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# Almost sure central limit theorems on the Wiener space

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**Abstract**

In this paper, we study almost sure central limit theorems for sequences of functionals of general Gaussian fields. We apply our result to non-linear functions of stationary Gaussian sequences. We obtain almost sure central limit theorems for these non-linear functions when they converge in law to a normal distribution.


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*Keywords:* Almost sure limit theorem; Multiple stochastic integrals; Fractional Brownian motion; Hermite power variation

## 1. Introduction

Let $\{X_n\}_{n\geqslant 1}$ be a sequence of real-valued independent identically distributed random variables with $E[X_n]=0$ and $E[X_n^2]=1$, and define

$$S_n=\frac{1}{\sqrt{n}}\sum_{k=1}^{n}X_k.$$

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The celebrated almost sure central limit theorem (ASCLT) states that the sequence of random empirical measures given by

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\delta_{S_k}$$

converges almost surely to the $\mathcal{N}(0,1)$ distribution as $n\to\infty$. In other words, if $N$ is a $\mathcal{N}(0,1)$ random variable, then, almost surely, for all $x\in\mathbb{R}$,

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbf{1}_{\{S_k\leqslant x\}}\longrightarrow P(N\leqslant x),\quad \text{as } n\to\infty,$$

or, equivalently, almost surely, for any bounded and continuous function $\varphi:\mathbb{R}\to\mathbb{R}$,

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\varphi(S_k)\longrightarrow E[\varphi(N)],\quad \text{as } n\to\infty. \tag{1.1}$$

The ASCLT was stated first by Lévy [14] without proof. It was then forgotten for half a century. It was rediscovered by Brosamler [7] and Schatte [20] and proven, in its present form, by Lacey and Philipp [13]. We refer the reader to Berkes and Csáki [1] for a universal ASCLT covering a large class of limit theorems for partial sums, extremes, empirical distribution functions and local times associated with independent random variables $\{X_n\}$, as well as to the work of Gonchigdanzan [10], where extensions of the ASCLT to weakly dependent random variables are studied, for example in the context of strong mixing or $\rho$-mixing. Ibragimov and Lifshits [12,11] have provided a criterion for (1.1) which does not require the sequence $\{X_n\}$ of random variables to be necessarily independent or the sequence $\{S_n\}$ to take the specific form of partial sums. This criterion is stated in Theorem 3.1.

Our goal in the present paper is to investigate the ASCLT for a sequence of functionals of general Gaussian fields. Conditions ensuring the convergence in law of this sequence to the standard $\mathcal{N}(0,1)$ distribution may be found in [15,16] by Nourdin, Peccati and Reinert. Here, we shall propose a suitable criterion for this sequence of functionals to satisfy also the ASCLT. As an application, we shall consider some non-linear functions of strongly dependent Gaussian random variables.

The paper is organized as follows. In Section 2, we present the basic elements of Gaussian analysis and Malliavin calculus used in this paper. An abstract version of our ASCLT is stated and proven in Section 3, as well as an application to partial sums of non-linear functions of a strongly dependent Gaussian sequence. In Section 4, we apply our ASCLT to discrete-time fractional Brownian motion. In Section 5, we consider applications to partial sums of Hermite polynomials of strongly dependent Gaussian sequences, when the limit in distribution is Gaussian. Finally, in Section 6, we discuss the case where the limit in distribution is non-Gaussian.

## 2. Elements of Malliavin calculus

We shall now present the basic elements of Gaussian analysis and Malliavin calculus that are used in this paper. The reader is referred to the monograph by Nualart [17] for any unexplained definition or result.

Let $\mathfrak{H}$ be a real separable Hilbert space. For any $q\geqslant 1$, let $\mathfrak{H}^{\otimes q}$ be the $q$th tensor product of $\mathfrak{H}$ and denote by $\mathfrak{H}^{\odot q}$ the associated $q$th symmetric tensor product. We write $X=\{X(h),h\in\mathfrak{H}\}$

to indicate an isonormal Gaussian process over $\mathfrak{H}$, defined on some probability space $(\Omega, \mathcal{F}, P)$. This means that $X$ is a centered Gaussian family, whose covariance is given in terms of the scalar product of $\mathfrak{H}$ by $E[X(h)X(g)] = \langle h, g\rangle_{\mathfrak{H}}$.

For every $q \geqslant 1$, let $\mathcal{H}_q$ be the $q$th Wiener chaos of $X$, that is, the closed linear subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the family of random variables $\{H_q(X(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where $H_q$ is the $q$th Hermite polynomial defined as

$$H_q(x) = (-1)^q \mathrm{e}^{\frac{x^2}{2}} \frac{\mathrm{d}^q}{\mathrm{d}x^q}\big(\mathrm{e}^{-\frac{x^2}{2}}\big). \tag{2.2}$$

The first few Hermite polynomials are $H_1(x) = x$, $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x$. We write by convention $\mathcal{H}_0 = \mathbb{R}$ and $I_0(x) = x$, $x \in \mathbb{R}$. For any $q \geqslant 1$, the mapping $I_q(h^{\otimes q}) = H_q(X(h))$ can be extended to a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot q}$ equipped with the modified norm $\|\cdot\|_{\mathfrak{H}^{\odot q}} = \sqrt{q!}\,\|\cdot\|_{\mathfrak{H}^{\otimes q}}$ and the $q$th Wiener chaos $\mathcal{H}_q$. Then

$$E[I_p(f)I_q(g)] = \delta_{p,q} \times p!\langle f, g\rangle_{\mathfrak{H}^{\otimes p}} \tag{2.3}$$

where $\delta_{p,q}$ stands for the usual Kronecker symbol, for $f \in \mathfrak{H}^{\odot p}$, $g \in \mathfrak{H}^{\odot q}$ and $p, q \geqslant 1$. Moreover, if $f \in \mathfrak{H}^{\otimes q}$, we have

$$I_q(f) = I_q(\widetilde{f}), \tag{2.4}$$

where $\widetilde{f} \in \mathfrak{H}^{\odot q}$ is the symmetrization of $f$.

It is well known that $L^2(\Omega, \mathcal{F}, P)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_q$. Therefore, any square integrable random variable $G \in L^2(\Omega, \mathcal{F}, P)$ admits the following Wiener chaotic expansion:

$$G = E[G] + \sum_{q=1}^{\infty} I_q(f_q), \tag{2.5}$$

where the $f_q \in \mathfrak{H}^{\odot q}$, $q \geqslant 1$, are uniquely determined by $G$.

Let $\{e_k,\ k \geqslant 1\}$ be a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, for every $r = 0, \ldots, p \wedge q$, the *contraction* of $f$ and $g$ of order $r$ is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined by

$$f \otimes_r g = \sum_{i_1,\ldots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_r}\rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_r}\rangle_{\mathfrak{H}^{\otimes r}}. \tag{2.6}$$

Since $f \otimes_r g$ is not necessarily symmetric, we denote its symmetrization by $f\widetilde{\otimes}_r g \in \mathfrak{H}^{\odot(p+q-2r)}$. Observe that $f \otimes_0 g = f \otimes g$ equals the tensor product of $f$ and $g$ while, for $p = q$, $f \otimes_q g = \langle f, g\rangle_{\mathfrak{H}^{\otimes q}}$, namely the scalar product of $f$ and $g$. In the particular case $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$, where $(A, \mathcal{A})$ is a measurable space and $\mu$ is a $\sigma$-finite and non-atomic measure, one has that $\mathfrak{H}^{\odot q} = L^2_s(A^q, \mathcal{A}^{\otimes q}, \mu^{\otimes q})$ is the space of symmetric and square integrable functions on $A^q$. In this case, (2.6) can be rewritten as

$$(f \otimes_r g)(t_1, \ldots, t_{p+q-2r}) = \int_{A^r} f(t_1, \ldots, t_{p-r}, s_1, \ldots, s_r) \times g(t_{p-r+1}, \ldots, t_{p+q-2r}, s_1, \ldots, s_r)\mathrm{d}\mu(s_1)\ldots\mathrm{d}\mu(s_r),$$

that is, we identify $r$ variables in $f$ and $g$ and integrate them out. We shall make use of the following lemma whose proof is a straightforward application of the definition of contractions and the Fubini theorem.

