MAXIMAL REGULARITY FOR NON-AUTONOMOUS EVOLUTION EQUATIONS

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ABSTRACT. We consider the maximal regularity problem for non-autonomous evolution equations

$$\begin{cases} u'(t) + A(t)\,u(t) &=& f(t),\ t \in (0,\tau] \\ u(0) &=& u_0. \end{cases} \tag{0.1}$$

Each operator $A(t)$ is associated with a sesquilinear form $\mathfrak{a}(t;\cdot,\cdot)$ on a Hilbert space H. We assume that these forms all have the same domain and satisfy some regularity assumption with respect to t (e.g., piecewise α-Hölder continuous for some $\alpha > 1/2$). We prove maximal L_p- regularity for all u_0 in the real-interpolation space $(H, \mathcal{D}(A(0)))_{1-1/p,p}$. The particular case where $p = 2$ improves previously known results and gives a positive answer to a question of J.L. Lions [16] on the set of allowed initial data u_0.

1. INTRODUCTION AND MAIN RESULTS

Let H be a real or complex Hilbert space and let V be another Hilbert space with dense and continuous embedding $V \hookrightarrow H$. We denote by V' the (anti-)dual of V and by $[\cdot \,|\, \cdot]_H$ the scalar product of H and $\langle \cdot, \cdot \rangle$ the duality pairing $V' \times V$. The latter satisfies (as usual) $\langle v, h \rangle = [v \,|\, h]_H$ whenever $v \in H$ and $h \in V$. By the standard identification of H with H' we then obtain continuous and dense embeddings $V \hookrightarrow H \approx H' \hookrightarrow V'$. We denote by $\|.\|_V$ and $\|.\|_H$ the norms of V and H, respectively.

We are concerned with the non-autonomous evolution equation

$$\begin{cases} u'(t) + A(t)\,u(t) &=& f(t),\ t \in (0,\tau] \\ u(0) &=& u_0, \end{cases} \tag{P}$$

where each operator $A(t)$, $t \in [0,\tau]$, is associated with a sesquilinear form $\mathfrak{a}(t;\cdot,\cdot)$. Throughout this article we will assume that

[H1] (constant form domain) $\mathcal{D}(\mathfrak{a}(t;\cdot,\cdot)) = V$.

[H2] (uniform boundedness) there exists $M > 0$ such that for all $t \in [0,\tau]$ and $u,v \in V$, we have $|\mathfrak{a}(t;u,v)| \leq M\|u\|_V \|v\|_V$.

[H3] (uniform quasi-coercivity) there exist $\alpha > 0$, $\delta \in \mathbb{R}$ such that for all $t \in [0,\tau]$ and all $u,v \in V$ we have $\alpha\|u\|_V^2 \leq \operatorname{Re}\mathfrak{a}(t;u,u) + \delta\|u\|_H^2$.

Recall that $u \in H$ is in the domain $\mathcal{D}(A(t))$ if there exists $h \in H$ such that for all $v \in V$: $\mathfrak{a}(t;u,v) = [h \,|\, v]_H$. We then set $A(t)u := h$. We mention that equality of the form domains, i.e., $\mathcal{D}(\mathfrak{a}(t;\cdot,\cdot)) = V$ for $t \in [0,\tau]$ does not imply equality of the domains $\mathcal{D}(A(t))$ of the corresponding operators. For each fixed $u \in V$, $\phi := \mathfrak{a}(t;u,\cdot)$ defines a continuous (anti-)linear functional on V, i.e. $\phi \in V'$, then it induces a linear operator $\mathcal{A}(t) : V \to V'$ such that $\mathfrak{a}(t;u,v) = \langle \mathcal{A}(t)u,v \rangle$ for all $u,v \in V$. Observe that for $u \in V$,

$$\|\mathcal{A}(t)u\|_{V'} = \sup_{v \in V, \|v\|_V = 1} |\langle \mathcal{A}(t)u,v \rangle| = \sup_{v \in V, \|v\|_V = 1} |\mathfrak{a}(t;u,v)| \leq M\|u\|_V$$

so that $\mathcal{A}(t) \in \mathcal{B}(V, V')$. The operator $\mathcal{A}(t)$ can be seen as an unbounded operator on V' with domain V for all $t \in [0, \tau]$. The operator $A(t)$ is then the part of $\mathcal{A}(t)$ on H, that is,

$$\mathcal{D}(A(t)) = \{ u \in V, \ \mathcal{A}(t)u \in H \}, \qquad A(t)u = \mathcal{A}(t)u.$$

