduced in Section 3.1. Also observe that the i-MCMC chain $X^{(l+1)}$ is a time inhomogeneous model of the form (3.3) with a collection of transitions $M^{(l+1)}_{\eta_n^{(l)}}$ that depend on the flow of occupation measures $\eta_n^{(l)}$ associated with the i-MCMC chain at level l. In this scenario, condition (3.4) is satisfied with

$$\forall n \geq 0 \quad \|\eta_n^{(l)} - \eta_{n-1}^{(l)}\| \leq \tau^{(l)}(n) := \frac{2}{n+1}$$

and we have

$$\overline{\tau}^{(l)}(n) := \sum_{0 \leq p \leq n} \tau^{(l)}(p) = 2 \sum_{0 \leq p \leq n} \frac{1}{p+1} \leq 2(1 + \log(n+1)).$$

Finally, we recall that for any $m \geq 1$ we have

$$\forall l \geq 0 \, \forall f \in \mathcal{B}_1(S^{(l)}) \quad \mathbb{E}(|\eta_n^{(l)}(f) - \pi^{(l)}(f)|^m)^{\frac{1}{m}} \leq b(m)c(l)\frac{1}{\sqrt{n+1}} \tag{4.1}$$

for a collection of finite constants $b(m)$ whose values only depend on the parameter m (see for instance [9]). Using Theorem 3.5, we can prove that the random fields

$$V_n^{(l)} := W_n^{(l)}(\delta^{(l)}) \tag{4.2}$$

associated with a given weight array function $w^{(l)} \in \mathcal{W}$ converges in law to a Gaussian random field $V^{(l)}$ as $n \to \infty$ such that

$$\forall (f, g) \in \mathcal{B}(S^{(l)})^2 \quad \mathbb{E}(V^{(l)}(f)V^{(l)}(g)) = \pi^{(l)}[C^{(l)}(f, g)]. \tag{4.3}$$

Here the covariance functions $C^{(l)}(f, g)$ are defined in terms of the resolvent operator $P_{\pi^{(l-1)}}^{(l)}$ associated to the Markov transition $M_{\pi^{(l-1)}}^{(l)}$ and the fixed point measure $\Phi^{(l)}(\pi^{(l-1)}) = \pi^{(l)}$ with the following formula

$$C^{(l)}(f, g) := M_{\pi^{(l-1)}}^{(l)}[(P_{\pi^{(l-1)}}^{(l)}(f) - M_{\pi^{(l-1)}}^{(l)} P_{\pi^{(l-1)}}^{(l)}(f))(P_{\pi^{(l-1)}}^{(l)}(g) - M_{\pi^{(l-1)}}^{(l)} P_{\pi^{(l-1)}}^{(l)}(g))].$$

The main objective of this section is to prove the following theorem.

Theorem 4.2. *We consider a collection of weight array functions $(w^{(l)})_{l \geq 0} \in \mathcal{W}^{\mathbb{N}}$. In this situation, the corresponding flow of weighted random fields $(V_n^{(l)})_{l \geq 0}$ defined in (4.2), converges in law, as n tends to infinity and in the sense of finite dimensional distributions, to a sequence of independent and centered Gaussian fields $(V^{(l)})_{l \geq 0}$ with covariance functions defined in (4.3).*

Using this result, the proof of the multivariate central limit Theorem 2.1 follows exactly the same arguments as the ones we used in the proof of Theorem 2.1 in [5, Section 6].

