Infinitely small & large

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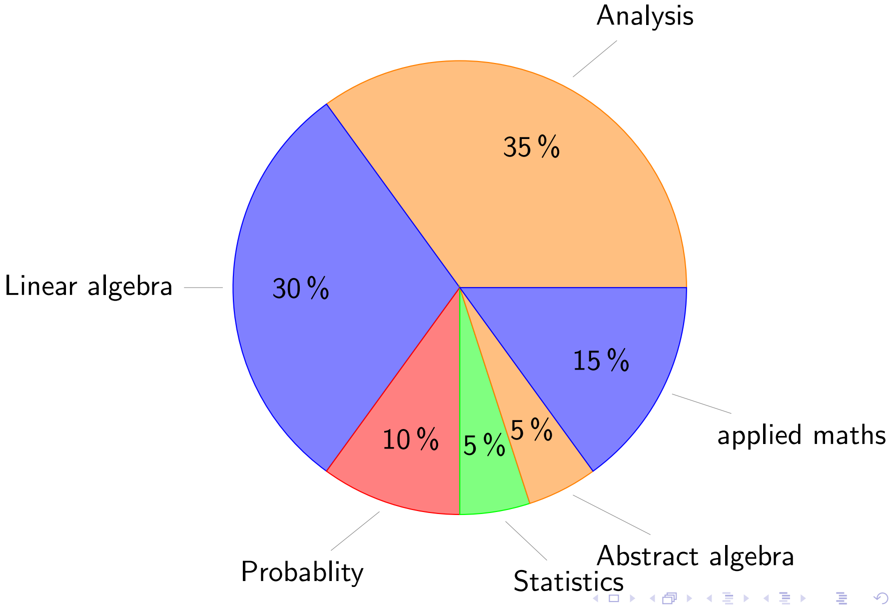
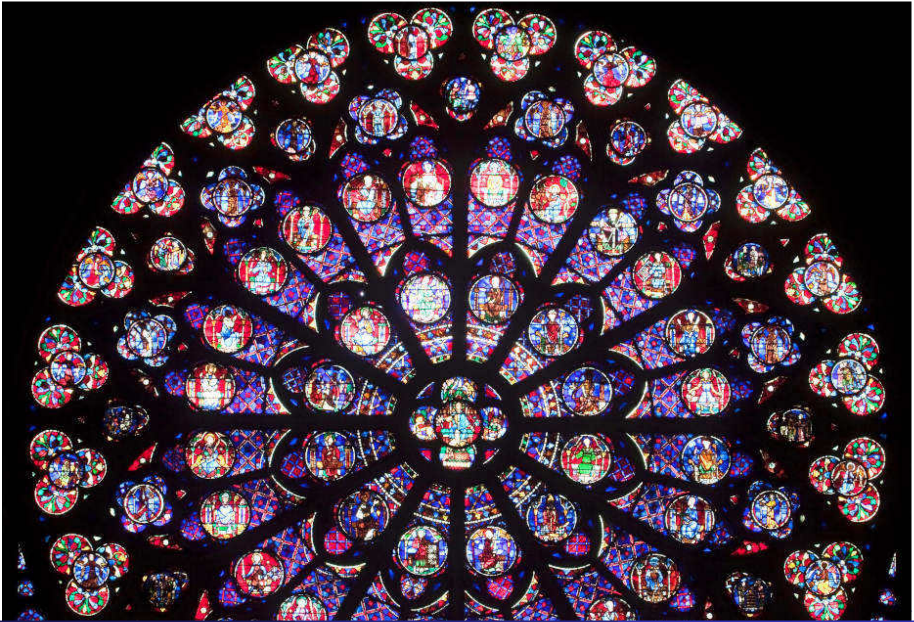
Your Conception of mathematics ??

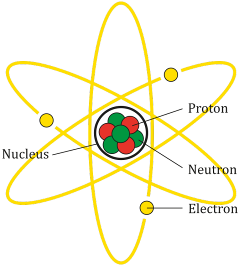


Mathematics ? It's rather like this!



Schrödinger's Question

Question: Is random behaviour at atomic scale consistent with deterministic mechanics?



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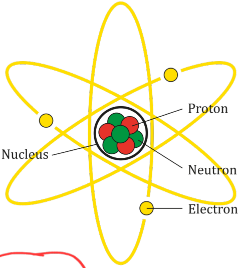
random $\longleftarrow ? \longrightarrow$ deterministic

X_i random

$\oplus$ . . . .