It is a known fact that $-A(t)$ and $-\mathcal{A}(t)$ both generate holomorphic semigroups $(e^{-s\,A(t)})_{s \geq 0}$ and $(e^{-s\,\mathcal{A}(t)})_{s \geq 0}$ on H and V', respectively. For each $s \geq 0$, $e^{-s\,A(t)}$ is the restriction of $e^{-s\,\mathcal{A}(t)}$ to H. For all this, we refer to Ouhabaz [22, Chapter 1].

The notion of maximal L_p–regularity for the above Cauchy problem is defined as follows:

Definition 1.1. Fix u_0. We say that (P) has maximal L_p–regularity (in H) if for each $f \in L_p(0, \tau; H)$ there exists a unique $u \in W_p^1(0, \tau; H)$, such that $u(t) \in \mathcal{D}(A(t))$ for almost all t, which satisfies (P) in the L_p–sense. Here $W_p^1(0, \tau; H)$ denotes the classical L_p–Sobolev space of order one of functions defined on $(0, \tau)$ with values in H.

Maximal regularity of an evolution equation on a Banach space E depends on the operators involved in the equation, the space E and the initial data u_0. The initial data has to be in an appropriate space. In the autonomous case, i.e., $A(t) = A$ for all $t \in [0, \tau]$, maximal L_p–regularity is well understood and it is also known that u_0 has to be in the real-interpolation space $(E, \mathcal{D}(A))_{1 - 1/p, p}$, see [8]. We refer the reader further to the survey of Denk, Hieber and Prüss [10] and the references given therein. Note also that maximal regularity turns to be an important tool to study quasi-linear equations, see e.g. the monograph of Amann [2].

For the non-autonomous case we consider here, we first recall that if the evolution equation is considered in V', then Lions proved maximal L_2–regularity for all initial data $u_0 \in H$, see e.g. [16], [25, page 112]. This powerful result means that for every $u_0 \in H$ and $f \in L_2(0, \tau; V')$, the equation

$$\begin{cases} u'(t) + \mathcal{A}(t)\,u(t) &=& f(t) \\ u(0) &=& u_0 \end{cases} \tag{P'}$$

has a unique solution $u \in W_2^1(0, \tau; V') \cap L_2(0, \tau; V)$. Note that this implies the continuity of $u(\cdot)$ as an H–valued function, see [9, XVIII Chapter 3, p. 513]. It is a remarkable fact that Lions's theorem does not require any regularity assumption (with respect to t) on the sesquilinear forms apart from measurability. The apparently additional information $u \in L_2(0, \tau; V)$ follows from maximal regularity and the equation as follows: for almost all t, $u(t) \in V$. For these $t \in (0, \tau)$

$$\mathrm{Re}\,\mathfrak{a}(t; u(t), u(t)) = -\mathrm{Re}\,\langle u'(t), u(t) \rangle + \mathrm{Re}\,\langle f(t), u(t) \rangle$$

$$\leq \|u(t)\|_V \|u'(t)\|_{V'} + \|u(t)\|_V \|f(t)\|_{V'}.$$

Suppose now that the forms are coercive (i.e., $\delta = 0$ in [H3]). Then for some constant $c > 0$ independent of t,

$$\|u(t)\|_V^2 \leq c \left[\|u'(t)\|_{V'}^2 + \|f(t)\|_{V'}^2 \right]. \tag{1.1}$$

Therefore, $u \in L_2(0, \tau; V)$ whenever $u \in W_2^1(0, \tau; V')$ and $f \in L_2(0, \tau; V')$. If the forms are merely quasi-coercive, we note that if $u(t)$ is the solution of (P') then $u(t)e^{-\delta t}$ is the solution of the same problem with $\mathcal{A}(t) + \delta$ instead of $\mathcal{A}(t)$ and $f(t)e^{-\delta t}$ instead of $f(t)$. We apply now the previous estimate (1.1) to $u(t)e^{-\delta t}$ and $f(t)e^{-\delta t}$ and obtain

$$\|u(t)\|_V^2 \leq c \left[\|u'(t)\|_{V'}^2 + \|u(t)\|_{V'}^2 + \|f(t)\|_{V'}^2 \right]$$

for some constant c' independent of t.