Proof of Theorem 2.1. We let $\mathbb{S}^k := \mathbb{S}\mathbb{S}^{k-1}$ be the k-th iterate of the mapping $\mathbb{S} : \eta \in \mathcal{M}(S^{(l)})^{\mathbb{N}} \mapsto \mathbb{S}(\eta) = (\mathbb{S}_n(\eta))_{n \geq 0} \in \mathcal{M}(S^{(l)})^{\mathbb{N}}$ defined for any $\eta = (\eta_n)_{n \geq 0} \in \mathcal{M}(S^{(l)})^{\mathbb{N}}$ by

$$\forall n \geq 0 \quad \mathbb{S}_n(\eta) = \frac{1}{n+1} \sum_{0 \leq p \leq n} \eta_p.$$

We observe that the time averaged semigroup $\mathbb{S}^k$ can be rewritten in terms of the following weighted summations

$$\mathbb{S}_n^k(\eta) = \frac{1}{n+1} \sum_{0 \leq p \leq n} s_n^{(k)}(p)\eta_p$$

with the weight array functions $s_n^{(k)} := (s_n^{(k)}(p))_{0\le p\le n}$ defined by

$$\forall k \ge 1\ \forall 0 \le p \le n \quad s_n^{(k+1)}(p) = \sum_{p\le q\le n} \frac{1}{(q+1)} s_n^{(k)}(q) \quad \text{and} \quad s_n^{(1)}(p) := 1.$$

We also know from Proposition 6.1 in [5] that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{0\le q\le n} s_n^{(k+1)}(q)^2 = (2k)!/k!^2$$

and, for any $k \ge 1$, the weight array functions $w^{(k)}$ defined by

$$\forall n \ge 0\ \forall 0 \le p \le n \quad w_n^{(k)}(p) := s_n^{(k)}(p) \Big/ \sqrt{\sum_{0\le q\le n} s_n^{(k)}(q)^2}$$

belong to the set $\mathcal{W}$ introduced in Definition 3.2.

Moreover, using Proposition 5.2 in [5], we have the following multilevel expansion

$$\eta_n^{(k)} - \pi^{(k)} = \sum_{0\le l\le k} \mathbb{S}_n^{(l+1)}(\delta^{(k-l)}) D_{(k-l)+1,k} + \Xi_n^{(k)} \tag{4.4}$$

where $\Xi^{(k)} = (\Xi_n^{(k)})_{n\ge 0}$ is a flow of signed random measures such that

$$\forall m \ge 1 \qquad \sup_{f\in\mathcal{B}_1(S^{(k)})} \mathbb{E}(|\Xi_n^{(k)}(f)|^m)^{\frac{1}{m}} \le b(m)c(k)(\log(n+1))^k/(n+1).$$

Here $b(m)$ stands for some constant whose value only depends on the parameter m. This multilevel expansion implies that

$$\sqrt{(n+1)}[\eta_n^{(k)} - \pi^{(k)}] = \sum_{0\le l\le k} \sqrt{\frac{1}{n+1} \sum_{0\le q\le n} s_n^{(l+1)}(q)^2} \times W_n^{(k-l)}(\delta^{(k-l)}) D_{(k-l)+1,k} + \overline{\Xi}_n^{(k)}$$

with the weighted distribution flow mappings $W^{(k-l)}$ associated to the weight functions $w^{(l+1)}$ and a remainder signed measure $\overline{\Xi}_n^{(k)}$ such that

$$\sup_{f\in\mathcal{B}_1(S^{(k)})} \mathbb{E}(|\overline{\Xi}_n^{(k)}(f)|) \le c(k)(\log(n+1))^k/\sqrt{(n+1)}.$$

The proof of Theorem 2.1 is now a direct consequence of Theorem 4.2. □

4.2. A martingale limit theorem

This section is mainly concerned with the proof of Theorem 4.2. We following the same lines of arguments as the ones used in Section 3.3 devoted to the fluctuations of weighted occupation measures associated with time inhomogeneous Markov chains.

First, we introduce a few notations. For any $k \ge 0$ and any $\mu \in \mathcal{P}(S^{(k-1)})$, let $P_\mu^{(k)}$ be the resolvent operator associated to the Markov transition $M_\mu^{(k)}$ and its invariant measure $\Phi^{(k)}(\mu) \in \mathcal{P}(S^{(k)})$. We also set

$$p^{(k)}(n_k) := 2n_k/(1 - m_k(n_k))$$

with the pair of parameters (n_k, m_k) defined in (2.4).