**Lemma 2.1.** *Let $f, g \in \mathfrak{H}^{\odot 2}$. Then $\|f \otimes_1 g\|^2_{\mathfrak{H}^{\otimes 2}} = \langle f \otimes_1 f, g \otimes_1 g \rangle_{\mathfrak{H}^{\otimes 2}}$.*

Let us now introduce some basic elements of the Malliavin calculus with respect to the isonormal Gaussian process $X$. Let $\mathcal{S}$ be the set of all cylindrical random variables of the form

$$G = \varphi\left(X(h_1), \ldots, X(h_n)\right), \tag{2.7}$$

where $n \geqslant 1$, $\varphi : \mathbb{R}^n \to \mathbb{R}$ is an infinitely differentiable function with compact support and $h_i \in \mathfrak{H}$. The *Malliavin derivative* of $G$ with respect to $X$ is the element of $L^2(\Omega, \mathfrak{H})$ defined as

$$DG = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}\left(X(h_1), \ldots, X(h_n)\right) h_i. \tag{2.8}$$

By iteration, one can define the $m$th derivative $D^m G$, which is an element of $L^2(\Omega, \mathfrak{H}^{\odot m})$, for every $m \geqslant 2$. For instance, for $G$ as in (2.7), we have

$$D^2 G = \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(X(h_1), \ldots, X(h_n)) h_i \otimes h_j.$$

For $m \geqslant 1$ and $p \geqslant 1$, $\mathbb{D}^{m,p}$ denotes the closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{m,p}$, defined by the relation

$$\|G\|^p_{m,p} = E\left[|G|^p\right] + \sum_{i=1}^{m} E\left(\|D^i G\|^p_{\mathfrak{H}^{\otimes i}}\right). \tag{2.9}$$

In particular, $DX(h) = h$ for every $h \in \mathfrak{H}$. The Malliavin derivative $D$ verifies moreover the following *chain rule*. If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with bounded partial derivatives and if $G = (G_1, \ldots, G_n)$ is a vector of elements of $\mathbb{D}^{1,2}$, then $\varphi(G) \in \mathbb{D}^{1,2}$ and

$$D\varphi(G) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(G) DG_i.$$

Let now $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$ with $\mu$ non-atomic. Then an element $u \in \mathfrak{H}$ can be expressed as $u = \{u_t, t \in A\}$ and the Malliavin derivative of a multiple integral $G$ of the form $I_q(f)$ (with $f \in \mathfrak{H}^{\odot q}$) is the element $DG = \{D_t G, t \in A\}$ of $L^2(A \times \Omega)$ given by

$$D_t G = D_t\left[I_q(f)\right] = q I_{q-1}\left(f(\cdot, t)\right). \tag{2.10}$$

Thus the derivative of the random variable $I_q(f)$ is the stochastic process $q I_{q-1}\big(f(\cdot, t)\big)$, $t \in A$. Moreover,

$$\|D\left[I_q(f)\right]\|^2_{\mathfrak{H}} = q^2 \int_A I_{q-1}\left(f(\cdot, t)\right)^2 \mu(\mathrm{d}t).$$

For any $G \in L^2(\Omega, \mathcal{F}, P)$ as in (2.5), we define

$$L^{-1} G = -\sum_{q=1}^{\infty} \frac{1}{q} I_q(f_q). \tag{2.11}$$

It is proven in [15] that for every centered $G \in L^2(\Omega, \mathcal{F}, P)$ and every $\mathcal{C}^1$ and Lipschitz function $h : \mathbb{R} \to \mathbb{C}$,

$$E[Gh(G)] = E[h'(G)\langle DG, -DL^{-1}G\rangle_{\mathfrak{H}}]. \tag{2.12}$$

In the particular case $h(x) = x$, we obtain from (2.12) that

$$\mathrm{Var}[G] = E[G^2] = E[\langle DG, -DL^{-1}G\rangle_{\mathfrak{H}}], \tag{2.13}$$

where 'Var' denotes the variance. Moreover, if $G \in \mathbb{D}^{2,4}$ is centered, then it is shown in [16] that

$$\mathrm{Var}[\langle DG, -DL^{-1}G\rangle] \leqslant \frac{5}{2} E[\|DG\|_{\mathfrak{H}}^4]^{\frac{1}{2}} E[\|D^2G \otimes_1 D^2G\|_{\mathfrak{H}^{\otimes 2}}^2]^{\frac{1}{2}}. \tag{2.14}$$

Finally, we shall also use the following bound, established in a slightly different way in [16, Corollary 4.2], for the difference between the characteristic functions of a centered random variable in $\mathbb{D}^{2,4}$ and of a standard Gaussian random variable.

**Lemma 2.2.** *Let $G \in \mathbb{D}^{2,4}$ be centered. Then, for any $t \in \mathbb{R}$, we have*

$$\begin{aligned} \left|E[e^{itG}] - e^{-t^2/2}\right| &\leqslant |t| \left|1 - E[G^2]\right| \\ &\quad + \frac{|t|}{2}\sqrt{10}\, E[\|D^2G \otimes_1 D^2G\|_{\mathfrak{H}^{\otimes 2}}^2]^{\frac{1}{4}} E[\|DG\|_{\mathfrak{H}}^4]^{\frac{1}{4}}. \end{aligned} \tag{2.15}$$

**Proof.** For all $t \in \mathbb{R}$, let $\varphi(t) = e^{t^2/2}E[e^{itG}]$. It follows from (2.12) that

$$\varphi'(t) = te^{t^2/2}E[e^{itG}] + ie^{t^2/2}E[Ge^{itG}] = te^{t^2/2}E[e^{itG}(1 - \langle DG, -DL^{-1}G\rangle_{\mathfrak{H}})].$$

Hence, we obtain that

$$|\varphi(t) - \varphi(0)| \leqslant \sup_{u\in[0,t]} |\varphi'(u)| \leqslant |t|e^{t^2/2}E\big[|1 - \langle DG, -DL^{-1}G\rangle_{\mathfrak{H}}|\big],$$

which leads to

$$\left|E[e^{itG}] - e^{-t^2/2}\right| \leqslant |t|\, E\big[|1 - \langle DG, -DL^{-1}G\rangle_{\mathfrak{H}}|\big].$$

Consequently, we deduce from (2.13) together with the Cauchy–Schwarz inequality that

$$\begin{aligned} \left|E[e^{itG}] - e^{-t^2/2}\right| &\leqslant |t| \left|1 - E[G^2]\right| + |t|\, E\big[|E[G^2] - \langle DG, -DL^{-1}G\rangle_{\mathfrak{H}}|\big] \\ &\leqslant |t| \left|1 - E[G^2]\right| + |t|\sqrt{\mathrm{Var}\big(\langle DG, -DL^{-1}G\rangle_{\mathfrak{H}}\big)}. \end{aligned}$$

We conclude the proof of Lemma 2.2 by using (2.14). □

## 3. A criterion for ASCLT on the Wiener space

The following result, due to Ibragimov and Lifshits [12], gives a sufficient condition for extending convergence in law to ASCLT. It will play a crucial role in all of the sequel.

**Theorem 3.1.** Let $\{G_n\}$ be a sequence of random variables converging in distribution towards a random variable $G_\infty$, and set

$$\Delta_n(t) = \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\big(e^{itG_k} - E(e^{itG_\infty})\big). \tag{3.16}$$

If, for all $r > 0$,

$$\sup_{|t|\leqslant r} \sum_n \frac{E|\Delta_n(t)|^2}{n \log n} < \infty, \tag{3.17}$$

then, almost surely, for all continuous and bounded functions $\varphi : \mathbb{R} \to \mathbb{R}$, we have

$$\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \varphi(G_k) \longrightarrow E[\varphi(G_\infty)], \quad \text{as } n \to \infty.$$

The following theorem is the main abstract result of this section. It provides a suitable criterion for an ASCLT for normalized sequences in $\mathbb{D}^{2,4}$.

**Theorem 3.2.** Retain the notation of Section 2. Let $\{G_n\}$ be a sequence in $\mathbb{D}^{2,4}$ satisfying, for all $n \geqslant 1$, $E[G_n] = 0$ and $E[G_n^2] = 1$. Assume that

$$(A_0) \quad \sup_{n\geqslant 1} E\big[\|DG_n\|_{\mathfrak{H}}^4\big] < \infty,$$

and

$$E[\|D^2 G_n \otimes_1 D^2 G_n\|^2_{\mathfrak{H}^{\otimes 2}}] \to 0, \quad \text{as } n \to \infty.$$

Then, $G_n \xrightarrow{\text{law}} N \sim \mathscr{N}(0, 1)$ as $n \to \infty$. Moreover, assume that the two following conditions also hold:

$$(A_1) \quad \sum_{n\geqslant 2} \frac{1}{n \log^2 n} \sum_{k=1}^n \frac{1}{k} E[\|D^2 G_k \otimes_1 D^2 G_k\|^2_{\mathfrak{H}^{\otimes 2}}]^{\frac{1}{4}} < \infty,$$

$$(A_2) \quad \sum_{n\geqslant 2} \frac{1}{n \log^3 n} \sum_{k,l=1}^n \frac{|E(G_k G_l)|}{kl} < \infty.$$

Then, $\{G_n\}$ satisfies an ASCLT. In other words, almost surely, for all continuous and bounded functions $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \varphi(G_k) \longrightarrow E[\varphi(N)], \quad \text{as } n \to \infty.$$

**Remark 3.3.** If there exists $\alpha > 0$ such that $E[\|D^2 G_k \otimes_1 D^2 G_k\|^2_{\mathfrak{H}^{\otimes 2}}] = O(k^{-\alpha})$, as $k \to \infty$, then $(A_1)$ is clearly satisfied. On the other hand, if there exists $C, \alpha > 0$ such that $|E[G_k G_l]| \leqslant C \left(\frac{k}{l}\right)^\alpha$ for all $k \leqslant l$, then, for some positive constants $a, b$ independent of $n$, we have

$$\sum_{n\geqslant 2} \frac{1}{n \log^3 n} \sum_{l=1}^n \frac{1}{l} \sum_{k=1}^l \frac{|E[G_k G_l]|}{k} \leqslant C \sum_{n\geqslant 2} \frac{1}{n \log^3 n} \sum_{l=1}^n \frac{1}{l^{1+\alpha}} \sum_{k=1}^l k^{\alpha-1}$$
$$\leqslant a \sum_{n\geqslant 2} \frac{1}{n \log^3 n} \sum_{l=1}^n \frac{1}{l} \leqslant b \sum_{n\geqslant 2} \frac{1}{n \log^2 n} < \infty,$$

which means that $(A_2)$ is also satisfied.