Note however that maximal regularity in V' is not satisfactory in applications to elliptic boundary value problems. For example, in order to identify the boundary condition one has to consider the evolution equation in H rather than in V'. For symmetric forms (equivalently, self-adjoint operators $A(t)$) and $u_0 = 0$, Lions [16, p. 65], proved maximal L_2–regularity in H under the additional assumption that $t \mapsto \mathfrak{a}(t; u, v)$ is C^1 on $[0, \tau]$. For $u(0) = u_0 \in \mathcal{D}(A(0))$ Lions [16, p. 95] proved maximal L_2–regularity in H for (P) provided $t \mapsto \mathfrak{a}(t; u, v)$ is C^2. If the forms are symmetric and C^1 with respect to t, Lions proved maximal L_2-regularity for $u(0) = u_0 \in V$ (one has to combine [16, Theorem 1.1, p. 129 and Theorem 5.1, p. 138] to see this). He asked the following problems.

Problem 1: Does the maximal L_2–regularity in H hold for (P) with $u_0 = 0$ when $t \mapsto \mathfrak{a}(t; u, v)$ is continuous or even merely measurable ?

Problem 2: For $u(0) = u_0 \in \mathcal{D}(A(0))$, does the maximal L_2–regularity in H hold under the weaker assumption that $t \mapsto \mathfrak{a}(t; u, v)$ is C^1 rather than C^2 ?

The problem 1 is still open although some progress has been made. We mention here Ouhabaz and Spina [23] who prove maximal L_p–regularity for (P) when $u(0) = 0$ and $t \mapsto \mathfrak{a}(t; u, v)$ is α-Hölder continuous for some $\alpha > \frac{1}{2}$. More recently, Arendt et al. [3] prove maximal L_2–regularity in H for

$$\begin{cases} B(t)u'(t) + A(t)\, u(t) & = & f(t),\ t \in (0, \tau] \\ u(0) & = & 0 \end{cases}$$

in the case where $t \mapsto \mathfrak{a}(t; u, v)$ is piecewise Lipschitz and $B(t)$ are bounded measurable operators satisfying $\gamma \|v\|_H^2 \le \mathrm{Re}\, [B(t)v \,|\, v]_H \le \gamma' \|v\|_H^2$ for some positive constants γ and γ' and all $v \in H$. The multiplicative perturbation by $B(t)$ was motivated there by applications to some quasi-linear evolution equations.

Concerning the problem with $u_0 \ne 0$ and forms which are not necessarily symmetric, Bardos [6] gave a partial positive answer to Problem 2 in the sense that one can take the initial data u_0 in V under the assumptions that the domains of both $A(t)^{1/2}$ and $A(t)^{*1/2}$ coincide with V as spaces and topologically with constants independent of t, and that $\mathcal{A}(\cdot)^{1/2}$ is continuously differentiable with values in $\mathcal{L}(V, V')$. Note however that the property $\mathcal{D}(A(t)^{1/2}) = \mathcal{D}(A(t)^{*1/2})$ is not always true; this equality is equivalent to the Kato's square root property: $\mathcal{D}(A(t)^{1/2}) = V$. The result of [6] was extended in Arendt et al. [3] by including the multiplication $B(t)$ above and also weakening the regularity of $\mathcal{A}(\cdot)^{1/2}$ from continuously differentiable to piecewise Lipschitz. As in [6], it is also assumed in [3] that the domains of $A(t)^{1/2}$ and $A(t)^{*1/2}$ coincide with V as spaces and topologically with constants independent of t.

We emphasize that the above results from [3, 6, 16] on maximal L_2–regularity do not give any information on maximal L_p–regularity when $p \ne 2$ since the techniques used there are based on a representation lemma in (pre-) Hilbert spaces.

In the present paper we prove maximal L_p–regularity for (P) for all $p \in (1, \infty)$. We extend the results mentioned above and give a complete treatment of the problem with initial data $u_0 \ne 0$ even when the forms are not necessarily symmetric. In particular, we obtain a positive answer to Problem 2 under even more general assumptions.

Our main result is the following.

