MATHEMATICAL ANALYSIS OF THE MOTION OF A RIGID BODY IN A COMPRESSIBLE NAVIER-STOKES-FOURIER FLUID.

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ABSTRACT. We study an initial and boundary value problem modelling the motion of a rigid body in a heat conducting gas. The solid is supposed to be a perfect thermal insulator. The gas is described by the compressible Navier-Stokes-Fourier equations, whereas the motion of the solid is governed by Newton's laws. The main results assert the existence of strong solutions, in an L^p-L^q setting, both locally in time and globally in time for small data. The proof is essentially using the maximal regularity property of associated linear systems. This property is checked by proving the $\mathcal{R}$-sectoriality of the corresponding operators, which in turn is obtained by a perturbation method.

Key words. Compressible Navier-Stokes-Fourier System, fluid-particle interaction, strong solutions, $\mathcal{R}$-sectorial operators, maximal regularity.

AMS subject classifications. 35Q30, 76D05, 76N10

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Part 1. Introduction and Statement of the Main Results

1. Introduction

The purpose of this work is to provide existence and uniqueness results for a coupled PDEs-ODEs system which models the motion of a rigid body in a viscous heat conducting gas. The rigid body is assumed to be a perfect insulator. As far as we know, this system has not been studied in the literature in three space dimensions. A related problem in one space dimension, the so-called *adiabatic piston problem* has been studied in [15].

Let us now mention some related works from the literature. The one-dimensional piston problem with homogenous boundary conditions has been studied by Shelukhin [23, 24]. Maity, Takahashi and Tucsnak [20] proved existence and uniqueness of global in time strong solutions with nonhomogeneous boundary conditions in a Hilbert space setting. Local in time existence and uniqueness of a heat conducting piston in L^p-L^q framework is studied by Maity and Tucsnak [21]. Concerning three-dimensional models, global existence of weak solutions for compressible fluid and rigid body interaction problems was studied by Desjardins and Esteban [10] and Feireisl [14]. Boulakia and Guerrero [5] proved global existence and uniqueness of strong solutions for small initial data within the Hilbert space framework. Hieber and Murata [17] proved local in time existence and uniqueness in a L^p-L^q setting. Let us also mention that an important influence on the methods in this work comes from several recent advances on the L^p-L^q theory of viscous compressible fluids (without structure), see Enomoto and Shibata [13] and Murata and Shibata [25].

In this work we are interested in strong solutions and the main novelties we bring in are:

- The full nonlinear free boundary system coupling the compressible Navier-Stokes-Fourier system with the ODE system for the solid has not, at our knowledge, been studied in the literature.
- The existence and uniqueness results are proved in a L^p-L^q setting, which, at least as global existence is concerned, is new even in the case when the fluid is barotropic.

The methodologies we employ for the local in time, versus the global in time (for small data), existence results are quite different. More precisely, in the proof of the local existence theorem, we begin by considering a linear "cascade" system. The corresponding operator is proved to have the maximal regularity property in appropriate spaces by combining various existing maximal L^p-L^q regularity results for parabolic equations. This allows us to develop a quite simple fixed point procedure to obtain the local in time existence and uniqueness of solutions. More precisely, using this associated linear system we estimate the nonlinear terms with a coefficient involving the length of the considered time interval, see Proposition 6.4 below. This is why this method is suitable for local existence results (but not relevant if we are interested in global existence for small data).

The strategy developed in proving global existence and uniqueness for small initial data is more involved. More precisely, in this case it is essential to linearize around a stationary solution and to prove that the corresponding linear system is exponentially stable. To prove this property we use a "monolithic" approach, which means that the linear system preserves the coupling between fluid and structure. However, in order to obtain the maximal regularity property we repeatedly use a perturbation argument. Roughly speaking, this means we deduce

the maximal regularity property for the coupled system from the corresponding properties of the linearized fluid equations with homogeneous boundary conditions.

The plan of the paper is as follows. In the next section, we introduce the governing equations and we state our main results. Section 3 is devoted to notation, which introduces, in particular, several function spaces playing an important role in the remaining part of this work.

Part 2 is devoted to the proof of a local in time existence result. More precisely, in Section 4 we rewrite the governing equations in Lagrangian coordinates, in Section 5 we prove the maximal L^p-L^q regularity for an associate "cascade" type linear system, whereas in Section 6, we derive the required estimates and Lipschitz properties of the nonlinear terms in order to apply a fixed-point procedure.

In Part 3 we prove global existence and uniqueness of solutions for small initial data. This is divided into several sections. In Section 7, we linearize the system around a constant steady state and we rewrite the system in the reference configuration. In Section 8 we recall some results concerning maximal L^p regularity for abstract Cauchy problems and its connections with the $\mathcal{R}$-sectoriality property. In Section 9 and in Section 10 we prove the maximal L^p-L^q regularity of a linearized coupled fluid-structure interaction problem on time interval $[0, \infty)$. We prove Lipschitz properties of the nonlinear terms in Section 11 and finally in Section 12 we prove the global existence theorem.

2. Statement of the Main Results

We consider a rigid structure immersed in a viscous heat conducting gas and we denote by $\Omega_S(t)$ the domain occupied by the solid at time $t \geqslant 0$. We assume that the fluid and rigid are contained in a smooth bounded domain $\Omega \subset \mathbb{R}^3$. Moreover, we suppose that $\Omega_S(0)$ has a smooth boundary and that

$$\operatorname{dist}(\Omega_S(0), \partial\Omega) \geqslant \nu > 0. \tag{2.1}$$

For any time $t \geqslant 0$, $\Omega_F(t) = \Omega \setminus \overline{\Omega_S(t)}$ denotes the region occupied by the fluid. The motion of the fluid is given by

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 & \text{in } (0, T) \times \Omega_F(t), \\ \rho(\partial_t u + (u \cdot \nabla)u) - \operatorname{div} \sigma(u, p) = 0 & \text{in } (0, T) \times \Omega_F(t), \quad (2.2) \\ c_v \rho \left(\partial_t \vartheta + u \cdot \nabla\vartheta\right) + p \operatorname{div} u - \kappa\Delta\vartheta = \alpha(\operatorname{div} u)^2 + 2\mu Du : Du & \text{in } (0, T) \times \Omega_F(t), \end{cases}$$

where

$$\sigma(u, p) = 2\mu Du + (\alpha \operatorname{div} u - p)I_3,$$

$$Du = \frac{1}{2}(\nabla u + \nabla u^\top),$$

$$\mu > 0 \text{ and } \alpha + \frac{2}{3}\mu > 0, \tag{2.3}$$

$$A : B = \sum_{i,j} a_{ij}b_{ij} \qquad \text{is the canonical scalar product of two } n \times n \text{ matrices,}$$

$$p = R\rho\vartheta, \qquad R \text{ is the universal gas constant.}$$

Note that we denote by $M^\top$ the transpose of a matrix M.

At time $t \geqslant 0$, let $a(t) \in \mathbb{R}^3$, $Q(t) \in SO_3(\mathbb{R})$ and $\omega(t) \in \mathbb{R}^3$ denote the position of the center of mass, the orthogonal matrix giving the orientation of the solid and the angular velocity of the rigid body. Therefore we have,

$$\dot{Q}(t)Q(t)^{-1}y = A(\omega(t))y = \omega(t) \times y, \quad \forall y \in \mathbb{R}^3,$$

where the skew-symmetric matrix $A(\omega)$ is given by

$$A(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \qquad \omega \in \mathbb{R}^3$$

and where $\dot{f}$ denotes the time derivative of f.

Without loss of generality we can assume that

$$a(0) = 0 \quad \text{and} \quad Q(0) = I_3. \tag{2.4}$$

Thus the domain occupied by the structure $\Omega_S(t)$ is given by

$$\Omega_S(t) = a(t) + Q(t)y, \quad t \geqslant 0, y \in \Omega_S(0). \tag{2.5}$$

We denote by $m > 0$ the mass of rigid structure and $J(t) \in \mathcal{M}_{3\times 3}(\mathbb{R})$ its tensor of inertia at time t. The equations of the structures are given by

$$\begin{cases} m\dfrac{d^2}{dt^2}a = -\displaystyle\int_{\partial\Omega_S(t)} \sigma(u,p)n \; d\gamma & \text{in } (0,T), \\ J\dfrac{d}{dt}\omega = (J\omega) \times \omega - \displaystyle\int_{\partial\Omega_S(t)} (x - a(t)) \times \sigma(u,p)n \; d\gamma & \text{in } (0,T), \end{cases} \tag{2.6}$$

where $n(t,x)$ is the unit normal to $\partial\Omega_S(t)$ at the point x directed toward the interior of the rigid body. We assume that the fluid velocity satisfies the no-slip boundary conditions:

$$u(t,x) = 0, \quad x \in \partial\Omega,$$
$$u(t,x) = \dot{a}(t) + \omega(t) \times (x - a(t)) \qquad (x \in \partial\Omega_S(t)). \tag{2.7}$$

We also suppose that the structure is thermally insulating:

$$\frac{\partial\vartheta}{\partial n}(t,x) = 0 \qquad\qquad (t \in (0,T), \; x \in \partial\Omega_F(t)). \tag{2.8}$$

The above system is completed by the following initial conditions

$$\rho(0,\cdot) = \rho_0, \quad u(0,\cdot) = u_0, \quad \vartheta(0,\cdot) = \vartheta_0 \qquad (\text{in } \Omega_F(0)),$$
$$a(0) = 0, \quad \dot{a}(0) = \ell_0, \quad Q(0) = I_3, \quad \omega(0) = \omega_0. \tag{2.9}$$

To state our main results we introduce some notation. Firstly $W^{s,q}(\Omega)$, with $s \geqslant 0$ and $q > 1$, denote the usual Sobolev spaces. Let $k \in \mathbb{N}$. For every $0 < s < k$, $1 \leqslant p < \infty$, $1 \leqslant q < \infty$, we define the Besov spaces by real interpolation of Sobolev spaces

$$B_{q,p}^s(\Omega) = (L^q(\Omega), W^{k,q}(\Omega))_{s/k,p} .$$

We refer to [1] and [28] for a detailed presentation of Besov spaces. We also need some notation specific to our problem.

Let $2 < p < \infty$ and $3 < q < \infty$ such that $\dfrac{1}{p} + \dfrac{1}{2q} \neq \dfrac{1}{2}$. We set

$$\mathcal{I}_{p,q} = \left\{ (\rho_0, u_0, \vartheta_0, \ell_0, \omega_0) \mid \rho_0 \in W^{1,q}(\Omega_F(0)), \; u_0 \in B_{q,p}^{2(1-1/p)}(\Omega_F(0))^3, \right.$$

$$\vartheta_0 \in B_{q,p}^{2(1-1/p)}(\Omega_F(0)), \ell_0 \in \mathbb{R}^3, \ \omega_0 \in \mathbb{R}^3, \ \min_{\overline{\Omega_F(0)}} \rho_0 > 0$$

$$u_0 = 0 \text{ on } \partial\Omega, \quad u_0(y) = \ell_0 + \omega_0 \times y \quad y \in \partial\Omega_S(0)\Big\}\Big\}. \quad (2.10)$$

Such a definition has a sense since from the Sobolev embedding we have

$$W^{1,q}(\Omega_F(0)) \hookrightarrow C(\overline{\Omega_F(0)}).$$

Moreover since $\dfrac{1}{p} + \dfrac{1}{2q} < 1$, if $f \in B_{q,p}^{2(1-1/p)}(\Omega_F(0))$, then (see, for instance, [28, p.200]), f admits a trace on $\partial\Omega_F(0)$ with $f_{|\partial\Omega_F(0)} \in B_{q,p}^{2(1-1/p)-1/q}(\partial\Omega_F(0))$.

The norm of $\mathcal{I}_{p,q}$ is the norm of

$$W^{1,q}(\Omega_F(0)) \times B_{q,p}^{2(1-1/p)}(\Omega_F(0))^3 \times B_{q,p}^{2(1-1/p)}(\Omega_F(0)) \times \mathbb{R}^6.$$

We introduce the set of initial data

$$\mathcal{I}_{p,q}^{cc} = \begin{cases} \mathcal{I}_{p,q} & \text{if } \dfrac{1}{2} < \dfrac{1}{p} + \dfrac{1}{2q} < 1, \\ \left\{ (\rho_0, u_0, \vartheta_0, \ell_0, \omega_0) \in \mathcal{I}_{p,q} \mid \dfrac{\partial\vartheta_0}{\partial n} = 0, \text{ on } \partial\Omega_F(0) \right\} & \text{if } \dfrac{1}{p} + \dfrac{1}{2q} < \dfrac{1}{2}. \end{cases} \quad (2.11)$$

Again, the normal derivative in the above definition is well-defined due to the trace theorem for Besov spaces (see, for instance [28, p.200]).

We also need a definition of Sobolev spaces in the time dependent domain $\Omega_F(t)$. Let $\Lambda(t, \cdot)$ be a C^1-diffeomorphism from $\Omega_F(0)$ onto $\Omega_F(t)$ such that all the derivatives up to second order in space variable and all the derivatives up to first order in time variable exist. For all functions $v(t, \cdot) : \Omega_F(t) \mapsto \mathbb{R}$, we denote $\widehat{v}(t, y) = v(t, \Lambda(t, y))$ Then for any $1 < p, q < \infty$ we define

$$L^p(0, T; L^q(\Omega_F(\cdot))) = \{v \mid \widehat{v} \in L^p(0, T; L^q(\Omega_F(0)))\},$$
$$L^p(0, T; W^{2,q}(\Omega_F(\cdot))) = \{v \mid \widehat{v} \in L^p(0, T; W^{2,q}(\Omega_F(0)))\},$$
$$W^{1,p}(0, T; L^q(\Omega_F(\cdot))) = \{v \mid \widehat{v} \in W^{1,p}(0, T; L^q(\Omega_F(0)))\},$$
$$C([0, T]; W^{1,q}(\Omega_F(\cdot))) = \{v \mid \widehat{v} \in C([0, T]; W^{1,q}(\Omega_F(0)))\},$$
$$C([0, T]; B_{q,p}^{2(1-1/p)}(\Omega_F(\cdot))) = \left\{v \mid \widehat{v} \in C([0, T]; B_{q,p}^{2(1-1/p)}(\Omega_F(0))\right\}.$$

We are now in a position to state our first main result.

Theorem 2.1. *Let $2 < p < \infty$ and $3 < q < \infty$ satisfying the condition $\dfrac{1}{p} + \dfrac{1}{2q} \neq \dfrac{1}{2}$. Assume that* (2.1) *is satisfied and $(\rho_0, u_0, \vartheta_0, \ell_0, \omega_0)$ belongs to $\mathcal{I}_{p,q}^{cc}$. Let $M > 0$ be such that*

$$\|(\rho_0, u_0, \vartheta_0, \ell_0, \omega_0)\|_{\mathcal{I}_{p,q}} \leqslant M, \quad \frac{1}{M} \leqslant \rho_0(x) \leqslant M \text{ for } x \in \Omega_F(0). \quad (2.12)$$

Then, there exists $T > 0$, depending only on M and ν such that the system (2.2) - (2.9) *admits a unique strong solution*

$$\rho \in W^{1,p}(0, T; W^{1,q}(\Omega_F(\cdot))) \cap C([0, T]; W^{1,q}(\Omega_F(\cdot))),$$
$$u \in L^p(0, T; W^{2,q}(\Omega_F(\cdot))^3) \cap W^{1,p}(0, T; L^q(\Omega_F(\cdot))^3) \cap C([0, T]; B_{q,p}^{2(1-1/p)}(\Omega_F(\cdot))^3),$$
$$\vartheta \in L^p(0, T; W^{2,q}(\Omega_F(\cdot))) \cap W^{1,p}(0, T; L^q(\Omega_F(\cdot))) \cap C([0, T]; B_{q,p}^{2(1-1/p)}(\Omega_F(\cdot))),$$

$$a \in W^{2,p}(0,T;\mathbb{R}^3), \quad \omega \in W^{1,p}(0,T;\mathbb{R}^3).$$

Moreover, there exists a constant $M_T > 0$ such that $\dfrac{1}{M_T} \leqslant \rho(t,x) \leqslant M_T$ for all $t \in (0,T), x \in \Omega_F(t)$ and $\mathrm{dist}(\Omega_S(t), \partial\Omega) \geqslant \nu/2$ for all $t \in [0,T]$.

Our second main result asserts global existence and uniqueness under a smallness condition on the initial data.

Theorem 2.2. *Let $2 < p < \infty$ and $3 < q < \infty$ satisfying the condition $\dfrac{1}{p} + \dfrac{1}{2q} \neq \dfrac{1}{2}$. Assume that (2.1) is satisfied. Let $\overline{\rho} > 0$ and $\overline{\vartheta} > 0$ be two given constants. Then there exists $\eta_0 > 0$ such that, for all $\eta \in (0, \eta_0)$ there exist two constants $\delta_0 > 0$ and $C > 0$, such that, for all $\delta \in (0, \delta_0)$ and for any $(\rho_0, u_0, \vartheta_0, \ell_0, \omega_0)$ in $\mathcal{I}_{p,q}^{cc}$ with*

$$\frac{1}{|\Omega_F(0)|} \int_{\Omega_F(0)} \rho_0 \ \mathrm{d}x = \overline{\rho}, \tag{2.13}$$

and

$$\|(\rho_0 - \overline{\rho}, u_0, \vartheta_0 - \overline{\vartheta}, \ell_0, \omega_0)\|_{\mathcal{I}_{p,q}} \leqslant \delta,$$

the system (2.2) - (2.9) admits a unique strong solution $(\rho, u, \vartheta, \ell, \omega)$ in the class of functions satisfying

$$\|(\rho - \overline{\rho})\|_{L^\infty(0,\infty;W^{1,q}(\Omega_F(\cdot)))} + \|e^{\eta(\cdot)}\nabla\rho\|_{W^{1,p}(0,\infty;L^q(\Omega_F(\cdot)))} + \|e^{\eta(\cdot)}\partial_t\rho\|_{L^p(0,\infty;L^q(\Omega_F(\cdot)))}$$

$$+ \|e^{\eta(\cdot)}\,u\|_{L^p(0,\infty;W^{2,q}(\Omega_F(\cdot))^3)} + \|e^{\eta(\cdot)}\partial_t u\|_{L^p(0,\infty;L^q(\Omega_F(\cdot))^3)} + \|e^{\eta(\cdot)}u\|_{L^\infty(0,\infty;B_{q,p}^{2(1-1/p)}(\Omega_F(\cdot))^3)}$$

$$+ \|e^{\eta(\cdot)}\partial_t\vartheta\|_{L^p(0,\infty;L^q(\Omega_F(\cdot)))} + \|e^{\eta(\cdot)}\nabla\vartheta\|_{L^p(0,\infty;L^q(\Omega_F(\cdot)))} + \|e^{\eta(\cdot)}\nabla^2\vartheta\|_{L^p(0,\infty;L^q(\Omega_F(\cdot)))}$$

$$+ \|(\vartheta - \overline{\vartheta})\|_{L^\infty(0,\infty;B_{q,p}^{2(1-1/p)}(\Omega_F(\cdot)))} + \|e^{\eta(\cdot)}\dot{a}\|_{L^p(0,\infty;\mathbb{R}^3)} + \|e^{\eta(\cdot)}\ddot{a}\|_{L^p(0,\infty;\mathbb{R}^3)}$$

$$+ \|a\|_{L^\infty(0,\infty;\mathbb{R}^3)} + \|e^{\eta(\cdot)}\,\omega\|_{W^{1,p}(0,\infty;\mathbb{R}^3)} \leqslant C\delta. \tag{2.14}$$

Moreover, $\rho(t,x) \geqslant \dfrac{\overline{\rho}}{2}$ for all $t \in (0,\infty), \ x \in \Omega_F(t)$ and $\mathrm{dist}(\Omega_S(t), \partial\Omega) \geqslant \nu/2$ for all $t \in [0,\infty)$.

As shown in Section 12, a simple consequence of the above theorem is:

Corollary 2.3. *With the assumptions and notation in Theorem 2.2 we have*

$$\|u(t,\cdot)\|_{B_{q,p}^{2(1-1/p)}(\Omega_F(t))^3} + \|\dot{a}(t)\|_{\mathbb{R}^3} + \|\omega(t)\|_{\mathbb{R}^3} \leqslant C\delta e^{-\eta t},$$

$$\|\rho(t,\cdot) - \overline{\rho}\|_{W^{1,q}(\Omega_F(\cdot))} \leqslant C\delta e^{-\eta t}, \tag{2.15}$$

where the constant C is independent of $t > 0$.

To prove Theorem 2.1 and Theorem 2.2 we follow a strategy which is widely used in the literature on existence and uniqueness of solutions for fluid-solid interaction models, which is:

- **Step 1.** Since the domain of the fluid equation is one of the unknowns, we first rewrite the system in a fixed spatial domain. This can be achieved either by a "geometric" change of variables (see [5]), by using Lagrangian coordinates (see [20]) or by combining these two change of coordinates (see [17]). In the present work, we found more convenient to use Lagrangian variables. Apart from allowing to rewrite the coupled

system in a fixed cylindrical domain this allows us to tackle the term $u \cdot \nabla \rho$ in the density equation.

- **Step 2.** Next we associate to the original nonlinear problem a linear one, involving source terms. A crucial step here is to establish the L^p-L^q regularity property for this linear problem. This is done by proving that the associated linear operators are $\mathcal{R}$-sectorial in an appropriate Banach spaces.
- **Step 3.** We estimate the nonlinear terms in the governing equations and we use the Banach fixed point theorem to prove existence and uniqueness results in the reference configuration.
- **Step 4.** In the final step we come back to the original configuration.

3. Notation

In this section, we fix some notations that we use throughout this paper. For $s \in (0,1)$ and a Banach space U, $F_{p,q}^s(0,T,U)$ stands for U valued Lizorkin-Triebel space. For precise definition of such spaces we refer to [28, 22]. If $T < \infty$, this spaces can be characterised as follows (see [31])

$$F_{p,q}^s(0,T;U) = \left\{ f \in L^p(0,T,U) \mid |f|_{F_{p,q}^s(0,T;U)} < \infty \right\},$$

where

$$|f|_{F_{p,q}^s(0,T;U)} = \left(\int_0^T \left(\int_0^{T-t} h^{-1-sq} \|f(t+h) - f(t)\|_U^q \, \mathrm{d}h \right)^{p/q} \mathrm{d}t \right)^{1/p}.$$

These spaces endowed with the natural norm

$$\|f\|_{F_{p,q}^s(0,T;U)} = \|f\|_{L^p(0,T;U)} + |f|_{F_{p,q}^s(0,T;U)}. \tag{3.1}$$

If $T \in (0,\infty]$, we set $Q_T^F = (0,T) \times \Omega_F(0)$ and

$$W_{q,p}^{2,1}(Q_T^F) = L^p(0,T;W^{2,q}(\Omega_F(0))) \cap W^{1,p}(0,T;L^q(\Omega_F(0))),$$

with

$$\|u\|_{W_{q,p}^{2,1}(Q_T^F)} = \|u\|_{L^p(0,T;W^{2,q}(\Omega_F(0)))} + \|u\|_{W^{1,p}(0,T;L^q(\Omega_F(0)))}.$$

For $T \in (0,\infty]$, the space $\mathcal{S}_{T,p,q}$ is defined by

$$\mathcal{S}_{T,p,q} = \left\{ (\rho, u, \vartheta, \ell, \omega) \mid \rho \in W^{1,p}(0,T;W^{1,q}(\Omega_F(0))), \quad u \in \left(W_{q,p}^{2,1}(Q_T^F)\right)^3, \right.$$

$$\left. \vartheta \in W_{q,p}^{2,1}(Q_T^F), \quad \ell \in W^{1,p}(0,T;\mathbb{R}^3), \quad \omega \in W^{1,p}(0,T;\mathbb{R}^3) \right\} \tag{3.2}$$

and

$$\|(\rho, u, \vartheta, \ell, \omega)\|_{\mathcal{S}_{T,p,q}} = \|\rho\|_{W^{1,p}(0,T;W^{1,q}(\Omega_F(0)))} + \|u\|_{W_{q,p}^{2,1}(Q_T^F)} + \|\vartheta\|_{W_{q,p}^{2,1}(Q_T^F)}$$

$$+ \|\ell\|_{W^{1,p}(0,T)} + \|\omega\|_{W^{1,p}(0,T)}.$$

For any $T < \infty$ or $T = \infty$ we define $\mathcal{B}_{T,p,q}$ as follows

$$\mathcal{B}_{T,p,q} = \left\{ (f_1, f_2, f_3, h, g_1, g_2) \mid f_1 \in L^p(0,T,W^{1,q}(\Omega_F(0))), f_2 \in L^p(0,T;L^q(\Omega_F(0)))^3, \right.$$

$$f_3 \in L^p(0,T;L^q(\Omega_F(0))), h \in F_{p,q}^{(1-1/q)/2}(0,T;L^q(\partial\Omega_F(0))) \cap L^p(0,T;W^{1-1/q,q}(\partial\Omega_F(0))),$$

$$g_1 \in L^p(0,T), g_2 \in L^p(0,T) \text{ with } h(0,y) = 0 \text{ for all } y \in \partial\Omega_F(0) \text{ if } \frac{1}{p} + \frac{1}{2q} < \frac{1}{2}\Big\}, \quad (3.3)$$

with

$$\|(f_1, f_2, f_3, h, g_1, g_2)\|_{\mathcal{B}_{T,p,q}} = \|f_1\|_{L^p(0,T;W^{1,q})} + \|f_2\|_{L^p(0,T;L^q)} + \|f_3\|_{L^p(0,T;L^q)}$$

$$+\|h\|_{F_{p,q}^{(1-1/q)/2}(0,T;L^q(\partial\Omega_F(0)))\cap L^p(0,T;W^{1-1/q,q}(\partial\Omega_F(0)))} + \|g_1\|_{L^p(0,T)} + \|g_2\|_{L^p(0,T)}.$$

Part 2. **Local in Time Existence and Uniqueness**

4. Lagrangian Change of Variables

In this section, we describe a change of variables, obtained by a slight variation of the usual passage to Lagrangian coordinates, which allows us to rewrite the governing equations in a fixed spatial domain and to preserve the linear form of the transmission condition for the velocity field. More precisely, we consider the characteristics X associated to the fluid velocity u, that is the solution of the Cauchy problem

$$\begin{cases} \partial_t X(t,y) = u(t, X(t,y)) & (t > 0), \\ X(0,y) = y \in \overline{\Omega_F(0)}. \end{cases} \quad (4.1)$$