**Proof of Theorem 3.2.** The fact that $G_n \xrightarrow{\text{law}} N \sim \mathscr{N}(0, 1)$ follows from [16, Corollary 4.2]. In order to prove that the ASCLT holds, we shall verify the sufficient condition (3.17), that is, the

Ibragimov–Lifshits criterion. For simplicity, let $g(t) = E(\mathrm{e}^{\mathrm{i}tN}) = \mathrm{e}^{-t^2/2}$. Then, we have

$$\begin{aligned}E|\Delta_n(t)|^2 &= \frac{1}{\log^2 n}\sum_{k,l=1}^{n}\frac{1}{kl}E\left[\left(\mathrm{e}^{\mathrm{i}tG_k}-g(t)\right)\left(\mathrm{e}^{-\mathrm{i}tG_l}-g(t)\right)\right]\\ &= \frac{1}{\log^2 n}\sum_{k,l=1}^{n}\frac{1}{kl}\left[E\left(\mathrm{e}^{\mathrm{i}t(G_k-G_l)}\right)-g(t)\left(E\left(\mathrm{e}^{\mathrm{i}tG_k}\right)+E\left(\mathrm{e}^{-\mathrm{i}tG_l}\right)\right)+g^2(t)\right]\\ &= \frac{1}{\log^2 n}\sum_{k,l=1}^{n}\frac{1}{kl}\left[\left(E\left(\mathrm{e}^{\mathrm{i}t(G_k-G_l)}\right)-g^2(t)\right)-g(t)\left(E\left(\mathrm{e}^{\mathrm{i}tG_k}\right)-g(t)\right)\right.\\ &\quad\left.-g(t)\left(E\left(\mathrm{e}^{-\mathrm{i}tG_l}\right)-g(t)\right)\right].\end{aligned}\tag{3.18}$$

Let $t \in \mathbb{R}$ and $r > 0$ be such that $|t| \leqslant r$. It follows from inequality (2.15) together with assumption $(A_0)$ that

$$\left|E\left(\mathrm{e}^{\mathrm{i}tG_k}\right)-g(t)\right| \leqslant \frac{r\xi}{2}\sqrt{10}\,E[\|D^2G_k\otimes_1 D^2G_k\|^2_{\mathfrak{H}^{\otimes 2}}]^{\frac14}\tag{3.19}$$

where $\xi = \sup_{n\geqslant 1} E\left[\|DG_n\|^4_{\mathfrak{H}}\right]^{\frac14}$. Similarly,

$$\left|E\left(\mathrm{e}^{-\mathrm{i}tG_l}\right)-g(t)\right| \leqslant \frac{r\xi}{2}\sqrt{10}\,E[\|D^2G_l\otimes_1 D^2G_l\|^2_{\mathfrak{H}^{\otimes 2}}]^{\frac14}.\tag{3.20}$$

On the other hand, we also have via (2.15) that

$$\begin{aligned}\left|E\left(\mathrm{e}^{\mathrm{i}t(G_k-G_l)}\right)-g^2(t)\right| &= \left|E\left(\mathrm{e}^{\mathrm{i}t\sqrt{2}\,\frac{G_k-G_l}{\sqrt{2}}}\right)-g(\sqrt{2}\,t)\right|\\ &\leqslant \sqrt{2}r\left|1-\frac12 E[(G_k-G_l)^2]\right| + r\xi\sqrt{5}\,E[\|D^2(G_k-G_l)\otimes_1 D^2(G_k-G_l)\|^2_{\mathfrak{H}^{\otimes 2}}]^{\frac14}\\ &\leqslant \sqrt{2}r|E[G_kG_l]| + r\xi\sqrt{5}\,E[\|D^2(G_k-G_l)\otimes_1 D^2(G_k-G_l)\|^2_{\mathfrak{H}^{\otimes 2}}]^{\frac14}.\end{aligned}$$

Moreover

$$\begin{aligned}\|D^2(G_k-G_l)\otimes_1 D^2(G_k-G_l)\|^2_{\mathfrak{H}^{\otimes 2}} &\leqslant 2\|D^2G_k\otimes_1 D^2G_k\|^2_{\mathfrak{H}^{\otimes 2}}\\ &\quad+2\|D^2G_l\otimes_1 D^2G_l\|^2_{\mathfrak{H}^{\otimes 2}}+4\|D^2G_k\otimes_1 D^2G_l\|^2_{\mathfrak{H}^{\otimes 2}}.\end{aligned}$$

In addition, we infer from Lemma 2.1 that

$$\begin{aligned}E\left[\|D^2G_k\otimes_1 D^2G_l\|^2_{\mathfrak{H}^{\otimes 2}}\right] &= E\left[\langle D^2G_k\otimes_1 D^2G_k, D^2G_l\otimes_1 D^2G_l\rangle_{\mathfrak{H}^{\otimes 2}}\right]\\ &\leqslant \left(E\left[\|D^2G_k\otimes_1 D^2G_k\|^2_{\mathfrak{H}^{\otimes 2}}\right]\right)^{\frac12}\left(E\left[\|D^2G_l\otimes_1 D^2G_l\|^2_{\mathfrak{H}^{\otimes 2}}\right]\right)^{\frac12}\\ &\leqslant \frac12 E\left[\|D^2G_k\otimes_1 D^2G_k\|^2_{\mathfrak{H}^{\otimes 2}}\right]+\frac12 E\left[\|D^2G_l\otimes_1 D^2G_l\|^2_{\mathfrak{H}^{\otimes 2}}\right].\end{aligned}$$

Consequently, we deduce from the elementary inequality $(a+b)^{\frac14} \leqslant a^{\frac14}+b^{\frac14}$ that

$$\begin{aligned}\left|E\left(\mathrm{e}^{\mathrm{i}t(G_k-G_l)}\right)-g^2(t)\right| &\leqslant \sqrt{2}r|E[G_kG_l]| + r\xi\sqrt{10}\left(E\left[\|D^2G_k\otimes_1 D^2G_k\|^2_{\mathfrak{H}^{\otimes 2}}\right]^{\frac14}\right.\\ &\quad\left.+E\left[\|D^2G_l\otimes_1 D^2G_l\|^2_{\mathfrak{H}^{\otimes 2}}\right]^{\frac14}\right).\end{aligned}\tag{3.21}$$

Finally, (3.17) follows from the conjunction of $(A_1)$ and $(A_2)$ together with (3.18)–(3.21), which completes the proof of Theorem 3.2. □

We now provide an explicit application of Theorem 3.2.

**Theorem 3.4.** Let $X = \{X_n\}_{n\in\mathbb{Z}}$ denote a centered stationary Gaussian sequence with unit variance, such that $\sum_{r\in\mathbb{Z}} |\rho(r)| < \infty$, where $\rho(r) = E[X_0 X_r]$. Let $f : \mathbb{R} \to \mathbb{R}$ be a symmetric real function of class $\mathcal{C}^2$, and let $N \sim \mathcal{N}(0, 1)$. Assume moreover that $f$ is not constant and that $E[f''(N)^4] < \infty$. For any $n \geqslant 1$, let

$$G_n = \frac{1}{\sigma_n \sqrt{n}} \sum_{k=1}^{n} \big(f(X_k) - E[f(X_k)]\big)$$

where $\sigma_n$ is the positive normalizing constant which ensures that $E[G_n^2] = 1$. Then, as $n \to \infty$, $G_n \xrightarrow{\text{law}} N$ and $\{G_n\}$ satisfies an ASCLT. In other words, almost surely, for any continuous and bounded function $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi(G_k) \longrightarrow E[\varphi(N)], \quad \text{as } n \to \infty.$$

**Remark 3.5.** We can replace the assumption '$f$ is symmetric and non-constant' with

$$\sum_{q=1}^{\infty} \frac{1}{q!} \big(E[f(N) H_q(N)]\big)^2 \sum_{r\in\mathbb{Z}} |\rho(r)|^q < \infty \quad \text{and}$$
$$\sum_{q=1}^{\infty} \frac{1}{q!} \big(E[f(N) H_q(N)]\big)^2 \sum_{r\in\mathbb{Z}} \rho(r)^q > 0.$$

Indeed, it suffices to replace the monotone convergence argument used to prove (3.22) by a bounded convergence argument. However, this new assumption seems rather difficult to check in general, except of course when the sum with respect to $q$ is finite, that is, when $f$ is a polynomial.