Theorem 1.2. *Suppose that the forms $(\mathfrak{a}(t; \cdot, \cdot))_{0 \le t \le \tau}$ satisfy the standing hypotheses [H1]– [H3] and the regularity condition*

$$|\mathfrak{a}(t; u, v) - \mathfrak{a}(s; u, v)| \le \omega(|t-s|)\, \|u\|_V \|v\|_V \tag{1.2}$$

where $\omega : [0, \tau] \to [0, \infty)$ is a non-decreasing function such that

$$\int_0^\tau \frac{\omega(t)}{t^{3/2}}\, \mathrm{d}t < \infty. \tag{1.3}$$

Then the Cauchy problem (P) with $u_0 = 0$ has maximal L_p–regularity in H for all $p \in (1, \infty)$. If in addition ω satisfies the p–Dini condition

$$\int_0^\tau \left(\frac{\omega(t)}{t}\right)^p \mathrm{d}t < \infty, \tag{1.4}$$

then (P) has maximal L_p–regularity for all $u_0 \in (H, \mathcal{D}(A(0)))_{1-1/p, p}$. Moreover there exists a positive constant C such that

$$\|u\|_{L_p(0,\tau;H)} + \|u'\|_{L_p(0,\tau;H)} + \|A(\cdot)u(\cdot)\|_{L_p(0,\tau;H)} \le C \left[\|f\|_{L_p(0,\tau;H)} + \|u_0\|_{(H, \mathcal{D}(A(0)))_{1-1/p, p}} \right].$$

In this theorem, $(H, \mathcal{D}(A(0)))_{1-1/p, p}$ denotes the classical real-interpolation space, see [26, Chapter 1.13] or [17, Proposition 6.2].

We wish to point out that Y. Yamamoto* states a result which resembles to our previous theorem in the setting of operators satisfying the so-called Acquistapace-Terreni conditions on the corresponding resolvents. Unfortunately the proofs in his paper present inaccuracies at several places, see for example the proof of Lemma 4.3 at the beginning of page 306, the end of the proof of Proposition 2 at page 307 and a poorly justified estimate after formula (5.5) on page 310 in the proof of Lemma 3.4.

We have the following corollaries.

Corollary 1.3. *Under the assumptions of the previous theorem, the Cauchy problem (P) with $u_0 = 0$ has maximal L_2–regularity in H. If in addition ω satisfies*

$$\int_0^\tau \left(\tfrac{\omega(t)}{t} \right)^2 \mathrm{d}t < \infty, \tag{1.5}$$

then (P) has maximal L_2–regularity for all $u_0 \in \mathcal{D}((\delta + A(0))^{1/2})$. In addition, there exists a positive constant C such that

$$\|u\|_{W_2^1(0,\tau;H)} + \|A(\cdot)u(\cdot)\|_{L_2(0,\tau;H)} \leq C \left[\|f\|_{L_2(0,\tau;H)} + \|u_0\|_{\mathcal{D}((\delta+A(0))^{1/2})} \right].$$

Obviously, if $A(0)$ is accretive then $A(0)^{1/2}$ is well defined and $\mathcal{D}((\delta + A(0))^{1/2}) = \mathcal{D}(A(0)^{1/2})$.
This corollary solves Problem 2 even under more general conditions than conjectured by J.L. Lions

Corollary 1.4. *Assume that additionally to the standing assumptions [H1]– [H3] that the form $\mathfrak{a}(t;\cdot,\cdot)$ is piecewise α–Hölder continuous for some $\alpha > 1/2$. That is, there exist $t_0 = 0 < t_1 < ... < t_k = \tau$ such that on each interval (t_i, t_{i+1}) the form is the restriction of a α–Hölder continuous form on $[t_i, t_{i+1}]$. Assume in addition that at the discontinuity points, we have $\mathcal{D}((\delta + A(t_j^-))^{1/2}) = \mathcal{D}((\delta + A(t_j^+))^{1/2})$. Then the Cauchy problem (P) has maximal L_2–regularity for all $u_0 \in \mathcal{D}((\delta + A(0))^{1/2})$ and there exists a positive constant C such that*

$$\|u\|_{W_2^1(0,\tau;H)} + \|A(\cdot)u(\cdot)\|_{L_2(0,\tau;H)} \leq C \left[\|f\|_{L_2(0,\tau;H)} + \|u_0\|_{\mathcal{D}((\delta+A(0))^{1/2})} \right].$$

