Exercise 1 Are the following functions differentiable on an open subset of $\mathbb{R}^n$? Calculate the (Frechet) derivative if possible.

a) $f : \mathbb{R}^3 \to \mathbb{R}, \quad f(x,y,z) = xy + yz + zx.$

b) $f : \mathbb{R}^3 \to \mathbb{R}^2, \quad f(x,y,z) = (x^2 z - 2xy, z^3 + xyz).$

c) $f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x,y) = (y\sin(x), \cos(x)).$

Exercise 2 Consider the following functions from $\mathbb{R}^2 \to \mathbb{R}$. Are they f partially differentiable? Fréchet differentiable? Even of class $C^1(\mathbb{R}^2)$?

$$f_1(x,y) = \begin{cases} \frac{\sin(x^3+y^3)}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

$$f_2(x,y) = \begin{cases} \frac{x^2y+2xy^2)}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

$$f_3(x,y) = \begin{cases} \frac{x\sin(y)}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

$$f_4(x,y) = \begin{cases} (x^2+xy+y^2)\sin(\frac{1}{x^2+y^2}) & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

$$f_5(x,y) = \begin{cases} \frac{\sin^2(x)(e^y-1)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

$$f_6(x,y) = \begin{cases} \frac{x^3y+xy^3}{x^4+y^4} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

Exercise 3 Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and that $f(tx) = tf(x)$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Show that there exists some matrix $A \in \mathbb{R}^{n\times n}$ such that $f(x) = A \cdot x$.

Exercise 4 Let $f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x) = \|x\|_2$. Is f differentiable on an open subset of $\mathbb{R}^2$? Which one? Calculate the derivative (same question for $\|x\|_1$ and $\|x\|_\infty$, if you like).

Exercise 5* Let $\Omega = \mathrm{GL}_n(\mathbb{R}) \subset \mathbb{R}^{n\times n}$.

a) Show that $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is continuous.

b) Infer that $\Omega \subset \mathbb{R}^{n\times n}$ is open.

c) Show that there exists some $\delta > 0$ such that $I + H$ is invertible for $\|H\| < \delta$.

d) Use $(I+H)(I-H) = I - H^2$ to show that $(I+H)^{-1} = I - H - H^2(I-H)^{-1}$ for $\|H\| < \delta$.

e) Let $f : \Omega \to \Omega$ be given by $f(M) = M^{-1}$. Show that f is differentiable at I. Infer that f is differentiable on Ω and that $(Df)(M)(H) = -M^{-1}HM^{-1}$.

Exercise 6 Let $f(x,y) = xe^{xy}$, $g(t) = (t^2, t-1)$. Calculate the derivative of $h = f \circ g$, by (1) using the chain rule and (2) direcly, by simplifying $h(t)$. Same question for

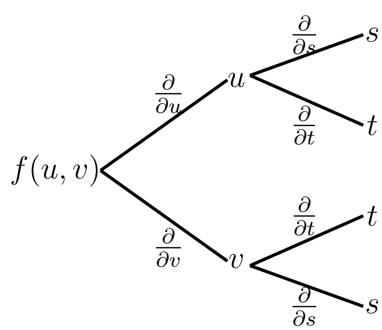
$$\begin{array}{lll} f(x,y) = x^2y^3 + y\cos(x) & \text{and} & g(t) = (\ln(t^2), \sin(4t)) \\ f(x,y) = \sin(x^2+y^2) & \text{and} & g(t) = (t^2+3, t^3) \\ f(x,y) = x^2y & \text{and} & g(t) = (\sin(t), t^2+1) \\ f(x,y) = x^2y^2 & \text{and} & g(s,t) = (st, t^2 - s^2) \end{array}$$

Exercise 7 Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$ and $\mathcal{O} = \{(u,v,w) \in \mathbb{R}^3 : w > 0\}$ Let $f : \Omega \to \mathbb{R}^3$ and $g : \mathcal{O} \to \mathbb{R}$ be given by

$$f(x,y) = (\ln(xy), \cos(x^2+y), e^x) \qquad \text{and} \qquad g(u,v,w) = e^u + vw + \ln(w)$$

Show that $h = g \circ f$ is differentiable in Ω and calculate its derivative (1) using the chain rule and (2) directly, by simplifying $h(x,y)$.

Exercise 8 Let $f, u, v : \mathbb{R}^2 \to \mathbb{R}$. Write out the chain rule for $\frac{\partial}{\partial s} f(u(s,t), v(s,t))$. Write $\frac{\partial f}{\partial u} = \partial_1 f$ and $\frac{\partial f}{\partial v} = \partial_2 f$. Compare the chain rule with the diagram. This explains the expression "chain rule".



Exercise 9 Let $f(x,y) = \int_0^x \sin(yt)\,dt$. Show that f is of class C^1 and calculate its derivative. Infer the derivative of

$$h(x) = \int_0^x \sin(xt)\,dt$$

Exercise 10*

a) Calculate the derivative of trace : $\mathbb{R}^{n\times n} \to \mathbb{R}$, $\text{trace}(A) = \sum_i a_{ii}$. (hint: work out some properties of the trace map first: sums, products, commutativity...).

b) Calculate the derivative of $f : A \mapsto \text{trace}(A^2)$, once by using the chain rule and the map $g : \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \to \mathbb{R}$ given by $g(X,Y) = XY$, another time by decomposing

$$\text{trace}((A+H)^2) = \text{trace}(A) + \ell(H) + r(H).$$

for a suitable linear functional $\ell : \mathbb{R}^{n\times n} \to \mathbb{R}$.

c) Same question for $\text{trace}(A^n)$ where $n \in \mathbb{N}^*$.