**Proof of Theorem 3.4.** First, note that a consequence of [16, inequality (3.19)] is that we automatically have $E[f'(N)^4] < \infty$ and $E[f(N)^4] < \infty$. Let us now expand $f$ in terms of Hermite polynomials. Since $f$ is symmetric, we can write

$$f = E[f(N)] + \sum_{q=1}^{\infty} c_{2q} H_{2q},$$

where the real numbers $c_{2q}$ are given by $(2q)! c_{2q} = E[f(N) H_{2q}(N)]$. Consequently,

$$\begin{aligned}\sigma_n^2 &= \frac{1}{n} \sum_{k,l=1}^{n} \mathrm{Cov}[f(X_k), f(X_l)] = \sum_{q=1}^{\infty} c_{2q}^2 (2q)! \frac{1}{n} \sum_{k,l=1}^{n} \rho(k-l)^{2q} \\ &= \sum_{q=1}^{\infty} c_{2q}^2 (2q)! \sum_{r\in\mathbb{Z}} \rho(r)^{2q} \left(1 - \frac{|r|}{n}\right) \mathbf{1}_{\{|r|\leqslant n\}}.\end{aligned}$$

Hence, it follows from the monotone convergence theorem that

$$\sigma_n^2 \longrightarrow \sigma_\infty^2 = \sum_{q=1}^{\infty} c_{2q}^2 (2q)! \sum_{r\in\mathbb{Z}} \rho(r)^{2q}, \quad \text{as } n \to \infty. \tag{3.22}$$

Since $f$ is not constant, one can find some $q \geqslant 1$ such that $c_{2q} \neq 0$. Moreover, we also have $\sum_{r \in \mathbb{Z}} \rho(r)^{2q} \geqslant \rho(0)^{2q} = 1$. Hence, $\sigma_\infty > 0$, which implies in particular that the infimum of the sequence $\{\sigma_n\}_{n \geqslant 1}$ is positive.

The Gaussian space generated by $X = \{X_k\}_{k \in \mathbb{Z}}$ can be identified with an isonormal Gaussian process of the type $X = \{X(h) : h \in \mathfrak{H}\}$, for $\mathfrak{H}$ defined as follows: (i) denote by $\mathcal{E}$ the set of all sequences indexed by $\mathbb{Z}$ with finite support; (ii) define $\mathfrak{H}$ as the Hilbert space obtained by closing $\mathcal{E}$ with respect to the scalar product

$$\langle u, v \rangle_{\mathfrak{H}} = \sum_{k,l \in \mathbb{Z}} u_k v_l \rho(k-l). \tag{3.23}$$

In this setting, we have $X(\varepsilon_k) = X_k$ where $\varepsilon_k = \{\delta_{kl}\}_{l \in \mathbb{Z}}$, $\delta_{kl}$ standing for the Kronecker symbol. In view of (2.8), we have

$$DG_n = \frac{1}{\sigma_n \sqrt{n}} \sum_{k=1}^{n} f'(X_k) \varepsilon_k.$$

Hence

$$\|DG_n\|_{\mathfrak{H}}^2 = \frac{1}{\sigma_n^2\, n} \sum_{k,l=1}^{n} f'(X_k) f'(X_l) \langle \varepsilon_k, \varepsilon_l \rangle_{\mathfrak{H}} = \frac{1}{\sigma_n^2\, n} \sum_{k,l=1}^{n} f'(X_k) f'(X_l) \rho(k-l),$$

and so

$$\|DG_n\|_{\mathfrak{H}}^4 = \frac{1}{\sigma_n^4\, n^2} \sum_{i,j,k,l=1}^{n} f'(X_i) f'(X_j) f'(X_k) f'(X_l) \rho(i-j) \rho(k-l).$$

We deduce from Cauchy–Schwarz inequality that

$$\left| E[f'(X_i) f'(X_j) f'(X_k) f'(X_l)] \right| \leqslant (E[f'(N)^4])^{\frac{1}{4}},$$

which leads to

$$E[\|DG_n\|_{\mathfrak{H}}^4] \leqslant \frac{1}{\sigma_n^4} \Big( E[f'(N)^4] \Big)^{\frac{1}{4}} \left( \sum_{r \in \mathbb{Z}} |\rho(r)| \right)^2. \tag{3.24}$$

On the other hand, we also have

$$D^2 G_n = \frac{1}{\sigma_n \sqrt{n}} \sum_{k=1}^{n} f''(X_k) \varepsilon_k \otimes \varepsilon_k,$$

and therefore

$$D^2 G_n \otimes_1 D^2 G_n = \frac{1}{\sigma_n^2\, n} \sum_{k,l=1}^{n} f''(X_k) f''(X_l) \rho(k-l) \varepsilon_k \otimes \varepsilon_l.$$

Hence

$$\begin{aligned} &E\big[\|D^2 G_n \otimes_1 D^2 G_n\|_{\mathfrak{H}^{\otimes 2}}^2\big], \\ &\quad = \frac{1}{\sigma_n^4\, n^2} \sum_{i,j,k,l=1}^{n} E\big[f''(X_i) f''(X_j) f''(X_k) f''(X_l)\big] \rho(k-l) \rho(i-j) \rho(k-i) \rho(l-j) \end{aligned}$$

$$\leqslant \frac{(E\big[f''(N)^4\big])^{\frac{1}{4}}}{\sigma_n^4\, n} \sum_{u,v,w\in\mathbb{Z}} |\rho(u)\|\rho(v)\|\rho(w)\|\rho(-u+v+w)|$$

$$\leqslant \frac{(E\big[f''(N)^4\big])^{\frac{1}{4}}\|\rho\|_\infty}{\sigma_n^4\, n} \left(\sum_{r\in\mathbb{Z}}|\rho(r)|\right)^3 < \infty. \tag{3.25}$$

By virtue of Theorem 3.2 together with the fact that $\inf_{n\geqslant 1}\sigma_n > 0$, the inequalities (3.24) and (3.25) imply that $G_n \xrightarrow{\text{law}} N$. Now, in order to show that the ASCLT holds, we shall also check that conditions $(A_1)$ and $(A_2)$ in Theorem 3.2 are fulfilled. First, still because $\inf_{n\geqslant 1}\sigma_n > 0$, $(A_1)$ holds since we have $E\big[\|D^2G_n\otimes_1 D^2G_n\|^2_{\mathfrak{H}^{\otimes 2}}\big] = O(n^{-1})$ by (3.25); see also Remark 3.3. Therefore, it only remains to prove $(A_2)$. Gebelein's inequality (see e.g. identity (1.7) in [3]) states that

$$\big|\mathrm{Cov}[f(X_i), f(X_j)]\big| \leqslant E[X_iX_j]\sqrt{\mathrm{Var}[f(X_i)]}\sqrt{\mathrm{Var}[f(X_j)]} = \rho(i-j)\mathrm{Var}[f(N)].$$

Consequently,

$$|E[G_kG_l]| = \frac{1}{\sigma_k\sigma_l\sqrt{kl}}\left|\sum_{i=1}^k\sum_{j=1}^l \mathrm{Cov}[f(X_i), f(X_j)]\right| \leqslant \frac{\mathrm{Var}[f(N)]}{\sigma_k\sigma_l\sqrt{kl}}\sum_{i=1}^k\sum_{j=1}^l|\rho(i-j)|$$

$$= \frac{\mathrm{Var}[f(N)]}{\sigma_k\sigma_l\sqrt{kl}}\sum_{i=1}^k\sum_{r=i-l}^{i-1}|\rho(r)| \leqslant \frac{\mathrm{Var}[f(N)]}{\sigma_k\sigma_l}\sqrt{\frac{k}{l}}\sum_{r\in\mathbb{Z}}|\rho(r)|.$$

Finally, via the same arguments as in Remark 3.3, $(A_2)$ is satisfied, which completes the proof of Theorem 3.4. □

The following result specializes Theorem 3.2, by providing a criterion for an ASCLT for *multiple stochastic integrals* of fixed order $q\geqslant 2$. It is expressed in terms of the kernels of these integrals.

**Corollary 3.6.** *Retain the notation of Section 2. Fix $q\geqslant 2$, and let $\{G_n\}$ be a sequence of the form $G_n = I_q(f_n)$, with $f_n\in\mathfrak{H}^{\odot q}$. Assume that $E[G_n^2] = q!\|f_n\|^2_{\mathfrak{H}^{\otimes q}} = 1$ for all n, and that*

$$\|f_n\otimes_r f_n\|_{\mathfrak{H}^{\otimes 2(q-r)}} \to 0 \quad \text{as } n\to\infty, \quad \text{for every } r = 1,\ldots,q-1. \tag{3.26}$$

*Then, $G_n \xrightarrow{\text{law}} N\sim\mathscr{N}(0,1)$ as $n\to\infty$. Moreover, if the following two conditions are also satisfied:*

$$(A_1') \qquad \sum_{n\geqslant 2}\frac{1}{n\log^2 n}\sum_{k=1}^n\frac{1}{k}\|f_k\otimes_r f_k\|_{\mathfrak{H}^{\otimes 2(q-r)}} < \infty \quad \text{for every } r = 1,\ldots,q-1,$$

$$(A_2') \qquad \sum_{n\geqslant 2}\frac{1}{n\log^3 n}\sum_{k,l=1}^n\frac{\big|\langle f_k, f_l\rangle_{\mathfrak{H}^{\otimes q}}\big|}{kl} < \infty,$$

*then $\{G_n\}$ satisfies an ASCLT. In other words, almost surely, for all continuous and bounded functions $\varphi:\mathbb{R}\to\mathbb{R}$,*

$$\frac{1}{\log n}\sum_{k=1}^n\frac{1}{k}\varphi(G_k) \longrightarrow E[\varphi(N)], \quad \text{as } n\to\infty.$$

