Exercise 1 Are the following functions differentiable? Calculate the (Frechet) derivative when possible.

$f(x,y) = 2x^4 - 3x^2y^2 + x^3y$, $f(x,y) = (y^3 + 2x^2y + 3)^2$, $f(x,y) = \frac{y}{x} + \frac{x}{y}$

$f(x,y) = \frac{x}{\sqrt{x^2+y^2}}$, $f(x,y) = \log(x + \sqrt{x^2+y^2})$, $f(x,y) = \arctan \frac{x+y}{x-y}$,

$f(x,y,z) = \sqrt{x^2+y^2+z^2}$, $f(x,y,z) = e^{xy\sin z}$.

Exercise 2 Suppose that $f : R^3 \to R^2$ is defined by $f(x,y,z) = (x^2 + yz, \sin(xyz) + z)$

a) Why is f differentiable on R^3? Compute the Jacobian matrix at $a = (-1, 0, 1)$

b) Are there any directions in which the directional derivative at a is zero? If so, find them.

c) Same question for the functions $g(x,y) = 3x^2 + 5y^2$ at $(1,1)$ and for $h(x,y) = x\sin(x+y)$ at $(\frac{\pi}{4}, \frac{\pi}{4})$. *Special question**: can you prove (without calculating) that in each of these cases, some directional derivative must vanish? Give a geometric argument, and an algebraic one.

Exercise 3 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable in $a \in \mathbb{R}^2$. Let $u = \frac{1}{\sqrt{2}}(1,1)$ and $v = (0,-1)$. The following directional derivatives are given:

$$\frac{\partial f}{\partial u}(a) = \sqrt{8} \qquad \frac{\partial f}{\partial v}(a) = -3$$

Calculate $\nabla f(a)$ and $\frac{\partial f}{\partial w}(a)$ where $w = \frac{1}{\sqrt{5}}(1,2)$.

Exercise 4 Find a function (if one exists) whose gradient is

a) $(\ y^2/x + 2xy^3 - \frac{1}{1+x^2},\ 2y\ln(x) + 3x^2y^2 - \sin y\ )$

b) $(\ 4x^3y - \frac{1}{1+x^2} + e^y,\ x^4 + xe^y + x\ )$

c) $(\ y^3 + 2xy + 3x^2 + 2xy^2,\ 4y^3 + x^2 + 2x^2y + 3xy^2\ )$.

d) $(\ x^2 \arcsin y,\ \frac{x^3}{3\sqrt{1-y^2}} - \ln(y)\ )$.

e) $(\ \ln(x) + 2xye^y + \frac{x}{\sqrt{1-x^2}},\ x^2(1+y)e^y + \frac{1}{\sqrt{1-y^2}}\ )$.

Exercise 5 Let $f(x,y) = x^2 + y^3 + \cos(x)$. Find a constant $C > 0$ such that

$$|f(x,y) - 1| \le C(x^2+y^2)^{1/2}$$

for all x, y such that $x^2 + y^2 \le 1$. Hint: use the mean value theorem.

Exercise 6

a) Let A be a square matrix and $\mathbb{R}^n$ equipped with the Euclidean norm.

Show that $\|A\|_{2\to 2} := \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$ is dominated by the matrix norm $N(A) = \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2}$.

b) Let

$$f(x,y) = \left(\ e^{-x^2/4}\cos(y/2),\ \sin(x/2)\cos(y/3)\ \right)^{\mathtt{t}}$$

Show that f is a strict contraction on $\mathbb{R}^2$, i.e. $\|f(x,y) - f(u,v)\|_2 \le q\|(x,y)-(u,v)\|_2$ for some $q < 1$.

Exercise 7 Calculate the second-degree Taylor polynomial of $f(x,y) = e^{-x^2-y^2}$ at the point $(0,0)$ and at the point $(1,2)$.

Exercise 8 Find the second-degree Taylor polynomial for functions $f(x,y) = \sin(2x) + \cos(y)$ for (x,y) near the point $(0,0)$ and for the function $g(x,y) = xe^y + 1$ for (x,y) near the point $(1,0)$.

Exercise 9 Show that for h and k small enough, the values of

$$\cos(\pi/4 + h)\sin(\pi/4 + k) \qquad \text{and} \qquad \tfrac{1}{2}(1 - h + k)$$

agree to 3 decimal places.

Exercise 10* Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is a function such that all partial derivatives of order 3 exist and are continuous. Write down (explicitly in terms of partial derivatives of f) a polynomial $P(x,y)$ of degree 2 in x and y such that

$$|f(x,y) - P(x,y)| \leq C(x^2 + y^2)^{3/2}$$

for all (x,y) in some small neighbourhood of $(0,0)$, where C is a number that may depend on f but not on x or y. (hint: use Taylor's formula of order 3).

Exercise 11 Find the local extrema of the following functions defined on $\mathbb{R}^2$ (if there exist any)

$$\begin{aligned} f_1(x,y) &= x^3 + x^2y - y^2 - 4y, & f_2(x,y) &= x^2y + x^2 + y^2, \\ f_3(x,y) &= x^3 + 3xy^2 - 15x - 12y, & f_4(x,y) &= \sin(x)\sin(y), \\ f_5(x,y) &= (x - y^2)e^{-x^2-y^2}, & f_6(x,y) &= (x^4 + y^2)e^{1-x^2} \end{aligned}$$

Homework: for each of these functions, visualise the surface given by their graph (i.e. the set of points $\{(x, y, f_k(x,y)) : (x,y) \in \mathbb{R}^2\}$) using adequate software (like GNU octave, scilab, wolframalpha, geogebra, or others).

Exercise 12 Let $\Omega = \{(x,y) : x + y > 0\}$ and $f : \Omega \to \mathbb{R}$ given by $f(x,y) = xy\ln(x+y)$. Find its local extrema (if there exist any).

Exercise 13 Prove that

$$\tfrac{1}{4}(x^2 + y^2) \leq e^{x+y-2}$$

for all $x, y \in \mathbb{R}$. Hint: consider $f(x,y) = (x^2 + y^2)e^{-x-y}$.