Assume that $X(t,\cdot)$ is a C^1-diffeomorphism from $\Omega_F(0)$ onto $\Omega_F(t)$ for all $t \in (0,T)$ (see (6.17)). For each $t \in (0,T)$, we denote by $Y(t,\cdot) = [X(t,\cdot)]^{-1}$ the inverse of $X(t,\cdot)$. We consider the following change of variables

$$\widetilde{\rho}(t,y) = \rho(t, X(t,y)), \qquad \widetilde{u}(t,y) = Q^{-1}(t)u(t, X(t,y)),$$
$$\widetilde{\vartheta}(t,y) = \vartheta(t, X(t,y)), \qquad \widetilde{p} = R\widetilde{\rho}\widetilde{\vartheta}, \qquad (4.2)$$
$$\widetilde{\ell}(t) = Q^{-1}(t)\dot{a}(t), \qquad \widetilde{\omega}(t) = Q^{-1}(t)\omega(t),$$

for $(t,y) \in (0,T) \times \Omega_F(0)$. In particular,

$$\rho(t,x) = \widetilde{\rho}(t, Y(t,x)), \quad u(t,x) = Q(t)\widetilde{u}(t, Y(t,x)), \quad \vartheta(t,x) = \widetilde{\vartheta}(t, Y(t,x)), \quad (4.3)$$

for $(t,x) \in (0,T) \times \Omega_F(t)$. This change of variables implies that $(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}, a, Q)$ satisfies

$$\begin{cases} \partial_t\widetilde{\rho} + \rho_0 \operatorname{div}\widetilde{u} = \mathcal{F}_1 & \text{in } (0,T) \times \Omega_F(0), \\ \widetilde{\rho}(0,\cdot) = \rho_0 & \text{in } \Omega_F(0), \end{cases} \quad (4.4)$$

$$\begin{cases} \partial_t\widetilde{u} - \dfrac{\mu}{\rho_0}\Delta\widetilde{u} - \dfrac{\alpha+\mu}{\rho_0}\nabla(\operatorname{div}\widetilde{u}) = \mathcal{F}_2 & \text{in } (0,T) \times \Omega_F(0), \\ \widetilde{u} = 0 & \text{on } (0,T) \times \partial\Omega, \\ \widetilde{u} = \widetilde{\ell} + \widetilde{\omega} \times y & \text{on } (0,T) \times \partial\Omega_S(0), \\ \widetilde{u}(0,\cdot) = u_0 & \text{in } \Omega_F(0), \end{cases} \quad (4.5)$$

$$\begin{cases} m\dfrac{d}{dt}\widetilde{\ell} = \mathcal{G}_1 & \text{in } (0,T), \\ J(0)\dfrac{d}{dt}\widetilde{\omega} = \mathcal{G}_2 & \text{in } (0,T), \\ \widetilde{\ell}(0) = \ell_0, \quad \widetilde{\omega}(0) = \omega_0. \end{cases} \quad (4.6)$$

$$\begin{cases} \partial_t \widetilde{\vartheta} - \dfrac{\kappa}{\rho_0 c_v} \Delta \widetilde{\vartheta} = \mathcal{F}_3 & \text{in } (0,T) \times \Omega_F(0), \\[2mm] \dfrac{\partial \widetilde{\vartheta}}{\partial n} = \mathcal{H} \cdot n & \text{on } (0,T) \times \partial \Omega_F(0), \\[2mm] \widetilde{\vartheta}(0, \cdot) = \vartheta_0 & \text{in } \Omega_F(0), \end{cases} \tag{4.7}$$

$$\begin{cases} \dot{a} = Q\widetilde{\ell} & \text{in } (0,T), \\ \dot{Q} = Q A(\widetilde{\omega}) & \text{in } (0,T), \\ a(0) = 0, \quad Q(0) = I_3, \end{cases} \tag{4.8}$$

where

$$X(t,y) = y + \int_0^t Q(s)\widetilde{u}(s,y)\,\mathrm{d}s, \quad \text{and} \quad \nabla Y(t, X(t,y)) = [\nabla X]^{-1}(t,y), \tag{4.9}$$

for every $y \in \Omega_F(0)$ and $t \geqslant 0$. Using the notation

$$Z(t,y) = (Z_{i,j})_{1 \leqslant i,j \leqslant 3} = [\nabla X]^{-1}(t,y) \qquad (t \geqslant 0, \ y \in \Omega_F(0)), \tag{4.10}$$

the remaining terms in (4.4)–(4.7) are defined by:

$$\mathcal{F}_1(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) = -(\widetilde{\rho} - \rho_0)\operatorname{div}\widetilde{u} - \widetilde{\rho}(\nabla \widetilde{u}) : \left[(ZQ)^\top - I_3 \right], \tag{4.11}$$

$$\begin{aligned} (\mathcal{F}_2)_i(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) = &-\frac{\widetilde{\rho}}{\rho_0}(\widetilde{\omega} \times Q\widetilde{u})_i + \left(1 - \frac{\widetilde{\rho}}{\rho_0}\right)(\partial_t \widetilde{u})_i - \frac{\widetilde{\rho}}{\rho_0}\left[(Q - I)\partial_t \widetilde{u}\right]_i \\ &+ \frac{\mu}{\rho_0}\sum_{l,j,k} \frac{\partial^2(Q\widetilde{u})_i}{\partial y_l \partial y_k}\left(Z_{k,j} - \delta_{k,j}\right)Z_{l,j} + \frac{\mu}{\rho_0}\sum_{l,k}\frac{\partial^2(Q\widetilde{u})_i}{\partial y_l \partial y_k}\left(Z_{l,k} - \delta_{l,k}\right) \\ &+ \frac{\mu}{\rho_0}\left[(Q - I)\Delta \widetilde{u}\right]_i + \frac{\mu}{\rho_0}\sum_{l,j,k} Z_{l,j}\frac{\partial(Q\widetilde{u})_i}{\partial y_k}\frac{\partial Z_{k,j}}{\partial y_l} \\ &+ \frac{\mu + \alpha}{\rho_0}\sum_{l,j,k}\frac{\partial^2(Qu)_j}{\partial y_l \partial y_k}\left(Z_{k,j} - \delta_{k,j}\right)Z_{l,i} + \frac{\mu + \alpha}{\rho_0}\sum_{l,j}\frac{\partial^2(Qu)_j}{\partial y_l \partial y_j}\left(Z_{l,i} - \delta_{l,i}\right) \\ &+ \frac{\alpha + \mu}{\rho_0}\frac{\partial}{\partial y_i}\left[\nabla \widetilde{u} : (Q^\top - I_3)\right] + \frac{\alpha + \mu}{\rho_0}\sum_{l,j,k} Z_{l,i}\frac{\partial(Qu)_j}{\partial y_k}\frac{\partial Z_{k,j}}{\partial y_l} \\ &- R\frac{\widetilde{\vartheta}}{\rho_0}\left(Z^\top \nabla \widetilde{\rho}\right)_i - R\frac{\widetilde{\rho}}{\rho_0}\left(Z^\top \nabla \widetilde{\vartheta}\right)_i, \end{aligned} \tag{4.12}$$

$$\begin{aligned} \mathcal{F}_3(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) = &\left(\frac{\rho_0 - \widetilde{\rho}}{c_v \rho_0}\right)\partial_t \widetilde{\vartheta} - \frac{R\widetilde{\vartheta}\widetilde{\rho}}{c_v \rho_0}[ZQ]^\top : \nabla \widetilde{u} + \frac{\kappa}{c_v \rho_0}\sum_{l,j,k} Z_{l,j}\frac{\partial \widetilde{\vartheta}}{\partial y_k}\frac{\partial Z_{k,j}}{\partial y_l} \\ &+ \frac{\kappa}{c_v \rho_0}\sum_{j,k,l}\frac{\partial^2 \widetilde{\vartheta}}{\partial y_k \partial y_l}\left(Z_{k,j} - \delta_{k,j}\right)Z_{l,j} + \frac{\kappa}{c_v \rho_0}\sum_{k,l}\frac{\partial^2 \widetilde{\vartheta}}{\partial y_k \partial y_l}\left(Z_{l,k} - \delta_{l,k}\right) \\ &+ \frac{\alpha}{c_v \rho_0}\left([ZQ]^\top : \nabla \widetilde{u}\right)^2 + \frac{\mu}{2c_v \rho_0}\left|\nabla \widetilde{u}ZQ + (\nabla \widetilde{u}ZQ)^\top\right|^2, \end{aligned} \tag{4.13}$$

$$\mathcal{H}(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) = \mathbb{1}_{\partial \Omega}\mathcal{H}_F + \mathbb{1}_{\partial \Omega_S(0)}\mathcal{H}_S, \tag{4.14}$$

$$\mathcal{H}_F(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{u}) = (I_3 - Z^\top)\nabla \widetilde{\vartheta} \qquad \mathcal{H}_S(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) = (I_3 - (ZQ)^\top)\nabla \widetilde{\vartheta},$$

$$\mathcal{G}_0(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) = \mu \left[\nabla \widetilde{u} Z Q + (\nabla \widetilde{u} Z Q)^\top \right] + \alpha \left([ZQ]^\top : \nabla \widetilde{u} \right) I_3 + R \widetilde{\rho} \widetilde{\vartheta} I_3, \qquad (4.15)$$

$$\mathcal{G}_1(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) = -m(\widetilde{\omega} \times \widetilde{\ell}) - \int_{\partial \Omega_S(0)} \mathcal{G}_0 n \; \mathrm{d}\gamma, \qquad (4.16)$$

$$\mathcal{G}_2(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) = J(0) \widetilde{\omega} \times \widetilde{\omega} - \int_{\partial \Omega_S(0)} y \times \mathcal{G}_0 n \; \mathrm{d}\gamma. \qquad (4.17)$$

Using the above change of variables, our main result in Theorem 2.1 can be rephrased as:

Theorem 4.1. *Let $2 < p < \infty$ and $3 < q < \infty$ satisfying the condition $\dfrac{1}{p} + \dfrac{1}{2q} \neq \dfrac{1}{2}$. Assume that $(\rho_0, u_0, \vartheta_0, \ell_0, \omega_0)$ belongs to $\mathcal{I}^{cc}_{p,q}$ such that (2.12) holds. Then, there exists $\widetilde{T} > 0$ such that for any $T \in (0, \widetilde{T})$, the system (4.4) - (4.17) admits a unique strong solution*

$$
\begin{aligned}
&\widetilde{\rho} \in W^{1,p}(0, T; W^{1,q}(\Omega_F(0))), \\
&\widetilde{u} \in L^p(0, T; W^{2,q}(\Omega_F(0))^3) \cap W^{1,p}(0, T; L^q(\Omega_F(0))^3) \cap C([0, T]; B^{2(1-1/p)}_{q,p}(\Omega_F(0))^3), \\
&\widetilde{\vartheta} \in L^p(0, T; W^{2,q}(\Omega_F(0))) \cap W^{1,p}(0, T; L^q(\Omega_F(0))) \cap C([0, T]; B^{2(1-1/p)}_{q,p}(\Omega_F(0))), \\
&\widetilde{\ell} \in W^{1,p}(0, T; \mathbb{R}^3), \quad \widetilde{\omega} \in W^{1,p}(0, T; \mathbb{R}^3), \\
&a \in W^{2,p}(0, T; \mathbb{R}^3), \quad Q \in W^{2,p}(0, T; SO(3)), \\
&X \in W^{1,p}(0, T; W^{2,q}(\Omega_F(0))) \cap W^{2,p}(0, T; L^q(\Omega_F(0))), \\
&X(t, \cdot) : \Omega_F(0) \to \Omega_F(t) \text{ is a } C^1 - \text{diffeormorphim for all } t \in [0, T].
\end{aligned}
\qquad (4.18)
$$

Moreover, there exists a constant $M_T > 0$, such that $\dfrac{1}{M_T} \leqslant \widetilde{\rho}(t, y) \leqslant M_T$, for all $t \in (0, T), y \in \Omega_F(0)$.

The proof of the above theorem relies on a fixed point theorem and a linearization. We describe below the main steps of the proof using the maximal regularity of an associated linear problem and some estimates of the non linear terms involved in the fixed point procedure. For the clarity of the presentation , we postpone the detailed proofs of the maximal regularity and the estimates of the non linear terms technical results in Section 5 and Section 6 respectively.

Proof of Theorem 4.1. Assume

$$(\rho_0, u_0, \vartheta_0, \ell_0, \omega_0) \in \mathcal{I}^{cc}_{p,q},$$

is given (see (2.11)) and (2.12) holds with $M > 0$.

We consider the following linear problem.

$$
\begin{cases}
\partial_t \widetilde{\rho} + \rho_0 \, \mathrm{div} \, \widetilde{u} = f_1 & \text{in } (0, T) \times \Omega_F(0), \\
\widetilde{\rho}(0, \cdot) = \rho_0 & \text{in } \Omega_F(0),
\end{cases}
\qquad (4.19)
$$

$$
\begin{cases}
\partial_t \widetilde{u} - \dfrac{\mu}{\rho_0} \Delta \widetilde{u} - \dfrac{\alpha + \mu}{\rho_0} \nabla(\mathrm{div} \, \widetilde{u}) = f_2 & \text{in } (0, T) \times \Omega_F(0), \\
\widetilde{u} = 0 & \text{on } (0, T) \times \partial \Omega, \\
\widetilde{u} = \widetilde{\ell} + \widetilde{\omega} \times y & \text{on } (0, T) \times \partial \Omega_S(0), \\
\widetilde{u}(0, \cdot) = u_0 & \text{in } \Omega_F(0),
\end{cases}
\qquad (4.20)
$$

$$\begin{cases} m\dfrac{d}{dt}\widetilde{\ell} = g_1 & \text{in } (0,T), \\[2mm] J(0)\dfrac{d}{dt}\widetilde{\omega} = g_2 & \text{in } (0,T), \\[2mm] \widetilde{\ell}(0) = \ell_0, \quad \widetilde{\omega}(0) = \omega_0. \end{cases} \tag{4.21}$$

$$\begin{cases} \partial_t\widetilde{\vartheta} - \dfrac{\kappa}{\rho_0 c_v}\Delta\widetilde{\vartheta} = f_3 & \text{in } (0,T)\times\Omega_F(0), \\[2mm] \dfrac{\partial\widetilde{\vartheta}}{\partial n} = h & \text{on } (0,T)\times\partial\Omega_F(0), \\[2mm] \widetilde{\vartheta}(0,\cdot) = \vartheta_0 & \text{in } \Omega_F(0). \end{cases} \tag{4.22}$$

where

$$(\rho_0, u_0, \vartheta_0, \ell_0, \omega_0) \in \mathcal{I}_{p,q}^{cc}, \quad (f_1, f_2, f_3, h, g_1, g_2) \in \mathcal{B}_{T,p,q}$$

are given (see (2.11) and (3.3)) and such that the initial conditions satisfy (2.12) with $M > 0$.

In the following we shall denote for each $T_* \in (0,T]$, by $\widetilde{\mathcal{B}}_{T_*}$ the closed unit ball in $\mathcal{B}_{T_*,p,q}$.

In Section 5 we will construct a solution to (4.19)—(4.22) providing a bounded 'solution operator'

$$F : \begin{cases} \mathcal{B}_{T,p,q} & \to & \mathcal{S}_{T,p,q} \\ (f_1, f_2, f_3, h, g_1, g_2) & \mapsto & (\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}), \end{cases}$$

where we recall that $\mathcal{B}_{T,p,q}$ and $\mathcal{S}_{T,p,q}$ are defined in (3.3) and in (3.2).

In Section 6 we then prove norm estimates for the nonlinear terms, $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{H}_F, \mathcal{H}_S, \mathcal{G}_1, \mathcal{G}_2$. More precisely, assuming that

$$T \in (0,\widetilde{T}), \quad \text{with } \widetilde{T} < 1 \quad \text{small enough,}$$

we show that the obtained norm bounds here depend, up to a constant, on T^δ where δ depends on p, q only. This allows us to define the operator

$$\mathcal{N} : \begin{cases} \mathcal{B}_{T,p,q} & \to & \mathcal{B}_{T,p,q} \\ (f_1, f_2, f_3, h, g_1, g_2) & \mapsto & (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{H}, \mathcal{G}_1, \mathcal{G}_2), \end{cases}$$

and to show that, for sufficiently small $\widetilde{T}$, it becomes a self-map of the closed ball

$$\{(f_1, f_2, f_3, h, g_1, g_2) \in \mathcal{B}_{T,p,q} \mid \|(f_1, f_2, f_3, h, g_1, g_2)\|_{\mathcal{B}_{T,p,q}} \leqslant 1\}.$$

Finally, in Proposition 6.5 a Lipschitz estimate for $\mathcal{N}$ is proved, again with a Lipschitz constant depending on T^δ, provided that $T \in (0,\widetilde{T})$. This allows us to enforce a strict contraction on the above closed ball and hence a fixed point, that provides a solution to (4.4)—(4.17) satisfying (4.18). The bound of $\widetilde{\rho}$ will be obtained from the estimate (6.8). $\square$

From Theorem 4.1 we can now deduce Theorem 2.1.

Proof of Theorem 2.1. Let us assume that $(\rho_0, u_0, \vartheta_0, \ell_0, \omega_0) \in \mathcal{I}_{p,q}^{cc}$ satisfies the condition (2.12).

Let $T < \widetilde{T}$ with $\widetilde{T}$ as in Theorem 4.1. In particular, there exists a unique solution $(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{g}, \widetilde{\omega})$ to the system (4.4) - (4.15) satisfying

$$\widetilde{\rho} \in W^{1,p}(0,T; W^{1,q}(\Omega_F(0)))$$
$$\widetilde{u} \in L^p(0,T; W^{2,q}(\Omega_F(0))^3) \cap W^{1,p}(0,T; L^q(\Omega_F(0))^3)$$
$$\widetilde{\vartheta} \in L^p(0,T; W^{2,q}(\Omega_F(0))) \cap W^{1,p}(0,T; L^q(\Omega_F(0)))$$

$$\widetilde{\ell} \in W^{1,p}(0,T;\mathbb{R}^3), \quad \widetilde{\omega} \in W^{1,p}(0,T;\mathbb{R}^3).$$

Since $T \leqslant \widetilde{T}$, $X(t,\cdot)$ is C^1- diffeomorphism from $\Omega_F(0)$ into $\Omega_F(t)$, we set $Y(t,\cdot) = X^{-1}(t,\cdot)$ and for $x \in \Omega_F(t), \ t \geqslant 0$

$$\begin{aligned}
\rho(t,x) &= \widetilde{\rho}(t,Y(t,x)), \\
u(t,x) &= Q(t)\widetilde{u}(t,Y(t,x)), \\
\vartheta(t,x) &= \widetilde{\vartheta}(t,Y(t,x)), \\
\dot{a}(t) &= Q(t)\widetilde{\ell}(t), \quad \omega(t) = Q(t)\widetilde{\omega}(t).
\end{aligned}$$

We can then check that $(\rho, u, \vartheta, a, \omega)$ satisfies the original system (2.2) - (2.9) and

$$\rho \in W^{1,p}(0,T;W^{1,q}(\Omega_F(\cdot))) \cap C([0,T];W^{1,q}(\Omega_F(\cdot))),$$

$$u \in L^p(0,T;W^{2,q}(\Omega_F(\cdot))^3) \cap W^{1,p}(0,T;L^q(\Omega_F(\cdot))^3) \cap C([0,T];B_{q,p}^{2(1-1/p)}(\Omega_F(\cdot))^3),$$

$$\vartheta \in L^p(0,T;W^{2,q}(\Omega_F(\cdot))) \cap W^{1,p}(0,T;L^q(\Omega_F(\cdot))) \cap C([0,T];B_{q,p}^{2(1-1/p)}(\Omega_F(\cdot))),$$

$$a \in W^{2,p}(0,T;\mathbb{R}^3), \quad \omega \in W^{1,p}(0,T;\mathbb{R}^3).$$

The uniqueness for the solution of (2.2) - (2.9) follows from uniqueness of solution to the system (4.4) - (4.15). Since $a(t)$ and $\omega(t)$ belongs to $C([0,T];\mathbb{R}^3)$, using (2.1) we obtain $\mathrm{dist}(\Omega_S(t), \partial\Omega) \geqslant \nu/2$ for all $t \in [0,T]$ if T is small enough. This completes the proof of Theorem 2.1. □

5. Maximal $L^p - L^q$ Regularity for a Linear Problem.

In this section, we fix $\widetilde{T} > 0$, and $1 < p, q < \infty$ such that $\dfrac{1}{p} + \dfrac{1}{2q} \neq 1$ and $\dfrac{1}{p} + \dfrac{1}{2q} \neq \dfrac{1}{2}$. We also take $T \in (0, \widetilde{T}]$. We consider the linear system (4.19)—(4.22) associated with (4.4)–(4.7), where we replace the terms in the right-hand side by given source terms. The initial data for the system (4.19)—(4.22) satisfies the following properties:

$$\rho_0 \in W^{1,q}(\Omega_F(0)) \cap C(\overline{\Omega_F(0)}), \quad \min_{\overline{\Omega_F(0)}} \rho_0 \geqslant M,$$

$$u_0 \in B_{q,p}^{2(1-1/p)}(\Omega_F(0))^3, \quad \vartheta_0 \in B_{q,p}^{2(1-1/p)}(\Omega_F(0)), \quad \ell_0 \in \mathbb{R}^3, \ \omega_0 \in \mathbb{R}^3, \qquad (5.1)$$

$$u_0 = 0 \text{ on } \partial\Omega, \quad u_0(y) = \ell_0 + \omega_0 \times y \quad y \in \partial\Omega_S(0) \text{ if } \frac{1}{p} + \frac{1}{2q} < 1,$$

$$\frac{\partial\vartheta_0}{\partial n} = 0, \text{ on } \partial\Omega_F(0) \text{ if } \frac{1}{p} + \frac{1}{2q} < \frac{1}{2}.$$

Observe that the linear system can be solved "in cascades": Equation (4.21) can be solved independently and admits a unique solution

$$(\widetilde{\ell}, \widetilde{\omega}) \in W^{1,p}(0,T)^3 \times W^{1,p}(0,T)^3.$$

Moreover there exists a constant $C = C(\widetilde{T})$ independent of T such that

$$\|\widetilde{\ell}\|_{W^{1,p}(0,T)^3} + \|\widetilde{\omega}\|_{W^{1,p}(0,T)^3} \leqslant C\Big(\|\ell_0\|_{\mathbb{R}^3} + \|\omega_0\|_{\mathbb{R}^3} + \|g_1\|_{L^p(0,T)^3} + \|g_2\|_{L^p(0,T)^3}\Big). \quad (5.2)$$

We also note that if we show that system (4.20) admits a unique solution $\widetilde{u} \in W_{q,p}^{2,1}(Q_T^F)^3$, we can use the Sobolev embedding to prove that system (4.19) admits a unique solution

$\widetilde{\rho} \in W^{1,p}(0,T;W^{1,q}(\Omega_F(0)))$. There exists again a constant $C = C(\widetilde{T})$ independent of T such that

$$\|\widetilde{\rho}\|_{W^{1,p}(0,T;W^{1,q}(\Omega_F(0)))} \leqslant C \left(\|\widetilde{u}\|_{W^{2,1}_{q,p}(Q^F_T)^3} + \|f_1\|_{L^p(0,T;W^{1,q}(\Omega_F(0)))} + \|\rho_0\|_{W^{1,q}(\Omega_F(0))} \right). \tag{5.3}$$

Consequently, in order to solve (4.19)–(4.22), we need to solve the two parabolic systems (4.20) and (4.22). This is done below by using [9, Theorem 2.3].

Proposition 5.1. *With the above notation, assume $T \in (0,\widetilde{T}]$ and $u_0 \in B^{2(1-1/p)}_{q,p}(\Omega_F(0))^3$ with*

$$u_0 = 0 \ on \ \partial\Omega, \quad u_0 = \ell_0 + \omega_0 \times y \ on \ \partial\Omega_S(0) \quad if \quad \frac{1}{p} + \frac{1}{2q} < 1. \tag{5.4}$$

Then for any $f_2 \in L^p(0,T;L^q(\Omega_F(0)))^3$, system (4.20) admits a unique strong solution $\widetilde{u} \in W^{2,1}_{q,p}(Q^F_T)^3$. Moreover, there exists a constant $C > 0$ depending only on M, $\widetilde{T}$ and $\Omega_F(0)$ such that

$$\|\widetilde{u}\|_{W^{2,1}_{q,p}(Q^F_T)^3} \leqslant C \Big(\|u_0\|_{B^{2(1-1/p)}_{q,p}(\Omega_F(0))^3} + \|\ell_0\|_{\mathbb{R}^3} + \|\omega_0\|_{\mathbb{R}^3}$$
$$+ \|g_1\|_{L^p(0,T)^3} + \|g_2\|_{L^p(0,T)^3} + \|f_2\|_{L^p(0,T;L^q(\Omega_F(0)))^3} \Big). \tag{5.5}$$

Proof. We take $\eta \in C^\infty(\overline{\Omega_F(0)})$ such that

$$\eta = 0 \quad \text{on } \partial\Omega, \quad \eta = 1 \quad \text{on } \partial\Omega_S(0).$$

For $(t,y) \in (0,T) \times \Omega_F(0)$, we set $w(t,y) = \eta(y)(\widetilde{\ell}(t) + \widetilde{\omega}(t) \times y)$. Therefore, using (5.2) we see that there exists a positive constant C depending on $\widetilde{T}$ and $\Omega_F(0)$ such that

$$\|w(0,\cdot)\|_{B^{2(1-1/p)}_{q,p}(\Omega_F(0))^3} + \|w\|_{W^{2,1}_{q,p}(Q^F_T)^3}$$
$$\leqslant C \Big(\|\ell_0\|_{\mathbb{R}^3} + \|\omega_0\|_{\mathbb{R}^3} + \|g_1\|_{L^p(0,T)^3} + \|g_2\|_{L^p(0,T)^3} \Big). \tag{5.6}$$

We look for the solution of (4.20) of the form $\widetilde{u} = v + w$, where v is the solution of

$$\begin{cases} \partial_t v - \dfrac{\mu}{\rho_0}\Delta v - \dfrac{\alpha+\mu}{\rho_0}\nabla(\text{div } v) = \widehat{f_2}, & \text{in } (0,T) \times \Omega_F(0), \\ v = 0 & \text{on } (0,T) \times \partial\Omega_F(0), \\ v(0,\cdot) = v_0 & \text{in } \Omega_F(0), \end{cases} \tag{5.7}$$

with $v_0 = u_0 - w(0,\cdot)$ and with

$$\widehat{f_2} = f_2 - \partial_t w + \frac{\mu}{\rho_0}\Delta w + \frac{\alpha+\mu}{\rho_0}\nabla(\text{div } w).$$

We can check that $\widehat{f_2}$ belongs to $L^p(0,T;L^q(\Omega_F(0)))^3$ and that there exists a constant depending only on M such that

$$\|\widehat{f_2}\|_{L^p(0,T;L^q(\Omega_F(0)))^3} \leqslant C \Big(\|f_2\|_{L^p(0,T;L^q(\Omega_F(0)))^3} + \|w\|_{W^{2,1}_{q,p}(Q^F_T)^3} \Big). \tag{5.8}$$

Moreover, since $v_0 \in B^{2(1-1/p)}_{q,p}(\Omega_F(0))^3$ with $v_0 = 0$ on $\partial\Omega_F(0)$ if $\dfrac{1}{p} + \dfrac{1}{2q} < 1$, we can apply [9, Theorem 2.3] and deduce that (5.7) admits unique solution $v \in W^{2,1}_{q,p}(Q^F_T)^3$.

More precisely, here the ellipticity of the interior symbol can be checked since $\rho_0(y) \geqslant \frac{1}{M} > 0$ for all $y \in \Omega_F(0)$, $\mu > 0$ and $\alpha + \frac{2}{3}\mu \geqslant 0$. The Lopatinsky-Shapiro condition also can be verified (see [2, Section 6]).

This yields that (4.20) admits a unique solution $\widetilde{u} \in W_{q,p}^{2,1}(Q_T^F)^3$. In order to prove estimate (5.5), we apply again [9, Theorem 2.3] on the system

$$\begin{cases} \partial_t \widehat{v} - \dfrac{\mu}{\rho_0}\Delta \widehat{v} - \dfrac{\alpha + \mu}{\rho_0}\nabla(\text{div } \widehat{v}) = \widehat{f}_T, & \text{in } (0, \widetilde{T}) \times \Omega_F(0), \\ \widehat{v} = 0 & \text{on } (0, \widetilde{T}) \times \partial\Omega_F(0) \\ \widehat{v}(0) = v_0 & \text{in } \Omega_F(0). \end{cases}$$

In particular, for any $\widehat{f} \in L^p(0, \widetilde{T}; L^q(\Omega_F(0))^3)$, there exists a unique solution

$$\widehat{v} \in L^p(0, \widetilde{T}; W^{2,q}(\Omega_F(0))^3) \cap W^{1,p}(0, \widetilde{T}, L^q(\Omega_F(0))^3)$$

of the above system and by the closed graph theorem, there exists a constant $C_{\widetilde{T}} > 0$ such that

$$\|\widehat{v}\|_{L^p(0,\widetilde{T};W^{2,q}(\Omega_F(0))^3)} + \|\widehat{v}\|_{W^{1,p}(0,\widetilde{T},L^q(\Omega_F(0))^3)} \leqslant C_{\widetilde{T}}\Big(\|v_0\|_{B_{q,p}^{2(1-1/p)}(\Omega_F(0))^3} + \|\widehat{f}\|_{L^p(0,\widetilde{T};L^q(\Omega_F(0))^3)}\Big).$$

Then we take

$$\widehat{f} = \begin{cases} \widehat{f}_2 & \text{if } 0 < t \leqslant T, \\ 0 & \text{if } T < t \leqslant \widetilde{T}, \end{cases}$$

and by the uniqueness of the solution, we note that $\widehat{v} = v$ for all $t \in [0, T]$. Thus the above estimate, (5.6) and (5.8) yield (5.5). □

Next we consider system (4.22).

Proposition 5.2. *With the above notation, assume $T \in (0, \widetilde{T}]$, $\vartheta_0 \in B_{q,p}^{2(1-1/p)}(\Omega_F(0))$ and $h \in F_{p,q}^{(1-1/q)/2}(0, T; L^q(\partial\Omega_F(0))) \cap L^p(0, T; W^{1-1/q,q}(\partial\Omega_F(0)))$ with the compatibility condition*

$$\frac{\partial\vartheta_0}{\partial n} = h(0, \cdot) = 0 \quad on \; \partial\Omega_F(0) \quad if \;\; \frac{1}{p} + \frac{1}{2q} < \frac{1}{2}. \tag{5.9}$$

Then for any $f_3 \in L^p(0, T; L^q(\Omega_F(0)))$, system (4.22) admits a unique strong solution $\widetilde{\vartheta} \in W_{q,p}^{2,1}(Q_T^F)$. Moreover, there exists a constant $C > 0$, depending only on M and $\widetilde{T}$, such that

$$\|\widetilde{\vartheta}\|_{W_{q,p}^{2,1}(Q_T^F)} \leqslant C\Big(\|\vartheta_0\|_{B_{q,p}^{2(1-1/p)}(\Omega_F(0))} + \|f_3\|_{L^p(0,T;L^q(\Omega_F(0)))} + \|h\|_{F_{p,q}^{(1-1/q)/2}(0,T;L^q(\partial\Omega_F(0)))}$$

$$+ \|h\|_{L^p(0,T;W^{1-1/q,q}(\partial\Omega_F(0)))}\Big). \tag{5.10}$$

Proof. The existence and the regularity results follow from [9, Theorem 2.3].