In this corollary, $A(t_j^-)$ is the operator associated with the extension of the form at the left of point t_j. Similarly for $A(t_j^+)$.
We mention that Fuje and Tanabe [12] constructed an evolution family associated with the non-autonomous problem considered here when the form $\mathfrak{a}(t;\cdot,\cdot)$ is α–Hölder continuous for some $\alpha > 1/2$. This is of independent interest but it is not clear if one obtains maximal regularity from any property of the corresponding evolution family.
Now we explain briefly the strategy of the proof. A starting point is a representation formula for the solution u (recall that u exists in V' by Lions theorem), which already appeared in the work of Acquistapace and Terreni [1], namely

$$\begin{aligned} u(t) = & \int_0^t e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t) - \mathcal{A}(s))u(s)\,\mathrm{d}s \\ & + \int_0^t e^{-(t-s)\mathcal{A}(t)} f(s)\,\mathrm{d}s + e^{-t\,\mathcal{A}(t)}u_0. \end{aligned} \tag{1.6}$$

This allows us to write $\mathcal{A}(t)u(t) = (Q\mathcal{A}(\cdot)u(\cdot))(t) + (Lf)(t) + (Ru_0)(t)$, where

$$(Qg)(t) := \int_0^t \mathcal{A}(t)e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t) - \mathcal{A}(s))\,\mathcal{A}(s)^{-1}g(s)\,\mathrm{d}s$$

$$(Lg)(t) := \mathcal{A}(t)\int_0^t e^{-(t-s)\mathcal{A}(t)}g(s)\,\mathrm{d}s \quad \text{and} \quad (Ru_0)(t) := \mathcal{A}(t)e^{-t\,\mathcal{A}(t)}u_0.$$

Condition (1.3) allows us to prove invertibility of the operator $(I-Q)$ on $L_p(0,\tau;H)$. The operator L is seen as a pseudo-differential operator with an operator-valued symbol. We prove an L_2-boundedness result for such operators in Section 4, see Theorem 4.1. For operators with scalar-valued symbols, this result is due to Muramatu and Nagase [19]. We adapt their arguments to our

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*see "Solutions in L^p of abstract parabolic equations in Hilbert spaces", J. Math. Kyoto Univ. **33** (1993), no. 2, 299–314.

setting of operator-valued symbols. This theorem is then used to prove $L_2(0, \tau; H)$-boundedness of L. In order to extend L to a bounded operator on $L_p(0, \tau; H)$, for $p \in (1, \infty)$, we look at L as a singular integral operator with an operator-valued kernel. We show that both L and its adjoint L^* satisfy the well known Hörmander integral condition. Finally, we treat the operator R by taking the difference with $\mathcal{A}(0)e^{-t\mathcal{A}(0)}u_0$ and using the functional calculus for accretive operators on Hilbert spaces. In order to handle this difference we use (1.4), the remaining term, $t \mapsto A(0)e^{-tA(0)}u_0$ is then in $L_p(0, \tau; H)$ if and only if $u_0 \in (H, \mathcal{D}(A(0)))_{1-1/p,p}$.

Although most of the arguments outlined here use heavily the fact that H is a Hilbert space, the strategy justifies some hope that the results extend to other situations such as L_q–spaces. One would then hope to prove $L_p(L_q)$ a priori estimates for parabolic equations with time dependent coefficients. In the last section of this paper we give some applications and prove $L_p(L_2)$–a priori estimates.

Maximal regularity may fail even for ordinary differential equations, letting $H = \mathbb{R}$. We illustrate this by an example which is essentially taken from Batty, Chill and Srivastava [7].

Example 1.5. Consider $\varphi(t) = |t|^{-1/p}$. Then φ in $L_{q,\mathrm{loc}}(\mathbb{R})$ for $1 \leq q < p$ but $\varphi \notin L_p([0, \varepsilon])$ for $\varepsilon > 0$. Chose a dense sequence (t_n) of $[0, 1]$ and a positive, summable sequence (c_n). Define $a(t) := 1 + \sum c_n \varphi(t - t_n)$. Then $a \in L_q([0, 1])$ for $1 \leq q < p$ but $a \notin L_p(I)$ for any interval $I \subset [0, 1]$. Consider the non-autonomous equation

$$\begin{cases} x'(t) + a(t)x(t) &= 1 \\ x(0) &= 0 \end{cases} \tag{1.7}$$

Then by variation of constants formula, $x(t) = \int_0^t \exp\left(-\int_s^t a(r)\,\mathrm{d}r\right)\,\mathrm{d}s$. Since $a(r) \geq 0$,

$$|a(t)x(t)| = a(t)\int_0^t \exp\left(-\int_s^t a(r)\,\mathrm{d}r\right)\,\mathrm{d}s$$

$$\geq a(t)\int_0^t \exp\left(-\int_0^1 a(r)\,\mathrm{d}r\right)\,\mathrm{d}s = Ct\,a(t).$$