Using Proposition 3.4, we find that the weighted measures $W_n^{(k)}(\delta^{(k)})$ satisfy the following decomposition

$$W_n^{(k)}(\delta^{(k)})(f) = \sum_{0\leq p\leq n} w_n^{(k)}(p)\Delta\mathbb{M}_{p+1}^{(k)}(f) + \mathbb{L}_n^{(k)}(f)$$

for any $f \in \mathcal{B}(S^{(k)})$ with the martingale increments

$$\Delta\mathbb{M}_{p+1}^{(k)}(f) = \mathbb{M}_{p+1}^{(k)}(f) - \mathbb{M}_p^{(k)}(f) = \left(P_{\eta_{p-1}^{(k-1)}}^{(k)}(f)(X_{p+1}^{(k)}) - M_{\eta_p^{(k-1)}}^{(k)} P_{\eta_{p-1}^{(k-1)}}^{(k)}(f)(X_p^{(k)})\right)$$

and the remainder signed measure $\mathbb{L}_n^{(k)}$ which are such that

$$\|\mathbb{L}_n^{(k)}\| \leq \{w_n^{(k)}(0)(1+2p^{(k)}(n_k)(1+\log(n+1)))(2+p^{(k)}(n_k))\Lambda\} \longrightarrow_{n\to\infty} 0.$$

We consider a sequence of functions $f = (f^i)_{1\leq i\leq d}$, with $d \geq 1$, and $f^i = (f_k^i)_{k\geq 0} \in \prod_{k\geq 0}\mathcal{B}(S^{(k)})$, and we let $\mathcal{W}^{(n)}(f) = (\mathcal{W}^{(n)}(f^i))_{1\leq i\leq d}$ be the $\mathbb{R}^d$-valued and $\mathcal{F}^{(n)}$-adapted process defined for any $l \geq 0$ and any $1 \leq i \leq d$ by

$$\mathcal{W}_l^{(n)}(f^i) := \sum_{0\leq k\leq l} W_n^{(k)}(\delta^{(k)})(f_k^i).$$

From the previous discussion, we find that

$$\mathcal{W}_l^{(n)}(f^i) = \mathcal{M}_l^{(n)}(f^i) + \mathcal{L}_l^{(n)}(f^i)$$

with the $\mathcal{F}^{(n)}$-martingale $\mathcal{M}_l^{(n)}(f^i)$ given below

$$\mathcal{M}_l^{(n)}(f^i) := \sum_{0\leq k\leq l} \Delta\mathcal{M}_k^{(n)}(f^i) \quad \text{with } \Delta\mathcal{M}_k^{(n)}(f^i) := \sum_{0\leq p\leq n} w_n^{(k)}(p)\Delta\mathbb{M}_{p+1}^{(k)}(f) \tag{4.5}$$

and the remainder bias type measure $\mathcal{L}_l^{(n)} = \sum_{0\leq k\leq l}\mathbb{L}_n^{(k)}$, such that

$$\lim_{n\to\infty}\|\mathcal{L}_l^{(n)}\| = 0.$$

Theorem 4.2 is now a direct consequence of the following proposition (see for instance the arguments used in Section 4.2 in [5]).

Proposition 4.3. *The sequence of martingales $\mathcal{M}_l^{(n)}(f)$ defined in (4.5) converges in law as $n \to \infty$ to an $\mathbb{R}^d$-valued Gaussian martingale $\mathcal{M}_l(f) = (\mathcal{M}_l(f^i))_{1\leq i\leq d}$ such that for any $l \geq 0$ and any pair of indexes $1 \leq i, j \leq d$*

$$\langle\mathcal{M}(f^i), \mathcal{M}(f^j)\rangle_l = \sum_{0\leq k\leq l}\pi^{(k)}[C^{(k)}(f^i, f^j)]$$

with the local covariance functions $\pi^{(k)}[C^{(k)}(f^i, f^j)]$ defined in (4.3).

