Exercise 1 For each of the following functions, justify that a minimum and maximum under the given constraint exists, and use Lagrange multipliers to find them.

function		constraint	
$f_1(x,y) =$	$4xy$	x^2+y^2	$=1$
$f_2(x,y) =$	x^2y	x^2+2y^2	$=1$
$f_3(x,y) =$	$x^2+y^2+2x-2y+1$	x^2+y^2	$=2$
$f_4(x,y) =$	x^2+y^2	xy	$=1$
$f_5(x,y) =$	x^2-y^2	$x-2y+6$	$=0$
$f_6(x,y) =$	x^2+y^2	$x+2y-5$	$=0$
$f_7(x,y) =$	x^2+y^2	$(x-1)^2+4y^2$	$=4$
$f_8(x,y) =$	$4x^3+y^2$	$2x^2+y^2$	$=1$

Exercise 2 Prove that the functions $f(x,y,z) := x^4+y^4+z^4$ and $g(x,y,z) := xyz$ admit a maximum on $S = \{x^2+y^2+z^2=1\}$. Calculate it.

Exercise 3 Show that $f(x,y,z) = yz+xy$ admit a minimum and maximum on the set $S = \{xy=1, y^2+z^2=1\}$. Find it.

Exercise 4 Consider $f(x,y,z) = x^2+y^2+z^2$ and $S = \{x+y+z=9, x+2y+3z=20\}$. Does the minimum/maximum on S exist? If so, calculate it in two different ways.

Exercise 5 Let $T = \{(x,y,z) \in \mathbb{R}^3 : (x-1)^2+y^2=0, y+z=0\}$ and $f(x,y,z) = x^2+y^2-z$. Prove that f admits a minimum and a maximum on T and find them.

Exercise 6 Let $C \subset \mathbb{R}$ be a closed set and $f : C \to C$ a self-map. Suppose that f is differentiable, and that $|f'(x)| \leq q < 1$. Show that f admits a fixed point in C. Example: $f(x) = \ln(2x+1)$ on $C = [1,2]$.

Exercise 7 Consider the function $g(x) = x^2/4 + 5x/4 - 1/2$.

a) g has two fixed points – what are they?

b) For each of these, find the largest region around them such that g is a contraction on that region.

Exercise 8 Let $f(x) = x + \frac{1}{x}$ on $A = [0,\infty)$. Show that f is a self-map of A and that $|f(x)-f(y)| < |x-y|$. Does f admit a fixed point?

Exercise 9* Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x,y) = (x^3-3xy^2, -3x^2y+y^3)$. Show that f is a strict contraction on $A := B[(0,0), \frac{1}{2}]$ by calculating both eigenvalues of the Jacobian matrix at (x,y). Justify all steps. Give the fixed point of f on A.

Exercise 10 Let (x_n) be the sequence defined by $x_1 = \sqrt{3}$, $x_2 = \sqrt{3+\sqrt{3}}$, $x_3 = \sqrt{3+\sqrt{3+\sqrt{3}}}$, $x_4 = \sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3}}}}, \dots$ Prove that (x_n) converges and give its limit.

Exercise 11* **(Newton iteration)**

a) Let $f \in C^2(I)$ and assume that $f(x^*) = 0$, but $f'(x^*) \neq 0$. Let $N(x) = x - (f'(x))^{-1} f(x)$. Prove that N is a contraction in a suitable closed neighbourhood C of x^*. Deduce that the sequence, defined by

$$x_{n+1} := N(x_n) \tag{1}$$

converges to x^* for any $x_0 \in C$.

b)** Now let Ω be an open set in $\mathbb{R}^n$ and let $F : \Omega \to \mathbb{R}^n$ be C^2 mapping (i.e., all first and second partial derivatives of all components of F are continuous). Let us assume that the equation $F(x) = 0$ has a solution $x^* \in \Omega$ such that the derivative $D_F(x^*)$ is invertible.

i) Show that the derivative $D_F(x)$ is invertible in some ball $B(x^*, r)$, $r > 0$.

ii) Let $N(x) = x - D_F(x)^{-1}(F(x))$ defined on $B(x^*, r)$. Show "by hand" that N is differentiable in x^* with $D_N(x^*) = 0$: to this end write

$$N(x^* + h) = N(x^*) + h - D_F(x^*)^{-1} f(x^* + h) - (D_F(x^* + h)^{-1} - D_F(x^*)^{-1})(f(x^* + h))$$

and use that $f(x^* + h) = 0 + D_F(x^*)(h) + r(h)$, with $\frac{\|r(h)\|}{\|h\|^2} \to 0$. Hint: the identity $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$ may be useful.

iii) Show now that in a suitable ball $C := B[x^*, R]$ (for some $0 < R < r$), N is a strictly contractive self-map from C to itself. Infer that the iteration scheme (1) converges to x^* for any starting point $x_0 \in C$.

Exercise 12

a) Formulate the local inversion theorem in the case of a C^1-function in $\mathbb{R}$ (dimension one), and give a simple proof. Compare your proof with the "general case" you saw in the lecture.

b) Consider $f(x) = x + x^2 \sin(\frac{1}{x})$. Show that f is differentiable in $x = 0$, but that f' is discontinuous in $x = 0$.

c)* Can one find a local inverse in a neighbourhood of $x = 0$?

Exercise 13 Let $f(x, y) = (x^2 - y^2, 2xy)$. Show that f is locally invertibel at each point. Is there a global inverse? Think complex!

Exercise 14 Let $\Omega = \mathbb{R}^2 \backslash \{(0, 0)\}$ and $f : \Omega \to \Omega$ be given by $f(x, y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$. Show that f is locally invertible in each point $\omega \in \Omega$. Provide explicitly the inverse function and discuss global invertibility.

Exercise 15 Let $f(x, y) = (x^2 + y^2, \sin(\cos(y)))$ and $g(x, y) = (x + y^3, e^x)$. In which points are these functions locally invertible?

Exercise 16 Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by $(x, y, z) \mapsto (x + y + z, xy + yz + zx, xyz)$. Show that f is locally invertible in a a neighbourhood of (a, b, c) if and only if $(a - b)(b - c)(c - a) \neq 0$. Provide in this case the Jacobian matrix of the inverse function g at the point $y = f(a, b, c)$.