Since we need a constant C in (5.10) independent of $T \in (0, \widetilde{T}]$ and this fact is not explicitly stated in [9], we provide below a short argument showing that the constant C can indeed be chosen to be uniform for $T \in (0, \widetilde{T}]$. To this aim, we decompose $\widetilde{\vartheta}$ in the form $\widetilde{\vartheta} = \widetilde{\vartheta}_1 + \widetilde{\vartheta}_2$, where $\widetilde{\vartheta}_1$ solves

$$\begin{cases} \partial_t \widetilde{\vartheta}_1 - \dfrac{\kappa}{\rho_0 c_v}\Delta\widetilde{\vartheta}_1 = f_3 & \text{in } (0, T) \times \Omega_F(0), \\ \dfrac{\partial\widetilde{\vartheta}_1}{\partial n} = 0 & \text{on } (0, T) \times \partial\Omega_F(0), \\ \widetilde{\vartheta}_1(0) = \vartheta_0 & \text{in } \Omega_F(0), \end{cases} \tag{5.11}$$

and $\widetilde{\vartheta}_2$ solves

$$\begin{cases} \partial_t\widetilde{\vartheta}_2 - \dfrac{\kappa}{\rho_0 c_v}\Delta\widetilde{\vartheta}_2 \ = 0 & \text{in } (0,T)\times\Omega_F(0), \\[2mm] \dfrac{\partial\widetilde{\vartheta}_2}{\partial n} = h & \text{on } (0,T)\times\partial\Omega_F(0), \\[2mm] \vartheta_2(0) = 0 & \text{in } \Omega_F(0). \end{cases} \tag{5.12}$$

Proceeding as in the proof of Proposition 5.1, we first obtain

$$\|\widetilde{\vartheta}_1\|_{W^{2,1}_{q,p}(Q^F_T)} \leqslant C\Big(\|\vartheta_0\|_{B^{2(1-1/p)}_{q,p}(\Omega_F(0))} + \|f_3\|_{L^p(0,T;L^q(\Omega_F(0)))}\Big), \tag{5.13}$$

where the constant C may depend on $\widetilde{T}$ but is independent of T. Let us set

$$\widehat{h} = \begin{cases} h & \text{if } 0 < t \leqslant T, \\ 0 & \text{if } T-\widetilde{T} < t \leqslant 0. \end{cases}$$

We first verify that

$$\widehat{h} \in F^{(1-1/q)/2}_{p,q}(T-\widetilde{T},T;L^q(\partial\Omega_F(0))) \cap L^p(T-\widetilde{T},T;W^{1-1/q,q}(\partial\Omega_F(0))).$$

Obviously $\widehat{h}$ belongs to $L^p(T-\widetilde{T},T;W^{1-1/q,q}(\partial\Omega_F(0)))$. The fact that $\widehat{h}$ belongs to $F^{(1-1/q)/2}_{p,q}(T-\widetilde{T},T;L^q(\partial\Omega_F(0)))$ follows from [28, Remark 2, Section 3.4.3, p.211]. Moreover

$$\|\widehat{h}\|_{F^{(1-1/q)/2}_{p,q}(T-\widetilde{T},T;L^q(\partial\Omega_F(0)))} + \|\widehat{h}\|_{L^p(T-\widetilde{T},T;W^{1-1/q,q}(\partial\Omega_F(0)))}$$
$$= \|h\|_{F^{(1-1/q)/2}_{p,q}(0,T;L^q(\partial\Omega_F(0)))} + \|h\|_{L^p(0,T;W^{1-1/q,q}(\partial\Omega_F(0)))}.$$

We consider the system

$$\begin{cases} \partial_t\widehat{\vartheta}_2 - \dfrac{\kappa}{\rho_0 c_v}\Delta\widehat{\vartheta}_2 \ = 0 & \text{in } (T-\widetilde{T},T)\times\Omega_F(0), \\[2mm] \dfrac{\partial\widehat{\vartheta}_2}{\partial n} = \widehat{h} & \text{on } (T-\widetilde{T},T)\times\partial\Omega_F(0), \\[2mm] \widehat{\vartheta}_2(T-\widetilde{T}) = 0 & \text{in } \Omega_F(0). \end{cases} \tag{5.14}$$

Note that, $\widehat{\vartheta}_2 = 0$ for all $t \in [T-\widetilde{T},0]$ and $\widehat{\vartheta}_2 = \widetilde{\vartheta}_2$ for all $t \in [0,T]$. Therefore, we have

$$\|\widetilde{\vartheta}_2\|_{L^p(0,T;W^{2,q}(\Omega_F(0)))} + \|\widetilde{\vartheta}_2\|_{W^{1,p}(0,T,L^q(\Omega_F(0)))}$$
$$= \|\widehat{\vartheta}_2\|_{L^p(T-\widetilde{T},T;W^{2,q}(\Omega_F(0)))} + \|\widehat{\vartheta}_2\|_{W^{1,p}(T-\widetilde{T},T,L^q(\Omega_F(0)))}$$
$$\leqslant C_{\widetilde{T}}\Big(\|\widehat{h}\|_{F^{(1-1/q)/2}_{p,q}(T-\widetilde{T},T;L^q(\partial\Omega_F(0)))} + \|\widehat{h}\|_{L^p(T-\widetilde{T},T;W^{1-1/q,q}(\partial\Omega_F(0)))}\Big)$$
$$\leqslant C_{\widetilde{T}}\Big(\|h\|_{F^{(1-1/q)/2}_{p,q}(0,T;L^q(\partial\Omega_F(0)))} + \|h\|_{L^p(0,T;W^{1-1/q,q}(\partial\Omega_F(0)))}\Big).$$

This completes the proof of the proposition. $\qquad\square$

Combining Proposition 5.1 and Proposition 5.2, we obtain the following result

Theorem 5.3. *Let $\widetilde{T}$ be an arbitrary fixed given time. Let $1 < p < \infty$ and $1 < q < \infty$ satisfying the conditions $\dfrac{1}{p} + \dfrac{1}{2q} \neq 1$ and $\dfrac{1}{p} + \dfrac{1}{2q} \neq \dfrac{1}{2}$. Let $(\rho_0, u_0, \vartheta_0, \ell_0, \omega_0)$ satisfy the assumptions (5.1). Then for any $(f_1, f_2, f_3, h, g_1, g_2) \in \mathcal{B}_{T,p,q}$ and $T \in (0,\widetilde{T})$ the system (4.19)–(4.22) admits a unique solution $(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) \in \mathcal{S}_{T,p,q}$ and there exists a constant $C > 0$ depending on $p, q, M, \widetilde{T}$ and independent of T such that*

$$\|(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega})\|_{\mathcal{S}_{T,p,q}} \leqslant C\Big(\|(\rho_0, u_0, \vartheta_0, \ell_0, \omega_0)\|_{\mathcal{I}_{p,q}} + \|(f_1, f_2, f_3, h, g_1, g_2)\|_{\mathcal{B}_{T,p,q}}\Big). \tag{5.15}$$

6. Estimating the Nonlinear Terms

In order to prepare the forthcoming fixed point argument, we provide in this section estimates of $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3, \mathcal{G}_1, \mathcal{G}_2$, $\mathcal{H}_F$ and $\mathcal{H}_S$ defined in (4.11)-(4.17) where $(f_1, f_2, f_3, h, g_1, g_2)$ are given and where $(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega})$ is the corresponding solution of (4.19)—(4.22) given by Theorem 5.3.

Assume $2 < p < \infty$ and $3 < q < \infty$ satisfy $\dfrac{1}{p} + \dfrac{1}{2q} \neq \dfrac{1}{2}$. Let p' denote the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. We will frequently use the following immediate consequences of Hölder's inequality.

$$\|f\|_{L^p(0,T)} \leqslant T^{1/p - 1/r} \|f\|_{L^r(0,T)}, \qquad \text{for all } f \in L^r(0,T),\ r > p, \tag{6.1}$$

$$\|f\|_{L^\infty(0,T)} \leqslant T^{1/p'} \|f\|_{W^{1,p}(0,T)}, \qquad \text{for all } f \in W^{1,p}(0,T), f(0) = 0. \tag{6.2}$$

In the following, when no confusion is possible, we will use the notation

$$\| \cdot \|_{W^{r,p}(0,T;W^{s,q})} = \| \cdot \|_{W^{r,p}(0,T;W^{s,q}(\Omega_F(0)))}.$$

We next recall three estimates which play an essential role in the remaining part of this section. For the first two estimates we refer to the relevant literature, whereas for the third one we provide a short proof.

Proposition 6.1. [25, Lemma 4.2] *Let $1 < p, q < \infty$ and T be any positive number. Let Ω be a smooth domain in $\mathbb{R}^n$. Then for any $u \in W_{q,p}^{2,1}((0,T) \times \Omega)$,*

$$\sup_{t \in (0,T)} \|u(t)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leqslant C \left(\|u(0)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|u\|_{W_{q,p}^{2,1}((0,T) \times \Omega)} \right), \tag{6.3}$$

where the constant C is independent of T.

To state the second estimate, we use the Lizorkin-Triebel space $F_{p,q}^s(0, T; X)$ defined in (3.1).

Proposition 6.2. [9, Proposition 6.4] *Let $1 < p, q < \infty$ and T be any positive number. Let Ω be a smooth domain in $\mathbb{R}^n$. Then for any $u \in W_{q,p}^{2,1}((0,T) \times \Omega)$, $\nabla u|_{\partial \Omega}$ belongs to $F_{p,q}^{(1-1/q)/2}(0, T; L^q(\partial \Omega)) \cap L^p(0, T; W^{1-1/q,q}(\partial \Omega))$. Moreover,*

$$\left\| \nabla u \cdot n \right\|_{F_{p,q}^{(1-1/q)/2}(0,T;L^q(\partial\Omega)) \cap L^p(0,T;W^{1-1/q,q}(\partial\Omega))} \leqslant C \left(\|u(0)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|u\|_{W_{q,p}^{2,1}((0,T) \times \Omega)} \right), \tag{6.4}$$

where the constant C is independent of time T.

The third one of the estimates mentioned above is given in the following result.

Proposition 6.3. *Let U_1, U_2 and U_3 be three Banach spaces and $\Phi : U_1 \times U_2 \to U_3$ a bounded bilinear map. Let us assume that $f \in F_{p,q}^s(0, T; U_1)$ and $g \in W^{1,p}(0, T; U_2)$ for some $s \in (0, 1)$, $p > 2$ and $q > 3$. Let us assume that $g(0) = 0$. If $s + \frac{1}{p} < 1$, then we have*

$$\|\Phi(f, g)\|_{F_{p,q}^s(0,T;U_3)} \leqslant CT^\delta \|g\|_{W^{1,p}(0,T;U_2)} \|f\|_{F_{p,q}^s(0,T;U_2)}, \tag{6.5}$$

for some positive constant δ depending only on p, q and s and the constant C is independent of time T.

Proof. From the boundedness of Φ and (6.2) we infer

$$\|\Phi(f,g)\|_{L^p(0,T;U_3)} \leqslant C\|f\|_{L^p(0,T;U_1)}\|g\|_{L^\infty(0,T;U_2)} \leqslant CT^{1/p'}\|f\|_{L^p(0,T;U_1)}\|g\|_{W^{1,p}(0,T;U_2)}, \quad (6.6)$$

since $g(0) = 0$. Using again the boundedness of Φ,

$$|\Phi(f,g)|^p_{F^s_{p,q}(0,T;U_3)} = \int_0^T \left(\int_0^{T-t} h^{-1-sq}\|\Phi(f(t+h),g(t+h)) - \Phi(f(t),g(t))\|^q_{U_3}\, dh\right)^{p/q}\, dt$$

$$\leqslant C_{p,q}\int_0^T \left(\int_0^{T-t} h^{-1-sq}\|(f(t+h) - f(t)\|^q_{U_1}\|g(t+h)\|^q_{U_2}\, dh\right)^{p/q}\, dt$$

$$+\ C_{p,q}\int_0^T \left(\int_0^{T-t} h^{-1-sq}\|f(t)\|^q_{U_1}\|g(t+h) - g(t)\|^q_{U_2}\, dh\right)^{p/q}\, dt$$

$$= I_1 + I_2.$$

We estimate I_1 using (6.2)

$$I_1 \leqslant C_{p,q}\|g\|^p_{L^\infty(0,T;U_2)}|f|^p_{F^s_{p,q}(0,T;U_1)} \leqslant C_{p,q}T^{p/p'}\|g\|^p_{W^{1,p}(0,T;U_2)}\|f\|^p_{F^s_{p,q}(0,T;U_1)}.$$

Since $g \in W^{1,p}(0,T;U_2)$, by using Hölder's inequality we have

$$\|g(t+h,\cdot) - g(t,\cdot)\|_{U_2} \leqslant h^{1/p'}\|g\|_{W^{1,p}(0,T;U_2)}, \quad \text{for all } h \in (0, T-t), t \in (0,T).$$

Using the above estimate and the fact that $0 < s + 1/p < 1$, we get

$$I_2 \leqslant C_{p,q}\|g\|^p_{W^{1,p}(0,T;U_2)}\int_0^T \|f(t)\|^p_{U_1}\left(\int_0^{T-t} h^{-1-sq}\, h^{q/p'}\, dh\right)^{p/q}\, dt$$

$$\leqslant C_{p,q}\|g\|^p_{W^{1,p}(0,T;U_2)}\int_0^T \|f(t)\|^p_{U_1}\left(\frac{(T-t)^{q(1-1/p-s)}}{q(1-1/p-s)}\right)^{p/q}\, dt$$

$$\leqslant C_{p,q,s}T^{p(1-1/p-s)}\|g\|^p_{W^{1,p}(0,T;U_2)}\|f\|^p_{L^p(0,T;U_1)}.$$

Combining the above estimates, we obtain (6.5). $\qquad\square$

Our aim is to estimate the non linear terms in (4.11)-(4.17):

Proposition 6.4. *Let $2 < p < \infty$ and $3 < q < \infty$ satisfying the condition $\frac{1}{p} + \frac{1}{2q} \neq \frac{1}{2}$. Let $(\rho_0, u_0, \vartheta_0, \ell_0, \omega_0) \in \mathcal{I}^{cc}_{p,q}$ such that (2.12) holds. There exist $\widetilde{T} < 1$, a constant $\delta > 0$ depending only on p and q, and a constant $C > 0$ depending only on $p, q, M, \widetilde{T}$ such that for $T \in (0, \widetilde{T}]$ and for $(f_1, f_2, f_3, h, g_1, g_2) \in \mathcal{B}_{T,p,q}$ satisfying*

$$\|(f_1, f_2, f_3, h, g_1, g_2)\|_{\mathcal{B}_{T,p,q}} \leqslant 1,$$

the solution of $(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) \in \mathcal{S}_{T,p,q}$ of (4.19)—(4.22) verifies

$$\|(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{H}, \mathcal{G}_1, \mathcal{G}_2)\|_{\mathcal{B}_{T,p,q}} \leqslant CT^\delta.$$

Proof. We consider $\widetilde{T} < 1$ and we assume that $T \in (0, \widetilde{T}]$. The constants C appearing in this proof depend only on M.

From (5.15) in Theorem 5.3, we first obtain

$$\|(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega})\|_{\mathcal{S}_{T,p,q}} \leqslant C. \tag{6.7}$$

Combining (6.2) and (6.7), we deduce

$$\|\widetilde{\rho} - \rho_0\|_{L^\infty(0,T;W^{1,q}(\Omega_F(0)))} \leqslant CT^{1/p'} \tag{6.8}$$

and

$$\|\widetilde{\rho}\|_{L^\infty(0,T;W^{1,q})} \leqslant C. \tag{6.9}$$

In a similar manner, we can obtain

$$\|\widetilde{\omega}\|_{L^\infty(0,T)}, \quad \|\widetilde{\ell}\|_{L^\infty(0,T)} \leqslant C, \tag{6.10}$$

and combining these estimates with (6.1) yields

$$\|\widetilde{\rho}\|_{L^p(0,T;W^{1,q})} + \|\widetilde{\omega}\|_{L^p(0,T)} + \|\widetilde{\ell}\|_{L^p(0,T)} \leqslant CT^{1/p}. \tag{6.11}$$

Since $2 < p < \infty$, one has $B_{q,p}^{2(1-1/p)}(\Omega_F(0)) \hookrightarrow W^{1,q}(\Omega_F(0))$. Therefore, using Proposition 6.1 and (6.7) we get

$$\|\widetilde{u}\|_{L^\infty(0,T;W^{1,q})} + \|\widetilde{\vartheta}\|_{L^\infty(0,T;W^{1,q})} \leqslant C. \tag{6.12}$$

For all $s \in (0,1)$ we have by complex interpolation

$$\|\widetilde{u}(t,\cdot)\|_{W^{1+s,q}(\Omega_F(0))} \leqslant C\|\widetilde{u}(t,\cdot)\|_{W^{2,q}(\Omega_F(0))}^{(1+s)/2}\|\widetilde{u}(t,\cdot)\|_{L^q(\Omega_F(0))}^{(1-s)/2},$$

and thus

$$\|\widetilde{u}\|_{L^p(0,T,W^{1+s,q}(\Omega_F(0))} \leqslant CT^{(1-s)/2p}\|\widetilde{u}\|_{L^\infty(0,T;L^q(\Omega_F(0))}^{1-s/2}\|\widetilde{u}\|_{L^p(0,T;W^{2,q}(\Omega_F(0)))}^{1+s/2}.$$

Therefore, using (6.1), (6.7) and (6.12), we get

$$\|\widetilde{u}\|_{L^p(0,T;W^{1+s,q})} + \|\widetilde{\vartheta}\|_{L^p(0,T;W^{1+s,q})} \leqslant CT^{(1-s)/2p}, \quad s \in (0,1). \tag{6.13}$$

Combining the above estimate with the fact that $W^{s,q}(\Omega_F(0))) \hookrightarrow L^\infty(\Omega_F(0))$ for $s \in (3/q, 1)$, we deduce

$$\|\widetilde{u}\|_{L^p(0,T,L^\infty)} + \|\nabla\widetilde{u}\|_{L^p(0,T,L^\infty)} + \|\nabla\widetilde{\vartheta}\|_{L^p(0,T,L^\infty)} \leqslant CT^{(1-s)/2p}, \quad s \in (3/q, 1). \tag{6.14}$$

The solution of (4.8) satisfies $Q \in SO(3)$ and thus $|Q(t)| = 1$ for all t. In particular, $|\dot{Q}| \leqslant C|\widetilde{\omega}|$ and we deduce from (6.2) and (6.7)

$$\|Q\|_{L^\infty(0,T;\mathbb{R}^{3\times 3})} \leqslant C \qquad \text{and} \qquad \|Q - I_3\|_{L^\infty(0,T;\mathbb{R}^{3\times 3})} \leqslant CT^{1/p'}. \tag{6.15}$$

Let X be defined as in (4.9). Then

$$\sup_{t\in(0,T)} \|\nabla X(t,\cdot) - I_3\|_{W^{1,q}(\Omega_F(0))} \leqslant C\int_0^T \|\nabla\widetilde{u}\|_{W^{1,q}(\Omega_F(0))} \leqslant CT^{1/p'}\|\nabla\widetilde{u}\|_{L^p(0,T;W^{1,q})}.$$

Now using $W^{1,q}(\Omega_F(0)) \hookrightarrow L^\infty(\Omega_F(0))$ and (6.7) we deduce from the above estimate

$$\sup_{t\in(0,T)} \|\nabla X(t,\cdot) - I_3\|_{L^\infty(\Omega_F(0))} \leqslant CT^{1/p'} \tag{6.16}$$

In particular, there exists $\widetilde{T}$ such that

$$\|\nabla X(t,\cdot) - I_3\|_{L^\infty(\Omega_F(0))} \leqslant \frac{1}{2} \tag{6.17}$$

for all $0 < t < T \leqslant \widetilde{T}$. This implies that $\nabla X(t,\cdot)$ is invertible and we can thus define $Z = [\nabla X]^{-1}$. More precisely, combining

$$\partial_t \nabla X(t,y) = Q(t)\nabla\widetilde{u}(t,y),$$

and (6.7) and (6.15) we get

$$\|\partial_t \nabla X\|_{L^p(0,T;W^{1,q})} \leqslant \|Q\|_{L^\infty(0,T)} \|\nabla \widetilde{u}\|_{L^p(0,T;W^{1,q})} \leqslant C,$$

where C depends only on M. The above estimate and (6.16) yield

$$\|\nabla X\|_{W^{1,p}(0,T;W^{1,q}(\Omega_F(0)))} + \|\nabla X\|_{L^\infty(0,T;W^{1,q}(\Omega_F(0)))} \leqslant C.$$

Since $W^{1,p}(0,T;W^{1,q}(\Omega_F(0)))$ and $L^\infty(0,T;W^{1,q}(\Omega_F(0)))$ are algebras for $p > 2$ and $q > 3$, this implies

$$\|\det \nabla X\|_{W^{1,p}(0,T;W^{1,q}(\Omega_F(0)))} + \|\det \nabla X\|_{L^\infty(0,T;W^{1,q}(\Omega_F(0)))} \leqslant C, \tag{6.18}$$

$$\|\mathrm{Cof}\nabla X\|_{W^{1,p}(0,T;W^{1,q}(\Omega_F(0)))} + \|\mathrm{Cof}\nabla X\|_{L^\infty(0,T;W^{1,q}(\Omega_F(0)))} \leqslant C. \tag{6.19}$$

From (6.17), we deduce that $\det \nabla X \geqslant C > 0$ in $(0,T) \times \Omega_F(0)$ and thus from

$$Z = \frac{1}{\det \nabla X}(\mathrm{Cof}\nabla X)^\top,$$

we deduce

$$\|Z\|_{W^{1,p}(0,T;W^{1,q}(\Omega_F(0)))} + \|Z\|_{L^\infty(0,T;W^{1,q}(\Omega_F(0)))} \leqslant C. \tag{6.20}$$

$$\|\nabla X\|_{W^{1,p}(0,T;W^{1,q}(\Omega_F(0)))} + \|\nabla X\|_{L^\infty(0,T;W^{1,q}(\Omega_F(0)))} \leqslant C, \tag{6.21}$$

The above estimate combined with (6.15) and with (6.7) implies

$$\|QZ\|_{W^{1,p}(0,T;W^{1,q}(\Omega_F(0)))} + \|QZ\|_{L^\infty(0,T;W^{1,q}(\Omega_F(0)))} \leqslant C. \tag{6.22}$$

We are now in position to estimate the non linear terms in (4.11)-(4.17):

Estimate of $\mathcal{F}_1$.

$$\|\mathcal{F}_1\|_{L^p(0,T;W^{1,q}(\Omega_F(0)))} \leqslant CT^{1/p'}. \tag{6.23}$$

Since $W^{1,q}(\Omega_F(0))$ is an algebra for $q > 3$, we can write

$$\|\mathcal{F}_1\|_{L^p(0,T;W^{1,q}(\Omega_F(0)))} \leqslant C\|\widetilde{\rho} - \rho_0\|_{L^\infty(0,T;W^{1,q})} \|\mathrm{div}\ \widetilde{u}\|_{L^p(0,T;W^{1,q})}$$
$$+ C\|\widetilde{\rho}\|_{L^\infty(0,T;W^{1,q})} \|QZ - I_3\|_{L^\infty(0,T;W^{1,q})} \|\nabla\widetilde{u}\|_{L^p(0,T;W^{1,q})}. \tag{6.24}$$

Combining the above estimate with (6.9), (6.8), (6.7), (6.22) and (6.2), we deduce (6.23).

Estimate of $\mathcal{F}_2$.

$$\|\mathcal{F}_2\|_{L^p(0,T;W^{1,q})} \leqslant CT^{1/p}. \tag{6.25}$$

Let us recall the definition (4.12) of $\mathcal{F}_2$:

$$(\mathcal{F}_2)_i(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) = -\frac{\widetilde{\rho}}{\rho_0}(\widetilde{\omega} \times Q\widetilde{u})_i + \left(1 - \frac{\widetilde{\rho}}{\rho_0}\right)(\partial_t \widetilde{u})_i - \frac{\widetilde{\rho}}{\rho_0}[(Q-I)\partial_t\widetilde{u}]_i$$

$$+ \frac{\mu}{\rho_0}\sum_{l,j,k}\frac{\partial^2(Q\widetilde{u})_i}{\partial y_l \partial y_k}(Z_{k,j} - \delta_{k,j})Z_{l,j} + \frac{\mu}{\rho_0}\sum_{l,k}\frac{\partial^2(Q\widetilde{u})_i}{\partial y_l \partial y_k}(Z_{l,k} - \delta_{l,k})$$

$$+ \frac{\mu}{\rho_0}[(Q-I)\Delta\widetilde{u}]_i + \frac{\mu}{\rho_0}\sum_{l,j,k}Z_{l,j}\frac{\partial(Q\widetilde{u})_i}{\partial y_k}\frac{\partial Z_{k,j}}{\partial y_l}$$

$$+ \frac{\mu+\alpha}{\rho_0}\sum_{l,j,k}\frac{\partial^2(Qu)_j}{\partial y_l \partial y_k}(Z_{k,j} - \delta_{k,j})Z_{l,i} + \frac{\mu+\alpha}{\rho_0}\sum_{l,j}\frac{\partial^2(Qu)_j}{\partial y_l \partial y_j}(Z_{l,i} - \delta_{l,i})$$

$$+ \frac{\alpha+\mu}{\rho_0}\frac{\partial}{\partial y_i}\left[\nabla\widetilde{u} : (Q^\top - I_3)\right] + \frac{\alpha+\mu}{\rho_0}\sum_{l,j,k}Z_{l,i}\frac{\partial(Qu)_j}{\partial y_k}\frac{\partial Z_{k,j}}{\partial y_l}$$

$$- R\frac{\widetilde{\vartheta}}{\rho_0}\left(Z^\top\nabla\widetilde{\rho}\right)_i - R\frac{\widetilde{\rho}}{\rho_0}\left(Z^\top\nabla\widetilde{\vartheta}\right)_i$$

- Estimate of first term of $\mathcal{F}_2$: using (6.1),(6.9), (6.10) and (6.12), we have

$$\left\|\frac{\widetilde{\rho}}{\rho_0}(\widetilde{\omega}\times Q\widetilde{u})_i\right\|_{L^p(0,T;L^q)} \leqslant C\|\widetilde{\rho}\|_{L^\infty(0,T;W^{1,q})}\|\widetilde{\omega}\|_{L^\infty(0,T;\mathbb{R}^3)}\|\widetilde{u}\|_{L^p(0,T;L^q)}$$
$$\leqslant CT^{1/p}.$$

- Estimate of second term of $\mathcal{F}_2$: using (6.7) and (6.8) we have

$$\left\|\left(1-\frac{\widetilde{\rho}}{\rho_0}\right)(\partial_t\widetilde{u})_i\right\|_{L^p(0,T;L^q)} \leqslant C\|\widetilde{\rho}-\rho_0\|_{L^\infty(0,T;W^{1,q})}\|\partial_t\widetilde{u}\|_{L^p(0,T;L^q)}$$
$$\leqslant CT^{1/p'}.$$

- Estimate of third term of $\mathcal{F}_2$: using (6.7), (6.9) and (6.15)

$$\left\|\frac{\widetilde{\rho}}{\rho_0}[(Q-I)\partial_t\widetilde{u}]_i\right\|_{L^p(0,T;L^q)} \leqslant C\|\widetilde{\rho}\|_{L^\infty(0,T;W^{1,q})}\|Q-I_3\|_{L^\infty(0,T)}\|\widetilde{u}\|_{W^{1,p}(0,T;L^q)}$$
$$\leqslant CT^{1/p'}.$$

- Estimate of fourth term of $\mathcal{F}_2$ (the estimate of fifth, eighth and ninth therm of $\mathcal{F}_2$ are similar)

$$\left\|\frac{\mu}{\rho_0}\sum_{l,j,k}\frac{\partial^2(Q\widetilde{u})_i}{\partial y_l\partial y_k}\left(Z_{k,j}-\delta_{k,j}\right)Z_{l,j}\right\|_{L^p(0,T;L^q)}$$
$$\leqslant C\|\widetilde{u}\|_{L^p(0,T;W^{2,q})}\|Z-I_3\|_{L^\infty(0,T;W^{1,q})}\|Z\|_{L^\infty(0,T;W^{1,q})}$$
$$\leqslant CT^{1/p'},$$

by using (6.7), (6.20) and (6.2).

