Exercise 1 Check if the following limits exist:

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2}, \quad \lim_{(x,y)\to(0,0)} \frac{(x+2y)^3}{x^2+y^2}, \quad \lim_{(x,y)\to(1,0)} \frac{\log(x+e^y)}{\sqrt{x^2+y^2}},$$

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^3-xy}{x^4+y^2}, \quad \lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^4}.$$

Exercise 2 Study continuity (or not) of the functions

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \quad \text{and} \quad g(x,y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Exercise 3 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that

a) For all x, the function $y \mapsto f(x,y)$ is continuous.
b) For all y, the function $x \mapsto f(x,y)$ is uniformly continuous.

Show that f is continuous. Hint: consider f at the points (x,y), (x,y') and (x',y'). *An important Remark:* if we replace (b) by (b')*: "For all* y*, the function* $x \mapsto f(x,y)$ *is continuous"*, the conclusion becomes wrong! consider for example the function $f(x,y) = \frac{|x|-|y|}{|x|+|y|}$ on $\mathbb{R}^2\setminus\{(0,0)\}$ that admits no limit in $(0,0)$ while (a) and (b') are satisfied.

Exercise 4 Prove that the concepts "open", "closed", "bounded", "compact" for a set $A \subset \mathbb{R}^n$ do not depend on the chosen norm.

Exercise 5 Let $(E, \|.\|)$ be a normed vector space. Prove or disprove the following propositions.

i) The empty set is compact.
ii) Singletons are compact.
iii) Finite sets are compact (start thinking with a two-point set).
iv) Countable sets are compact.

Are the following sets compact in R^2?

$$A = \{(x,y) : 2x^2+3y^2 < 1\} \quad \text{and} \quad B = \{(x,y) : 0 \leq x, y, \quad xy \leq 1\}.$$

For subsets of $\mathbb{R}^n$, prove or disprove

a) Intersections of compact sets are compact.
b) Compact sets are closed.
c) A closed subset of a compact set is compact.
d) A union of finitely many compact sets is compact (start thinking with two compact sets).
e) An arbitrary union of compact sets is itself compact.

Exercise 6 Let $K \subset \mathbb{R}^n$ be compact and $r > 0$. Let $K_r = \bigcup_{x\in K} B[x,r]$. Prove that K_r is compact.

Exercise 7 Show that $\max(a,b) = \frac{1}{2}(a+b+|a+b|)$ and $\min(a,b) = \frac{1}{2}(a+b-|a+b|)$. What can be said on $\max(f,g)$, if f,g are continuous real-valued functions?

Exercise 8 Let $f(x,y) = \max(e^x \sin(\arctan(y)), \ln(1+x^2+y^4))$. Prove that f admits a maximum on the set $A = \{(x,y) : x^{16}+y^{16} = 64\}$ (do not try to calculate any numerical value!). Prove that f admits a global minimum on $\mathbb{R}^2$. Give it without calculating!

Exercise 9 Let $K \subset R^n$ be compact and $F \subset R^n$ be closed. Prove that

$$g(x) = \inf\{\|x-f\| : f \in F\}$$

is 1-Lipschitz, hence continuous. (hint: start with $\|x-f\| \leq \|x-y\| + \|y-f\|$). Deduce that if the sets K and F do not intersect, they must have a strictly positive distance. Show that compacity is needed here by providing an example of two disjoint closed sets whose distance is zero.

Exercise 10 Let $\gamma(t) = (r\cos(t), r\sin(t))$ for $t \in [0, 2\pi]$. Calculate the arc-length of the curve γ.

Exercise 11 Let $\gamma(t) = (\frac{2}{3}t^3, \sqrt{2}t^2)$ for $t \in [\sqrt{2}, \sqrt{7}]$. Calculate the arc-length of the curve γ.

Exercise 12 Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Parameterise its graph as a curve, and give an integral expression for the arc-length of the curve. Concrete examples (observe that these function are of class C^1):

$$f_1(t) = -t \quad \text{on } [-2,2], \qquad f_2(t) = \tfrac{2}{3}(t-1)^{3/2} \quad \text{on } [1,4] \qquad \text{and} \qquad f_3(t) = \cosh(t) \quad \text{on } [0,1]$$

Exercise 13 Calculate the length of one turn of the helix given by $\gamma(t) = (\cos(t), \sin(t), t)$. What is the length from $(1,0,0)$ to $\gamma(t)$? Find an arc length parameterization of the helix, i.e. a function $\widetilde{\gamma}$ suchh that the distance from $(1,0,0)$ to $\widetilde{\gamma}(t)$ is exacly t.

Exercise 14 Let $f(x,y) = 3x^2y^3$, and $(a,b) \in \mathbb{R}^2$. Let $\gamma_1(t) = (t,b)$ and $\gamma_2(t) = (a,t)$ two curves on a suitable interval I containing a and b. Calculate $g'(a)$ where $g = f \circ \gamma_1$ and $h'(b)$ where $h = f \circ \gamma_2$. Show that write $f_x(a,b) = g'(a)$ and $f_y(a,b) = h'(b)$.

Exercise 15 Find all of the first order partial derivatives for the following functions.

$$f(x,y) = x^3 + 2\sqrt{y} - 10 \qquad g(x,y,z) = x^2y - 8y^2z^3 + 42x - 12\tan(2y)$$
$$h(s,t) = t^7\ln(s^2) + 9t^3 - \sqrt[7]{s^4} \qquad k(x,y) = \cos(2/x)\exp(xy^2 - 3y^3)$$
$$u(x,y) = \frac{9x}{x^2+y} \qquad v(x,y,z) = \frac{x\sin(y)}{z^2} - \sqrt{x^2 + \ln(5z - 3y^2)}$$

Exercise 16 Calculate all second order derivatives of the functions in the preceding exercise.