Proof. The proof of the proposition is along the same lines as the proof of Theorem 3.5. First, we consider the decomposition

$$\mathcal{M}_l^{(n)}(f^i) = \sum_{i=0}^{l(n+1)+n}\mathcal{V}_i^{(n)}(f)$$

where for every $0 \le i \le l(n+1)+n$, with $i = k(n+1)+p$ for some $0 \le k \le l$, and $0 \le p \le n$

$$\mathcal{V}_i^{(n)}(f) := w_n^{(k)}(p)\Delta\mathbb{M}_{p+1}^{(k)}(f_k).$$

We further denote by $\mathcal{G}_i^{(n)}$ the σ-field generated by the pair of random variables $(X_p^{(k)}, X_{p+1}^{(k)})$ for any pair of parameters (k, p) such that $k(n+1)+p \le i$. By construction, for any flow of functions $f = (f_l)_{l\ge 0}$ and $g = (g_l)_{l\ge 0} \in \prod_{l\ge 0}\mathcal{B}(S^{(l)})$ and for every $0 \le i \le l(n+1)+n$, with $i = k(n+1)+p$ for some $0 \le k \le l$, and $0 \le p \le n$, we find that

$$\mathbb{E}(\mathcal{V}_i^{(n)}(f) \mid \mathcal{G}_{i-1}^{(n)}) = 0$$

$$\mathbb{E}(\mathcal{V}_i^{(n)}(f)\mathcal{V}_i^{(n)}(g) \mid \mathcal{G}_{i-1}^{(n)}) = w_n^{(k)}(p)^2 C_p^{(k)}(f, g)(X_p^{(k)})$$

with the local covariance function

$$C_p^{(k)}(f, g) := M^{(k)}_{\eta_p^{(k-1)}}\left[\left(P^{(k)}_{\eta_{p-1}^{(k-1)}}(f_k) - M^{(k)}_{\eta_p^{(k-1)}}P^{(k)}_{\eta_{p-1}^{(k-1)}}(f_k)\right)\right.$$
$$\left.\times\left(P^{(k)}_{\eta_{p-1}^{(k-1)}}(g_k) - M^{(k)}_{\eta_p^{(k-1)}}P^{(k)}_{\eta_{p-1}^{(k-1)}}(g_k)\right)\right].$$

Under our Lipschitz regularity conditions (2.4) and (2.5), Proposition 3.1 applies to the mappings $\Phi^{(k)}$ and the resolvent operators $P_\mu^{(k)}$. As in the proof of Theorem 3.5, after some tedious but elementary calculations, we obtain

$$\|C_p^{(k)}(f, g) - C^{(k)}(f, g)\| \le c(k)\left\{\int |[\eta_{p-1}^{(k-1)} - \pi^{(k-1)}](h)|\,\Upsilon^{(k),1}_{\pi^{(k)},\pi^{(k-1)}}((f_k, g_k), dh)\right.$$
$$\left. + \int |[\eta_p^{(k-1)} - \pi^{(k-1)}](h)|\,\Upsilon^{(k),2}_{\pi^{(k)},\pi^{(k-1)}}((f_k, g_k), dh)\right\}$$

where $\Upsilon^{(k),i}_{\pi^{(k)},\pi^{(k-1)}}$, $i = 1, 2$, is a pair of bounded integral operators from $\mathcal{B}(S^{(k)})^2$ into $\mathcal{B}(S^{(k-1)})$ such that

$$\int \|h\|\,\Upsilon^{(k),i}_{\pi^{(k)},\pi^{(k-1)}}((f_k, g_k), dh) \le c(k)\|f_k\|\,\|g_k\|.$$