**Proof of Corollary 3.6.** The fact that $G_n \xrightarrow{\text{law}} N \sim \mathcal{N}(0,1)$ follows directly from (3.26), which is the Nualart–Peccati [18] criterion of normality. In order to prove that the ASCLT holds, we shall apply once again Theorem 3.2. This is possible because a multiple integral is always an element of $\mathbb{D}^{2,4}$. We have, by (2.13),

$$1 = E[G_k^2] = E[\langle DG_k, -DL^{-1}G_k\rangle_{\mathfrak{H}}] = \frac{1}{q}\, E[\|DG_k\|_{\mathfrak{H}}^2],$$

where the last inequality follows from $-L^{-1}G_k = \frac{1}{q}G_k$, using the definition (2.11) of $L^{-1}$. In addition, as the random variables $\|DG_k\|_{\mathfrak{H}}^2$ live inside the finite sum of the first $2q$ Wiener chaoses (where all the $L^p$ norms are equivalent), we deduce that condition $(A_0)$ of Theorem 3.2 is satisfied. On the other hand, it is proven in [16, page 604] that

$$E\big[\|D^2G_k \otimes_1 D^2G_k\|_{\mathfrak{H}^{\otimes 2}}^2\big] \leqslant q^4(q-1)^4 \sum_{r=1}^{q-1}(r-1)!^2 \binom{q-2}{r-1}^4 (2q-2-2r)! \\ \times \|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes 2(q-r)}}^2.$$

Consequently, condition $(A_1')$ implies condition $(A_1)$ of Theorem 3.2. Furthermore, by (2.3), $E[G_kG_l] = E\big[I_q(f_k)I_q(f_l)\big] = q!\langle f_k, f_l\rangle_{\mathfrak{H}^{\otimes q}}$. Thus, condition $(A_2')$ is equivalent to condition $(A_2)$ of Theorem 3.2, and the proof of the corollary is done. □

In Corollary 3.6, we supposed $q \geqslant 2$, which implies that $G_n = I_q(f_n)$ is a multiple integral of order at least 2 and hence is not Gaussian. We now consider the Gaussian case $q = 1$.

**Corollary 3.7.** *Let $\{G_n\}$ be a centered Gaussian sequence with unit variance. If the condition $(A_2)$ in Theorem 3.2 is satisfied, then $\{G_n\}$ satisfies an ASCLT. In other words, almost surely, for all continuous and bounded functions $\varphi : \mathbb{R} \to \mathbb{R}$,*

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\,\varphi(G_k) \longrightarrow E[\varphi(N)], \quad \text{as } n \to \infty.$$

**Proof of Corollary 3.7.** Let $t \in \mathbb{R}$ and $r > 0$ be such that $|t| \leqslant r$, and let $\Delta_n(t)$ be defined as in (3.16). We have

$$\begin{aligned}
E|\Delta_n(t)|^2 &= \frac{1}{\log^2 n}\sum_{k,l=1}^{n}\frac{1}{kl}E\left[\left(e^{itG_k} - e^{-t^2/2}\right)\left(e^{-itG_l} - e^{-t^2/2}\right)\right]\\
&= \frac{1}{\log^2 n}\sum_{k,l=1}^{n}\frac{1}{kl}\left[E\left(e^{it(G_k-G_l)}\right) - e^{-t^2}\right]\\
&= \frac{1}{\log^2 n}\sum_{k,l=1}^{n}\frac{e^{-t^2}}{kl}\left(e^{E(G_kG_l)t^2} - 1\right)\\
&\leqslant \frac{r^2e^{r^2}}{\log^2 n}\sum_{k,l=1}^{n}\frac{|E(G_kG_l)|}{kl},
\end{aligned}$$

since $|e^x - 1| \leqslant e^{|x|}|x|$ and $|E(G_kG_l)| \leqslant 1$. Therefore, assumption $(A_2)$ implies (3.17), and the proof of the corollary is done. □

## 4. Application to discrete-time fractional Brownian motion

Let us apply Corollary 3.7 to the particular case $G_n = B^H_n/n^H$, where $B^H$ is a fractional Brownian motion with Hurst index $H \in (0, 1)$. We recall that $B^H = (B^H_t)_{t\geqslant 0}$ is a centered Gaussian process with continuous paths such that

$$E[B^H_t B^H_s] = \frac{1}{2}\Big(t^{2H} + s^{2H} - |t-s|^{2H}\Big), \quad s, t \geqslant 0.$$

The process $B^H$ is self-similar with stationary increments and we refer the reader to Nualart [17] and Samorodnitsky and Taqqu [19] for its main properties. The increments

$$Y_k = B^H_{k+1} - B^H_k, \quad k \geqslant 0,$$

called 'fractional Gaussian noise', are centered stationary Gaussian random variables with covariance

$$\rho(r) = E[Y_k Y_{k+r}] = \frac{1}{2}\big(|r+1|^{2H} + |r-1|^{2H} - 2|r|^{2H}\big), \quad r \in \mathbb{Z}. \tag{4.27}$$

This covariance behaves asymptotically as

$$\rho(r) \sim H(2H-1)|r|^{2H-2} \quad \text{as } |r| \to \infty. \tag{4.28}$$

Observe that $\rho(0) = 1$ and:

(1) For $0 < H < 1/2$, $\rho(r) < 0$ for $r \neq 0$,

$$\sum_{r\in\mathbb{Z}} |\rho(r)| < \infty \quad \text{and} \quad \sum_{r\in\mathbb{Z}} \rho(r) = 0.$$

(2) For $H = 1/2$, $\rho(r) = 0$ if $r \neq 0$.

(3) For $1/2 < H < 1$,

$$\sum_{r\in\mathbb{Z}} |\rho(r)| = \infty.$$

The Hurst index measures the strength of the dependence when $H \geqslant 1/2$: the larger $H$, the stronger the dependence.

A continuous-time version of the following result was obtained by Berkes and Horváth [2] via a different approach.

**Theorem 4.1.** For all $H \in (0, 1)$, we have, almost surely, for all continuous and bounded functions $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\frac{1}{\log n}\sum_{k=1}^{n} \frac{1}{k}\,\varphi(B^H_k/k^H) \longrightarrow E[\varphi(N)], \quad \text{as } n \to \infty.$$

**Proof of Theorem 4.1.** We shall make use of Corollary 3.7. The cases $H < 1/2$ and $H \geqslant 1/2$ are treated separately. From now on, the value of a constant $C > 0$ may change from line to line, and we set $\rho(r) = \frac{1}{2}\big(|r+1|^{2H} + |r-1|^{2H} - 2|r|^{2H}\big)$, $r \in \mathbb{Z}$.

*Case* $H < 1/2$. For any $b \geqslant a \geqslant 0$, we have

$$b^{2H} - a^{2H} = 2H\int_0^{b-a} \frac{\mathrm{d}x}{(x+a)^{1-2H}} \leqslant 2H\int_0^{b-a} \frac{\mathrm{d}x}{x^{1-2H}} = (b-a)^{2H}.$$

Hence, for $l \geqslant k \geqslant 1$, we have $l^{2H} - (l-k)^{2H} \leqslant k^{2H}$, so

$$|E[B_k^H B_l^H]| = \frac{1}{2}\big(k^{2H} + l^{2H} - (l-k)^{2H}\big) \leqslant k^{2H}.$$

Thus

$$\begin{aligned}\sum_{n\geqslant 2} \frac{1}{n\log^3 n} \sum_{l=1}^{n} \frac{1}{l} \sum_{k=1}^{l} \frac{|E[G_k G_l]|}{k} &= \sum_{n\geqslant 2} \frac{1}{n\log^3 n} \sum_{l=1}^{n} \frac{1}{l^{1+H}} \sum_{k=1}^{l} \frac{|E[B_k^H B_l^H]|}{k^{1+H}} \\ &\leqslant \sum_{n\geqslant 2} \frac{1}{n\log^3 n} \sum_{l=1}^{n} \frac{1}{l^{1+H}} \sum_{k=1}^{l} \frac{1}{k^{1-H}} \\ &\leqslant C \sum_{n\geqslant 2} \frac{1}{n\log^3 n} \sum_{l=1}^{n} \frac{1}{l} \leqslant C \sum_{n\geqslant 2} \frac{1}{n\log^2 n} < \infty.\end{aligned}$$

Consequently, condition $(A_2)$ in Theorem 3.2 is satisfied.