Therefore, for $0 < \alpha < \beta \leq 1$ we have $|a(t)x(t)| \geq \alpha C\,a(t)$ on $[\alpha, \beta]$ which implies that (1.7) cannot have maximal L_p–regularity.

On the other hand, if we replace the constant function 1 by f we obtain

$$|a(t)x(t)| = a(t)|\int_0^t f(s)\exp\left(-\int_s^t a(r)\,\mathrm{d}r\right)|\,\mathrm{d}s$$

$$\leq a(t)\int_0^t |f(s)|\,\mathrm{d}s \leq Ca(t)\|f\|_{L_q}$$

on $[0, 1]$ and this shows that (1.7) has maximal L_q-regularity for $q < p$.

Notice however, that letting $p=2$ this example is not a counterexample to the questions we raise, since our standing hypothesis [H2] is not satisfied here.

Observe also that the operators in this example are all bounded and commute. Thus, these last two properties are not enough to obtain maximal L_2-regularity for non-autonomous evolution equations.

2. Preparatory lemmas

In this section we prove most of the main arguments which we will need for the proofs of our results. The only missing argument here concerns boundedness of pseudo-differential operators with operator-valued symbols which we write in a separate section for clarity of exposition. We formulate our arguments in a series of lemmata.

Throughout this section we will suppose that [H1]– [H3] are satisfied. Let $\mu \in \mathbb{R}$ and set $v(t) := e^{-\mu t}u(t)$. If u satisfies (P), then v satisfies the evolution equation

$$\begin{cases} v'(t) + (\mu + A(t))\,v(t) &= f(t)e^{-\mu t} \\ u(0) &= u_0. \end{cases}$$

In addition, $v \in W_p^1(0, \tau; H)$ *if and only if* $u \in W_p^1(0, \tau; H)$. This shows that we may replace $\mathcal{A}(t)$ (resp. $A(t)$) by $\mathcal{A}(t) + \mu$ (resp. $A(t) + \mu$). Therefore, we may suppose without loss of generality that $\delta=0$ in [H3]. In particular, we may suppose that $A(t)$ and $\mathcal{A}(t)$ are boundedly invertible by choosing $\mu > 0$ large enough. We will do so in the sequel without further mentioning it.

It is known that $-A(t)$ generates a bounded holomorphic semigroup on H. The same is true for $-\mathcal{A}(t)$ on V'. We write this explicitly in the next proposition in order to point out that the constants involved in the estimates are uniform with respect to t. The arguments are standard and can be found e.g. in [22, Section 1.4]. Denote by S_θ the open sector $S_\theta = \{z \in \mathbb{C}^* : |\arg(z)| < \theta\}$ with vertex 0.

Proposition 2.1. *For any $t \in [0, \tau]$, the operators $-A(t)$ and $-\mathcal{A}(t)$ generate strongly continuous analytic semigroups of angle $\pi/2 - \arctan(M/\alpha)$ on H and V', respectively. In addition, there exist constants C and C_θ, independent of t, such that*

(a) $\|e^{-z\,A(t)}\|_{\mathcal{B}(H)} \leq 1$ *and* $\|e^{-z\,\mathcal{A}(t)}\|_{\mathcal{B}(V')} \leq C$ *for all* $z \in S_{\pi/2 - \arctan(M/\alpha)}$,

(b) $\|A(t)e^{-s\,A(t)}\|_{\mathcal{B}(H)} \leq \frac{C}{s}$ *and* $\|\mathcal{A}(t)e^{-s\,\mathcal{A}(t)}\|_{\mathcal{B}(V')} \leq \frac{C}{s}$,

(c) $\|e^{-s\,A(t)}x\|_V \leq \frac{C}{\sqrt{s}}\|x\|_H$ *and* $\|e^{-s\,\mathcal{A}(t)}\phi\|_H \leq \frac{C}{\sqrt{s}}\|\phi\|_{V'}$,

(d) $\|(z - A(t))^{-1}x\|_V \leq \frac{C_\theta}{\sqrt{|z|}}\|x\|_H$ *for* $z \notin S_\theta$ *and fixed* $\theta > \arctan(M/\alpha)$.