- Estimate of sixth and tenth term of $\mathcal{F}_2$:

$$\left\|\frac{\mu}{\rho_0}[(Q-I)\Delta\widetilde{u}]_i\right\|_{L^p(0,T;L^q)} + \left\|\frac{\alpha+\mu}{\rho_0}\frac{\partial}{\partial y_i}\left[\nabla\widetilde{u}:(Q^\top-I_3)\right]\right\|_{L^p(0,T;L^q)}$$
$$\leqslant C\|Q-I_3\|_{L^\infty(0,T,\mathbb{R}^{3\times3})}\|\widetilde{u}\|_{L^p(0,T;W^{2,q})}$$
$$\leqslant CT^{1/p'}. \qquad (\text{ using (6.7) and (6.15)})$$

- Estimate of seventh term and similarly, the eleventh term of $\mathcal{F}_2$: notice that for any $1\leqslant j,k,l\leqslant 3$ and all $y\in\Omega_F(0)$, $\dfrac{\partial Z_{k,j}}{\partial y_l}(0,y)=0$. Therefore, using (6.2) we have

$$\left\|\frac{\partial Z_{k,j}}{\partial y_l}\right\|_{L^\infty(0,T;L^q)} \leqslant T^{1/p'}\left\|\frac{\partial Z_{k,j}}{\partial y_l}\right\|_{W^{1,p}(0,T;L^q)} \leqslant T^{1/p'}\|Z\|_{W^{1,p}(0,T;W^{1,q})} \leqslant CT^{1/p'}.$$

Using this estimate, along with (6.7), (6.15) and (6.20) we infer

$$\left\|\frac{\mu}{\rho_0}\sum_{l,j,k}Z_{l,j}\frac{\partial(Q\widetilde{u})_i}{\partial y_k}\frac{\partial Z_{k,j}}{\partial y_l}\right\|_{L^p(0,T;L^q)}$$
$$\leqslant C\|Q\|_{L^\infty(0,T)}\|Z\|_{L^p(0,T;W^{1,q})}\|\nabla\widetilde{u}\|_{L^p(0,T;W^{1,q})}\sum_{j,k,l}\left\|\frac{\partial Z_{k,j}}{\partial y_l}\right\|_{L^\infty(0,T;L^q)}$$
$$\leqslant CT^{1/p'}.$$

- Estimate of twelfth term of $\mathcal{F}_2$:

$$\big\|R\frac{\widetilde{\vartheta}}{\rho_0}\left(Z^\top\nabla\widetilde{\rho}\right)_i\big\|_{L^p(0,T;L^q)} \leqslant C\|\widetilde{\vartheta}\|_{L^p(0,T;W^{1,q})}\|Z\|_{L^\infty(0,T;W^{1,q})}\|\nabla\widetilde{\rho}\|_{L^\infty(0,T;L^q)}$$
$$\leqslant CT^{1/p}. \qquad (\text{ using } (6.9), (6.20) \text{ and } (6.12))$$

- Estimate of last term of $\mathcal{F}_2$:

$$\big\|R\frac{\widetilde{\rho}}{\rho_0}\left(Z^\top\nabla\widetilde{\vartheta}\right)_i\big\|_{L^p(0,T;L^q)} \leqslant C\|\widetilde{\rho}\|_{L^\infty(0,T;W^{1,q})}\|Z\|_{L^\infty(0,T;W^{1,q})}\|\nabla\widetilde{\vartheta}\|_{L^p(0,T;L^q)}$$
$$\leqslant CT^{1/p}. \qquad (\text{ using } (6.9), (6.20) \text{ and } (6.12))$$

We deduce (6.25) by noticing that $1/p \leqslant 1/p'$.

Estimate of $\mathcal{F}_3$.

$$\|\mathcal{F}_3\|_{L^p(0,T;W^{1,q}(\Omega_F(0)))} \leqslant CT^{(1-s)/2p}, \quad s \in (3/q,1). \tag{6.26}$$

We recall that $\mathcal{F}_3$ is defined by (4.13). The estimate of first five terms of $\mathcal{F}_3$ are similar to estimates of terms of $\mathcal{F}_2$ and we skip their proofs. The estimate of the last two terms are similar and we only consider one of these terms: using (6.14), (6.22) yields, for $s \in (3/q,1)$,

$$\big\|[ZQ]^\top : \nabla\widetilde{u}\big\|_{L^p(0,T;L^\infty)} \leqslant \|ZQ\|_{L^\infty(0,T,W^{1,q})}\|\nabla\widetilde{u}\|_{L^p(0,T,L^\infty)} \leqslant CT^{(1-s)/2p}. \tag{6.27}$$

Using (6.22) and (6.12), we obtain

$$\big\|[ZQ]^\top : \nabla\widetilde{u}\big\|_{L^\infty(0,T;L^q)} \leqslant \|ZQ\|_{L^\infty(0,T,W^{1,q})}\|\nabla\widetilde{u}\|_{L^\infty(0,T,L^q)} \leqslant C. \tag{6.28}$$

where the constant C depends only on M. Combining (6.27) and (6.28) we obtain

$$\Big\|\frac{\alpha}{c_v\rho_0}\left([ZQ]^\top : \nabla\widetilde{u}\right)^2\Big\|_{L^p(0,T;L^q)} \leqslant C\big\|[ZQ]^\top : \nabla\widetilde{u}\big\|_{L^p(0,T;L^\infty)}\big\|[ZQ]^\top : \nabla\widetilde{u}\big\|_{L^\infty(0,T;L^q)}$$
$$\leqslant CT^{(1-s)/2p}, \quad s \in (3/q,1).$$

Estimate of $\mathcal{G}_1$ and $\mathcal{G}_2$.

$$\|\mathcal{G}_1\|_{L^p(0,T)} + \|\mathcal{G}_2\|_{L^p(0,T)} \leqslant CT^\delta, \tag{6.29}$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are defined by (4.16) and (4.17)

We first show that

$$\Big\|\int_{\partial\Omega_S(0)}\mathcal{G}_0 n\Big\|_{L^p(0,T)} + \Big\|\int_{\partial\Omega_S(0)} y\times\mathcal{G}_0 n\Big\|_{L^p(0,T)} \leqslant CT^\delta, \tag{6.30}$$

where $\mathcal{G}_0$ is defined by (4.15).

Using (6.13) and (6.22)

$$\|\nabla\widetilde{u}ZQ\|_{L^p(0,T;W^{s,q})} \leqslant C\|ZQ\|_{L^\infty(0,T;W^{1,q})}\|\widetilde{u}\|_{L^p(0,T;W^{1+s,q})} \leqslant CT^{(1-s)/2p}, \quad s \in (1/q,1).$$

Using the trace theorem, we deduce that

$$\left\|\int_{\partial\Omega_S(0)}\nabla\widetilde{u}ZQ\cdot n\,d\gamma\right\|_{L^p(0,T)} \leqslant CT^{(1-s)/2p}, \quad s \in (1/q,1).$$

The other terms can be estimated similarly.

On the other hand, from (6.10),

$$\|\widetilde{\omega}\times\widetilde{\ell}\|_{L^p(0,T)} + \|J(0)\widetilde{\omega}\times\widetilde{\omega}\|_{L^p(0,T)} \leqslant CT^{1/p}$$

and combining this with (6.30), we deduce (6.29).

Estimate of $\mathcal{H}_F$ and $\mathcal{H}_S$.

$$\|\mathcal{H}_F \cdot n\|_{F_{p,q}^{(1-1/q)/2}(0,T;L^q(\partial\Omega)) \,\cap\, L^p(0,T;W^{1-1/q,q}(\partial\Omega))} \leqslant CT^\delta, \tag{6.31}$$

$$\|\mathcal{H}_S \cdot n\|_{F_{p,q}^{(1-1/q)/2}(0,T;L^q(\partial\Omega_S(0))) \,\cap\, L^p(0,T;W^{1-1/q,q}(\partial\Omega_S(0)))} \leqslant CT^\delta, \tag{6.32}$$

where $\mathcal{H}_F$ and $\mathcal{H}_S$ are defined by (4.14).

Recall that, $\mathcal{H}_F = (I_3 - Z^\top)\nabla\widetilde{\vartheta}$. Using (6.7), (6.20) and (6.2), and recalling that $W^{1,q}(\Omega_F(0))$ is an algebra we first obtain

$$\begin{aligned}
\|\mathcal{H}_F \cdot n\|_{L^p(0,T;W^{1-1/q}(\partial\Omega))} &\leqslant C\|\mathcal{H}_F\|_{L^p(0,T;W^{1,q}(\Omega_F(0)))} \\
&\leqslant C\|(I_3 - Z^\top)\|_{L^\infty(0,T;W^{1,q}(\Omega_F(0)))}\|\nabla\widetilde{\vartheta}\|_{L^p(0,T;W^{1,q}(\Omega_F(0)))} \\
&\leqslant CT^{1/p'}.
\end{aligned}$$

To estimate the Lizorkin-Triebel norm of $\mathcal{H}_F \cdot n$, we shall use Proposition 6.3 with parameter $s = (1-1/q)/2$: for $U_1 = U_3 = L^q(\partial\Omega)$, $U_2 = W^{1-1/q,q}(\partial\Omega)$ and $\Phi(f,g) = f \cdot g$. Since $3 < q < \infty$, $W^{1-1/q,q}(\partial\Omega) \hookrightarrow L^\infty(\partial\Omega)$ and so the hypothesis of the proposition on Φ are met. Since $2 < p < \infty$, we also have $s + 1/p < 1$. We write

$$\mathcal{H}_F|_{\partial\Omega} \cdot n = \sum_{j,k}\Big[(\delta_{j,k} - Z_{j,k})\frac{\partial\widetilde{\vartheta}}{\partial y_k}\Big](t,y)n_j(y), \quad y \in \partial\Omega$$

By Proposition 6.2 and (6.7), there exists a constant C depending only on M such that

$$\Big\|\frac{\partial\widetilde{\vartheta}}{\partial y_k}n_j\Big\|_{F_{p,q}^{(1-1/q)/2}(0,T;L^q(\partial\Omega))} \leqslant C$$

for all $1 \leqslant j,k \leqslant 3$. On the other hand, using (6.20), one has

$$\|\delta_{j,k} - Z_{j,k}\|_{W^{1,p}(0,T;W^{1-1/q,q}(\partial\Omega))} \leqslant C\|\delta_{j,k} - Z_{j,k}\|_{W^{1,p}(0,T;W^{1,q}(\Omega_F(0)))} \leqslant C,$$

where the constant C depends only on M. Finally $(\delta_{j,k} - Z_{j,k})(0,y) = 0$ for all $1 \leqslant j,k \leqslant 3$. From Proposition 6.3 we obtain

$$\big\|\mathcal{H}_F|_{\partial\Omega} \cdot n\big\|_{F_{p,q}^{(1-1/q)/2}(0,T;L^q(\partial\Omega))} \leqslant CT^\delta.$$

The estimate of $\mathcal{H}_S \cdot n$ is similar. $\qquad\square$

Proposition 6.5. *Let* $2 < p < \infty$ *and* $3 < q < \infty$ *satisfying the condition* $\frac{1}{p} + \frac{1}{2q} \neq \frac{1}{2}$. *Let* $(\rho_0, u_0, \vartheta_0, \ell_0, \omega_0) \in \mathcal{I}_{p,q}^{cc}$ *such that* (2.12) *holds. There exists* $\widetilde{T} < 1$, *a constant* $\delta > 0$ *depending only on* p *and* q, *and a constant* $C > 0$ *depending only on* $p, q, M, \widetilde{T}$ *such that for* $T \in (0, \widetilde{T}]$ *we have the following property: for* $(f_1^j, f_2^j, f_3^j, h^j, g_1^j, g_2^j) \in \mathcal{B}_{T,p,q}$ *satisfying*

$$\|(f_1^j, f_2^j, f_3^j, h^j, g_1^j, g_2^j)\|_{\mathcal{B}_{T,p,q}} \leqslant 1,$$

for $j = 1, 2$, *let* $(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{\vartheta}^j, \widetilde{\ell}^j, \widetilde{\omega}^j)$ *be the solution of* (4.19)—(4.22) *corresponding to the source term* $(f_1^j, f_2^j, f_3^j, h^j, g_1^j, g_2^j)$. *Let us set*

$$\begin{aligned}
&\mathcal{F}_1^j = \mathcal{F}_1(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{\vartheta}^j, \widetilde{\ell}^j, \widetilde{\omega}^j), \quad \mathcal{F}_2^j = \mathcal{F}_2(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{\vartheta}^j, \widetilde{\ell}^j, \widetilde{\omega}^j), \quad \mathcal{F}_3^j = \mathcal{F}_3(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{\vartheta}^j, \widetilde{\ell}^j, \widetilde{\omega}^j) \\
&\mathcal{H}_F^j = \mathcal{H}_F(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{\vartheta}^j, \widetilde{\ell}^j, \widetilde{\omega}^j), \quad \mathcal{H}_S^j = \mathcal{H}_S(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{\vartheta}^j, \widetilde{\ell}^j, \widetilde{\omega}^j), \mathcal{H}^j = \mathbb{1}_{\partial\Omega}\mathcal{H}_F^j + \mathbb{1}_{\partial\Omega_S(0)}\mathcal{H}_S^j \\
&\mathcal{G}_1^j = \mathcal{G}_1(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{\vartheta}^j, \widetilde{\ell}^j, \widetilde{\omega}^j), \quad \mathcal{G}_2^j = \mathcal{G}_2(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{\vartheta}^j, \widetilde{\ell}^j, \widetilde{\omega}^j),
\end{aligned}$$

Then

$$\left\|(\mathcal{F}_1^1 - \mathcal{F}_1^2, \mathcal{F}_2^1 - \mathcal{F}_2^2, \mathcal{F}_3^1 - \mathcal{F}_3^2, \mathcal{H}^1 - \mathcal{H}^2, \mathcal{G}_1^1 - \mathcal{G}_1^2, \mathcal{G}_2^1 - \mathcal{G}_2^2)\right\|_{\mathcal{B}_{T,p,q}} \leqslant CT^\delta.$$

Proof. The proof of this proposition is similar to the proof of Proposition 6.4 and we only give here some few ideas. From (5.15) in Theorem 5.3, we first obtain

$$\|(\widetilde{\rho}^1, \widetilde{u}^1, \widetilde{\vartheta}^1, \widetilde{\ell}^1, \widetilde{\omega}^1) - (\widetilde{\rho}^2, \widetilde{u}^2, \widetilde{\vartheta}^2, \widetilde{\ell}^2, \widetilde{\omega}^2)\|_{\mathcal{S}_{T,p,q}}$$
$$\leqslant C\left\|(f_1^1, f_2^1, f_3^1, h^1, g_1^1, g_2^1) - (f_1^2, f_2^2, f_3^2, h^2, g_1^2, g_2^2)\right\|_{\mathcal{B}_{T,p,q}}. \quad (6.33)$$

Combining (6.2) and (6.33), we deduce

$$\|\widetilde{\rho}^1 - \widetilde{\rho}^2\|_{L^\infty(0,T;W^{1,q}(\Omega_F(0)))} \leqslant CT^{1/p'}\left\|(f_1^1, f_2^1, f_3^1, h^1, g_1^1, g_2^1) - (f_1^2, f_2^2, f_3^2, h^2, g_1^2, g_2^2)\right\|_{\mathcal{B}_{T,p,q}}. \quad (6.34)$$

In a similar manner, we can obtain

$$\|\widetilde{\omega}^1 - \widetilde{\omega}^2\|_{L^\infty(0,T)} + \|\widetilde{\ell}^1 - \widetilde{\ell}^2\|_{L^\infty(0,T)}$$
$$\leqslant CT^{1/p'}\left\|(f_1^1, f_2^1, f_3^1, h^1, g_1^1, g_2^1) - (f_1^2, f_2^2, f_3^2, h^2, g_1^2, g_2^2)\right\|_{\mathcal{B}_{T,p,q}}. \quad (6.35)$$

Since $2 < p < \infty$, one has $B_{q,p}^{2(1-1/p)}(\Omega_F(0)) \hookrightarrow W^{1,q}(\Omega_F(0))$. Therefore, using Proposition 6.1 and (6.33) we get

$$\|\widetilde{u}^1 - \widetilde{u}^2\|_{L^\infty(0,T;W^{1,q})} + \|\widetilde{\vartheta}^1 - \widetilde{\vartheta}^2\|_{L^\infty(0,T;W^{1,q})}$$
$$\leqslant C\left\|(f_1^1, f_2^1, f_3^1, h^1, g_1^1, g_2^1) - (f_1^2, f_2^2, f_3^2, h^2, g_1^2, g_2^2)\right\|_{\mathcal{B}_{T,p,q}}. \quad (6.36)$$

Proceeding as in the proof of Proposition 6.4, we can then deduce

$$\|\widetilde{u}^1 - \widetilde{u}^2\|_{L^p(0,T,L^\infty)} + \|\nabla\widetilde{u}^1 - \nabla\widetilde{u}^2\|_{L^p(0,T,L^\infty)} + \|\nabla\widetilde{\vartheta}^1 - \nabla\widetilde{\vartheta}^2\|_{L^p(0,T,L^\infty)}$$
$$\leqslant CT^{(1-s)/2p}\left\|(f_1^1, f_2^1, f_3^1, h^1, g_1^1, g_2^1) - (f_1^2, f_2^2, f_3^2, h^2, g_1^2, g_2^2)\right\|_{\mathcal{B}_{T,p,q}}, \quad s \in (3/q, 1). \quad (6.37)$$

Let us denote by Q^1 and Q^2 the solution of (4.8) associated with $\widetilde{\omega}^1, \widetilde{\omega}^2$. Then $Q^1 - Q^2$ satisfies

$$\begin{cases} \dfrac{d}{dt}\left(Q^1 - Q^2\right) = \left(Q^1 - Q^2\right) A(\widetilde{\omega}^1) + Q^2 A(\widetilde{\omega}^1 - \widetilde{\omega}^2) & \text{in } (0,T), \\ \left(Q^1 - Q^2\right)(0) = 0, \end{cases} \quad (6.38)$$

and thus from (6.10), (6.35) and Grönwall's lemma, we obtain

$$\|Q^1 - Q^2\|_{L^\infty(0,T;\mathbb{R}^{3\times3})} \leqslant CT\left\|(f_1^1, f_2^1, f_3^1, h^1, g_1^1, g_2^1) - (f_1^2, f_2^2, f_3^2, h^2, g_1^2, g_2^2)\right\|_{\mathcal{B}_{T,p,q}}. \quad (6.39)$$

Let X^1, X^2 be defined as in (4.9) with $(Q^1, \widetilde{u}^1)$ and $(Q^2, \widetilde{u}^2)$. Then

$$\partial_t \nabla(X^1 - X^2) = \left(Q^1 - Q^2\right)\nabla\widetilde{u}^1 + Q^2\nabla\left(\widetilde{u}^1 - \widetilde{u}^2\right), \quad \nabla(X^1 - X^2)(0,\cdot) = 0.$$

and from (6.7), (6.15), (6.33) and (6.39) we get

$$\|\partial_t \nabla(X^1 - X^2)\|_{L^p(0,T;W^{1,q})}$$
$$\leqslant \|Q^1 - Q^2\|_{L^\infty(0,T)}\|\nabla\widetilde{u}^2\|_{L^p(0,T;W^{1,q})} + \|Q^2\|_{L^\infty(0,T)}\|\nabla\left(\widetilde{u}^1 - \widetilde{u}^2\right)\|_{L^p(0,T;W^{1,q})}$$
$$\leqslant C\left\|(f_1^1, f_2^1, f_3^1, h^1, g_1^1, g_2^1) - (f_1^2, f_2^2, f_3^2, h^2, g_1^2, g_2^2)\right\|_{\mathcal{B}_{T,p,q}}.$$

The above estimate yields

$$\|\nabla(X^1 - X^2)\|_{W^{1,p}(0,T;W^{1,q}(\Omega_F(0)))}$$
$$\leqslant C\big\|(f_1^1, f_2^1, f_3^1, h^1, g_1^1, g_2^1) - (f_1^2, f_2^2, f_3^2, h^2, g_1^2, g_2^2)\big\|_{\mathcal{B}_{T,p,q}}.$$

the above estimate and (6.2) yield

$$\|\nabla(X^1 - X^2)\|_{L^\infty(0,T;W^{1,q}(\Omega_F(0)))}$$
$$\leqslant CT^{1/p'}\big\|(f_1^1, f_2^1, f_3^1, h^1, g_1^1, g_2^1) - (f_1^2, f_2^2, f_3^2, h^2, g_1^2, g_2^2)\big\|_{\mathcal{B}_{T,p,q}}.$$

The rest of the proof runs as the proof of Proposition 6.4. $\qquad\square$

Part 3. Global in Time Existence

7. Linearization and Lagrangian Change of Variables

In this section, we slightly modify the change of variables introduced in Section 4 and we rephrase the global existence and uniqueness result in Theorem 2.2 in terms of the functions issued from this change of variables. The reason of this modification is that, here we need to linearize the system around the constant steady state $(\overline{\rho}, 0, \overline{\vartheta}, 0, 0)$. More precisely, define

$$\widetilde{\rho}(t,y) = \rho(t, X(t,y)) - \overline{\rho}, \qquad \widetilde{u}(t,y) = Q^{-1}(t)u(t, X(t,y)), \tag{7.1}$$

$$\widetilde{\vartheta}(t,y) = \vartheta(t, X(t,y)) - \overline{\vartheta}, \qquad \widetilde{p} = R\widetilde{\rho}\widetilde{\vartheta}, \tag{7.2}$$

$$\widetilde{\ell}(t) = Q^{-1}(t)\dot{a}(t), \qquad \widetilde{\omega}(t) = Q^{-1}(t)\omega(t), \tag{7.3}$$

for $(t,y) \in (0,\infty) \times \Omega_F(0)$, where X has been defined as in (4.1).

Then $(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega})$ satisfies the following system

$$
\begin{aligned}
&\partial_t \widetilde{\rho} + \overline{\rho}\operatorname{div}\widetilde{u} = \mathcal{F}_1 && \text{in } (0,\infty) \times \Omega_F(0), \\
&\partial_t \widetilde{u} - \operatorname{div}\sigma_l(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}) = \mathcal{F}_2 && \text{in } (0,\infty) \times \Omega_F(0), \\
&\partial_t \widetilde{\vartheta} - \frac{\kappa}{\overline{\rho}c_v}\Delta\widetilde{\vartheta} + \frac{R\overline{\vartheta}}{c_v}\operatorname{div}\widetilde{u} = \mathcal{F}_3 && \text{in } (0,\infty) \times \Omega_F(0), \\
&\widetilde{u} = 0 && \text{on } (0,\infty) \times \partial\Omega, \\
&\widetilde{u} = \widetilde{\ell} + \widetilde{\omega} \times y && \text{on } (0,\infty) \times \partial\Omega_S(0) \qquad (7.4) \\
&\frac{\partial\widetilde{\vartheta}}{\partial n} = \mathcal{H} \cdot n && \text{on } (0,\infty) \times \partial\Omega_F(0), \\
&\frac{d}{dt}\widetilde{\ell} = -m^{-1}\int_{\partial\Omega_S(0)}\sigma_l(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta})n\, d\gamma + \mathcal{G}_1 && t \in (0,\infty) \\
&\frac{d}{dt}\widetilde{\omega} = -J(0)^{-1}\int_{\partial\Omega_S(0)} y \times \sigma_l(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta})n\, d\gamma + \mathcal{G}_2 && t \in (0,\infty) \\
&\widetilde{\rho}(0) = \rho_0 - \overline{\rho}, \quad \widetilde{u}(0) = u_0, \quad \widetilde{\vartheta}(0) = \vartheta_0 - \overline{\vartheta} && \text{in } \Omega_F(0), \\
&\widetilde{\ell}(0) = \ell_0, \quad \widetilde{\omega}(0) = \omega_0,
\end{aligned}
$$

where

$$\sigma_l(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}) = \frac{2\mu}{\overline{\rho}}D\widetilde{u} + \left(\frac{\alpha}{\overline{\rho}}\operatorname{div}\widetilde{u} - \frac{R\overline{\vartheta}}{\overline{\rho}}\widetilde{\rho} - R\widetilde{\vartheta}\right)I_3, \quad D(\widetilde{u}) = \frac{1}{2}(\nabla\widetilde{u} + \nabla\widetilde{u}^T), \tag{7.5}$$

$$\dot{Q} = QA(\widetilde{\omega}), \quad Q(0) = I_3 \tag{7.6}$$

and

$$X(t, y) = y + \int_0^t Q(s)\widetilde{u}(s, y) \, ds, \quad \text{and} \quad \nabla Y(t, X(t, y)) = [\nabla X]^{-1}(t, y), \tag{7.7}$$

for every $y \in \Omega_F(0)$ and $t \geqslant 0$. Using the notation

$$Z(t, y) = (Z_{i,j})_{1 \leqslant i, j \leqslant 3} = [\nabla X]^{-1}(t, y) \qquad (t \geqslant 0, \ y \in \Omega_F(0)), \tag{7.8}$$

the remaining terms in (7.4) are defined by

$$\mathcal{F}_1(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) = -\widetilde{\rho} \operatorname{div} \widetilde{u} - (\widetilde{\rho} + \overline{\rho})(Z^\top - I_3) : \nabla \widetilde{u} \tag{7.9}$$

$$(\mathcal{F}_2)_i(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) = -\frac{\widetilde{\rho} + \overline{\rho}}{\overline{\rho}}(\widetilde{\omega} \times Q\widetilde{u})_i + \widetilde{\rho}(\partial_t \widetilde{u})_i - \frac{\widetilde{\rho} + \overline{\rho}}{\overline{\rho}}\left[(Q - I)\partial_t \widetilde{u}\right]_i$$

$$+ \frac{\mu}{\overline{\rho}} \sum_{l,j,k} \frac{\partial^2 (Q\widetilde{u})_i}{\partial y_l \partial y_k}(Z_{k,j} - \delta_{k,j})Z_{l,j} + \frac{\mu}{\overline{\rho}} \sum_{l,k} \frac{\partial^2 (Q\widetilde{u})_i}{\partial y_l \partial y_k}(Z_{l,k} - \delta_{l,k})$$

$$+ \frac{\mu}{\overline{\rho}}\left[(Q - I)\Delta\widetilde{u}\right]_i + \frac{\mu}{\overline{\rho}} \sum_{l,j,k} Z_{l,j} \frac{\partial (Q\widetilde{u})_i}{\partial y_k} \frac{\partial Z_{k,j}}{\partial y_l}$$

$$+ \frac{\mu + \alpha}{\overline{\rho}} \sum_{l,j,k} \frac{\partial^2 (Qu)_j}{\partial y_l \partial y_k}(Z_{k,j} - \delta_{k,j})Z_{l,i} + \frac{\mu + \alpha}{\overline{\rho}} \sum_{l,j} \frac{\partial^2 (Qu)_j}{\partial y_l \partial y_j}(Z_{l,i} - \delta_{l,i})$$

$$+ \frac{\alpha + \mu}{\overline{\rho}} \frac{\partial}{\partial y_i}\left[\nabla\widetilde{u} : (Q^\top - I_3)\right] + \frac{\alpha + \mu}{\overline{\rho}} \sum_{l,j,k} Z_{l,i} \frac{\partial (Qu)_j}{\partial y_k} \frac{\partial Z_{k,j}}{\partial y_l}$$

$$- R\frac{\widetilde{\vartheta}}{\overline{\rho}}\left(Z^\top \nabla\widetilde{\rho}\right)_i - R\frac{\widetilde{\rho}}{\overline{\rho}}\left(Z^\top \nabla\widetilde{\vartheta}\right)_i - \frac{R\overline{\vartheta}}{\overline{\rho}}\left[\left(Z^\top - I\right)\nabla\widetilde{\rho}\right]_i$$

$$- R\left[\left(Z^\top - I\right)\nabla\widetilde{\vartheta}\right]_i, \tag{7.10}$$

$$\mathcal{F}_3(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) = -\frac{\widetilde{\rho}}{\overline{\rho}}\partial_t\widetilde{\vartheta} - \frac{R}{c_v\overline{\rho}}\left(\widetilde{\vartheta}\widetilde{\rho} + \overline{\rho}\widetilde{\vartheta} + \overline{\vartheta}\widetilde{\rho}\right)[ZQ]^\top : \nabla\widetilde{u} - \frac{R\overline{\vartheta}}{c_v}\left([ZQ]^\top - I_3\right) : \nabla\widetilde{u}$$

$$+ \frac{\kappa}{c_v\overline{\rho}} \sum_{l,j,k} Z_{l,j} \frac{\partial\widetilde{\vartheta}}{\partial y_k} \frac{\partial Z_{k,j}}{\partial y_l} + \frac{\kappa}{c_v\overline{\rho}} \sum_{k,l} \frac{\partial^2\widetilde{\vartheta}}{\partial y_k \partial y_l}(Z_{l,k} - \delta_{l,k})Z_{l,j}$$

$$+ \frac{\kappa}{c_v\overline{\rho}} \sum_{j,k,l} \frac{\partial^2\widetilde{\vartheta}}{\partial y_k \partial y_l}(Z_{k,j} - \delta_{k,j})Z_{l,j}$$

$$+ \frac{\alpha}{c_v\overline{\rho}}\left([ZQ]^\top : \nabla\widetilde{u}\right)^2 + \frac{\mu}{2c_v\overline{\rho}}\left|\nabla\widetilde{u}ZQ + (\nabla\widetilde{u}ZQ)^\top\right|^2, \tag{7.11}$$

$$\mathcal{H}(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) = \mathbb{1}_{\partial\Omega}\mathcal{H}_F + \mathbb{1}_{\partial\Omega_S(0)}\mathcal{H}_S$$

$$\mathcal{H}_F(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) = (I_3 - Z^\top)\nabla\widetilde{\vartheta}, \quad \mathcal{H}_S(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) = (I_3 - (ZQ)^\top)\nabla\widetilde{\vartheta}, \tag{7.12}$$

$$\mathcal{G}_0(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) = \frac{\mu}{\overline{\rho}}\left[\nabla\widetilde{u}(ZQ - I_3) + [(ZQ)^\top - I_3](\nabla\widetilde{u})^\top\right]$$

$$+ \frac{\alpha}{\overline{\rho}} \left(([ZQ]^\top - I_3) : \nabla \widetilde{u} \right) I_3 + R\widetilde{\rho}\widetilde{\vartheta} I_3, \tag{7.13}$$

$$\mathcal{G}_1(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) = -\frac{m}{\overline{\rho}}(\widetilde{\omega} \times \widetilde{\ell}) - \int_{\partial\Omega_S(0)} \mathcal{G}_0 n \, \mathrm{d}\gamma,$$

$$\mathcal{G}_2(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) = \frac{J(0)}{\overline{\rho}}\widetilde{\omega} \times \widetilde{\omega} - \int_{\partial\Omega_S(0)} y \times \mathcal{G}_0 n \, \mathrm{d}\gamma. \tag{7.14}$$

Using the above change of variables, Theorem 2.2 can be rephrased as follows.