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{n\to\infty} E(X)$$



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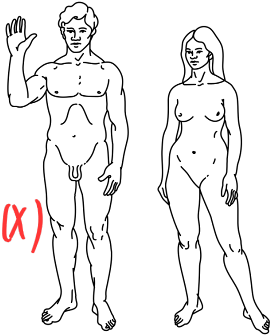

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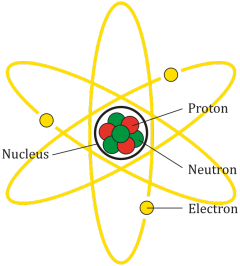

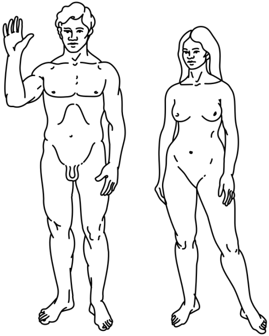

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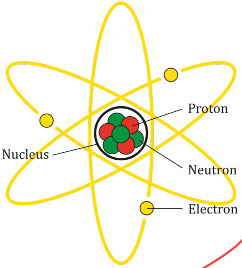

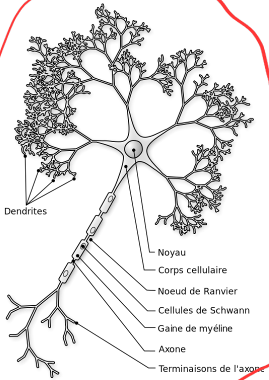

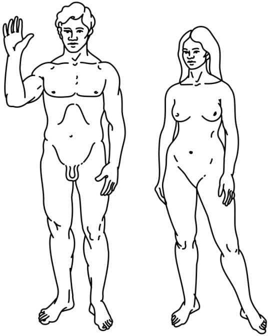

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Random variables and their sums

Probabilists have strange vocabulary: most important examples

Ω	$=$	probability space
Random variable	$=$	function
Expectation	$=$	integral
Event	$=$	subset of Ω

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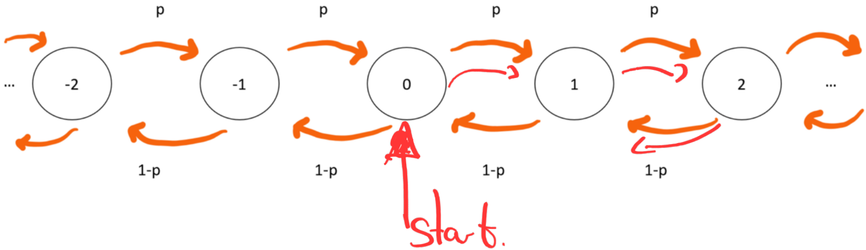
& Probabilists have strange way of thinking: a "random variable"

$$X : \underline{\Omega} \longrightarrow \mathbb{R}$$

is actually a **deterministic** function. The "random" comes from the fact that we don't know which $\omega \in \Omega$ is "selected", so we don't know $X(\omega)$. That sounds strange, but turns out very helpful.

$\omega \in \Omega$

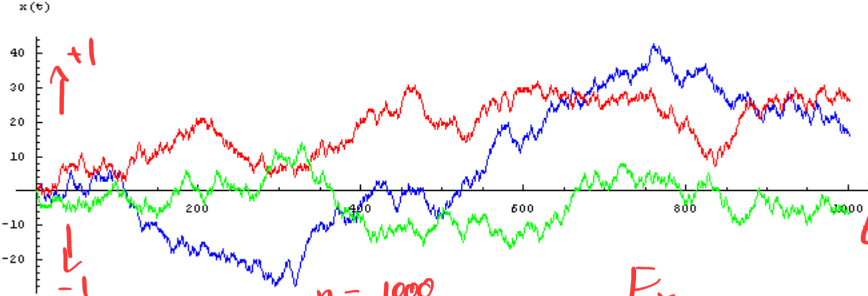
Let X_n be s sequence of independent random variables, taking only two values ± 1. So $\underbrace{\mathbb{P}(X_n = 1) = p}$ and $\mathbb{P}(X_n = -1) = 1 - p$.



Let $S_0 = 0$ and $S_n = X_1 + .. + X_n$. It is the position of a "random walk" (starting at 0), at time n (recall: all functions depending on ω!)

Random walk

3 different random walks – meaning: 3 different ω's. Each is 1000 "steps".



+1

-1

time

$n = 1000$

$(X_n(\omega))_{n \geq 0}$

3 diff. omega's.

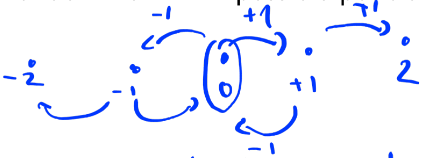
Ex.:

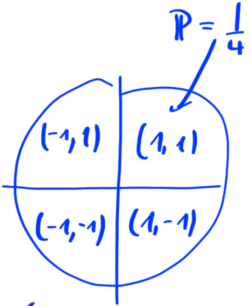
$X_n = \text{Sign}(\sin(2^n \pi \omega))$

$\omega \in [0,1] = \Omega$

Bionomial coefficients

Consider a random walk in $\mathbb{Z}$ possible positions at time $2n$??