Combining the generalized Minkowski integral inequality with (4.1) we prove the following almost sure convergence result

$$\lim_{p\to\infty}\|C_p^{(k)}(f, g) - C^{(k)}(f, g)\| = 0.$$

Arguing as in the proof of Theorem 3.5, we obtain the following almost sure convergence result

$$\lim_{n\to\infty}\sum_{p=0}^{n} w_n^{(k)}(p)^2 C_p^{(k)}(f, g)(X_p^{(k)}) = \lim_{n\to\infty}\sum_{p=0}^{n} w_n^{(k)}(p)^2 C^{(k)}(f, g)(X_p^{(k)})$$
$$= \lim_{n\to\infty}\sum_{p=0}^{n} w_n^{(k)}(p)^2\,\Phi^{(k)}(\eta_{p-1}^{(k-1)})(C^{(k)}(f, g))$$
$$= \Phi^{(k)}(\pi^{(k-1)})(C^{(k)}(f, g)) = \pi^{(k)}(C^{(k)}(f, g)).$$

This yields the almost sure convergence

$$\lim_{n\to\infty}\langle\mathcal{M}^{(n)}(f), \mathcal{M}^{(n)}(g)\rangle_l = \mathcal{C}_l^{(k)}(f, g) := \sum_{0\le k\le l}\pi^{(k)}(C^{(k)}(f, g)).$$

Using the same arguments as the ones used in the proof of Theorem 4.4 in [5], we conclude that the $\mathbb{R}^d$-valued martingale $\mathcal{M}_l^{(n)}(f)$ converges in law as n tends to infinity to a martingale $\mathcal{M}_l(f)$ with a predictable bracket given for any air of indexes $1 \leq j, j' \leq d$ by

$$\langle \mathcal{M}(f^j), \mathcal{M}(f^{j'}) \rangle_l = \mathcal{C}_l^{(k)}(f^j, f^{j'}).$$

This ends the proof of the proposition. □

5. Path space i-MCMC models

The aim of this final section is to show that the multivariate CLT discussed in Section 2 can be generalized directly to analyze the fluctuations of the occupation measures of $(X_n^{(k)})_{0\leq k\leq l}$ around the limiting tensor product measure $\otimes_{0\leq k\leq l}\pi^{(k)}$, for any time horizon $l \geq 0$. To do this, we check here that the regularity conditions discussed in Section 2.1 remain valid on path spaces.

Let us fix a final time horizon l for (1.1) and consider the path space model given by

$$\forall n \geq 0 \quad X_n^{[l]} := (X_n^{(0)}, \ldots, X_n^{(l)}) \in S^{[l]} := \prod_{0\leq k\leq l} S^{(k)}.$$

For every $0 \leq k \leq l$ we denote by $\eta^{(k)} \in \mathcal{P}(S^{(k)})$ the image measure of a measure $\eta \in \mathcal{P}(S^{[l]})$ on the k-th coordinate level set $S^{(k)}$ of the product space $S^{[l]} := \prod_{0\leq k\leq l} S^{(k)}$. Using this notation, it is easy to check that $X_n^{[l]}$ is an $S^{[l]}$-valued self-interacting Markov chain with transitions defined by

$$\mathbb{P}(X_{n+1}^{[l]} \in dx \mid (X_p^{[l-1]})_{0\leq p\leq n}, X_n^{[l]}) = M^{[l]}_{\eta_n^{[l-1]}}(X_n^{[l]}, dx) \tag{5.1}$$

with the occupation measures $\eta_n^{[l-1]}$ and the collection of transitions $M^{[l]}_{\eta_n^{[l-1]}}$ defined by the following formulae

$$\eta_n^{[l-1]} := \frac{1}{n+1}\sum_{p=0}^{n} \delta_{X_p^{[l-1]}} \quad \text{and} \quad M^{[l]}_{\eta_n^{[l-1]}}(X_n^{[l]}, dx) = \prod_{0\leq k\leq l} M^{(k)}_{\eta_n^{(k-1)}}(X_n^{(k)}, dx^l)$$