Theorem 7.1. *Let $2 < p < \infty$ and $3 < q < \infty$ satisfying the condition $\dfrac{1}{p} + \dfrac{1}{2q} \neq \dfrac{1}{2}$. Assume that (2.1) is satisfied. Let $\overline{\rho} > 0$ and $\overline{\vartheta} > 0$ be two given constants. Then there exists $\eta_0 > 0$ such that, for all $\eta \in (0, \eta_0)$ there exist two constants $\delta_0 > 0$ and $C > 0$ such that, for all $\delta \in (0, \delta_0)$ and for any $(\rho_0, u_0, \vartheta_0, \ell_0, \omega_0)$ belongs to $\mathcal{I}^{cc}_{p,q}$ satisfying*

$$\frac{1}{|\Omega_F(0)|} \int_{\Omega_F(0)} \rho_0 \, \mathrm{d}x = \overline{\rho}, \tag{7.15}$$

and

$$\|(\rho_0 - \overline{\rho}, u_0, \vartheta_0 - \overline{\vartheta}, \ell_0, \omega_0)\|_{\mathcal{I}_{p,q}} \leqslant \delta, \tag{7.16}$$

the system (7.4) - (7.13) admits a unique solution $(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega})$ with

$$\|\widetilde{\rho}\|_{L^\infty(0,\infty;W^{1,q}(\Omega_F(0)))} + \|e^{\eta(\cdot)}\nabla\widetilde{\rho}\|_{W^{1,p}(0,\infty;L^q(\Omega_F(0)))} + \|e^{\eta(\cdot)}\partial_t\widetilde{\rho}\|_{L^p(0,\infty;L^q(\Omega_F(0)))}$$

$$+ \|e^{\eta(\cdot)} \widetilde{u}\|_{L^p(0,\infty;W^{2,q}(\Omega_F(0))^3)} + \|e^{\eta(\cdot)}\partial_t\widetilde{u}\|_{L^p(0,\infty;L^q(\Omega_F(0))^3)} + \|e^{\eta(\cdot)}\widetilde{u}\|_{L^\infty(0,\infty;B^{2(1-1/p)}_{q,p}(\Omega_F(0))^3)}$$

$$+ \|e^{\eta(\cdot)}\partial_t\widetilde{\vartheta}\|_{L^p(0,\infty;L^q(\Omega_F(0)))} + \|e^{\eta(\cdot)}\nabla\widetilde{\vartheta}\|_{L^p(0,\infty;L^q(\Omega_F(0)))} + \|e^{\eta(\cdot)}\nabla^2\widetilde{\vartheta}\|_{L^p(0,\infty;L^q(\Omega_F(0)))}$$

$$+ \|\widetilde{\vartheta}\|_{L^\infty(0,\infty;B^{2(1-1/p)}_{q,p}(\Omega_F(0)))} + \|e^{\eta(\cdot)}\widetilde{\ell}\|_{W^{1,p}(0,\infty;\mathbb{R}^3)}$$

$$+ \|e^{\eta(\cdot)} \widetilde{\omega}\|_{W^{1,p}(0,\infty;\mathbb{R}^3)} \leqslant C\delta. \tag{7.17}$$

Moreover, $X \in L^\infty(0,\infty;W^{2,q}(\Omega_F(0)))^3 \cap W^{1,\infty}(0,\infty;W^{1,q}(\Omega_F(0)))$ and $X(t,\cdot) : \Omega_F(0) \to \Omega_F(t)$ is a C^1-diffeormorphim for all $t \in [0,\infty)$.

8. Some Background on $\mathcal{R}$ Sectorial Operators

In this section, we recall some definitions and results on maximal L^p–regularity and $\mathcal{R}$-boundedness. In what follows, we use Rademacher random variables, that is symmetric random variables with value in $\{-1, 1\}$. We first recall the notion of $\mathcal{R}$-boundedness.

Definition 8.1 ($\mathcal{R}$-bounded family of operators). Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is called $\mathcal{R}$–bounded if there exist $p \in [1, \infty)$ and a constant $C > 0$, such that for any integer $N \geqslant 1$, any $T_1, \ldots T_N \in \mathcal{T}$, any independent Rademacher random variables $r_1, \ldots, r_N$, and any $x_1, \ldots, x_N \in \mathcal{X}$,

$$\left(\mathbb{E} \Big\| \sum_{j=1}^N r_j T_j x_j \Big\|_{\mathcal{Y}}^p \right)^{1/p} \leqslant C \left(\mathbb{E} \Big\| \sum_{j=1}^N r_j x_j \Big\|_{\mathcal{X}}^p \right)^{1/p}.$$

The smallest constant C in the above inequality is called the $\mathcal{R}_p$-bound of $\mathcal{T}$ on $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ and is denoted by $\mathcal{R}_p(\mathcal{T})$. As usual we denote by $\mathbb{E}$ the expectation.

For more information on $\mathcal{R}$-boundedness we refer to [7, 8, 19] and references therein. In particular, it is proved in [8, p.26] that this definition is independent of $p \in [1, \infty)$.

We also recall some useful properties (see Proposition 3.4 in [8]):

$$\mathcal{R}_p(\mathcal{S} + \mathcal{T}) \leqslant \mathcal{R}_p(\mathcal{S}) + \mathcal{R}_p(\mathcal{T}), \quad \mathcal{R}_p(\mathcal{S}\mathcal{T}) \leqslant \mathcal{R}_p(\mathcal{T})\mathcal{R}_p(\mathcal{S}). \tag{8.1}$$

For any $\beta \in (0, \pi)$, we write

$$\Sigma_\beta = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg(\lambda)| < \beta\}.$$

We now come to the second central definition.

Definition 8.2 (sectorial and $\mathcal{R}$-sectorial operators)**.** *Let A be a densely defined closed linear operator on a Banach space $\mathcal{X}$ with domain $\mathcal{D}(A)$. We say that A is a sectorial operator of angle $\tau \in (0, \pi)$ if for any $\beta \in (\tau, \pi)$, $\Sigma_{\pi-\beta} \subset \rho(A)$ and*

$$R_\beta = \left\{ \lambda(\lambda - A)^{-1} : \ \lambda \in \Sigma_{\pi-\beta} \right\}$$

is bounded in $\mathcal{L}(\mathcal{X})$. In that case, we write

$$M_\beta(A) = \sup_{\lambda \in \Sigma_{\pi-\beta}} \|\lambda(\lambda - A)^{-1}\|_{\mathcal{L}(\mathcal{X})}.$$

Analogously, we say that A is a $\mathcal{R}$-sectorial operator of angle τ if A is a sectorial operator of angle τ and if for any $\beta \in (\tau, \pi)$, R_β is $\mathcal{R}$-bounded. We denote $\mathcal{R}_{p,\beta}(A)$ the $\mathcal{R}_p$-bound of R_β.

One can replace in the above definitions R_β by

$$\widetilde{R_\beta} = \left\{ A(\lambda - A)^{-1} : \ \lambda \in \Sigma_{\pi-\beta} \right\}.$$

In that case, we denote the uniform bound and the $\mathcal{R}$-bound by $\widetilde{M_\beta}(A)$ and $\widetilde{\mathcal{R}_{p,\beta}}(A)$.

The importance of $\mathcal{R}$-sectorial operators is explained by the following result:

Theorem 8.3 (Weis)**.** *Let $\mathcal{X}$ be a UMD Banach space and A a densely defined, closed linear operator on $\mathcal{X}$. Then the following assertions are equivalent*

(a) *For any $T \in \mathbb{R}_+^*$, $f \in L^p(0, T; \mathcal{X})$*

$$u' = Au + f \quad in \quad (0, T), \quad u(0) = 0 \tag{8.2}$$

admits a unique solution u satisfying the above equation almost everywhere and such that $Au \in L^p(0, T; \mathcal{X})$.

(b) *A is $\mathcal{R}$-sectorial of angle $\tau < \pi/2$.*

This result is due to [32] (see also [8, p.45]). We recall that $\mathcal{X}$ is a UMD Banach space if the Hilbert transform is bounded in $L^p(\mathbb{R}; \mathcal{X})$ for $p \in (1, \infty)$. In particular, the closed subspaces of $L^q(\Omega)$ for $q \in (1, \infty)$ are UMD Banach spaces. We refer the reader to [3, pp.141–147] for more information on UMD spaces.

We can also add an initial condition in (8.2) and consider the following system:

$$u' = Au + f \quad in \quad (0, \infty), \quad u(0) = u_0. \tag{8.3}$$

Corollary 8.4. *Let $\mathcal{X}$ be a UMD Banach space, $1 < p < \infty$ and let A be a closed, densely defined operator in $\mathcal{X}$ with domain $\mathcal{D}(A)$. Let us assume that A is a $\mathcal{R}$-sectorial operator of angle $\tau < \pi/2$ and that the semigroup generated by A has negative exponential type. Then for every $u_0 \in (\mathcal{X}, \mathcal{D}(A))_{1-1/p,p}$ and for every $f \in L^p(0, \infty; \mathcal{X})$, Eq. (8.3) admits a unique solution in $L^p(0, \infty; \mathcal{D}(A)) \cap W^{1,p}(0, \infty; \mathcal{X})$.*

Proof. The proof follows from the above theorem, [11, Theorem 2.4] and [27, Theorem 1.8.2]. $\square$

In view of the above results, it is natural to consider the perturbation theory of $\mathcal{R}$-sectoriality. The following result was obtained in [18, Corollary 2].

Proposition 8.5. *Let A be a $\mathcal{R}$-sectorial operator of angle τ on a Banach space $\mathcal{X}$ and let $\beta \in (\tau, \pi)$. Let B be a linear operator on $\mathcal{X}$ such that $\mathcal{D}(A) \subset \mathcal{D}(B)$ and*

$$\|Bx\|_{\mathcal{X}} \leqslant a\|Ax\|_{\mathcal{X}} + b\|x\|_{\mathcal{X}}, \tag{8.4}$$

for some $a, b \geqslant 0$. If $a < \left(\widetilde{M_\beta(A)} \widetilde{\mathcal{R}_{p,\beta}(A)} \right)^{-1}$ then $A + B - \lambda$ is $\mathcal{R}$-sectorial for each

$$\lambda > \lambda_0 = \frac{b\widetilde{M_\beta(A)}\widetilde{\mathcal{R}_{p,\beta}(A)}}{1 - a\widetilde{M_\beta(A)}\widetilde{\mathcal{R}_{p,\beta}(A)}}.$$

9. Linearized Fluid-Structure Interaction System

In this section we study the fluid-structure system linearized around $(\overline{\rho}, 0, \overline{\vartheta}, 0, 0)$, $\overline{\rho} > 0$, $\overline{\vartheta} > 0$. More precisely, we consider the following linear system

$$
\begin{aligned}
&\partial_t \rho + \overline{\rho} \operatorname{div} u = 0 && \text{in } (0, \infty) \times \Omega_F(0), \\
&\partial_t u - \operatorname{div} \sigma_l(\rho, u, \vartheta) = 0 && \text{in } (0, \infty) \times \Omega_F(0), \\
&\partial_t \vartheta - \frac{\kappa}{\overline{\rho} c_v} \Delta \vartheta + \frac{R\overline{\vartheta}}{c_v} \operatorname{div} u = 0 && \text{in } (0, \infty) \times \Omega_F(0), \\
&u = \ell + \omega \times y && \text{on } (0, \infty) \times \partial\Omega_S(0) \\
&u = 0 && \text{on } (0, \infty) \times \partial\Omega, \\
&\frac{\partial \vartheta}{\partial n} = 0 && \text{on } (0, \infty) \times \partial\Omega_F(0), \\
&\frac{d}{dt}\ell = -m^{-1} \int_{\partial\Omega_S(0)} \sigma_l(\rho, u, \vartheta) n \, d\gamma && t \in (0, \infty) \\
&\frac{d}{dt}\omega = -J(0)^{-1} \int_{\partial\Omega_S(0)} y \times \sigma_l(\rho, u, \vartheta) n \, d\gamma && t \in (0, \infty) \\
&\rho(0) = \rho_0, \quad u(0) = u_0, \quad \vartheta(0) = \vartheta_0 && \text{in } \Omega_F(0), \\
&\ell(0) = \ell_0, \quad \omega(0) = \omega_0,
\end{aligned}
\tag{9.1}
$$

where

$$\sigma_l(\rho, u, \vartheta) = \frac{2\mu}{\overline{\rho}} Du + \left(\frac{\alpha}{\overline{\rho}} \operatorname{div} u - \frac{R\overline{\vartheta}}{\overline{\rho}} \rho - R\vartheta \right) I_3, \quad D(u) = \frac{1}{2}(\nabla u + \nabla u^T). \tag{9.2}$$

Our aim is to show that the linearized operator is $\mathcal{R}$-sectorial in a suitable functional space. In order to do this, we first consider the case of a linearized compressible Navier-Stokes-Fourier system without rigid body and we use Proposition 8.5 in order to deal with the equations for the rigid body.

9.1. **Linearized compressible Navier-Stokes-Fourier system.** In this subsection we discuss some properties of the generator of the semigroup describing the linearization around an equilibrium state of the Navier-Stokes-Fourier system. Most of these properties follow from the corresponding linearized compressible Navier-Stokes system and of the heat equation, and in this case we just provide the appropriate references. Our contribution is to show that the coupling terms can be seen as perturbations and thus tackled using either direct estimates or abstract perturbation results for $\mathcal{R}$-sectorial operators.

Let us set

$$\mathcal{X} = W^{1,q}(\Omega_F(0)) \times L^q(\Omega_F(0))^3 \times L^q(\Omega_F(0)) \tag{9.3}$$

and consider the operator $A_F : \mathcal{D}(A_F) \to \mathcal{X}$ defined by

$$\mathcal{D}(A_F) = \left\{ (\rho, u, \vartheta) \in W^{1,q}(\Omega_F(0)) \times W^{2,q}(\Omega_F(0))^3 \times W^{2,q}(\Omega_F(0)) \mid \right.$$

$$\left. u = 0 \text{ on } \partial\Omega_F(0), \quad \frac{\partial\vartheta}{\partial n} = 0 \text{ on } \partial\Omega_F(0) \right\},$$

$$A_F = \begin{pmatrix} 0 & -\overline{\rho}\,\mathrm{div} & 0 \\ -\dfrac{R\overline{\vartheta}}{\overline{\rho}}\nabla & \dfrac{\mu}{\overline{\rho}}\Delta + \dfrac{\alpha+\mu}{\overline{\rho}}\nabla\,\mathrm{div} & -R\nabla \\ 0 & -\dfrac{R\overline{\vartheta}}{c_v}\,\mathrm{div} & \dfrac{\kappa}{\overline{\rho}c_v}\Delta \end{pmatrix}. \tag{9.4}$$

Let us first study some properties of the operator A_F.

Theorem 9.1. *Assume* $1 < q < \infty$. *Then there exists* $\gamma_0 > 0$ *such that* $A_F - \gamma_0$ *is an* $\mathcal{R}$*-sectorial operator in* $\mathcal{X}$ *of angle* $< \pi/2$.

Proof. We first define

$$\mathcal{D}(A_u) = \left\{ u \in W^{2,q}(\Omega_F(0))^3 \mid u = 0 \text{ on } \partial\Omega_F(0) \right\}, \quad A_u = \frac{\mu}{\overline{\rho}}\Delta + \frac{\alpha+\mu}{\overline{\rho}}\nabla\,\mathrm{div}, \tag{9.5}$$

and

$$\mathcal{D}(A_\vartheta) = \left\{ \vartheta \in W^{2,q}(\Omega_F(0)) \mid \frac{\partial\vartheta}{\partial n} = 0 \text{ on } \partial\Omega_F(0) \right\}, \quad A_\vartheta = \frac{\kappa}{\overline{\rho}c_v}\Delta. \tag{9.6}$$

From [8, Theorem 8.2], there exists $\gamma_\vartheta \in \mathbb{R}$ such that $A_\vartheta - \gamma_\vartheta$ is $\mathcal{R}$-sectorial of angle $< \pi/2$. Using [13, Theorem 2.5], we also obtain the existence of $\gamma_u \in \mathbb{R}$ such that $A_u - \gamma_u$ is $\mathcal{R}$-sectorial of angle $< \pi/2$.

In particular there exist γ and $\beta < \pi/2$ such that,

$$\mathcal{R}_p\left\{\lambda(\lambda - A_u)^{-1} : \lambda \in \gamma + \Sigma_{\pi-\beta}\right\} < \infty, \quad \mathcal{R}_p\left\{A_u(\lambda - A_u)^{-1} : \lambda \in \gamma + \Sigma_{\pi-\beta}\right\} < \infty, \tag{9.7}$$

and

$$\mathcal{R}_p\left\{\lambda(\lambda - A_\vartheta)^{-1} : \lambda \in \gamma + \Sigma_{\pi-\beta}\right\} < \infty \tag{9.8}$$

respectively in $\mathcal{L}(L^q(\Omega_F(0))^3)$ and $\mathcal{L}(L^q(\Omega_F(0)))$. We deduce from the properties (8.1) that

$$\mathcal{R}_p\left\{\mathrm{div}(\lambda - A_u)^{-1} : \lambda \in \gamma + \Sigma_{\pi-\beta}\right\} < \infty \tag{9.9}$$

in $\mathcal{L}(L^q(\Omega_F(0))^3, W^{1,q}(\Omega_F(0)))$.

We rewrite the operator A_F in the form $A_F = A_F^0 + B_F$, with

$$A_F^0 = \begin{pmatrix} 0 & -\overline{\rho}\,\mathrm{div} & 0 \\ 0 & A_u & 0 \\ 0 & 0 & A_\vartheta \end{pmatrix} \text{ and } B_F = \begin{pmatrix} 0 & 0 & 0 \\ -\dfrac{R\overline{\vartheta}}{\overline{\rho}}\nabla & 0 & -R\nabla \\ 0 & -\dfrac{R\overline{\vartheta}}{c_v}\,\mathrm{div} & 0 \end{pmatrix}.$$

Some standard calculation shows that for $\lambda \in \gamma + \Sigma_{\pi-\beta}$,

$$\lambda(\lambda I - A_F^0)^{-1} = \begin{pmatrix} I & -\overline{\rho}\,\mathrm{div}(\lambda I - A_u)^{-1} & 0 \\ 0 & \lambda(\lambda I - A_u)^{-1} & 0 \\ 0 & 0 & \lambda(\lambda I - A_\vartheta)^{-1} \end{pmatrix}.$$

Using again the properties (8.1), we deduce from the above formula and from (9.7)–(9.9) that

$$\mathcal{R}_p\left\{\lambda(\lambda - A_F^0)^{-1} :\ \lambda \in \gamma + \Sigma_{\pi-\beta}\right\} < \infty$$

in $\mathcal{L}(\mathcal{X})$.

Now for all $(\rho, u, \vartheta) \in \mathcal{D}(A_F)$ we have

$$\left\| B_F \begin{pmatrix} \rho \\ u \\ \vartheta \end{pmatrix} \right\|_{\mathcal{X}} \leqslant C \left(\|\rho\|_{W^{1,q}(\Omega_F(0))} + \|u\|_{W^{1,q}(\Omega_F(0))} + \|\vartheta\|_{W^{1,q}(\Omega_F(0))} \right). \tag{9.10}$$

Using the compactness of the embedding $W^{2,q}(\Omega_F(0)) \hookrightarrow W^{1,q}(\Omega_F(0))$ and a classical result (see [26, Chapter 3, Lemma 2.1]), we deduce that for any $\delta > 0$, there exists $C(\delta) > 0$ such that for all $f \in W^{2,q}(\Omega_F(0))$

$$\|f\|_{W^{1,q}(\Omega_F(0))} \leqslant \delta\|f\|_{W^{2,q}(\Omega_F(0))} + C(\delta)\|f\|_{L^q(\Omega_F(0))}.$$

Applying the above estimate to (9.10) we obtain, for every $\delta > 0$, there exists $C(\delta) > 0$ such that

$$\left\| B_F \begin{pmatrix} \rho \\ u \\ \vartheta \end{pmatrix} \right\|_{\mathcal{X}} \leqslant \delta \left\| A_F^0 \begin{pmatrix} \rho \\ u \\ \vartheta \end{pmatrix} \right\|_{\mathcal{X}} + C(\delta) \left\| \begin{pmatrix} \rho \\ u \\ \vartheta \end{pmatrix} \right\|_{\mathcal{X}}.$$

Therefore applying Proposition 8.5 we complete the proof of the theorem. $\qquad \square$

Now we want to show that the operator A_F is invertible in a suitable subspace of $\mathcal{X}$. For this purpose we consider the following problem

$$\begin{cases} \overline{\rho}\,\mathrm{div}\,u = f_1 & \text{in } \Omega_F(0), \\ -\dfrac{\mu}{\overline{\rho}}\Delta u - \dfrac{\alpha+\mu}{\overline{\rho}}\nabla(\mathrm{div}\ u) + \dfrac{R\overline{\vartheta}}{\overline{\rho}}\nabla\rho + R\nabla\vartheta = f_2 & \text{in } \Omega_F(0), \\ -\dfrac{\kappa}{\overline{\rho}c_v}\Delta\vartheta + \dfrac{R\overline{\vartheta}}{c_v}\,\mathrm{div}\,u = f_3 & \text{in } \Omega_F(0), \\ u = 0 \text{ on } \partial\Omega_F(0), \quad \dfrac{\partial\vartheta}{\partial n} = 0 \text{ on } \partial\Omega_F(0). \end{cases} \tag{9.11}$$

By integrating the first and the third equations of (9.11) and by using the boundary conditions, we see that we need to impose the following compatibility conditions on f_1 and f_3:

$$\int_{\Omega_F(0)} f_1 \, dx = \int_{\Omega_F(0)} f_3 \, dx = 0.$$

We thus define

$$L^q_m(\Omega_F(0)) = \left\{ f \in L^q(\Omega_F(0)) \mid \int_{\Omega_F(0)} f \ dx = 0 \right\}, \tag{9.12}$$

and

$$\mathcal{X}_m = \left[W^{1,q}(\Omega_F(0)) \cap L^q_m(\Omega_F(0)) \right] \times L^q(\Omega_F(0))^3 \times L^q_m(\Omega_F(0)). \tag{9.13}$$

Since $\mathcal{X}_m$ is invariant under $(e^{tA_F})_{t \geqslant 0}$ the operator A_F may be restricted to $\mathcal{X}_m$. The *part of A_F in $\mathcal{X}_m$* is the restriction of A_F to the domain $\mathcal{D}(A_F) \cap \mathcal{X}_m$ ([30, Definition 2.4.1]).

Theorem 9.2. *The part of A_F in $\mathcal{X}_m$ is invertible in $\mathcal{X}_m$: for every $(f_1, f_2, f_3) \in \mathcal{X}_m$, the system* (9.11) *admits a unique solution $(\rho, u, \vartheta) \in \mathcal{D}(A_F) \cap \mathcal{X}_m$ satisfying*

$$\|\rho\|_{W^{1,q}(\Omega_F(0))} + \|u\|_{W^{2,q}(\Omega_F(0))} + \|\vartheta\|_{W^{2,q}(\Omega_F(0))}$$
$$\leqslant C(\|f_1\|_{W^{1,q}(\Omega_F(0))} + \|f_2\|_{L^q(\Omega_F(0))^3} + \|f_3\|_{L^q(\Omega_F(0))}). \tag{9.14}$$

Proof. Replacing $\operatorname{div} u = f_1 / \overline{\rho}$ in the third equation of (9.11) yields

$$-\frac{\kappa}{\overline{\rho} c_v} \Delta \vartheta = f_3 - \frac{R\overline{\vartheta}}{\overline{\rho} c_v} f_1 \text{ in } \Omega_F(0), \quad \frac{\partial \vartheta}{\partial n} = 0 \text{ on } \partial \Omega_F(0).$$

Since $f_1, f_3 \in L^q_m(\Omega_F(0))$, by the standard elliptic theory (see for instance [29, chapter 3]), the above system admits a unique solution $\vartheta \in W^{2,q}(\Omega_F(0)) \cap L^q_m(\Omega_F(0))$ and we have the estimate

$$\|\vartheta\|_{W^{2,q}(\Omega_F(0))} \leqslant C(\|f_1\|_{W^{1,q}(\Omega_F(0))} + \|f_3\|_{L^q(\Omega_F(0))}). \tag{9.15}$$

Then, we are reduced to solve the Stokes type system

$$\begin{cases} \overline{\rho} \operatorname{div} u = f_1, \text{ in } \Omega_F(0), \\ -\dfrac{\mu}{\overline{\rho}} \Delta u + \dfrac{R\overline{\vartheta}}{\overline{\rho}} \nabla \rho = \overline{f}_2, \text{ in } \Omega_F(0), \\ u = 0 \text{ on } \partial \Omega_F(0), \end{cases}$$

with $\overline{f}_2 = f_2 - R\nabla\vartheta + \frac{\alpha+\mu}{\overline{\rho}^2} \nabla f_1$. In particular, $\overline{f}_2 \in L^q(\Omega_F(0))^3$ with

$$\|\overline{f}_2\|_{L^q(\Omega_F(0))^3} \leqslant C(\|f_1\|_{W^{1,q}(\Omega_F(0))} + \|f_2\|_{L^q(\Omega_F(0))^3} + \|\vartheta\|_{W^{2,q}(\Omega_F(0))}). \tag{9.16}$$

From [13, Theorem 2.9(1)], the above system admits a unique solution

$$(\rho, u) \in \left[W^{1,q}(\Omega_F(0)) \cap L^q_m(\Omega_F(0)) \right] \times W^{2,q}(\Omega_F(0))^3$$

satisfying the estimate

$$\|\rho\|_{W^{1,q}(\Omega_F(0))} + \|u\|_{W^{2,q}(\Omega_F(0))} \leqslant C \left(\|f_1\|_{W^{1,q}(\Omega_F(0))} + \|\overline{f}_2\|_{L^q(\Omega_F(0))^3} \right). \tag{9.17}$$

The proof follows from the estimates (9.15) - (9.17). $\qquad\square$

9.2. **Rewriting (9.1) in an operator form.** Let us consider the following problem

$$
\begin{cases}
-\mu\Delta u_s + R\overline{\vartheta}\nabla\rho_s = 0 \text{ in } \Omega_F(0), \quad \operatorname{div} u_s = 0 \text{ in } \Omega_F(0), \\
u_s = \ell + \omega \times y \text{ on } \partial\Omega_S(0), \quad u_s = 0 \text{ on } \partial\Omega, \\
\displaystyle\int_{\Omega_F(0)} \rho_s \, dy = 0.
\end{cases}
\tag{9.18}
$$

Lemma 9.3. *Let $(\ell, \omega) \in \mathbb{C}^3 \times \mathbb{C}^3$ and let $\{e_i\}$ denote the canonical basis in $\mathbb{C}^3$. Then the solution (ρ_s, u_s) of (9.18) can be expressed as follows*

$$
\rho_s = \sum_{i=1}^{3} \ell_i P_i + \sum_{i=4}^{6} \omega_{i-3} P_i, \quad u_s = \sum_{i=1}^{3} \ell_i U_i + \sum_{i=4}^{6} \omega_{i-3} U_i,
\tag{9.19}
$$

where (U_i, P_i), $i = 1, 2, \cdots, 6$ solves the following systems

$$
\begin{cases}
-\mu\Delta U_i + R\overline{\vartheta}\nabla P_i = 0 \text{ in } \Omega_F(0), \\
\operatorname{div} U_i = 0 \quad \text{ in } \Omega_F(0), \\
U_i = 0, \quad \text{ on } \partial\Omega, \\
U_i = e_i \text{ on } \partial\Omega_F(0), \quad (i = 1, 2, 3), \\
U_i = e_{i-3} \times y \text{ on } \partial\Omega_F(0), \quad (i = 4, 5, 6), \\
\displaystyle\int_{\Omega_F(0)} P_i \, dy = 0.
\end{cases}
\tag{9.20}
$$

Moreover,

$$
\left(
\begin{matrix}
\displaystyle\int_{\partial\Omega_S(0)} \sigma_l(\rho_s, u_s, 0)n \, d\gamma \\
\displaystyle\int_{\partial\Omega_S(0)} y \times \sigma_l(\rho_s, u_s, 0)n \, d\gamma
\end{matrix}
\right) = \mathbb{A}\begin{pmatrix}\ell \\ \omega\end{pmatrix},
$$

where

$$
\mathbb{A}_{i,j} = \frac{2\mu}{\overline{\rho}}\int_{\Omega_F(0)} DU_i : DU_j \, dx.
\tag{9.21}
$$

Proof. See [16, Chapter 5]. $\qquad\square$

Let us set

$$
\mathcal{Z} = W^{1,q}(\Omega_F(0)) \times W^{2,q}(\Omega_F(0))^3 \times W^{2,q}(\Omega_F(0)) \text{ and } \mathcal{Y} = \mathcal{X} \times \mathbb{C}^3 \times \mathbb{C}^3.
$$

We introduce the Dirichlet operator $D_s \in \mathcal{L}(\mathbb{C}^3 \times \mathbb{C}^3; \mathcal{Z})$ defined by

$$
D_s\begin{pmatrix}\ell \\ \omega\end{pmatrix} = \begin{pmatrix}\rho_s \\ u_s \\ 0\end{pmatrix},
\tag{9.22}
$$

where (ρ_s, u_s) is the solution of the system (9.18). In view of Lemma 9.3, the operator D_s can also be defined as

$$
D_s\begin{pmatrix}\ell \\ \omega\end{pmatrix} = \begin{pmatrix}P_1 & P_2 & \cdots & P_6 \\ U_1 & U_2 & \cdots & U_6 \\ 0 & 0 & \cdots & 0\end{pmatrix}\begin{pmatrix}\ell \\ \omega\end{pmatrix},
\tag{9.23}
$$

where (U_i, P_i), $i = 1, 2, \cdots, 6$ are the solutions of the systems (9.20). Let us recall the operator $(A_F, \mathcal{D}(A_F; \mathcal{X}))$ introduced in Section 9.1. By Theorem 9.1 we know that, the operator A_F generates a C^0 semigroup on $\mathcal{X}$. It is well-known that, the operator A_F has an extension, also denoted by A_F, such that $A_F \in \mathcal{L}(\mathcal{X}, \mathcal{D}(A_F^*)')$, where A_F^* denotes the adjoint operator of A_F and $\mathcal{D}(A_F^*)'$ denotes the dual of $\mathcal{D}(A_F^*)$ (see [12, Chapter 2, Section 5]). Let us now briefly describe, how to rewrite system (9.1) as an evolution equation. All the details can be found in [21].