First $2n$ must be even!

$n = 0$ of course $S_0 = 0$

$n = 2$

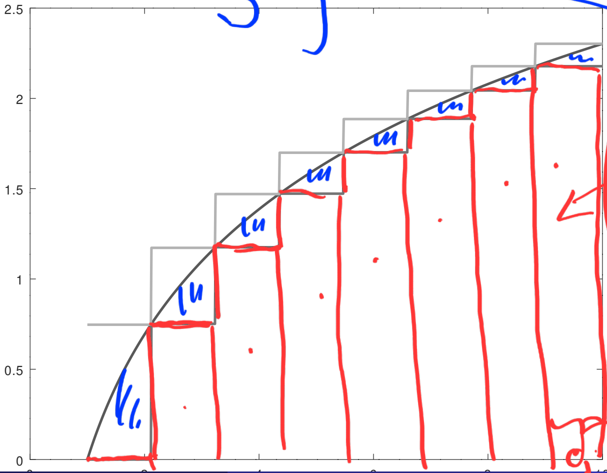
	-2	-1	0	1	2
$\mathbb{P}$:	$\frac{1}{4}$	0	$\frac{1}{2}$	0	$\frac{1}{4}$

If we follow this logic, $\frac{1}{4}(1, \cancel{0}, 2, \cancel{0}, 1)$

if $n = 4$: $\frac{1}{8}(1, \cancel{0}, 3, \cancel{0}, 3, \cancel{0}, 1)$

let $[a, b]$ be an interval, f a positive function. In order to estimate the "air under the curve" we use lower and upper estimates by rectangle decompositions:

$S \int \rightarrow \int_a^b f\,dx$



rectangle!

h_j air

$\sum_{j=1}^{n} \delta_j \cdot h_j$

δ_j

Dyadic Riemann sums

have a look at

https://www.math.u-bordeaux.fr/∼bhaak/enseignement/riemann_sums.gif

Random walk

Question: Probablity to get back to your origin in time $2n$?

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in $\mathbb{Z}$ we need a walk

$$(+1, +1, -1, +1, -1, -1, \dots, -1)$$

with $2n$ steps, exactly n steps $+1$ and n steps -1! i.e.

$$\mathbb{P}(S_{2n} = 0) = \left(\tfrac{1}{2}\right)^{2n} \binom{2n}{n} = \left(\tfrac{1}{2}\right)^{2n} \frac{(2n)!}{n!.n!}$$

What can be said about this formula?

$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$

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estimate.

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What can be said about this formula?

Generalise it! In $\mathbb{Z}^2$ we need $4n$ steps, exactly n steps "up", "down", "left" and "right". And in $\mathbb{Z}^3$ we need $6n$ steps ...
Here, a small student project starts!

Stirling formula

Question: estimate $n!$.

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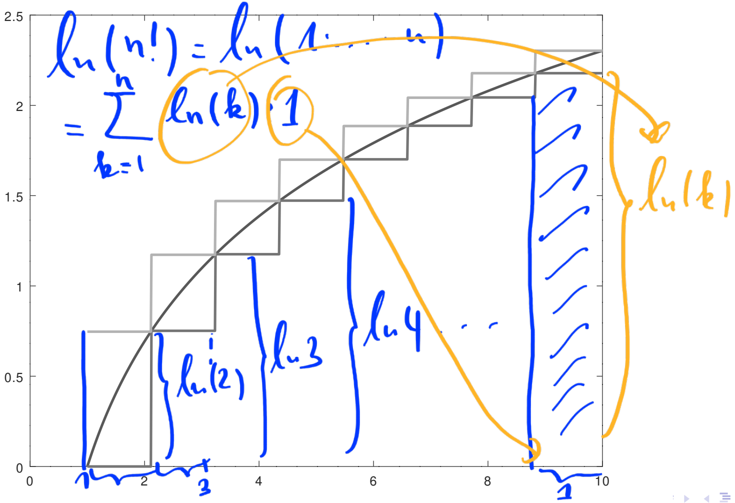

Stirling formula 2

$$\int_1^n \ln(t)dt \leq \ln(n!) \leq \int_1^n \ln(1+t)dt$$

Stirling formula 2

$\int \ln(t) = t \cdot \ln(t) - t$

$$\int_1^n \ln(t)dt \le \ln(n!) \le \int_1^n \ln(1+t)dt$$

Knowing the anti-derivative $t\ln(t) - t$ we get

$$n\ln(n) - n \quad \le \ln(n!) \le \quad (n+1)\ln(n+1) - (n-1) - 2\ln(2)$$

or, writing $1 = \ln(e)$ to get only logarithms,