*Case* $H \geqslant 1/2$. For $l \geqslant k \geqslant 1$, it follows from (4.27)–(4.28) that

$$\begin{aligned}|E[B_k^H B_l^H]| &= \left|\sum_{i=0}^{k-1}\sum_{j=0}^{l-1} E[(B_{i+1}^H - B_i^H)(B_{j+1}^H - B_j^H)]\right| \leqslant \sum_{i=0}^{k-1}\sum_{j=0}^{l-1} |\rho(i-j)| \\ &\leqslant k \sum_{r=-l+1}^{l-1} |\rho(r)| \leqslant Ckl^{2H-1}.\end{aligned}$$

The last inequality comes from the fact that $\rho(0) = 1$, $\rho(1) = \rho(-1) = (2^{2H} - 1)/2$ and, if $r \geqslant 2$,

$$\begin{aligned}|\rho(-r)| = |\rho(r)| = \left|E[(B_{r+1}^H - B_r^H)B_1^H]\right| &= H(2H-1)\int_0^1 \mathrm{d}u \int_r^{r+1} \mathrm{d}v (v-u)^{2H-2} \\ &\leqslant H(2H-1)\int_0^1 (r-u)^{2H-2}\mathrm{d}u \leqslant H(2H-1)(r-1)^{2H-2}.\end{aligned}$$

Consequently,

$$\begin{aligned}\sum_{n\geqslant 2} \frac{1}{n\log^3 n} \sum_{l=1}^{n} \frac{1}{l} \sum_{k=1}^{l} \frac{|E[G_k G_l]|}{k} &= \sum_{n\geqslant 2} \frac{1}{n\log^3 n} \sum_{l=1}^{n} \frac{1}{l^{1+H}} \sum_{k=1}^{l} \frac{|E[B_k^H B_l^H]|}{k^{1+H}} \\ &\leqslant C\sum_{n\geqslant 2} \frac{1}{n\log^3 n} \sum_{l=1}^{n} \frac{1}{l^{2-H}} \sum_{k=1}^{l} \frac{1}{k^{H}} \\ &\leqslant C \sum_{n\geqslant 2} \frac{1}{n\log^3 n} \sum_{l=1}^{n} \frac{1}{l} \leqslant C \sum_{n\geqslant 2} \frac{1}{n\log^2 n} < \infty.\end{aligned}$$

Finally, condition $(A_2)$ in Theorem 3.2 is satisfied, which completes the proof of Theorem 4.1. □

## 5. Partial sums of Hermite polynomials: the Gaussian limit case

Let $X = \{X_k\}_{k\in\mathbb{Z}}$ be a centered stationary Gaussian process and for all $r \in \mathbb{Z}$, set $\rho(r) = E[X_0 X_r]$. Fix an integer $q \geqslant 2$, and let $H_q$ stand for the Hermite polynomial of degree

$q$; see (2.2). We are interested in an ASCLT for

$$V_n = \sum_{k=1}^{n} H_q(X_k), \quad n \geqslant 1, \tag{5.29}$$

in cases where $V_n$, adequately normalized, converges to a normal distribution. Our result is as follows.

**Theorem 5.1.** Assume that $\sum_{r\in\mathbb{Z}} |\rho(r)|^q < \infty$, that $\sum_{r\in\mathbb{Z}} \rho(r)^q > 0$ and that there exists $\alpha > 0$ such that $\sum_{|r|>n} |\rho(r)|^q = O(n^{-\alpha})$, as $n \to \infty$. For any $n \geqslant 1$, define

$$G_n = \frac{V_n}{\sigma_n \sqrt{n}},$$

where $V_n$ is given by (5.29) and $\sigma_n$ denotes the positive normalizing constant which ensures that $E[G_n^2] = 1$. Then $G_n \xrightarrow{\text{law}} N \sim \mathscr{N}(0,1)$ as $n \to \infty$, and $\{G_n\}$ satisfies an ASCLT. In other words, almost surely, for all continuous and bounded functions $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi(G_k) \longrightarrow E[\varphi(N)], \quad \text{as } n \to \infty.$$

**Proof.** We shall make use of Corollary 3.6. Let $C$ be a positive constant, depending only on $q$ and $\rho$, whose value may change from line to line. We consider the real and separable Hilbert space $\mathfrak{H}$ as defined in the proof of Theorem 3.4, with the scalar product (3.23). Following the same line of reasoning as in the proof of (3.22), it is possible to show that $\sigma_n^2 \to q! \sum_{r\in\mathbb{Z}} \rho(r)^q > 0$. In particular, the infimum of the sequence $\{\sigma_n\}_{n\geqslant 1}$ is positive. On the other hand, we have $G_n = I_q(f_n)$, where the kernel $f_n$ is given by

$$f_n = \frac{1}{\sigma_n \sqrt{n}} \sum_{k=1}^{n} \varepsilon_k^{\otimes q},$$

with $\varepsilon_k = \{\delta_{kl}\}_{l\in\mathbb{Z}}$, $\delta_{kl}$ standing for the Kronecker symbol. For all $n \geqslant 1$ and $r = 1, \ldots, q-1$, we have

$$f_n \otimes_r f_n = \frac{1}{\sigma_n^2 n} \sum_{k,l=1}^{n} \rho(k-l)^r \varepsilon_k^{\otimes(q-r)} \otimes \varepsilon_l^{\otimes(q-r)}.$$

We deduce that

$$\|f_n \otimes_r f_n\|^2_{\mathfrak{H}^{\otimes(2q-2r)}} = \frac{1}{\sigma_n^4 n^2} \sum_{i,j,k,l=1}^{n} \rho(k-l)^r \rho(i-j)^r \rho(k-i)^{q-r} \rho(l-j)^{q-r}.$$

Consequently, as in the proof of (3.25), we obtain that $\|f_n \otimes_r f_n\|^2_{\mathfrak{H}^{\otimes(2q-2r)}} \leqslant A_n$ where

$$A_n = \frac{1}{\sigma_n^4 n} \sum_{u,v,w\in D_n} |\rho(u)|^r |\rho(v)|^r |\rho(w)|^{q-r} |\rho(-u+v+w)|^{q-r}$$

with $D_n = \{-n, \ldots, n\}$. Fix an integer $m \geqslant 1$ such that $n \geqslant m$. We can split $A_n$ into two terms $A_n = B_{n,m} + C_{n,m}$ where

$$B_{n,m} = \frac{1}{\sigma_n^4 n} \sum_{u,v,w\in D_m} |\rho(u)|^r |\rho(v)|^r |\rho(w)|^{q-r} |\rho(-u+v+w)|^{q-r},$$

$$C_{n,m} = \frac{1}{\sigma_n^4 n} \sum_{\substack{u,v,w \in D_n \\ |u| \vee |v| \vee |w| > m}} |\rho(u)|^r |\rho(v)|^r |\rho(w)|^{q-r} |\rho(-u+v+w)|^{q-r}.$$

We clearly have

$$B_{n,m} \leqslant \frac{1}{\sigma_n^4 n} \|\rho\|_\infty^{2q} (2m+1)^3 \leqslant \frac{Cm^3}{n}.$$

On the other hand, $D_n \cap \{|u| \vee |v| \vee |w| > m\} \subset D_{n,m,u} \cup D_{n,m,v} \cup D_{n,m,w}$ where the set $D_{n,m,u} = \{|u| > m, |v| \leqslant n, |w| \leqslant n\}$ with similar definitions for $D_{n,m,v}$ and $D_{n,m,w}$. Define

$$C_{n,m,u} = \frac{1}{\sigma_n^4 n} \sum_{u,v,w \in D_{n,m,u}} |\rho(u)|^r |\rho(v)|^r |\rho(w)|^{q-r} |\rho(-u+v+w)|^{q-r}$$

with similar expressions for $C_{n,m,v}$ and $C_{n,m,w}$. It follows from the Hölder inequality that

$$C_{n,m,u} \leqslant \frac{1}{\sigma_n^4 n} \left( \sum_{u,v,w \in D_{n,m,u}} |\rho(u)|^q |\rho(v)|^q \right)^{\frac{r}{q}} \times \left( \sum_{u,v,w \in D_{n,m,u}} |\rho(w)|^q |\rho(-u+v+w)|^q \right)^{1-\frac{r}{q}}. \tag{5.30}$$

However,

$$\sum_{u,v,w \in D_{n,m,u}} |\rho(u)|^q |\rho(v)|^q \leqslant (2n+1) \sum_{|u|>m} |\rho(u)|^q \sum_{v \in \mathbb{Z}} |\rho(v)|^q \leqslant Cn \sum_{|u|>m} |\rho(u)|^q.$$

Similarly,

$$\sum_{u,v,w \in D_{n,m,u}} |\rho(w)|^q |\rho(-u+v+w)|^q \leqslant (2n+1) \sum_{v \in \mathbb{Z}} |\rho(v)|^q \sum_{w \in \mathbb{Z}} |\rho(w)|^q \leqslant Cn.$$

Therefore, (5.30) and the last assumption of Theorem 5.1 imply that for $m$ large enough

$$C_{n,m,u} \leqslant C \left( \sum_{|u|>m} |\rho(u)|^q \right)^{\frac{r}{q}} \leqslant C m^{-\frac{\alpha r}{q}}.$$

We obtain exactly the same bound for $C_{n,m,v}$ and $C_{n,m,w}$. Combining all these estimates, we finally find that

$$\|f_n \otimes_r f_n\|^2_{\mathfrak{H}^{\otimes(2q-2r)}} \leqslant C \times \inf_{m \leqslant n} \left\{ \frac{m^3}{n} + m^{-\frac{\alpha r}{q}} \right\} \leqslant C n^{-\frac{\alpha r}{3q+\alpha r}}$$

by taking the value $m = n^{\frac{q}{3q+\alpha r}}$. This ensures that condition $(A_1')$ in Corollary 3.6 is met. Let us now prove $(A_2')$. We have

$$\langle f_k, f_l \rangle_{\mathfrak{H}^{\otimes q}} = \frac{1}{\sigma_k \sigma_l \sqrt{kl}} \left| \sum_{i=1}^k \sum_{j=1}^l \rho(i-j)^q \right| \leqslant \frac{1}{\sigma_k \sigma_l \sqrt{kl}} \sum_{i=1}^k \sum_{j=1}^l |\rho(i-j)|^q,$$
$$\leqslant \frac{1}{\sigma_k \sigma_l} \sqrt{\frac{k}{l}} \sum_{r \in \mathbb{Z}} |\rho(r)|^q,$$

so $(A_2')$ is also satisfied (see Remark 3.3), which completes the proof of Theorem 5.1. □

The following result contains an explicit situation where the assumptions in Theorem 5.1 are in order.