We introduce the operator $\mathcal{A}_{FS} : \mathcal{D}(\mathcal{A}_{FS}) \to \mathcal{Y}$ defined by

$$\mathcal{D}(\mathcal{A}_{FS}) = \left\{ (\rho, u, \vartheta, \ell, \omega) \in \mathcal{Z} \times \mathbb{C}^3 \times \mathbb{C}^3 \mid A_F \begin{pmatrix} \rho \\ u \\ \vartheta \end{pmatrix} - A_F D_s \begin{pmatrix} \ell \\ \omega \end{pmatrix} \in \mathcal{X} \right\},$$

$$\mathcal{A}_{FS} = \begin{pmatrix} A_F & -A_F D_s \\ C & 0 \end{pmatrix}, \tag{9.24}$$

where $C \in \mathcal{L}(\mathcal{Z}, \mathbb{C}^3 \times \mathbb{C}^3)$ is defined by

$$C \begin{pmatrix} \rho \\ u \\ \vartheta \end{pmatrix} = \begin{pmatrix} -m^{-1} \displaystyle\int_{\partial \Omega_S(0)} \sigma_l(\rho, u, \vartheta) n \ d\gamma \\ -J(0)^{-1} \displaystyle\int_{\partial \Omega_S(0)} y \times \sigma_l(\rho, u, \vartheta) n \ d\gamma \end{pmatrix}. \tag{9.25}$$

Proposition 9.4. *Let* $1 < p < \infty$ *and* $1 < q < \infty$. *Let* $\ell \in W^{1,p}(0, \infty; \mathbb{C}^3)$, $\omega \in W^{1,p}(0, \infty; \mathbb{C}^3)$, $\rho \in W^{1,p}(0, \infty; W^{1,q}(\Omega_F(0)))$, $u \in W_{q,p}^{2,1}(Q_F^\infty)^3$ *and* $\vartheta \in W_{q,p}^{2,1}(Q_F^\infty)$. *Then* $(\rho, v, \vartheta, \ell, \omega)$ *is a solution of the system* (9.1) *if and only if*

$$\frac{d}{dt} \begin{pmatrix} \rho \\ u \\ \vartheta \\ \ell \\ \omega \end{pmatrix} = \mathcal{A}_{FS} \begin{pmatrix} \rho \\ u \\ \vartheta \\ \ell \\ \omega \end{pmatrix} \quad in \ \mathcal{D}(A_F^*)' \times \mathbb{C}^3 \times \mathbb{C}^3, \quad \begin{pmatrix} \rho(0) \\ u(0) \\ \vartheta(0) \\ \ell(0) \\ \omega(0) \end{pmatrix} = \begin{pmatrix} \rho_0 \\ u_0 \\ \vartheta_0 \\ \ell_0 \\ \omega_0 \end{pmatrix}. \tag{9.26}$$

We skip the proof since it is standard. We end this subsection by recalling an equivalence of norms in $\mathcal{D}(\mathcal{A}_{FS})$ (see [21, Lemma 1.24]).

Lemma 9.5. *The map*

$$(\rho, u, \vartheta, \ell, \omega) \mapsto \|(\rho, u, \vartheta)\|_{\mathcal{Z}} + \|(\ell, \omega)\|_{\mathbb{C}^3 \times \mathbb{C}^3}$$

or equivalently the map

$$(\rho, u, \vartheta, \ell, \omega) \mapsto \|\rho\|_{W^{1,q}(\Omega_F(0))} + \|u\|_{W^{2,q}(\Omega_F(0))^3} + \|\vartheta\|_{W^{2,q}(\Omega_F(0))} + \|\ell\|_{\mathbb{C}^3} + \|\omega\|_{\mathbb{C}^3}$$

is a norm on $\mathcal{D}(\mathcal{A}_{FS})$ *equivalent to the graph norm.*

9.3. $\mathcal{R}$-**sectoriality of the operator** $\mathcal{A}_{FS}$. In this subsection we prove the following theorem

Theorem 9.6. *Let* $1 < q < \infty$. *Then there exists* $\gamma_3 > 0$ *such that* $\mathcal{A}_{FS} - \gamma_3$ *is an* $\mathcal{R}$-*sectorial operator in* $\mathcal{Y}$ *of angle* $< \pi/2$.

Proof. We write $\mathcal{A}_{FS}$ in the form $\mathcal{A}_{FS} = \mathcal{A}_{FS,1} + B_{FS}$ where

$$\mathcal{A}_{FS,1} = \begin{pmatrix} A_F & -A_F D_s \\ 0 & 0 \end{pmatrix}, \quad B_{FS} = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}.$$

Observe that
$$\lambda(\lambda I - \mathcal{A}_{FS,1})^{-1} = \begin{pmatrix} \lambda(\lambda I - A_F)^{-1} & -(\lambda I - A_F)^{-1}A_F D_s \\ 0 & I \end{pmatrix}.$$

Therefore by Theorem 9.1 and (8.1), there exists γ such that $\mathcal{A}_{FS,1} - \gamma$ is $\mathcal{R}$-sectorial.

Using standard trace result on Sobolev spaces (see, for instance [1, Theorem 5.36, p.164]) and the definition (9.2) of σ_l, we obtain that for any $s \in (1/q, 1)$ there exists a constant C such that

$$\|\sigma_l(\rho, u, \vartheta)\|_{L^q(\partial\Omega_S(0))} \leqslant C(\|\rho\|_{W^{s,q}(\Omega_F(0))} + \|u\|_{W^{1+s,q}(\Omega_F(0))} + \|\vartheta\|_{W^{s,q}(\Omega_F(0))}) \quad ((\rho, u, \vartheta) \in \mathcal{Z}).$$

Now, combining the above estimate with the definition (9.25) of the operator C, we deduce that that for any $s \in (1/q, 1)$ there exists a constant C such that for any $(\rho, u, \vartheta) \in \mathcal{Z}$,

$$\left\| C \begin{pmatrix} \rho \\ u \\ \vartheta \end{pmatrix} \right\|_{\mathbb{C}^3 \times \mathbb{C}^3} \leqslant C(\|\rho\|_{W^{s,q}(\Omega_F(0))} + \|u\|_{W^{1+s,q}(\Omega_F(0))} + \|\vartheta\|_{W^{s,q}(\Omega_F(0))}).$$

Since the embedding $W^{1,q}(\Omega_F(0)) \hookrightarrow W^{s,q}(\Omega_F(0))$ is compact for $s \in (1/q, 1)$, we obtain that for any $\delta > 0$ there exists $C(\delta) > 0$ such that

$$\left\| C \begin{pmatrix} \rho \\ u \\ \vartheta \end{pmatrix} \right\|_{\mathbb{C}^3 \times \mathbb{C}^3} \leqslant \delta \|(\rho, u, \vartheta)\|_{\mathcal{Z}} + C(\delta)\|(\rho, u, \vartheta)\|_{\mathcal{X}}.$$

By Lemma 9.5, this implies that for any $\delta > 0$ there exists $C(\delta) > 0$ such that

$$\left\| C \begin{pmatrix} \rho \\ u \\ \vartheta \end{pmatrix} \right\|_{\mathbb{C}^3 \times \mathbb{C}^3} \leqslant \delta \left\| \mathcal{A}_{FS,1} \begin{pmatrix} \rho \\ u \\ \vartheta \\ \ell \\ \omega \end{pmatrix} \right\|_{\mathcal{Y}} + C(\delta) \left\| \begin{pmatrix} \rho \\ u \\ \vartheta \\ \ell \\ \omega \end{pmatrix} \right\|_{\mathcal{Y}}.$$

Therefore the proof follows from Proposition 8.5.   □

9.4. **Exponential stability of the semigroup** $e^{t\mathcal{A}_{FS}}$**.** The aim of this subsection is to show the operator $\mathcal{A}_{FS}$ generates an analytic semigroup of negative type in a suitable subspace of $\mathcal{Y}$. Let us set
$$\mathcal{Y}_m = \mathcal{X}_m \times \mathbb{C}^3 \times \mathbb{C}^3,$$
where $\mathcal{X}_m$ is defined as in (9.13). One can check that the space $\mathcal{Y}_m$ is invariant under $(e^{t\mathcal{A}_{FS}})_{t\geqslant 0}$. We prove the following theorem

Theorem 9.7. *Let* $1 < q < \infty$. *Then the part of* $\mathcal{A}_{FS}$ *in* $\mathcal{Y}_m$ *generates an exponentially stable semigroup* $(e^{t\mathcal{A}_{FS}})_{t\geqslant 0}$ *on* $\mathcal{Y}_m$. *In other words, there exist constants* $C > 0$ *and* $\eta_0 > 0$ *such that*

$$\|e^{t\mathcal{A}_{FS}}(\rho_0, u_0, \vartheta_0, \ell_0, \omega_0)^\top\|_{\mathcal{Y}_m} \leqslant Ce^{-\eta_0 t}\|(\rho_0, u_0, \vartheta_0, \ell_0, \omega_0)^\top\|_{\mathcal{Y}_m}, \tag{9.27}$$

for all $(\rho_0, u_0, \vartheta_0, \ell_0, \omega_0)^\top \in \mathcal{Y}_m$.

We consider the following resolvent problem
$$\lambda\rho + \overline{\rho}\operatorname{div} u = f_1, \text{ in } \Omega_F(0),$$
$$\lambda u - \operatorname{div}\sigma_l(\rho, u, \vartheta) = f_2, \text{ in } \Omega_F(0),$$

$$\lambda\vartheta - \frac{\kappa}{\bar{\rho}c_v}\Delta\vartheta + \frac{R\bar{\vartheta}}{c_v}\operatorname{div}u = f_3, \text{ in } \Omega_F(0), \tag{9.28}$$

$$u = \ell + \omega \times y \text{ on } \partial\Omega_S(0) \quad u = 0 \text{ on } \partial\Omega, \quad \frac{\partial\vartheta}{\partial n} = 0 \text{ on } \partial\Omega_F(0),$$

$$\lambda\ell = -m^{-1}\int_{\partial\Omega_S(0)}\sigma_l(\rho, u, \vartheta)n \, d\gamma + g_1,$$

$$\lambda\omega = -J(0)^{-1}\int_{\partial\Omega_S(0)} y \times \sigma_l(\rho, u, \vartheta)n \, d\gamma + g_2.$$

We want to show that the set $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geqslant 0\}$, i.e. the entire right half plane is contained in the resolvent set of part of $\mathcal{A}_{FS}$ in $\mathcal{Y}_m$. This will be achieved in two steps. In the first step we show that 0 belongs to the resolvent of part of $\mathcal{A}_{FS}$ in $\mathcal{Y}_m$. In the second step, we show that the set $\{\lambda \in \mathbb{C} \setminus \{0\} \mid 0 \leqslant \operatorname{Re}\lambda\}$ is contained in the resolvent of part of $\mathcal{A}_{FS}$ in $\mathcal{Y}_m$.

Remark 9.8. *If $\lambda = 0$, integrating the first and third equations of* (9.28) *and using the boundary conditions of u and ϑ we obtain*

$$\int_{\Omega_F(0)} f_1 \, dy = \int_{\Omega_F(0)} f_3 \, dy = 0.$$

Therefore in order to show $\mathcal{A}_{FS}$ generates an exponentially stable semigroup it is necessary to consider $\mathcal{Y}_m$ instead of $\mathcal{Y}$.

Theorem 9.9. *Let $1 < q < \infty$ and $\lambda = 0$. Then for every $(f_1, f_2, f_3, g_1, g_2) \in \mathcal{Y}_m$ the system* (9.28) *admits a unique solution $(\rho, u, \vartheta, \ell, \omega) \in \mathcal{D}(\mathcal{A}_{FS}) \cap \mathcal{Y}_m$ satisfying the estimate*

$$\|\rho\|_{W^{1,q}(\Omega_F(0))} + \|u\|_{W^{2,q}(\Omega_F(0))} + \|\vartheta\|_{W^{2,q}(\Omega_F(0))} + \|\ell\|_{\mathbb{C}^3} + \|\omega\|_{\mathbb{C}^3} \leqslant C\|(f_1, f_2, f_3, g_1, g_2)\|_{\mathcal{Y}_m}. \tag{9.29}$$

Proof. When $\lambda = 0$ it is easy to see that (9.28) is equivalent to

$$\mathcal{A}_{FS}(\rho, u, \vartheta, \ell, \omega)^\top = (f_1, f_2, f_3, g_1, g_2)^\top.$$

Thus to prove the theorem, we first show that the operator $\mathcal{A}_{FS}$ is invertible. One can easily check that, if the operators A_F and CD_s are invertible then the operator $\mathcal{A}_{FS}$ is invertible and its inverse is given by the formula

$$\mathcal{A}_{FS}^{-1} = \begin{pmatrix} A_F^{-1} - D_s(CD_s)^{-1}CA_F^{-1} & D_s(CD_s)^{-1} \\ -(CD_s)^{-1}CA_F^{-1} & (CD_s)^{-1} \end{pmatrix}.$$

We know that A_F is invertible on $\mathcal{X}_m$ (Theorem 9.2). Thus to complete the proof we need to verify that CD_s is an invertible matrix. From Lemma 9.3, we can see that

$$CD_s = -\mathbb{M}^{-1}\mathbb{A} \quad \text{where} \quad \mathbb{M} = \begin{pmatrix} mI & 0 \\ 0 & J(0) \end{pmatrix}. \tag{9.30}$$

Since the matrix $\mathbb{A}$ is self-adjoint and positive, we deduce the result. $\qquad\square$

Theorem 9.10. *Assume $1 < q < \infty$ and $\lambda \in \mathbb{C} \setminus \{0\}$, with $\operatorname{Re}\lambda \geqslant 0$. Then for any $(f_1, f_2, f_3, g_1, g_2) \in \mathcal{Y}_m$, the system* (9.28) *admits a unique solution $(\rho, u, \vartheta, \ell, \omega) \in \mathcal{D}(\mathcal{A}_{FS}) \cap \mathcal{Y}_m$ satisfying the estimate*

$$\|\rho\|_{W^{1,q}(\Omega_F(0))} + \|u\|_{W^{2,q}(\Omega_F(0))} + \|\vartheta\|_{W^{2,q}(\Omega_F(0))} + \|\ell\|_{\mathbb{C}^3} + \|\omega\|_{\mathbb{C}^3}$$
$$\leqslant C\|(f_1, f_2, f_3, g_1, g_2)\|_{\mathcal{Y}_m}. \tag{9.31}$$

Proof. Let us fix $\lambda \in \mathbb{C} \setminus \{0\}$, with $\operatorname{Re} \lambda \geqslant 0$. By setting $\rho = \dfrac{1}{\lambda}(f_1 - \overline{\rho} \operatorname{div} u)$, the system (9.28) can be rewritten as

$$\begin{cases} \lambda u - \operatorname{div} \widehat{\sigma}_\lambda(u, \vartheta) = \widehat{f}_2, \ \text{in } \Omega_F(0), \\ \lambda \vartheta - \dfrac{\kappa}{\overline{\rho} c_v} \Delta \vartheta + \dfrac{R\overline{\vartheta}}{c_v} \operatorname{div} u = f_3, \ \text{in } \Omega_F(0), \\ u = \ell + \omega \times y \ \text{on } \partial\Omega_S(0) \quad u = 0 \ \text{on } \partial\Omega, \quad \dfrac{\partial \vartheta}{\partial n} = 0 \ \text{on } \partial\Omega_F(0), \\ \lambda \ell = -m^{-1} \displaystyle\int_{\partial\Omega_S(0)} \widehat{\sigma}_\lambda(u, \vartheta) \ n \ d\gamma + \widehat{g}_1 \\ \lambda \omega = -J(0)^{-1} \displaystyle\int_{\partial\Omega_S(0)} y \times \widehat{\sigma}_\lambda(u, \vartheta) n \ d\gamma + \widehat{g}_2 \end{cases} \tag{9.32}$$

where

$$\widehat{\sigma}_\lambda(u, \vartheta) = \frac{2\mu}{\overline{\rho}} D(u) + \left(\frac{1}{\overline{\rho}} \left(\alpha + \frac{R\overline{\vartheta}\overline{\rho}}{\lambda} \right) \operatorname{div} u - R\vartheta \right) I_3,$$

$$\widehat{f}_2 = f_2 - \frac{R\overline{\vartheta}}{\lambda \overline{\rho}} \nabla f_1, \quad \widehat{g}_1 = \left(g_1 + m^{-1} \frac{R\overline{\vartheta}}{\lambda \overline{\rho}} \int_{\partial\Omega_S(0)} f_1 n \ d\gamma \right),$$

$$\widehat{g}_2 = \left(g_2 + J(0)^{-1} \frac{R\overline{\vartheta}}{\lambda \overline{\rho}} \int_{\partial\Omega_S(0)} y \times f_1 n \ d\gamma \right).$$

If $(f_1, f_2, f_3, g_1, g_2) \in \mathcal{Y}_m$, the above formulas imply that $(\widehat{f}_2, \widehat{g}_1, \widehat{g}_2) \in L^q(\Omega_F(0))^3 \times \mathbb{C}^3 \times \mathbb{C}^3$. We introduce the following notation:

$$\widehat{\mathcal{Z}} = W^{2,q}(\Omega_F(0))^3 \times \left(W^{2,q}(\Omega_F(0)) \cap L_m^q(\Omega_F(0)) \right), \quad \widehat{\mathcal{X}} = L^q(\Omega_F(0))^3 \times L_m^q(\Omega_F(0))$$

$$\widehat{\mathcal{Y}} = \widehat{\mathcal{X}} \times \mathbb{C}^3 \times \mathbb{C}^3.$$

- $\widehat{D}_s \in \mathcal{L}(\mathbb{C}^3 \times \mathbb{C}^3; \widehat{\mathcal{Z}})$ and $\widehat{E}_s \in \mathcal{L}(\mathbb{C}^3 \times \mathbb{C}^3; \widehat{\mathcal{Z}})$ defined by

$$\widehat{D}_s \begin{pmatrix} \ell \\ \omega \end{pmatrix} = \begin{pmatrix} u_s \\ 0 \end{pmatrix}, \quad \widehat{E}_s \begin{pmatrix} \ell \\ \omega \end{pmatrix} = \begin{pmatrix} \dfrac{R\overline{\vartheta}}{\overline{\rho}} \nabla \rho_s \\ 0 \end{pmatrix},$$

  where (ρ_s, u_s) is the solution of the system (9.18).
- $\widehat{A}_\lambda$ defined by

$$\mathcal{D}(\widehat{A}_\lambda) = \left\{ (u, \vartheta) \in \widehat{\mathcal{Z}} \mid u = \frac{\partial \vartheta}{\partial n} = 0 \ \text{on } \partial\Omega_F(0) \right\}$$

$$\widehat{A}_\lambda \begin{pmatrix} u \\ \vartheta \end{pmatrix} = \begin{pmatrix} \operatorname{div} \widehat{\sigma}_\lambda(u, \vartheta) \\ \dfrac{\kappa}{\overline{\rho} c_v} \Delta \vartheta - \dfrac{R\overline{\vartheta}}{c_v} \operatorname{div} u \end{pmatrix}, \quad \begin{pmatrix} u \\ \vartheta \end{pmatrix} \in \mathcal{D}(\widehat{A}_\lambda).$$

- $\widehat{C}_\lambda \in \mathcal{L}(\mathcal{Z}, \mathbb{C}^3 \times \mathbb{C}^3)$ defined by

$$\widehat{C}_\lambda \begin{pmatrix} u \\ \vartheta \end{pmatrix} = \begin{pmatrix} -m^{-1} \displaystyle\int_{\partial\Omega_S(0)} \widehat{\sigma}_\lambda(u, \vartheta) n \ d\gamma \\ -J(0)^{-1} \displaystyle\int_{\partial\Omega_S(0)} y \times \widehat{\sigma}_\lambda(u, \vartheta) n \ d\gamma \end{pmatrix}.$$

- $\mathcal{A}_\lambda$ defined by

$$\mathcal{D}(\mathcal{A}_\lambda) = \left\{ (u, \vartheta, \ell, \omega) \in \widehat{\mathcal{Z}} \times \mathbb{C}^3 \times \mathbb{C}^3 \mid \widehat{A}_\lambda \begin{pmatrix} u \\ \vartheta \end{pmatrix} - \widehat{A}_\lambda \widehat{D}_s \begin{pmatrix} \ell \\ \omega \end{pmatrix} \in \widehat{\mathcal{X}} \right\},$$

$$\mathcal{A}_\lambda = \begin{pmatrix} \widehat{A}_\lambda & -\widehat{A}_\lambda \widehat{D}_s + \widehat{E}_s \\ \widehat{C}_\lambda & 0 \end{pmatrix}.$$

With the above notation, we can write (9.32) as

$$(\lambda I - \mathcal{A}_\lambda) \begin{pmatrix} u \\ \vartheta \\ \ell \\ \omega \end{pmatrix} = \begin{pmatrix} \widehat{f}_2 \\ f_3 \\ \widehat{g}_1 \\ \widehat{g}_2 \end{pmatrix}. \tag{9.33}$$

Proceeding as in Theorem 9.6, one can show that there exists $\widetilde{\lambda} > 0$ such that $\widetilde{\lambda} I - \mathcal{A}_\lambda$ is invertible. Consequently, we write (9.33) as

$$\left[I + (\lambda - \widetilde{\lambda})(\widetilde{\lambda} I - \mathcal{A}_\lambda)^{-1} \right] \begin{pmatrix} u \\ \vartheta \\ \ell \\ \omega \end{pmatrix} = (\widetilde{\lambda} I - \mathcal{A}_\lambda)^{-1} \begin{pmatrix} \widehat{f}_2 \\ f_3 \\ \widehat{g}_1 \\ \widehat{g}_2 \end{pmatrix} \tag{9.34}$$

and since $(\widetilde{\lambda} I - \mathcal{A}_\lambda)^{-1}$ is a compact operator, in view of the Fredholm alternative theorem, the existence and the uniqueness of a solution of system (9.34) are equivalent.