(e) *All the previous estimates hold for $A(t) + \mu$ with constants independent of μ for $\mu \geq 0$.*

Proof. Fix $t \in [0, \tau]$. By uniform boundedness and coercivity,

$$|\operatorname{Im} \mathfrak{a}(t; u, u)| \leq M\|u\|_V^2 \leq M/\alpha \operatorname{Re} \mathfrak{a}(t; u, u). \tag{2.1}$$

This means that $A(t)$ has numerical range contained in the closed sector $\overline{S_{\omega_0}}$ with $\omega_0 = \arctan(M/\alpha)$. This implies the first part of assertion (a), see e.g. [22, Theorem 1.54]. Let $u \in V$ and set $\varphi = (\lambda + \mathcal{A}(t))u \in V'$. Then

$$\langle \varphi, u \rangle = \lambda\|u\|_H^2 + \mathfrak{a}(t; u, u)$$

and so coercivity yields

$$\begin{aligned}
\|u\|_V^2 &\leq \tfrac{1}{\alpha}\operatorname{Re}\mathfrak{a}(t; u, u) \leq \tfrac{1}{\alpha}\big(|\langle \varphi, u \rangle| + |\lambda|\|u\|_H^2\big) \\
&\leq \tfrac{1}{\alpha}\big(\|\varphi\|_{V'}\|u\|_V + |\lambda|\|u\|_H^2\big).
\end{aligned} \tag{2.2}$$

We aim to estimate $|\lambda|\|u\|_H^2$ against $\|u\|_V\|\varphi\|_{V'}$. Since the numerical range of $\mathfrak{a}(t; \cdot, \cdot)$ in contained in $\overline{S_{\omega_0}}$ we have

$$\begin{aligned}
\operatorname{dist}(\lambda, -S_{\omega_0})\,\|u\|_H^2 &\leq \left|\lambda + \mathfrak{a}(t; \tfrac{u}{\|u\|}, \tfrac{u}{\|u\|})\right|\|u\|_H^2 \\
&\leq |\langle(\lambda I + \mathcal{A}(t))u, u\rangle| \leq \|u\|_V\|\varphi\|_{V'}.
\end{aligned}$$

Now let $\theta > \omega_0$ and $\lambda \notin S_\theta$. Then $\operatorname{dist}(\lambda, -S_{\omega_0}) \geq |\lambda|\sin(\theta - \omega_0)$ and therefore

$$|\lambda|\|u\|_H^2 \leq \tfrac{1}{\sin(\theta - \omega_0)}\|u\|_V\|\varphi\|_{V'}$$

as desired. From this and (2.2) it follows that

$$\|u\|_V \leq \tfrac{1}{\alpha}\big(1 + \tfrac{1}{\sin(\theta - \omega_0)}\big)\|(\lambda + \mathcal{A}(t))u\|_{V'} \tag{2.3}$$

uniformly for all $\lambda \notin S_\theta$, $\theta > \omega_0$. This implies that $(\lambda + \mathcal{A}(t))$ is invertible with a uniform norm bound on $S_\theta^{\complement}$, $\theta > \omega_0$. This is equivalent to $-\mathcal{A}(t)$ being the generator of a bounded analytic semigroup on V'. The bound is independent of t. This proves assertion (a).

Assertion (b) follows from the analyticity of the semigroups $(e^{-s\,A(t)})_{s\geq 0}$ on H and $(e^{-s\,\mathcal{A}(t)})_{s\geq 0}$ on V' and the Cauchy formula as usual.

For assertion (c), observe that for $x \in H$

$$\begin{aligned}
\alpha\|e^{-s\,A(t)}x\|_V^2 &\leq \operatorname{Re}\mathfrak{a}(t; e^{-s\,A(t)}x, e^{-s\,A(t)}x) \\
&= \operatorname{Re}\left[\mathcal{A}(t)e^{-s\,A(t)}x \,|\, e^{-s\,A(t)}x\right]_H \\
&\leq \tfrac{C}{s}\|x\|_H^2.
\end{aligned}$$

The second inequality in (c) follows by duality.