**Proposition 5.2.** *Assume that $\rho(r) \sim |r|^{-\beta} L(r)$, as $|r| \to \infty$, for some $\beta > 1/q$ and some slowly varying function $L$. Then $\sum_{r \in \mathbb{Z}} |\rho(r)|^q < \infty$ and there exists $\alpha > 0$ such that $\sum_{|r|>n} |\rho(r)|^q = O(n^{-\alpha})$, as $n \to \infty$.*

**Proof.** By a Riemann sum argument, it is immediate that $\sum_{r \in \mathbb{Z}} |\rho(r)|^q < \infty$. Moreover, by [4, Prop. 1.5.10], we have $\sum_{|r|>n} |\rho(r)|^q \sim \frac{2}{\beta q - 1} n^{1-\beta q} L^q(n)$ so we can choose $\alpha = \frac{1}{2}(\beta q - 1) > 0$ (for instance). $\square$

## 6. Partial sums of Hermite polynomials of increments of fractional Brownian motion

We focus here on increments of the fractional Brownian motion $B^H$ (see Section 4 for details about $B^H$). More precisely, for every $q \geqslant 1$, we are interested in an ASCLT for the $q$-Hermite power variation of $B^H$, defined as

$$V_n = \sum_{k=0}^{n-1} H_q(B_{k+1}^H - B_k^H), \quad n \geqslant 1, \tag{6.31}$$

where $H_q$ stands for the Hermite polynomial of degree $q$ given by (2.2). Observe that Theorem 4.1 corresponds to the particular case $q = 1$. That is why, from now on, we assume that $q \geqslant 2$. When $H \neq 1/2$, the increments of $B^H$ are not independent, so the asymptotic behavior of (6.31) is difficult to investigate because $V_n$ is not linear. In fact, thanks to the seminal works of Breuer and Major [6], Dobrushin and Major [8], Giraitis and Surgailis [9] and Taqqu [21], it is known (recall that $q \geqslant 2$) that, as $n \to \infty$:

- If $0 < H < 1 - \frac{1}{2q}$, then

$$G_n := \frac{V_n}{\sigma_n \sqrt{n}} \xrightarrow{\text{law}} \mathscr{N}(0, 1). \tag{6.32}$$

- If $H = 1 - \frac{1}{2q}$, then

$$G_n := \frac{V_n}{\sigma_n \sqrt{n \log n}} \xrightarrow{\text{law}} \mathscr{N}(0, 1). \tag{6.33}$$

- If $H > 1 - \frac{1}{2q}$, then

$$G_n := n^{q(1-H)-1} V_n \xrightarrow{\text{law}} G_\infty \tag{6.34}$$

where $G_\infty$ has a 'Hermite distribution'. Here, $\sigma_n$ denotes the positive normalizing constant which ensures that $E[G_n^2] = 1$. The proofs of (6.32) and (6.33), together with rates of convergence, can be found in [15] and [5], respectively. A short proof of (6.34) is given in Proposition 6.1 below. Notice that rates of convergence can be found in [5]. Our proof of (6.34) is based on the fact that, for *fixed* $n$, $Z_n$ defined in (6.35) below and $G_n$ share the same law, because of the self-similarity property of fractional Brownian motion.

**Proposition 6.1.** *Assume $H > 1 - \frac{1}{2q}$, and define $Z_n$ by*

$$Z_n = n^{q(1-H)-1} \sum_{k=0}^{n-1} H_q\big(n^H (B^H_{(k+1)/n} - B^H_{k/n})\big), \quad n \geqslant 1. \tag{6.35}$$

*Then, as $n \to \infty$, $\{Z_n\}$ converges almost surely and in $L^2(\Omega)$ to a limit denoted by $Z_\infty$, which belongs to the $q$th chaos of $B^H$.*

**Proof.** Let us first prove the convergence in $L^2(\Omega)$. For $n, m \geqslant 1$, we have

$$E[Z_n Z_m] = q!(nm)^{q-1} \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \Big( E\big[\big(B^H_{(k+1)/n} - B^H_{k/n}\big)\big(B^H_{(l+1)/m} - B^H_{l/m}\big)\big]\Big)^q .$$

Furthermore, since $H > 1/2$, we have for all $s, t \geqslant 0$,

$$E[B^H_s B^H_t] = H(2H-1) \int_0^t \mathrm{d}u \int_0^s \mathrm{d}v |u-v|^{2H-2}.$$

Hence

$$\begin{aligned} E[Z_n Z_m] &= q! H^q (2H-1)^q \\ &\quad \times \frac{1}{nm} \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \left( nm \int_{k/n}^{(k+1)/n} \mathrm{d}u \int_{l/m}^{(l+1)/m} \mathrm{d}v |v-u|^{2H-2} \right)^q . \end{aligned}$$

Therefore, as $n, m \to \infty$, we have

$$E[Z_n Z_m] \to q! H^q (2H-1)^q \int_{[0,1]^2} |u-v|^{(2H-2)q} \mathrm{d}u \mathrm{d}v,$$

and the limit is finite since $H > 1 - \frac{1}{2q}$. In other words, the sequence $\{Z_n\}$ is Cauchy in $L^2(\Omega)$, and hence converges in $L^2(\Omega)$ to some $Z_\infty$.

Let us now prove that $\{Z_n\}$ converges also almost surely. Observe first that, since $Z_n$ belongs to the $q$th chaos of $B^H$ for all $n$, since $\{Z_n\}$ converges in $L^2(\Omega)$ to $Z_\infty$ and since the $q$th chaos of $B^H$ is closed in $L^2(\Omega)$ by definition, we have that $Z_\infty$ also belongs to the $q$th chaos of $B^H$. In [5, Proposition 3.1], it is shown that $E[|Z_n - Z_\infty|^2] \leqslant C n^{2q-1-2qH}$, for some positive constant $C$ not depending on $n$. Inside a fixed chaos, all the $L^p$-norms are equivalent. Hence, for any $p > 2$, we have $E[|Z_n - Z_\infty|^p] \leqslant C n^{p(q-1/2-qH)}$. Since $H > 1 - \frac{1}{2q}$, there exists $p > 2$ large enough that $(q - 1/2 - qH)p < -1$. Consequently

$$\sum_{n \geqslant 1} E[|Z_n - Z_\infty|^p] < \infty,$$

leading, for all $\varepsilon > 0$, to

$$\sum_{n \geqslant 1} P[|Z_n - Z_\infty| > \varepsilon] < \infty.$$

Therefore, we deduce from the Borel–Cantelli lemma that $\{Z_n\}$ converges almost surely to $Z_\infty$. $\square$

We now want to see whether one can associate almost sure central limit theorems with the convergences in law (6.32)–(6.34). We first consider the case $H < 1 - \frac{1}{2q}$.

**Theorem 6.2.** Assume that $q \geqslant 2$ and that $H < 1 - \frac{1}{2q}$, and consider

$$G_n = \frac{V_n}{\sigma_n \sqrt{n}}$$

as in (6.32). Then, $\{G_n\}$ satisfies an ASCLT.

**Proof.** Since $2H - 2 > 1/q$, it suffices to combine (4.28), Proposition 5.2 and Theorem 5.1. □

Next, let us consider the critical case $H = 1 - \frac{1}{2q}$. In this case, $\sum_{r\in\mathbb{Z}} |\rho(r)|^q = \infty$. Consequently, as it is impossible to apply Theorem 5.1, we propose another strategy which relies on the following lemma established in [5].

**Lemma 6.3.** *Set $H = 1 - \frac{1}{2q}$. Let $\mathfrak{H}$ be the real and separable Hilbert space defined as follows:* (i) *denote by $\mathscr{E}$ the set of all $\mathbb{R}$-valued step functions on* $[0, \infty)$, (ii) *define $\mathfrak{H}$ as the Hilbert space obtained by closing $\mathscr{E}$ with respect to the scalar product*

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathfrak{H}} = E[B_t^H B_s^H].$$

*For any $n \geqslant 2$, let $f_n$ be the element of $\mathfrak{H}^{\odot q}$ defined by*

$$f_n = \frac{1}{\sigma_n \sqrt{n \log n}} \sum_{k=0}^{n-1} \mathbf{1}_{[k,k+1]}^{\otimes q}, \tag{6.36}$$

*where $\sigma_n$ is the positive normalizing constant which ensures that $q!\|f_n\|^2_{\mathfrak{H}^{\otimes q}} = 1$. Then, there exists a constant $C > 0$ depending only on $q$ and $H$ such that, for all $n \geqslant 1$ and $r = 1, \ldots, q-1$,*

$$\|f_n \otimes_r f_n\|_{\mathfrak{H}^{\otimes(2q-2r)}} \leqslant C(\log n)^{-1/2}.$$

We can now state and prove the following result.

**Theorem 6.4.** Assume that $q \geqslant 2$ and $H = 1 - \frac{1}{2q}$, and consider

$$G_n = \frac{V_n}{\sigma_n \sqrt{n \log n}}$$

as in (6.33). Then, $\{G_n\}$ satisfies an ASCLT.