Assume $(u, \vartheta, \ell, \omega) \in \mathcal{D}(\mathcal{A}_\lambda)$ satisfies

$$(\lambda I - \mathcal{A}_\lambda)(u, \ \vartheta, \ \ell, \ \omega)^\top = 0. \tag{9.35}$$

We first show that $(u, \vartheta) \in W^{2,2}(\Omega_F(0))^3 \times W^{2,2}(\Omega_F(0))$. If $q \geqslant 2$, this is a consequence of Hölder's estimates. Assume $1 < q < 2$. In that case, we can write (9.35) as

$$(\widetilde{\lambda} I - \mathcal{A}_\lambda)(u, \ \vartheta, \ \ell, \ \omega)^\top = (\widetilde{\lambda} - \lambda)(u, \ \vartheta, \ \ell, \ \omega)^\top \tag{9.36}$$

and since $W^{2,q}(\Omega_F(0)) \hookrightarrow L^2(\Omega_F(0))$, we deduce from the invertibility of $\widetilde{\lambda} I - \mathcal{A}_\lambda$ that $(u, \vartheta) \in W^{2,2}(\Omega_F(0))^3 \times W^{2,2}(\Omega_F(0))$. We then rewrite (9.35) as

$$\begin{cases} \lambda u - \operatorname{div} \widehat{\sigma}_\lambda(u, \vartheta) = 0, \ \text{in } \Omega_F(0), \\[2mm] \lambda \vartheta - \dfrac{\kappa}{\overline{\rho} c_v} \Delta \vartheta + \dfrac{R \overline{\vartheta}}{c_v} \operatorname{div} u = 0, \ \text{in } \Omega_F(0), \\[2mm] u = \ell + \omega \times y \text{ on } \partial \Omega_S(0) \quad u = 0 \text{ on } \partial \Omega, \quad \dfrac{\partial \vartheta}{\partial n} = 0 \text{ on } \partial \Omega_F(0), \\[2mm] \lambda \ell = -m^{-1} \displaystyle\int_{\partial \Omega_S(0)} \widehat{\sigma}_\lambda(u, \vartheta) \ n \ d\gamma \\[2mm] \lambda \omega = -J(0)^{-1} \displaystyle\int_{\partial \Omega_S(0)} y \times \widehat{\sigma}_\lambda(u, \vartheta) n \ d\gamma. \end{cases} \tag{9.37}$$

Multiplying the first equation of (9.37) by $\overline{u}$, the second equation by $\overline{\vartheta}$, the forth equation by $\overline{\ell}$ and the fifth equation by $\overline{\omega}$, we deduce after integration by parts,

$$\operatorname{Re} \lambda \int_{\Omega_F(0)} |u|^2 \ dy + \frac{2\mu}{\overline{\rho}} \int_{\Omega_F(0)} |D(u)|^2 \ dy + \left(\frac{\alpha}{\overline{\rho}} + \frac{R \overline{\vartheta} \ \operatorname{Re} \lambda}{|\lambda|^2} \right) \int_{\Omega_F(0)} |\operatorname{div} u|^2 \ dy$$

$$+ \frac{\operatorname{Re}\lambda\, c_v}{\overline{\vartheta}} \int_{\Omega_F(0)} |\vartheta|^2 \, dy + \frac{\kappa}{\overline{\rho}\,\overline{\vartheta}} \int_{\Omega_F(0)} |\nabla\vartheta|^2 \, dy + \operatorname{Re}\lambda m|\ell|^2 + \operatorname{Re}(\lambda J(0)\omega \cdot \overline{\omega}) = 0. \quad (9.38)$$

Using

$$|\operatorname{div} u|^2 \leqslant 3|D(u)|^2,$$

$\operatorname{Re}\lambda \geqslant 0$ and (2.3), we obtain

$$\int_{\Omega_F(0)} |D(u)|^2 \, dy + \int_{\Omega_F(0)} |\nabla\vartheta|^2 \, dy \leqslant 0.$$

The above estimate and the fact that $(u, \vartheta, \ell, \omega) \in \mathcal{D}(\mathcal{A}_\lambda)$ imply that $u = \vartheta = \ell = \omega = 0$. $\square$

Proof of Theorem 9.7. By virtue of Theorem 9.9 and Theorem 9.10, we deduce

$$\{\lambda \in \mathbb{C} \; ; \; \operatorname{Re}\lambda \geqslant 0\} \subset \rho(\mathcal{A}_{FS}).$$

Moreover, Theorem 9.6 yields the existence of $C_1 > 0$ such that for any $\lambda \in \gamma_3 + \Sigma_{\pi-\beta_3}$ with $\beta_3 < \pi/2$,

$$\|(\lambda - \mathcal{A}_{FS})^{-1}\|_{\mathcal{L}(\mathcal{Y}_m)} \leqslant C_1.$$

Since

$$\{\lambda \in \mathbb{C} \; ; \; \operatorname{Re}\lambda \geqslant 0\} \setminus [\gamma_3 + \Sigma_{\pi-\beta_3}]$$

is a compact set, we deduce the existence of $C > 0$ such that for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geqslant 0$

$$\|(\lambda - \mathcal{A}_{FS})^{-1}\|_{\mathcal{L}(\mathcal{Y}_m)} \leqslant C.$$

This yields that

$$\{\lambda \in \mathbb{C} \; ; \; \operatorname{Re}\lambda \geqslant -\eta\} \subset \rho(\mathcal{A}_{FS}),$$

for some $\eta > 0$. Applying standard results on analytic semigroups (see, for instance, Proposition 2.9 in [4, p.120]), we deduce the exponential stability of the semigroup generated by the part of $\mathcal{A}_{FS}$ in $\mathcal{Y}_m$. $\square$

10. Maximal L^p-L^q Regularity for the Linearized Fluid-Structure System

In this section, we study the maximal L^p-L^q regularity of the system (9.1) with source terms and boundary terms. More precisely, we consider the following system

$$
\begin{aligned}
&\partial_t \widetilde{\rho} + \overline{\rho}\operatorname{div}\widetilde{u} = f_1 && \text{in } (0, \infty) \times \Omega_F(0), \\
&\partial_t \widetilde{u} - \operatorname{div}\sigma_l(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}) = f_2 && \text{in } (0, \infty) \times \Omega_F(0), \\
&\partial_t \widetilde{\vartheta} - \frac{\kappa}{\overline{\rho}c_v}\Delta\widetilde{\vartheta} + \frac{R\overline{\vartheta}}{c_v}\operatorname{div}\widetilde{u} = f_3 && \text{in } (0, \infty) \times \Omega_F(0), \\
&\widetilde{u} = 0 && \text{on } (0, \infty) \times \partial\Omega, \\
&\widetilde{u} = \widetilde{\ell} + \widetilde{\omega} \times y && \text{on } (0, \infty) \times \partial\Omega_S(0) \qquad (10.1)\\
&\frac{\partial\widetilde{\vartheta}}{\partial n} = h && \text{on } (0, \infty) \times \partial\Omega_F(0), \\
&\frac{d}{dt}\widetilde{\ell} = -m^{-1}\int_{\partial\Omega_S(0)} \sigma_l(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta})n \, d\gamma + g_1 && t \in (0, \infty), \\
&\frac{d}{dt}\widetilde{\omega} = -J(0)^{-1}\int_{\partial\Omega_S(0)} y \times \sigma_l(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta})n \, d\gamma + g_2 && t \in (0, \infty) \\
&\widetilde{\rho}(0) = \rho_0 - \overline{\rho}, \quad \widetilde{u}(0) = u_0, \quad \widetilde{\vartheta}(0) = \vartheta_0 - \overline{\vartheta} && \text{in } \Omega_F(0),
\end{aligned}
$$

$$\widetilde{\ell}(0) = \ell_0, \quad \widetilde{\omega}(0) = \omega_0.$$

We want to combine Corollary 8.4 and Theorem 9.7. However the latter is stated in $\mathcal{Y}_m$, that is with the constraints that some quantities have to be with zero mean-value. As a consequence, we introduce the following standard decomposition: for any $f \in L^1(\Omega_F(0))$,

$$f = f_m + f_{avg}, \quad \text{with} \quad \int_{\Omega_F(0)} f_m \, dy = 0, \quad f_{avg} = |\Omega_F(0)|^{-1} \int_{\Omega_F(0)} f(y) \, dy. \qquad (10.2)$$

We use the same decomposition and the same notation for a function in $L^1(\partial\Omega_F(0))$. The next result is the main result of this section. We recall that $\mathcal{B}_{\infty,p,q}$ and $\mathcal{S}_{\infty,p,q}$ are defined in (3.2) and (3.3).

Assume $1 < p < \infty$ and $1 < q < \infty$ satisfying the conditions

$$\frac{1}{p} + \frac{1}{2q} \neq 1, \quad \frac{1}{p} + \frac{1}{2q} \neq \frac{1}{2}. \qquad (10.3)$$

We set

$$\mathcal{J}_{p,q} = \Big\{ (\rho_0, u_0, \vartheta_0, \ell_0, \omega_0) \mid \rho_0 \in W^{1,q}(\Omega_F(0)) \cap L_m^q(\Omega_F(0)), \; u_0 \in B_{q,p}^{2(1-1/p)}(\Omega_F(0))^3,$$

$$\vartheta_0 \in B_{q,p}^{2(1-1/p)}(\Omega_F(0)), \; \ell_0 \in \mathbb{R}^3, \; \omega_0 \in \mathbb{R}^3 \Big\}, \quad (10.4)$$

and we introduce the space of initial data

$$\mathcal{J}_{p,q}^{cc} = \mathcal{J}_{p,q} \quad \text{if} \quad \frac{1}{p} + \frac{1}{2q} \geqslant 1,$$

$$\mathcal{J}_{p,q}^{cc} = \Big\{ (\rho_0, u_0, \vartheta_0, \ell_0, \omega_0) \in \mathcal{J}_{p,q} \mid u_0 = 0 \text{ on } \partial\Omega, \; u_0(y) = \ell_0 + \omega_0 \times y \quad y \in \partial\Omega_S(0) \Big\}$$

$$\text{if} \quad \frac{1}{2} < \frac{1}{p} + \frac{1}{2q} < 1,$$

$$\mathcal{J}_{p,q}^{cc} = \Big\{ (\rho_0, u_0, \vartheta_0, \ell_0, \omega_0) \in \mathcal{J}_{p,q} \mid u_0 = 0 \text{ on } \partial\Omega, \; u_0(y) = \ell_0 + \omega_0 \times y \quad y \in \partial\Omega_S(0),$$

$$\frac{\partial\vartheta_0}{\partial n} = 0 \text{ on } \partial\Omega_F(0) \Big\} \quad \text{if} \quad \frac{1}{p} + \frac{1}{2q} < \frac{1}{2}.$$

The above definition is well-defined due to the trace theorem for Besov spaces (see, for instance [28, p.200]).

Theorem 10.1. *Let $1 < p < \infty$ and $1 < q < \infty$ satisfying (10.3). Let $\overline{\rho} > 0$ and $\overline{\vartheta} > 0$ and $\eta \in (0, \eta_0)$, where η_0 is the constant introduced in Theorem 9.7. Then for any*

$$(\rho_0 - \overline{\rho}, u_0, \vartheta_0 - \overline{\vartheta}, \ell_0, \omega_0) \in \mathcal{J}_{p,q}^{cc}$$

and for any $(f_1, f_2, f_3, h, g_1, g_2) \in e^{-\eta(\cdot)}\mathcal{B}_{\infty,p,q}$ with $(f_{1,avg}, f_{3,avg}, h_{avg}) \in L^1(0,\infty)^3$, the system (10.1) admits a unique solution $(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega})$ with

$$(\widetilde{\rho}_m, \widetilde{u}, \widetilde{\vartheta}_m, \widetilde{\ell}, \widetilde{\omega}) \in e^{-\eta(\cdot)}\mathcal{S}_{\infty,p,q}, \quad (\widetilde{\rho}_{avg}, \widetilde{\vartheta}_{avg}) \in L^\infty(0,\infty)^2. \qquad (10.5)$$

Moreover, there exists a positive constant C_L depending only on p, q and η such that

$$\left\| e^{\eta(\cdot)}(\widetilde{\rho}_m, \widetilde{u}, \widetilde{\vartheta}_m, \widetilde{\ell}, \widetilde{\omega}) \right\|_{\mathcal{S}_{\infty,p,q}} + \left\| (\widetilde{\rho}_{avg}, \widetilde{\vartheta}_{avg}) \right\|_{L^\infty(0,\infty)^2} + \left\| e^{\eta(\cdot)}(\partial_t\widetilde{\rho}_{avg}, \partial_t\widetilde{\vartheta}_{avg}) \right\|_{L^p(0,\infty)^2}$$

$$\leqslant C_L \Big(\big\|(\rho_0 - \overline{\rho}, u_0, \vartheta_0 - \overline{\vartheta}, \ell_0, \omega_0)\big\|_{\mathcal{J}_{p,q}} + \big\|e^{\eta(\cdot)}(f_1, f_2, f_3, h, g_1, g_2)\big\|_{\mathcal{B}_{\infty,p,q}}$$

$$+ \|(f_{1,avg}, f_{3,avg}, h_{avg})\|_{L^1(0,\infty)^3} \Big). \quad (10.6)$$

Proof. Let us first consider the case $\eta = 0$. We consider the following heat equation

$$\partial_t \varphi^1 + \mu' \varphi^1 - \frac{\kappa}{\overline{\rho} c_v} \Delta \varphi^1 = f_3 - f_{3,avg} - \frac{\kappa |\partial \Omega_F(0)|}{\overline{\rho} c_v |\Omega_F(0)|} h_{avg} \quad \text{in } (0,\infty) \times \Omega_F(0)$$

$$\frac{\partial \varphi^1}{\partial n} = h \qquad \text{on } (0,\infty) \times \partial \Omega_F(0),$$

$$\varphi^1(0) = \vartheta_0 - \overline{\vartheta} \qquad \text{on } \Omega_F(0),$$

with $\mu' > 0$. Using $(f_1, f_2, f_3, h, g_1, g_2) \in \mathcal{B}_{\infty,p,q}$ and applying Proposition 6.4 in [9] (taking $\mu' > 0$ large enough), we deduce that the above system admits a unique solution $\varphi^1 \in W^{2,1}_{q,p}(Q^F_\infty)$. Moreover, we have the estimate

$$\|\varphi^1\|_{W^{2,1}_{q,p}(Q^F_\infty)} \leqslant C \Big(\|f_3\|_{L^p(0,\infty;L^q(\Omega_F(0)))} + \|h\|_{F^{(1-1/q)/2}_{p,q}(0,\infty;L^q(\partial\Omega_F(0)))\cap L^p(0,\infty;W^{1-1/q,q}(\partial\Omega_F(0)))}$$

$$+ \|\vartheta_0 - \overline{\vartheta}\|_{B^{2(1-1/p)}_{q,p}(\Omega_F(0))} \Big). \quad (10.7)$$

Standard calculation on the above system yields

$$\partial_t \varphi^1_{avg} + \mu' \varphi^1_{avg} = 0, \quad \varphi^1_{avg}(0) = \vartheta_{0,avg} - \overline{\vartheta}.$$

Thus $\varphi^1_{avg}(t) = (\vartheta_{0,avg} - \overline{\vartheta})e^{-\mu' t}$ and $\varphi^1_{avg} \in L^r(0,\infty)$ for any $1 \leqslant r \leqslant \infty$. Next, we define

$$\varphi^2(t) = \int_0^t \left(f_{3,avg}(s) + \frac{\kappa |\partial \Omega_F(0)|}{\overline{\rho} c_v |\Omega_F(0)|} h_{avg}(s) + \mu' \varphi^1_{avg}(s) \right) ds.$$

Since $f_{3,avg}, h_{avg}, \varphi^1_{avg} \in L^1(0,\infty)$, we obtain $\varphi^2 \in L^\infty(0,\infty)$.

Integrating the first equation of (10.1) in $\Omega_F(0)$ and using the boundary condition of u, we deduce that $\widetilde{\rho}_{avg}$ is solution of the following system

$$\partial_t \widetilde{\rho}_{avg} = f_{1,avg} \quad t \in (0,\infty), \quad \widetilde{\rho}_{avg}(0) = 0. \quad (10.8)$$

As $f_{1,avg}$ belongs to $L^1(0,\infty)$ we have $\widetilde{\rho}_{avg} \in L^\infty(0,\infty)$. We set

$$\widetilde{\varphi} = \widetilde{\vartheta} - \varphi^1 - \varphi^2.$$

Then system (10.1) is transformed into the following system for $(\widetilde{\rho}_m, \widetilde{u}, \widetilde{\varphi}, \widetilde{\ell}, \widetilde{\omega})$:

$$\partial_t \widetilde{\rho}_m + \overline{\rho} \operatorname{div} \widetilde{u} = \widetilde{f}_1 \quad \text{in } (0,\infty) \times \Omega_F(0),$$

$$\partial_t \widetilde{u} - \operatorname{div} \sigma_l(\widetilde{\rho}_m, \widetilde{u}, \widetilde{\varphi}) = \widetilde{f}_2 \quad \text{in } (0,\infty) \times \Omega_F(0),$$

$$\partial_t \widetilde{\varphi} - \frac{\kappa}{\overline{\rho} c_v} \Delta \widetilde{\varphi} + \frac{R\overline{\vartheta}}{c_v} \operatorname{div} \widetilde{u} = \widetilde{f}_3 \quad \text{in } (0,\infty) \times \Omega_F(0),$$

$$\widetilde{u} = 0 \text{ on } (0,\infty) \times \partial\Omega, \quad \widetilde{u} = \widetilde{\ell} + \widetilde{\omega} \times y \text{ on } (0,\infty) \times \partial\Omega_S(0) \quad (10.9)$$

$$\frac{\partial \widetilde{\varphi}}{\partial n} = 0 \text{ on } (0,\infty) \times \partial\Omega_F(0),$$

$$\frac{d}{dt} \widetilde{\ell} = -m^{-1} \int_{\partial\Omega_S(0)} \sigma_l(\widetilde{\rho}_m, \widetilde{u}, \widetilde{\varphi}) n \, d\gamma + \widetilde{g}_1, \quad t \in (0,\infty)$$

$$\frac{d}{dt}\widetilde{\omega} = -J(0)^{-1}\int_{\partial\Omega_S(0)} y \times \sigma_l(\widetilde{\rho}_m, \widetilde{u}, \widetilde{\varphi})n \ d\gamma + \widetilde{g}_2, \quad t \in (0, \infty)$$

$$\widetilde{\rho}_m(0) = \rho_0 - \overline{\rho}, \quad \widetilde{u}(0) = u_0, \quad \widetilde{\varphi}(0) = 0, \quad \text{in } \Omega_F(0),$$

$$\widetilde{\ell}(0) = \ell_0, \quad \widetilde{\omega}(0) = \omega_0,$$

where

$$\widetilde{f}_1 = f_1 - f_{1,avg}, \quad \widetilde{f}_2 = f_2 - R\nabla\varphi^1, \quad \widetilde{f}_3 = \mu'(\varphi^1 - \varphi^1_{avg})$$

$$\widetilde{g}_1 = g_1 + m^{-1}R\int_{\partial\Omega_S(0)} \varphi^1 n \ d\gamma, \quad \widetilde{g}_2 = g_2 + J(0)^{-1}R\int_{\partial\Omega_S(0)} y \times \varphi^1 n \ d\gamma,$$

and where we recall that σ_l is defined by (9.2). Using that $(f_1, f_2, f_3, h, g_1, g_2) \in \mathcal{B}_{\infty,p,q}$ and that $\varphi^1 \in W^{2,1}_{q,p}(Q^F_\infty)$, one can show that $(\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3, \widetilde{g}_1, \widetilde{g}_2)$ belongs to $L^p(0, \infty; \mathcal{Y}_m)$.

From Theorem 9.6 and Theorem 9.7, we know that $\mathcal{A}_{FS}$ generates an analytic exponential stable semigroup on $\mathcal{Y}_m$ and is a $\mathcal{R}$-sectorial operator on $\mathcal{Y}_m$. Moreover, by hypothesis of Theorem 10.1, we have $(\rho_0 - \overline{\rho}, u_0, 0, \ell_0, \omega_0) \in (\mathcal{Y}_m, \mathcal{D}(\mathcal{A}_{FS}) \cap \mathcal{Y}_m)_{1-1/p,p}$. Therefore by Corollary 8.4, the system (10.9) admits a unique solution

$$(\widetilde{\rho}_m, \widetilde{u}, \widetilde{\varphi}, \widetilde{\ell}, \widetilde{\omega}) \in L^p(0, \infty; \mathcal{D}(\mathcal{A}_{FS}) \cap \mathcal{Y}_m) \cap W^{1,p}(0, \infty; \mathcal{Y}_m) \subset \mathcal{S}_{\infty,p,q}.$$

We recover (10.5) and (10.6) by remarking that

$$\widetilde{\vartheta}_m = \widetilde{\varphi} + \varphi^1_m, \quad \widetilde{\vartheta}_{avg} = \varphi^2 + \varphi^1_{avg}.$$

The case $\eta > 0$ can be reduced to the previous case by multiplying all the functions by $e^{\eta t}$ and using the fact that $\mathcal{A}_{FS} + \eta$ is a $\mathcal{R}$-sectorial operator with negative type. $\qquad\square$

11. Estimates of the Nonlinear Terms

In this section, we are going to estimate the nonlinear terms $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{H}, \mathcal{G}_1$ and $\mathcal{G}_2$ defined in (7.9) - (7.14).

Throughout this section we assume $2 < p < \infty$ and $3 < q < \infty$ satisfy $\frac{1}{p} + \frac{1}{2q} \neq \frac{1}{2}$. Let p' denote the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. In the following, when no confusion is possible, we use the notation

$$\|\cdot\|_{W^{r,p}(0,T;W^{s,q})} = \|\cdot\|_{W^{r,p}(0,T;W^{s,q}(\Omega_F(0)))}.$$

Let us fix $\eta \in (0, \eta_0)$, where η_0 is the constant introduced in Theorem 9.7. and we introduce the following ball

$$\widetilde{\mathcal{S}}_\varepsilon = \left\{ (\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) \mid \left\|(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega})\right\|_{\mathcal{S}} \leqslant \varepsilon \right\},$$

where

$$\left\|(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega})\right\|_{\mathcal{S}} = \left\|(e^{\eta(\cdot)}\widetilde{\rho}_m, e^{\eta(\cdot)}\widetilde{u}, e^{\eta(\cdot)}\widetilde{\vartheta}_m, e^{\eta(\cdot)}\widetilde{\ell}, e^{\eta(\cdot)}\widetilde{\omega})\right\|_{\mathcal{S}_{\infty,p,q}}$$

$$+ \|\widetilde{\rho}_{avg}, \widetilde{\vartheta}_{avg}\|_{L^\infty(0,\infty)^2} + \|e^{\eta(\cdot)}\partial_t\widetilde{\rho}_{avg}, e^{\eta(\cdot)}\partial_t\widetilde{\vartheta}_{avg}\|_{L^p(0,\infty)^2}, \quad (11.1)$$

and where we use the notation (10.2).

Let us first show that X be defined as in (7.7) is a C^1-diffeomorphism.

Lemma 11.1. *Let X be defined as in* (7.7)*. Then there exists a constant $C_X > 0$, depending only on p, q, η and $\Omega_F(0)$ such that, for every $(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) \in \widetilde{\mathcal{S}}_\varepsilon$, we have*

$$\|\nabla X - I_3\|_{L^\infty((0,\infty)\times\Omega_F(0))} + \|\nabla X - I_3\|_{L^\infty(0,\infty;W^{1,q})} \leqslant C_X\varepsilon, \tag{11.2}$$

$$|X(\cdot, y_1) - X(\cdot, y_2)| \geqslant (1 - C_X\varepsilon)|y_1 - y_2| \quad (y_1, y_2 \in \Omega_F(0)). \tag{11.3}$$

In particular for every $\varepsilon \in (0, \frac{1}{2C_X})$ and for every $(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) \in \widetilde{\mathcal{S}}_\varepsilon$, we have

$$\|\nabla X - I_3\|_{L^\infty((0,\infty)\times\Omega_F(0))} < \frac{1}{2}. \tag{11.4}$$

Proof. From the definition of X we obtain

$$\begin{aligned}
\|\nabla X - I_3\|_{L^\infty((0,\infty)\times\Omega_F(0))} &\leqslant C\|\nabla X - I_3\|_{L^\infty(0,\infty;W^{1,q}(\Omega_F(0)))} \\
&\leqslant C\int_0^\infty e^{-\eta t}e^{\eta t}\|\nabla\widetilde{u}(t, \cdot)\|_{W^{1,q}(\Omega_F(0))}\,dt \\
&\leqslant C\left(\int_0^\infty e^{-p'\eta t}\,dt\right)^{1/p'}\|e^{\eta(\cdot)}\widetilde{u}\|_{W_{q,p}^{2,1}(Q_\infty^F)} \\
&\leqslant C\left(\frac{1}{p'\eta}\right)^{1/p'}\varepsilon,
\end{aligned}$$

where C depends only on $\Omega_F(0)$. The proof of (11.3) is similar. This completes the proof of the lemma. $\qquad\square$

From now on we assume that

$$\varepsilon_0 = \min\left\{1, \frac{1}{2C_X}\right\}, \tag{11.5}$$

where C_X is the constant in Lemma 11.1.

In the following lemma we estimate some other norms of ∇X and $[\nabla X]^{-1}$ that we need to estimate the nonlinear terms.

Lemma 11.2. *Let X be defined as in* (7.7) *and Z defined by* (7.8)*. Then there exists a constant $C > 0$ depending only on p, q, η and $\Omega_F(0)$ such that, for every $\varepsilon \in (0, \varepsilon_0)$ and for every $(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) \in \widetilde{\mathcal{S}}_\varepsilon$, we have*

$$\begin{aligned}
\|\det(\nabla X - I_3)\|_{L^\infty(0,\infty;W^{1,q})} &+ \|\operatorname{Cof}(\nabla X - I_3)\|_{L^\infty(0,\infty;W^{1,q})} \\
+ \|Z - I_3\|_{L^\infty(0,\infty;W^{1,q})} + \|\partial_t\nabla X\|_{L^p(0,\infty;W^{1,q})} &+ \|\partial_t(Z - I_3)\|_{L^p(0,\infty;W^{1,q})} \\
&+ \|Z - I_3\|_{C^{1/p'}(0,\infty;W^{1,q})} \leqslant C\varepsilon. \tag{11.6}
\end{aligned}$$

Proof. The estimates of $\det(\nabla X - I_3)$ and $\operatorname{Cof}(\nabla X - I_3)$ follow from Lemma 11.1 and from the fact that the space $L^\infty(0,\infty;W^{1,q}(\Omega_F(0)))$ is an algebra for $q > 3$.

From (11.4), we deduce that $\det\nabla X \geqslant C > 0$ in $(0,\infty)\times\Omega_F(0)$ and thus from

$$Z = \frac{\operatorname{Cof}(\nabla X)}{\det(\nabla X)}$$

we obtain

$$\|Z\|_{L^\infty(0,\infty;W^{1,q}(\Omega_F(0)))} \leqslant C.$$

Therefore,

$$\|Z - I_3\|_{L^\infty(0,\infty;W^{1,q})} \leqslant \|Z\|_{L^\infty(0,\infty;W^{1,q}(\Omega_F(0)))}\|\nabla X - I_3\|_{L^\infty(0,\infty;W^{1,q})} \leqslant C\varepsilon.$$

Next notice that,

$$\partial_t \nabla X(t,y) = Q(t)\nabla\widetilde{u}(t,y).$$

Therefore

$$\|\partial_t \nabla X\|_{L^p(0,\infty;W^{1,q})} \leqslant \|e^{\eta(\cdot)}\nabla\widetilde{u}\|_{L^p(0,\infty;W^{1,q})} \leqslant C\varepsilon.$$

We also have

$$\partial_t(Z - I_3) = \partial_t Z = -\nabla X^{-1}\left(\partial_t \nabla X\right)\nabla X^{-1} = -Z\left(\partial_t \nabla X\right)Z.$$

Thus

$$\|\partial_t(Z - I_3)\|_{L^p(0,\infty;W^{1,q})} \leqslant \|Z\|_{L^\infty(0,\infty;W^{1,q})}^2 \|\partial_t \nabla X\|_{L^p(0,\infty;W^{1,q})} \leqslant C\varepsilon.$$

Finally, by Hölder's inequality we have

$$\|(Z - I_3)(t_2,\cdot) - (Z - I_3)(t_1,\cdot)\|_{W^{1,q}} \leqslant \int_{t_1}^{t_2}\|\partial_t(Z - I_3)(s,\cdot)\|_{W^{1,q}}$$

$$\leqslant |t_1 - t_2|^{1/p'}\|\partial_t(Z - I_3)\|_{L^p(0,\infty;W^{1,q})} \leqslant C\varepsilon|t_1 - t_2|^{1/p'}.$$

This completes the proof of the lemma. □

Now we are in a position to estimate the nonlinear terms in (7.9) - (7.14). More precisely, we prove the following

Proposition 11.3. *Let ε_0 be the constant defined as in* (11.5)*. Then there exists a constant C_N depending only on $p, q, \eta, \Omega_F(0)$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and for every $(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) \in \widetilde{\mathcal{S}}_\varepsilon$, we have*

$$\left\|(e^{\eta(\cdot)}\mathcal{F}_1, e^{\eta(\cdot)}\mathcal{F}_2, e^{\eta(\cdot)}\mathcal{F}_3, e^{\eta(\cdot)}\mathcal{H}, e^{\eta(\cdot)}\mathcal{G}_1, e^{\eta(\cdot)}\mathcal{G}_2)\right\|_{\mathcal{B}_{\infty,p,q}}$$

$$+ \left\|(\mathcal{F}_{1,avg}, \mathcal{F}_{3,avg}, \mathcal{H}_{avg})\right\|_{L^1(0,\infty)^3} \leqslant C_N\varepsilon^2.$$

Proof. The constants C appearing in this proof depend only on $p, q, \eta, \Omega_F(0)$ and are independent of ε.