where $x := (x^0, \ldots, x^l) \in S^{[l]}$, $dx := dx^0 \times \cdots \times dx^l$ and we have used here the convention $M^{(0)}_{\eta_n^{(-1)}} = M^{(0)}$. It is straightforward to check that (5.1) coincides with the i-MCMC algorithm associated to the limiting evolution equation

$$\pi^{[l]} = \Phi^{[l]}(\pi^{[l-1]}) \quad \text{with } \pi^{[l]} := \pi^{(0)} \otimes \cdots \otimes \pi^{(l)}$$

and the invariant measure mapping

$$\Phi^{[l]} : \mu \in \mathcal{P}(S^{[l-1]}) \mapsto \Phi^{[l]}(\mu) := \pi^{(0)} \otimes \Phi^{(1)}(\mu^{(0)}) \otimes \cdots \otimes \Phi^{(l)}(\mu^{(l-1)}) \in \mathcal{P}(S^{[l]}).$$

To describe the main result of this section, we need to introduce some additional notation. For any $0 \leq k_1 \leq k_2$, we set

$$S^{[k_1,k_2]} := \prod_{k_1\leq k\leq k_2} S^{(k)} \quad \text{and} \quad \pi^{[k_1,k_2]} := \otimes_{k_1\leq k\leq k_2}\pi^{(k)} \in \mathcal{P}(S^{[k_1,k_2]}).$$

For any $0 \leq k < l$, any pair $(\mu_1, \mu_2) \in \mathcal{P}(S^{[0,k]}) \times \mathcal{P}(S^{[k+2,l+1]})$ and any integral operator D from $S^{(k)}$ into $S^{(k+1)}$, we denote by $\mu_1 \otimes D \otimes \mu_2$ the operator from $S^{[l]}$ into $S^{[l+1]}$

$$(\mu_1 \otimes D \otimes \mu_2)((x_1, x_2, x_3), dy_1 \times dy_2 \times dy_3) = \mu_1(dy_1)D(x_1, dy_2)\mu_2(dy_3)$$

where $(x_1, x_2, x_3) \in S^{[l]} = (S^{[0,k-1]} \otimes S^{(k)} \otimes S^{[k+1,l]})$ and $(y_1, y_2, y_3) \in S^{[l+1]} = (S^{[0,k]} \otimes S^{(k+1)} \otimes S^{[k+2,l+1]})$.

Proposition 5.1. *For any $l \geq 0$, the mappings $\Phi^{[l]}$ and the collection of Markov transition kernels $M^{[l]}_{\mu^{[l-1]}}$ satisfy the set of regularity conditions (2.1), (2.4), and (2.5) as long as the corresponding conditions are met for the marginal mappings $\Phi^{(k)}$ and the transitions $M^{(k)}_{\mu^{(k-1)}}$ where $1 \leq k \leq l$. In addition, the mappings $\Phi^{[l+1]}$ satisfy the first order decomposition (2.1) with bounded integral operators $D_{[l+1]}$ from $S^{[l]}$ into $S^{[l+1]}$ given by*

$$D_{[l+1]} = \pi^{[0,l]} \otimes D_{l+1} + \sum_{0 \leq k < l} \pi^{[0,k]} \otimes D_{k+1} \otimes \pi^{[k+2,l+1]}.$$

Before presenting the proof of this proposition, we emphasize that this latter directly implies that the multivariate CLT stated in Section 2.2 is also valid for the path space i-MCMC algorithm discussed above. In other words, for every $k \geq 0$, the sequence of random fields $(U_n^{[k]})_{n \geq 0}$ on $\mathcal{B}(S^{[k]})$ defined by

$$U_n^{[k]} := \sqrt{n}[\eta_n^{[k]} - \pi^{[k]}] = \frac{1}{\sqrt{n+1}} \sum_{p=0}^{n} [\delta_{(X_p^{(0}, \ldots, X_p^{(k)})} - (\pi^{(0)} \otimes \cdots \otimes \pi^{(k)})]$$

converges in law, as n tends to infinity and in the sense of finite dimensional distributions, to a sequence of Gaussian random fields $U^{[k]}$ defined as $U^{(k)}$ by replacing the semigroups D_{l_1,l_2} and the limiting measures $\pi^{(l)}$ by the corresponding objects on path spaces.