**Proof of Theorem 6.4.** We shall make use of Corollary 3.6. Let $C$ be a positive constant, depending only on $q$ and $H$, whose value may change from line to line. We consider the real and separable Hilbert space $\mathfrak{H}$ as defined in Lemma 6.3. We have $G_n = I_q(f_n)$ with $f_n$ given by (6.36). According to Lemma 6.3, we have for all $k \geqslant 1$ and $r = 1, \ldots, q-1$ that $\|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes(2q-2r)}} \leqslant C(\log k)^{-1/2}$. Hence

$$\begin{aligned}\sum_{n\geqslant 2} \frac{1}{n \log^2 n} \sum_{k=1}^{n} \frac{1}{k} \|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes(2q-2r)}} &\leqslant C \sum_{n\geqslant 2} \frac{1}{n \log^2 n} \sum_{k=1}^{n} \frac{1}{k\sqrt{\log k}} \\ &\leqslant C \sum_{n\geqslant 2} \frac{1}{n \log^{3/2} n} < \infty.\end{aligned}$$

Consequently, assumption $(A_1')$ is satisfied. As regards $(A_2')$, note that

$$\langle f_k, f_l \rangle_{\mathfrak{H}^{\otimes q}} = \frac{1}{\sigma_k \sigma_l \sqrt{k \log k}\sqrt{l \log l}} \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} \rho(j-i)^q.$$

We deduce from Lemma 6.5 below that $\sigma_n^2 \to \sigma_\infty^2 > 0$. Hence, for all $l \geqslant k \geqslant 1$

$$\begin{aligned}
\left|\langle f_k, f_l\rangle_{\mathfrak{H}^{\otimes q}}\right| &\leqslant \frac{C}{\sqrt{k\log k}\sqrt{l\log l}}\sum_{i=0}^{k-1}\sum_{j=0}^{l-1}|\rho(j-i)|^q\\
&= \frac{C}{\sqrt{k\log k}\sqrt{l\log l}}\sum_{i=0}^{k-1}\sum_{r=-i}^{l-1-i}|\rho(r)|^q\\
&\leqslant C\frac{\sqrt{k}}{\sqrt{\log k}\sqrt{l\log l}}\sum_{r=-l}^{l}|\rho(r)|^q \leqslant C\sqrt{\frac{k\log l}{l\log k}}.
\end{aligned}$$

The last inequality follows from the fact that $\sum_{r=-l}^{l}|\rho(r)|^q \leqslant C\log l$ since, by (4.28), as $|r| \to \infty$,

$$\rho(r) \sim \left(1-\frac{1}{q}\right)\left(1-\frac{1}{2q}\right)|r|^{-1/q}.$$

Finally, assumption $(A_2')$ is also satisfied as

$$\begin{aligned}
\sum_{n\geqslant 2}\frac{1}{n\log^3 n}\sum_{k,l=2}^{n}\frac{\left|\langle f_k, f_l\rangle_{\mathfrak{H}^{\otimes q}}\right|}{kl} &\leqslant 2\sum_{n\geqslant 2}\frac{1}{n\log^3 n}\sum_{l=2}^{n}\sum_{k=2}^{l}\frac{\left|\langle f_k, f_l\rangle_{\mathfrak{H}^{\otimes q}}\right|}{kl}\\
&\leqslant C\sum_{n\geqslant 2}\frac{1}{n\log^3 n}\sum_{l=2}^{n}\frac{\sqrt{\log l}}{l^{3/2}}\sum_{k=2}^{l}\frac{1}{\sqrt{k\log k}}\\
&\leqslant C\sum_{n\geqslant 2}\frac{1}{n\log^3 n}\sum_{l=2}^{n}\frac{1}{l} \leqslant C\sum_{n\geqslant 2}\frac{1}{n\log^2 n} < \infty. \quad \square
\end{aligned}$$

In the previous proof, we used the following lemma.

**Lemma 6.5.** *Assume that $q \geqslant 2$ and $H = 1 - \frac{1}{2q}$. Then,*

$$\sigma_n^2 \to 2q!\left(1-\frac{1}{q}\right)^q\left(1-\frac{1}{2q}\right)^q > 0, \quad \text{as } n \to \infty.$$

**Proof.** We have $E[(B_{k+1}^H - B_k^H)(B_{l+1}^H - B_l^H)] = \rho(k-l)$ where $\rho$ is given in (4.27). Hence,

$$\begin{aligned}
E[V_n^2] &= \sum_{k,l=0}^{n-1} E\big(H_q(B_{k+1}^H - B_k^H)H_q(B_{l+1}^H - B_l^H)\big) = q!\sum_{k,l=0}^{n-1}\rho(k-l)^q\\
&= q!\sum_{l=0}^{n-1}\sum_{r=-l}^{n-1-l}\rho(r)^q = q!\sum_{|r|<n}\big(n-1-|r|\big)\rho(r)^q\\
&= q!\left(n\sum_{|r|<n}\rho(r)^q - \sum_{|r|<n}\big(|r|+1\big)\rho(r)^q\right).
\end{aligned}$$

On the other hand, as $|r| \to \infty$,

$$\rho(r)^q \sim \left(1-\frac{1}{q}\right)^q\left(1-\frac{1}{2q}\right)^q\frac{1}{|r|}.$$

Therefore, as $n \to \infty$,

$$\sum_{|r|<n} \rho(r)^q \sim \left(1-\frac{1}{2q}\right)^q \left(1-\frac{1}{q}\right)^q \sum_{0<|r|<n} \frac{1}{|r|} \sim 2\left(1-\frac{1}{2q}\right)^q \left(1-\frac{1}{q}\right)^q \log n$$

and

$$\sum_{|r|<n} \big(|r|+1\big)\rho(r)^q \sim \left(1-\frac{1}{2q}\right)^q \left(1-\frac{1}{q}\right)^q \sum_{|r|<n} 1 \sim 2n\left(1-\frac{1}{2q}\right)^q \left(1-\frac{1}{q}\right)^q .$$

Consequently, as $n \to \infty$,

$$\sigma_n^2 = \frac{E[V_n^2]}{n\log n} \to 2q!\left(1-\frac{1}{q}\right)^q \left(1-\frac{1}{2q}\right)^q . \quad \square$$

Finally, we consider

$$G_n = n^{q(1-H)-1} V_n \tag{6.37}$$

with $H > 1 - \frac{1}{2q}$. We face in this case some difficulties. First, since the limit of $\{G_n\}$ in (6.34) is not Gaussian, we cannot apply our general criterion Corollary 3.6 to obtain an ASCLT. To modify the criterion adequately, we would need a version of Lemma 2.2 for random variables with a Hermite distribution, a result which is not currently available. Thus, an ASCLT associated with the convergence in law (6.34) falls outside the scope of this paper. We can nevertheless make a number of observations. First, changing the nature of the random variables without changing their law has no impact on CLTs as in (6.34), but may have a great impact on an ASCLT. To see this, observe that for each fixed $n$, the ASCLT involves not only the distribution of the single variable $G_n$, but also the joint distribution of the vector $(G_1, \ldots, G_n)$.

Consider, moreover, the following example. Let $\{G_n\}$ be a sequence of random variables converging in law to a limit $G_\infty$. According to a theorem of Skorohod, there is a sequence $\{G_n^*\}$ such that for any fixed $n$, $G_n^* \overset{\text{law}}{=} G_n$ and such that $\{G_n^*\}$ converges almost surely, as $n \to \infty$, to a random variable $G_\infty^*$ with $G_\infty^* \overset{\text{law}}{=} G_\infty$. In this case, we say that $G_n^*$ is a Skorohod version of $G_n$. Then, for any bounded continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, we have $\varphi(G_n^*) \longrightarrow \varphi(G_\infty^*)$ a.s. which clearly implies the almost sure convergence

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi(G_k^*) \longrightarrow \varphi(G_\infty^*).$$

This limit is, in general, different from $E[\varphi(G_\infty^*)]$ or equivalently $E[\varphi(G_\infty)]$, that is, different from the limit if one had an ASCLT.

Consider now the sequence $\{G_n\}$ defined by (6.37).

**Proposition 6.6.** *A Skorohod version of*

$$G_n = n^{q(1-H)-1} \sum_{k=0}^{n-1} H_q(B_{k+1}^H - B_k^H) \tag{6.38}$$

*is given by*

$$G_n^* = Z_n = n^{q(1-H)-1} \sum_{k=0}^{n-1} H_q\big(n^H (B_{(k+1)/n}^H - B_{k/n}^H)\big). \tag{6.39}$$

**Proof.** Just observe that $G_n^* \stackrel{\text{law}}{=} G_n$ and $G_n^*$ converges almost surely by Proposition 6.1. $\square$

Hence, in the case of Hermite distributions, by suitably modifying the argument of the Hermite polynomial $H_q$ in a way which does not change the limit in law, namely by considering $Z_n$ in (6.39) instead of $G_n$ in (6.38), we obtain the almost sure convergence

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\varphi(Z_k) \longrightarrow \varphi(Z_\infty).$$

The limit $\varphi(Z_\infty)$ is, in general, different from the limit expected under an ASCLT, namely $E[\varphi(Z_\infty)]$, because $Z_\infty$ is a non-constant random variable with a Hermite distribution [8,21]. Thus, knowing the law of $G_n$ in (6.38), for a fixed $n$, does not allow us to determine whether an ASCLT holds or not.

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