For every $(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) \in \widetilde{\mathcal{S}}_\varepsilon$, we have

$$\|\widetilde{\rho}\|_{L^\infty(0,\infty;W^{1,q})} \leqslant \|e^{\eta(\cdot)}\widetilde{\rho}_m\|_{L^\infty(0,\infty;W^{1,q})} + \|\widetilde{\rho}_{avg}\|_{L^\infty(0,\infty)}$$

$$\leqslant C\|e^{\eta(\cdot)}\widetilde{\rho}_m\|_{W^{1,p}(0,\infty;W^{1,q})} + \|\widetilde{\rho}_{avg}\|_{L^\infty(0,\infty)} \leqslant C\varepsilon. \tag{11.7}$$

Since $2 < p < \infty$, using the the following continuous embedding

$$W_{q,p}^{2,1}(Q_\infty^F) \hookrightarrow L^\infty(0,\infty; B_{q,p}^{2(1-1/p)}(\Omega_F(0))) \hookrightarrow L^\infty(0,\infty; W^{1,q}(\Omega_F(0))),$$

we can similarly deduce that

$$\|\widetilde{u}\|_{L^\infty(0,\infty;W^{1,q})^3} + \|\widetilde{\vartheta}\|_{L^\infty(0,\infty;W^{1,q})} \leqslant C\varepsilon. \tag{11.8}$$

Using the fact that $L^\infty(\Omega_F(0)) \hookrightarrow W^{1,q}(\Omega_F(0))$ for $q > 3$ it is easy to see that

$$\|\widetilde{u}\|_{L^p(0,\infty;L^\infty(\Omega_F(0)))} + \|\nabla\widetilde{u}\|_{L^p(0,\infty;L^\infty(\Omega_F(0)))} + \|\nabla\widetilde{\vartheta}\|_{L^p(0,\infty;L^\infty(\Omega_F(0)))} \leqslant C\varepsilon. \tag{11.9}$$

Let Q be defined as in (7.6). Then

$$\|Q - I_3\|_{L^\infty(0,\infty;\mathbb{R}^{3\times3})} \leqslant \|Q\|_{L^\infty(0,\infty;\mathbb{R}^{3\times3})} \|e^{\eta(\cdot)}\widetilde{\omega}\|_{L^\infty(0,\infty;\mathbb{R}^3)} \int_0^\infty e^{-\eta s} ds$$

$$\leqslant \frac{C}{\eta} \|e^{\eta(\cdot)}\widetilde{\omega}\|_{W^{1,p}(0,\infty;\mathbb{R}^3)} \leqslant C\varepsilon. \tag{11.10}$$

Now we estimate the nonlinear terms in (7.9) - (7.14).
Estimates of $\mathcal{F}_1$ and $\mathcal{F}_{1,avg}$.

$$\|e^{\eta(\cdot)}\mathcal{F}_1\|_{L^p(0,\infty;W^{1,q})} + \|\mathcal{F}_{1,avg}\|_{L^1(0,\infty)} \leqslant C\varepsilon^2. \tag{11.11}$$

Let us recall

$$e^{\eta t}\mathcal{F}_1(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) = -e^{\eta t}(\widetilde{\rho} + \overline{\rho})(Z^\top - I_3) : \nabla\widetilde{u} - e^{\eta t}\widetilde{\rho}\operatorname{div}\widetilde{u}.$$

Using the Lemma 11.2 and estimates (11.7)-(11.9), we obtain

$$\left\| -e^{\eta t}(\widetilde{\rho} + \overline{\rho})(Z^\top - I_3) : \nabla\widetilde{u} - e^{\eta t}\widetilde{\rho}\operatorname{div}\widetilde{u} \right\|_{L^p(0,\infty;W^{1,q})}$$

$$\leqslant \|(\widetilde{\rho} + \overline{\rho})\|_{L^\infty(0,\infty;W^{1,q})} \left\| Z^\top - I_3 \right\|_{L^\infty(0,\infty;W^{1,q})} \left\| e^{\eta(\cdot)}\nabla\widetilde{u} \right\|_{L^p(0,\infty;W^{1,q})}$$

$$+ \|\widetilde{\rho}\|_{L^\infty(0,\infty;W^{1,q})} \left\| e^{\eta(\cdot)}\operatorname{div}\widetilde{u} \right\|_{L^p(0,\infty;W^{1,q})}$$

$$\leqslant C\varepsilon^2.$$

We have

$$\mathcal{F}_{1,avg} = -\frac{1}{|\Omega_F(0)|} \int_{\Omega_F(0)} (\widetilde{\rho} + \overline{\rho})(Z^\top - I_3) : \nabla\widetilde{u}\ dy - \frac{1}{|\Omega_F(0)|} \int_{\Omega_F(0)} \widetilde{\rho}\operatorname{div}\widetilde{u}\ dy.$$

We estimate the first term of $\mathcal{F}_{1,avg}$ as follows

$$\left\| \int_{\Omega_F(0)} (\widetilde{\rho} + \overline{\rho})(Z^\top - I_3) : \nabla\widetilde{u}\ dy \right\|_{L^1(0,\infty)}$$

$$\leqslant \|\widetilde{\rho} + \overline{\rho}\|_{L^\infty((0,\infty)\times\Omega_F(0))} \|(Z^\top - I_3)\|_{L^\infty((0,\infty)\times\Omega_F(0))} \int_0^\infty \int_{\Omega_F(0)} |\nabla\widetilde{u}|\ dy dt$$

$$\leqslant C\|\widetilde{\rho} + \overline{\rho}\|_{L^\infty(0,\infty;W^{1,q})} \left\| Z^\top - I_3 \right\|_{L^\infty(0,\infty;W^{1,q})} \int_0^\infty e^{-\eta t} e^{\eta t} \|\nabla\widetilde{u}(t,\cdot)\|_{L^q(\Omega_F(0))}\ dt$$

$$\leqslant C\varepsilon \left(\int_0^\infty e^{-p'\eta t}\ dt \right)^{1/p'} \left\| e^{\eta(\cdot)}\nabla\widetilde{u} \right\|_{L^p(0,\infty;L^q(\Omega_F(0)))} \leqslant C\varepsilon^2.$$

The other estimate can be obtained similarly.
Estimates of $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_{3,avg}$.

$$\|e^{\eta(\cdot)}\mathcal{F}_2\|_{L^p(0,\infty;L^q)} + \|e^{\eta(\cdot)}\mathcal{F}_3\|_{L^p(0,\infty;L^q)} + \|\mathcal{F}_{3,avg}\|_{L^1(0,\infty)} \leqslant C\varepsilon^2. \tag{11.12}$$

The proof is similar to the proof of (11.11). Note that the terms of $\mathcal{F}_2$ and $\mathcal{F}_3$ are at least quadratic functions of $\widetilde{\rho}$, $\widetilde{u}$, $\widetilde{\vartheta}$, $Z^\top - I_3$ and $Q - I_3$. Therefore using Lemma 11.2 and estimates (11.7) - (11.10), we obtain (11.12).

Estimates of $\mathcal{H}_F$ and $\mathcal{H}_S$.

$$\|e^{\eta(\cdot)}\mathcal{H}_F \cdot n\|_{F_{p,q}^{(1-1/q)/2}(0,\infty;L^q(\partial\Omega))\cap L^p(0,\infty;W^{1-1/q,q}(\partial\Omega))} + \|\mathcal{H}_{F,avg}\|_{L^1(0,\infty)}$$
$$+ \|e^{\eta(\cdot)}\mathcal{H}_S \cdot n\|_{F_{p,q}^{(1-1/q)/2}(0,\infty;L^q(\partial\Omega_S(0)))\cap L^p(0,\infty;W^{1-1/q,q}(\partial\Omega_S(0)))} + \|\mathcal{H}_{S,avg}\|_{L^1(0,\infty)} \leqslant C\varepsilon^2. \tag{11.13}$$

Recall that

$$e^{\eta t}\mathcal{H}_F = e^{\eta t}(I_3 - Z^\top)\nabla\widetilde{\vartheta}.$$

Using Lemma 11.2 and estimates (11.8)-(11.9), we first obtain

$$\left\| e^{\eta(\cdot)}(I_3 - Z^\top)\nabla\widetilde{\vartheta} \right\|_{L^p(0,\infty;W^{1-1/q,q}(\partial\Omega))}$$
$$\leqslant C \left\| (I_3 - Z^\top)e^{\eta(\cdot)}\nabla\widetilde{\vartheta} \right\|_{L^p(0,\infty;W^{1,q}(\Omega_F(0)))}$$
$$\leqslant C \left\| (I_3 - Z^\top) \right\|_{L^\infty(0,\infty;W^{1,q}(\Omega_F(0)))} \left\| e^{\eta(\cdot)}\nabla\widetilde{\vartheta} \right\|_{L^p(0,\infty;W^{1,q}(\Omega_F(0)))} \leqslant C\varepsilon^2.$$

We write

$$e^{\eta t}\mathcal{H}_F|_{\partial\Omega} \cdot n = \sum_{j,k} \left[(\delta_{j,k} - Z_{j,k})\, e^{\eta t}\frac{\partial\widetilde{\vartheta}}{\partial y_k} \right](t,y)n_j(y), \quad y \in \partial\Omega.$$

We know $e^{\eta t}\widetilde{\vartheta} \in W_{q,p}^{2,1}(Q_F^\infty)$. Thus by [9, Proposition 6.4]

$$\left\| e^{\eta t}\frac{\partial\widetilde{\vartheta}}{\partial y_k}\Big|_{\partial\Omega} \right\|_{F_{p,q}^{(1-1/q)/2}(0,\infty;L^q(\partial\Omega))} \leqslant C\|e^{\eta(\cdot)}\widetilde{\vartheta}\|_{W_{q,p}^{2,1}(Q_F^\infty)} \leqslant C\varepsilon, \text{ for } k = 1,2,3.$$

Also from Lemma 11.2 we have

$$\|(\delta_{j,k} - Z_{j,k})\,|_{\partial\Omega}\|_{C^{1/p'}([0,\infty);W^{1-1/q,q}(\partial\Omega))} \leqslant \left\| Z^\top - I_3 \right\|_{C^{1/p'}([0,\infty);W^{1,q}(\Omega_F(0)))} \leqslant C\varepsilon.$$

Therefore by [28, Theorem 2.8.2(ii)], we obtain

$$\left\| e^{\eta(\cdot)}\mathcal{H}_F \cdot n \right\|_{F_{p,q}^{(1-1/q)/2}(0,\infty;L^q(\partial\Omega))}$$
$$\leqslant C\sum_{j,k} \left\| e^{\eta t}\frac{\partial\widetilde{\vartheta}}{\partial y_k}\Big|_{\partial\Omega} \right\|_{F_{p,q}^{(1-1/q)/2}(0,\infty;L^q(\partial\Omega))} \|(\delta_{j,k} - Z_{j,k})\,|_{\partial\Omega}\|_{C^{1/p'}([0,\infty);W^{1-1/q,q}(\partial\Omega))}$$
$$\leqslant C\varepsilon^2.$$

The other estimates can be obtained similarly.

Estimates of $\mathcal{G}_1$ and $\mathcal{G}_2$.

$$\left\| e^{\eta(\cdot)}\mathcal{G}_1 \right\|_{L^p(0,\infty)} + \left\| e^{\eta(\cdot)}\mathcal{G}_2 \right\|_{L^p(0,\infty)} \leqslant C\varepsilon^2. \tag{11.14}$$

The proof is easy and left to the reader. $\qquad\square$

Proposition 11.4. *Let ε_0 be the constant defined as in* (11.5). *Let us set*

$$\mathcal{F}_1^j = \mathcal{F}_1(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{\vartheta}^j, \widetilde{\ell}^j, \widetilde{\omega}^j), \quad \mathcal{F}_2^j = \mathcal{F}_2(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{\vartheta}^j, \widetilde{\ell}^j, \widetilde{\omega}^j), \quad \mathcal{F}_3^j = \mathcal{F}_3(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{\vartheta}^j, \widetilde{\ell}^j, \widetilde{\omega}^j)$$
$$\mathcal{H}_F^j = \mathcal{H}_F(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{\vartheta}^j, \widetilde{\ell}^j, \widetilde{\omega}^j), \quad \mathcal{H}_S^j = \mathcal{H}_S(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{\vartheta}^j, \widetilde{\ell}^j, \widetilde{\omega}^j), \mathcal{H}^j = \mathbb{1}_{\partial\Omega}\mathcal{H}_F^j + \mathbb{1}_{\partial\Omega_S(0)}\mathcal{H}_S^j$$

$$\mathcal{G}_1^j = \mathcal{G}_1(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{\vartheta}^j, \widetilde{\ell}^j, \widetilde{\omega}^j), \quad \mathcal{G}_2^j = \mathcal{G}_2(\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{\vartheta}^j, \widetilde{\ell}^j, \widetilde{\omega}^j),$$

Then there exists a constant $C_{lip} > 0$ depending only on $p, q, \eta, \Omega_F(0)$ such that for every for every $\varepsilon \in (0, \varepsilon_0)$ and for all $(\widetilde{\rho}^1, \widetilde{u}^1, \widetilde{\vartheta}^1, \widetilde{\ell}^1, \widetilde{\omega}^1) \in \widetilde{\mathcal{S}}_\varepsilon$ and $(\widetilde{\rho}^2, \widetilde{u}^2, \widetilde{\vartheta}^2, \widetilde{\ell}^2, \widetilde{\omega}^2) \in \widetilde{\mathcal{S}}_\varepsilon$, we have

$$\left\| e^{\eta(\cdot)} \left(\mathcal{F}_1 - \mathcal{F}_1^2, \mathcal{F}_2^1 - \mathcal{F}_2^2, \mathcal{F}_3^1 - \mathcal{F}_3^2, \mathcal{H}^1 - \mathcal{H}^2, \mathcal{G}_1^1 - \mathcal{G}_2^2, \mathcal{G}_2^1 - \mathcal{G}_2^2 \right) \right\|_{\mathcal{B}_{\infty,p,q}}$$
$$+ \left\| \left(\mathcal{F}_{1,avg}^1 - \mathcal{F}_{1,avg}^2, \mathcal{F}_{3,avg}^1 - \mathcal{F}_{3,avg}^2, \mathcal{H}_{avg}^1 - \mathcal{H}_{avg}^2 \right) \right\|_{L^1(0,\infty)^3}$$
$$\leqslant C_{lip}\varepsilon \left\| (\widetilde{\rho}^1, \widetilde{u}^1, \widetilde{\vartheta}^1, \widetilde{\ell}^1, \widetilde{\omega}^1) - (\widetilde{\rho}^2, \widetilde{u}^2, \widetilde{\vartheta}^2, \widetilde{\ell}^2, \widetilde{\omega}^2) \right\|_{\mathcal{S}}. \quad (11.15)$$

Proof. The proof of this proposition is similar to the proof of Proposition 11.3. $\qquad\square$

12. PROOF OF THE GLOBAL EXISTENCE THEOREM

This section is devoted to the proof of Theorem 2.2 and Corollary 2.3. First we prove a global existence theorem for (7.4) - (7.13). More precisely, we prove the following theorem, which implies Theorem 7.1.

Theorem 12.1. *Let $2 < p < \infty$ and $3 < q < \infty$ satisfying the condition $\dfrac{1}{p} + \dfrac{1}{2q} \neq \dfrac{1}{2}$. Assume that (2.1) is satisfied. Let $\overline{\rho} > 0$ and $\overline{\vartheta} > 0$ be two given constants and $\eta \in (0, \eta_0)$, where η_0 is the constant introduced in Theorem 9.7. Then there exists a constant $\widetilde{\varepsilon}_0 > 0$ such that, for all $\varepsilon \in (0, \widetilde{\varepsilon}_0)$ and for any $(\rho_0, u_0, \vartheta_0, \ell_0, \omega_0)$ belongs to $\mathcal{I}_{p,q}^{cc}$ satisfying*

$$\frac{1}{|\Omega_F(0)|} \int_{\Omega_F(0)} \rho_0 \, \mathrm{d}x = \overline{\rho},$$

and

$$\|(\rho_0 - \overline{\rho}, u_0, \vartheta_0 - \overline{\vartheta}, \ell_0, \omega_0)\|_{\mathcal{I}_{p,q}} \leqslant \frac{\varepsilon}{2C_L},$$

where C_L is the continuity constant appearing in Theorem 10.1, the system (7.4) - (7.13) admits a unique solution $(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega})$ with

$$\left\| (e^{\eta t}\widetilde{\rho}_m, e^{\eta t}\widetilde{u}, e^{\eta t}\widetilde{\vartheta}_m, e^{\eta t}\widetilde{\ell}, e^{\eta t}\widetilde{\omega}) \right\|_{\mathcal{S}_{\infty,p,q}} + \left\| \widetilde{\rho}_{avg}, \widetilde{\vartheta}_{avg} \right\|_{L^\infty(0,\infty)}$$
$$+ \left\| e^{\eta t}\partial_t\widetilde{\rho}_{avg}, e^{\eta t}\partial_t\widetilde{\vartheta}_{avg} \right\|_{L^p(0,\infty)^2} \leqslant \varepsilon.$$

Moreover, $X \in L^\infty(0,\infty; W^{2,q}(\Omega_F(0)))^3 \cap W^{1,\infty}(0,\infty; W^{1,q}(\Omega_F(0)))$ and $X(t, \cdot) : \Omega_F(0) \to \Omega_F(t)$ is a C^1-diffeormorphim for all $t \in [0, \infty)$.

Proof. Let us set

$$\widetilde{\varepsilon}_0 = \min \left\{ \varepsilon_0, \frac{1}{2C_L C_N}, \frac{1}{2C_L C_{lip}} \right\}, \quad (12.1)$$

where ε_0 is defined as in (11.5) and C_L, C_N and C_{lip} are the constants appearing in Theorem 10.1, Proposition 11.3 and Proposition 11.4 respectively. Let us choose $\varepsilon \in (0, \widetilde{\varepsilon}_0)$ and $(\sigma, v, \varphi, k, \tau) \in \widetilde{\mathcal{S}}_\varepsilon$, where $\widetilde{\mathcal{S}}_\varepsilon$ is defined as in (11.1). We consider the following problem

$$\partial_t \widetilde{\rho} + \overline{\rho} \operatorname{div} \widetilde{u} = \mathcal{F}_1(\sigma, v, \varphi, k, \tau), \text{ in } (0, \infty) \times \Omega_F(0),$$

$$\partial_t \widetilde{u} - \operatorname{div} \sigma_l(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}) = \mathcal{F}_2(\sigma, v, \varphi, k, \tau), \text{ in } (0, \infty) \times \Omega_F(0),$$

$$\partial_t \widetilde{\vartheta} - \frac{\kappa}{\overline{\rho} c_v} \Delta \widetilde{\vartheta} + \frac{R\overline{\vartheta}}{c_v} \operatorname{div} \widetilde{u} = \mathcal{F}_3(\sigma, v, \varphi, k, \tau), \text{ in } (0, \infty) \times \Omega_F(0),$$

$$\widetilde{u} = 0 \text{ on } (0, \infty) \times \partial\Omega, \quad \widetilde{u} = \widetilde{\ell} + \widetilde{\omega} \times y \text{ on } (0, \infty) \times \partial\Omega_S(0) \tag{12.2}$$

$$\frac{\partial \widetilde{\vartheta}}{\partial n} = \mathcal{H}_F(\sigma, v, \varphi, k, \tau) \cdot n \text{ on } (0, \infty) \times \partial\Omega, \quad \frac{\partial \widetilde{\vartheta}}{\partial n} = \mathcal{H}_S(\sigma, v, \varphi, k, \tau) \cdot n \text{ on } (0, \infty) \times \partial\Omega_S(0),$$

$$\frac{d}{dt} \widetilde{\ell} = -m^{-1} \int_{\partial\Omega_S(0)} \sigma_l(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}) n \; d\gamma + \mathcal{G}_1(\sigma, v, \varphi, k, \tau), \quad t \in (0, \infty)$$

$$\frac{d}{dt} \omega = -J(0)^{-1} \int_{\partial\Omega_S(0)} y \times \sigma_l(\rho, u, \vartheta) n \; d\gamma + \mathcal{G}_2(\sigma, v, \varphi, k, \tau), \quad t \in (0, \infty)$$

$$\widetilde{\rho}(0) = \rho_0 - \overline{\rho}, \quad u(0) = u_0, \quad \vartheta(0) = \vartheta_0 - \overline{\vartheta}, \quad \text{in } \Omega_F(0),$$

$$\ell(0) = \ell_0, \quad \omega(0) = \omega_0.$$

We are going to show, the mapping

$$\mathcal{N} : (\sigma, v, \varphi, k, \tau) \mapsto (\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega})$$

where $(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega})$ is the solution to the system (12.2), is a contraction in $\widetilde{\mathcal{S}}_\varepsilon$. Since $(\sigma, v, \varphi, k, \tau) \in \widetilde{\mathcal{S}}_\varepsilon$, we can apply Theorem 10.1 and Proposition 11.3 to the system (12.2) and using (7.16) we obtain

$$\left\| (\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega}) \right\|_{\mathcal{S}} \leqslant C_L \left\| (\rho_0 - \overline{\rho}, u_0, \vartheta_0 - \overline{\vartheta}, \ell_0, \omega_0) \right\|_{\mathcal{I}_{p,q}} + C_L C_N \varepsilon^2 \leqslant \varepsilon.$$

Thus $\mathcal{N}$ is a mapping from $\widetilde{\mathcal{S}}_\varepsilon$ to itself for all $\varepsilon \in (0, \widetilde{\varepsilon}_0)$.

Let $(\sigma^1, v^1, \varphi^1, k^1, \tau^1)$ and $(\sigma^2, v^2, \varphi^2, k^2, \tau^2)$ belong to $\widetilde{\mathcal{S}}_\varepsilon$. For $j = 1, 2$, we set $\mathcal{N}(\sigma^j, v^j, \varphi^j, k^j, \tau^j) = (\widetilde{\rho}^j, \widetilde{u}^j, \widetilde{\vartheta}^j, \widetilde{\ell}^j, \widetilde{\omega}^j)$. Using Theorem 10.1 and Proposition 11.4, we obtain

$$\left\| (\widetilde{\rho}^1, \widetilde{u}^1, \widetilde{\vartheta}^1, \widetilde{\ell}^1, \widetilde{\omega}^1) - (\widetilde{\rho}^2, \widetilde{u}^2, \widetilde{\vartheta}^2, \widetilde{\ell}^2, \widetilde{\omega}^2) \right\|_{\mathcal{S}}$$

$$\leqslant C_L C_{lip} \varepsilon \left\| (\sigma^1, v^1, \varphi^1, k^1, \tau^1) - (\sigma^2, v^2, \varphi^2, k^2, \tau^2) \right\|_{\mathcal{S}} \tag{12.3}$$

Using the definition of $\widetilde{\varepsilon}_0$ one can easily check that the mapping $\mathcal{N}$ is a contraction in $\widetilde{\mathcal{S}}_\varepsilon$. This completes the proof of the theorem. $\qquad\square$

Proof of Theorem 2.2. Let $(\widetilde{\rho}, \widetilde{u}, \widetilde{\vartheta}, \widetilde{\ell}, \widetilde{\omega})$ be the solution of (7.4) - (7.13) constructed in Theorem 12.1. Since $X(t, \cdot)$ is C^1- diffeomorphism from $\Omega_F(0)$ into $\Omega_F(t)$, we set $Y(t, \cdot) = X^{-1}(t, \cdot)$ and for $x \in \Omega_F(t), \; t \geqslant 0$

$$\rho(t, x) = \widetilde{\rho}(t, Y(t, x)) + \overline{\rho}, \quad u(t, x) = Q(t)\widetilde{u}(t, Y(t, x)), \quad \vartheta(t, x) = \widetilde{\vartheta}(t, Y(t, x)) + \overline{\vartheta},$$

$$\dot{a}(t) = Q(t)\widetilde{\ell}(t), \quad \omega(t) = Q(t)\widetilde{\omega}(t). \tag{12.4}$$

We can easily check that $(\rho, u, \vartheta, a, \omega)$ solves the original system (2.2) - (2.9) satisfying the estimate (2.14). By choosing δ_0 sufficiently small, from (2.14) it is easy to see that $\rho(t, x) \geqslant \dfrac{\overline{\rho}}{2}$ for all $(t, x) \in (0, \infty) \times \Omega_F(t)$. Finally from (2.5) and (2.14), we obtain

$$\operatorname{dist}(\Omega_S(t), \Omega_S(0)) \leqslant \|a(t)\|_{\mathbb{R}^3} + \|Q(t) - I_3\|_{\mathbb{R}^{3\times3}} |y| < \frac{\nu}{2} \quad \text{ for all } t \geqslant 0,$$

for sufficiently small δ_0. Therefore, $\operatorname{dist}(\Omega_S(t), \partial\Omega) \geqslant \nu/2$ for all $t \in [0, \infty)$. $\qquad\square$

Proof of Corollary 2.3. The first estimate obviously follows from (2.14). To prove (2.15) we integrate the density equation (2.2) over $\Omega_F(t)$ and using the boundary conditions and (2.13) we obtain

$$\frac{1}{|\Omega_F(t)|} \int_{\Omega_F(t)} \rho(t,x) \; dx = \overline{\rho} \qquad (t \geqslant 0).$$

Since $\text{dist}(\Omega_S(t), \partial\Omega) > \nu/2$ and $\Omega_S(t)$ has smooth boundary for every $t \in [0, \infty)$, by Poincaré-Wirtinger inequality we obtain

$$\|\rho(t,x) - \overline{\rho}\|_{L^q(\Omega_F(t))} \leqslant C \|\nabla\rho\|_{L^q(\Omega_F(t))}, \tag{12.5}$$

where the constant C can be chosen uniformly with respect to t (see for instance [6, Theorem 1]). Thus we have

$$\|e^{\eta(\cdot)}(\rho - \overline{\rho})\|_{W^{1,p}(0,\infty;W^{1,q}(\Omega_F(\cdot)))} \leqslant C \|e^{\eta(\cdot)}\nabla\rho\|_{W^{1,p}(0,\infty;L^q(\Omega_F(\cdot)))} \leqslant C\delta.$$

and consequently (2.15) follows. $\qquad\qquad\square$

References

[1] R. A. ADAMS AND J. J. F. FOURNIER, *Sobolev spaces*, vol. 140 of Pure and Applied Mathematics (Amsterdam), Elsevier/Academic Press, Amsterdam, second ed., 2003.

[2] M. S. AGRANOVICH, *Elliptic boundary problems*, in Partial differential equations, IX, vol. 79 of Encyclopaedia Math. Sci., Springer, Berlin, 1997, pp. 1–144, 275–281. Translated from the Russian by the author.

[3] H. AMANN, *Linear and quasilinear parabolic problems. Vol. I*, vol. 89 of Monographs in Mathematics, Birkhäuser Boston, Inc., Boston, MA, 1995. Abstract linear theory.

[4] A. BENSOUSSAN, G. DA PRATO, M. C. DELFOUR, AND S. K. MITTER, *Representation and control of infinite dimensional systems*, Systems & Control: Foundations & Applications, Birkhäuser Boston, Inc., Boston, MA, second ed., 2007.

[5] M. BOULAKIA AND S. GUERRERO, *A regularity result for a solid-fluid system associated to the compressible Navier-Stokes equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 26 (2009), pp. 777–813.

[6] A. BOULKHEMAIR AND A. CHAKIB, *On the uniform Poincaré inequality*, Comm. Partial Differential Equations, 32 (2007), pp. 1439–1447.

[7] P. CLÉMENT AND J. PRÜSS, *An operator-valued transference principle and maximal regularity on vector-valued L_p-spaces*, in Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998), vol. 215 of Lecture Notes in Pure and Appl. Math., Dekker, New York, 2001, pp. 67–87.

[8] R. DENK, M. HIEBER, AND J. PRÜSS, *ℛ-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Mem. Amer. Math. Soc., 166 (2003), pp. viii+114.

[9] ———, *Optimal L^p-L^q-estimates for parabolic boundary value problems with inhomogeneous data*, Math. Z., 257 (2007), pp. 193–224.

[10] B. DESJARDINS AND M. J. ESTEBAN, *On weak solutions for fluid-rigid structure interaction: compressible and incompressible models*, Comm. Partial Differential Equations, 25 (2000), pp. 1399–1413.

[11] G. DORE, *L^p regularity for abstract differential equations*, in Functional analysis and related topics, 1991 (Kyoto), vol. 1540 of Lecture Notes in Math., Springer, Berlin, 1993, pp. 25–38.

[12] K.-J. ENGEL AND R. NAGEL, *One-parameter semigroups for linear evolution equations*, vol. 194 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[13] Y. ENOMOTO AND Y. SHIBATA, *On the ℛ-sectoriality and the initial boundary value problem for the viscous compressible fluid flow*, Funkcial. Ekvac., 56 (2013), pp. 441–505.

[14] E. FEIREISL, *On the motion of rigid bodies in a viscous compressible fluid*, Arch. Ration. Mech. Anal., 167 (2003), pp. 281–308.

[15] E. FEIREISL, V. MÁCHA, V. S. NEČASOVÁ, AND M. TUCSNAK, *Analysis of the adiabatic piston problem via methods of continuum mechanics.* Submitted, 2016.

[16] J. Happel and H. Brenner, *Low Reynolds number hydrodynamics with special applications to particulate media*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1965.

[17] M. Hieber and M. Murata, *The L^p-approach to the fluid-rigid body interaction problem for compressible fluids*, Evol. Equ. Control Theory, 4 (2015), pp. 69–87.

[18] P. C. Kunstmann and L. Weis, *Perturbation theorems for maximal L_p−-regularity*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 30 (2001), pp. 415–435.

[19] P. C. Kunstmann and L. Weis, *Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus*, in Functional analytic methods for evolution equations, vol. 1855 of Lecture Notes in Math., Springer, Berlin, 2004, pp. 65–311.

[20] D. Maity, T. Takahashi, and M. Tucsnak, *Analysis of a system modelling the motion of a piston in a viscous gas*, J. Math. Fluid Mech., 19 (2017), pp. 551–579.

[21] D. Maity and M. Tucsnak, *A Maximal Regularity Approach to the Analysis of Some Particulate Flows*, Springer International Publishing, Cham, 2017, pp. 1–75.

[22] B. Scharf, H.-J. Schmeiss er, and W. Sickel, *Traces of vector-valued Sobolev spaces*, Math. Nachr., 285 (2012), pp. 1082–1106.

[23] V. V. Shelukhin, *Unique solvability of the problem of the motion of a piston in a viscous gas*, Dinamika Sploshn. Sredy, (1977), pp. 132–150.

[24] ——, *Motion with a contact discontinuity in a viscous heat conducting gas*, Dinamika Sploshn. Sredy, (1982), pp. 131–152.

[25] Y. Shibata and M. Murata, *On the global well-posedness for the compressible Navier-Stokes equations with slip boundary condition*, J. Differential Equations, 260 (2016), pp. 5761–5795.

[26] R. Temam, *Navier-Stokes equations*, vol. 2 of Studies in Mathematics and its Applications, North-Holland Publishing Co., Amsterdam-New York, revised ed., 1979. Theory and numerical analysis, With an appendix by F. Thomasset.

[27] H. Triebel, *Interpolation theory, function spaces, differential operators*, Johann Ambrosius Barth, Heidelberg, second ed., 1995.

[28] ——, *Theory of function spaces*, Modern Birkhäuser Classics, Birkhäuser/Springer Basel AG, Basel, 2010. Reprint of 1983 edition [MR0730762], Also published in 1983 by Birkhäuser Verlag [MR0781540].

[29] G. M. Troianiello, *Elliptic differential equations and obstacle problems*, The University Series in Mathematics, Plenum Press, New York, 1987.

[30] M. Tucsnak and G. Weiss, *Observation and control for operator semigroups*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 2009.

[31] P. Vaǐ demaǐer, *Lizorkin-Triebel spaces of vector-valued functions and sharp trace theory for functions in Sobolev spaces with a mixed L_p-norm in parabolic problems*, Mat. Sb., 196 (2005), pp. 3–16.

[32] L. Weis, *Operator-valued Fourier multiplier theorems and maximal L_p-regularity*, Math. Ann., 319 (2001), pp. 735–758.

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