Proof of Proposition 5.1. With some obvious notation, we have

$$\forall n \geq 1 \quad (M^{[l]}_{\mu^{[l-1]}})^n = \otimes_{0 \leq k \leq l} (M^{(k)}_{\mu^{(k-1)}})^n.$$

Using the fact that

$$\| \otimes_{0 \leq k \leq l} \mu^{(k)} - \otimes_{0 \leq k \leq l} \eta^{(k)} \| \leq \sum_{0 \leq k \leq l} \|\mu^{(k)} - \eta^{(k)}\|$$

for any sequence of probability measures $\mu^{(k)}, \eta^{(k)} \in \mathcal{P}(S^{(k)})$, with $0 \leq k \leq l$, we prove that

$$\beta((M^{[l]}_{\mu^{[l-1]}})^n) \leq \sum_{0 \leq k \leq l} \beta((M^{(k)}_{\mu^{(k-1)}})^n) \leq \frac{1}{l+1} \sum_{0 \leq k \leq l} m_k(n_k) < 1$$

as soon as

$$n \geq n_{[l]} := (\vee_{0 \leq k \leq l}\ n_k) \times \left(1 + \frac{\log(l+1)}{\wedge_{0 \leq k \leq l} \log(1/m_k(n_k))}\right).$$

We prove the pair of regularity conditions (2.1) and (2.5) by induction on the parameter l. We use the notation $D_{[l]}$, $\Xi_{[l]}$, $\Xi_{[l]}$ $m_{[l]}$, $n_{[l]}$ and $\Gamma_{[l],\mu}$ the corresponding objects introduced in the statement of conditions (2.1), (2.4), and (2.5). The results are clearly true for $m = 0$ with

$$\Phi^{[0]}(\mu) := \pi^{(0)} \quad \text{and} \quad M^{[0]}_{\eta_n^{[-1]}} := M^{(0)}.$$

In this case, we readily find that

$$D_{[0]} = D_0 = 0, \qquad \Xi_{[l]} = 0, \qquad m_{[0]} = m_0, \qquad n_{[0]} = n_0 \quad \text{and} \quad \Gamma_{[0],\mu} = \Gamma_{0,\mu} = 0.$$

Assume now that the result has been proved at some rank l. For any measure μ on $S^{[l]} = S^{[l-1]} \times S^{(l)}$, we denote by $\mu^{[l-1]}$ and $\mu^{(l)}$ its image measures on $S^{[l-1]}$ and $S^{(l)}$. In this notation we have

$$\Phi^{[l+1]}(\mu) = \Phi^{[l]}(\mu^{[l-1]}) \otimes \Phi^{(l+1)}(\mu^{(l)})$$

and

$$M^{[l+1]}_{\mu^{[l]}}((u,v), dx \times dy) = M^{[l]}_{\mu^{[l-1]}}(u, dx) \times M^{(l+1)}_{\mu^{(l)}}(v, dy)$$

for any $(u, v) \in S^{[l+1]} = (S^{[l]} \times S^{(l)})$ and $(x, y) \in S^{[l+1]} = (S^{[l]} \times S^{(l+1)})$. After some elementary computations, using the decomposition

$$\begin{aligned}
[\Phi^{[l+1]}(\mu) - \Phi^{[l+1]}(\pi^{[l]})] &= \Phi^{[l]}(\mu^{[l-1]}) \otimes \Phi^{(l+1)}(\mu^{(l)}) - \Phi^{[l]}(\pi^{[l-1]}) \otimes \Phi^{(l+1)}(\pi^{(l)}) \\
&= \Phi^{[l]}(\pi^{[l-1]}) \otimes [\Phi^{(l+1)}(\mu^{(l)}) - \Phi^{(l+1)}(\pi^{(l)})] \\
&\quad + [\Phi^{[l]}(\mu^{[l-1]}) - \Phi^{[l]}(\pi^{[l-1]})] \otimes \Phi^{(l+1)}(\pi^{(l)}) \\
&\quad + [\Phi^{[l]}(\mu^{[l-1]}) - \Phi^{[l]}(\pi^{[l-1]})] \\
&\quad \otimes [\Phi^{(l+1)}(\mu^{(l)}) - \Phi^{(l+1)}(\pi^{(l)})]
\end{aligned}$$

we find that the first order condition (2.1) is satisfied with an integral operator $D_{[l+1]}$ from $S^{[l]}$ into $S^{[l+1]}$ defined for any $f \in$ by

$$\begin{aligned}
D_{[l+1]}(u, d(x,y)) &= \Phi^{[l]}(\pi^{[l-1]})(dx) D_{(l+1)}(u, dy) + D_{[l]}(u, dx) \Phi^{(l+1)}(\pi^{(l)})(dy) \\
&= \pi^{[l]}(dx) D_{(l+1)}(u, dy) + D_{[l]}(u, dx) \pi^{(l+1)}(dy).
\end{aligned}$$

Condition (2.5) is proved using the same type of arguments. This ends the proof of the proposition. □

References

[1] C. Andrieu, A. Jasra, A. Doucet, P. Del Moral, On nonlinear Markov chain Monte Carlo, Bernoulli 17 (2011) 987–1014.

[2] C. Andrieu, E. Moulines, On the ergodicity properties of some adaptive Markov chain Monte Carlo algorithms, Ann. Appl. Probab. 16 (2006) 1462–1505.

[3] C. Andrieu, J. Thoms, A tutorial on adaptive Markov chain Monte Carlo, Stat. Comput. 18 (2008) 343–373.

[4] Y. Atchadé, A cautionary tale on the efficiency of some adaptive Monte Carlo schemes, Ann. Appl. Probab. 20 (2010) 841–868.

[5] B. Bercu, P. Del Moral, A. Doucet, A functional central limit theorem for a class of interacting Markov chain Monte Carlo methods, Electron. J. Probab. 14 (2009) 2130–2155.

[6] B. Bercu, P. Del Moral, A. Doucet, Fluctuations of interacting Markov chain Monte Carlo methods, Technical Report INRIA RR-6438, February 2008. Available at http://hal.inria.fr/inria-00227536/fr/.

[7] A. Brockwell, P. Del Moral, A. Doucet, Sequentially interacting Markov chain Monte Carlo methods, Ann. Statist. 38 (2010) 3387–3411.

[8] P. Del Moral, Feynman–Kac Formulae. Genealogical and Interacting Particle Systems, with Applications, in: Series: Probability and its Applications, Springer Verlag, New York, 2004.

[9] P. Del Moral, A. Doucet, Interacting Markov chain Monte Carlo methods for solving nonlinear measure-valued equations, Ann. Appl. Probab. 20 (2010) 593–639.

[10] A. Doucet, N. de Freitas, N. Gordon, Sequential Monte Carlo Methods in Practice, in: Series: Statistics for Engineering and Information Science, Springer Verlag, New York, 2001.

[11] S. Kou, Q. Zhou, W. Wong, Equi-energy sampler with applications in statistical inference and statistical mechanics (with discussion), Ann. Statist. 3 (4) (2006) 1581–1652.

[12] E. Lyman, D.M. Zuckerman, Resolution exchange simulation with incremental coarsening, J. Chem. Theory Comput. 2 (2006) 656–666.

[13] K.L. Mengersen, R. Tweedie, Rates of convergence of the Hastings and Metropolis algorithms, Ann. Statist. 24 (1996) 101–121.

[14] G.O. Roberts, J.S. Rosenthal, Coupling and ergodicity of adaptive Markov chain Monte Carlo algorithms, J. Appl. Probab. 44 (2007) 458–475.