The estimate (d) follows in a a natural way from (a) and (c) by writing the resolvent as the Laplace transform of the semigroup. Finally, in order to prove assertion (e), we notice that for a constant $\mu \geq 0$ we have

$$\|e^{-z\,(A(t)+\mu)}x\|_H \leq \|e^{-z\,A(t)}x\|_H,$$

for all $z \in S_{\pi/2-\arctan(M/\alpha)}$. The same estimate also holds when replacing the norm of H by the norm of V or the norm of V'. Now we use the Cauchy formula to obtain (b) for $A(t)+\mu$. Assertion (d) for $A(t) + \mu$ follows again by the Laplace transform. The other estimates are obvious. $\square$

Finally we mention the following easy corollary of the proposition.

Corollary 2.2. *Let $\omega : \mathbb{R} \to \mathbb{R}_+$ be some function and assume that*

$$|\mathfrak{a}(t; u, v) - \mathfrak{a}(s; u, v)| \leq \omega(|t-s|)\|u\|_V \|v\|_V$$

for all $u, v \in V$. Then

$$\|R(z, A(t)) - R(z, A(s))\|_{\mathcal{B}(H)} \leq \tfrac{c_\theta}{|z|}\omega(|t-s|)$$

for all $z \notin S_\theta$ with any fixed $\theta > \arctan(M/\alpha)$.

Proof. Observe that for $u, v \in V$,

$$\left|[R(z, A(t))u - R(z, A(s))u \mid v]_H\right|$$
$$= \left|[R(z, A(t))(A(s) - A(t))R(z, A(s))u \mid v]_H\right|$$
$$= \left|[A(s)R(z, A(s))u \mid R(z, A(t))^* v]_H - [A(t)R(z, A(s))u \mid R(z, A(t))^* v]_H\right|$$
$$= \left|\mathfrak{a}(s; R(z, A(s))u, R(z, A(t))^* v) - \mathfrak{a}(t; R(z, A(s))u, R(z, A(t))^* v)\right|$$
$$\leq \tfrac{c_\theta}{|z|}\omega(|s-t|)\,\|u\|_H\,\|v\|_H,$$

where we used Proposition 2.1(d). $\square$

Next we come to a formula for the solution u of (P') in V'. Recall that u exists by Lions' theorem mentioned in the introduction. This formula already appears in Acquistapace-Terreni [1]. Fix $f \in C_c^\infty(0, \tau; H)$ and $u_0 \in H$.

Lemma 2.3. *For almost every $t \in (0, \tau)$, we have, in V',*

$$u(t) = \int_0^t e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t) - \mathcal{A}(s))u(s)\,\mathrm{d}s$$
$$+ \int_0^t e^{-(t-s)\mathcal{A}(t)}f(s)\,\mathrm{d}s + e^{-t\,\mathcal{A}(t)}u_0. \tag{2.4}$$

Proof. Recall that $u \in W_2^1(0, \tau; V')$ by Lions' theorem and hence u has a continuous representative. Fix $t \in (0, \tau)$ such that $\mathcal{D}(\mathcal{A}(t)) = V$ (recall that this is true for almost all t). Set $v(s) := e^{-(t-s)\,\mathcal{A}(t)}u(s)$ for $0 < s \leq t < \tau$. Recall that $-\mathcal{A}(t)$ generates a bounded semigroup $e^{-s\mathcal{A}(t)}$ on V', see Proposition 2.1 (b). Since $u \in W_2^1(0, \tau; V')$, v has a distributional derivative in V' which satisfies

$$v'(s) = \mathcal{A}(t)e^{-(t-s)\mathcal{A}(t)}u(s) + e^{-(t-s)\mathcal{A}(t)}(f(s) - \mathcal{A}(s)u(s))$$
$$= e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t) - \mathcal{A}(s))u(s) + e^{-(t-s)\mathcal{A}(t)}f(s).$$

Using the fact that $u \in L_2(0, \tau; V)$, it follows that $v \in W_2^1(0, \tau; V')$. Hence

$$v(t) - v(0) = \int_0^t v'(s)\,\mathrm{d}s = \int_0^t e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t) - \mathcal{A}(s))u(s)\,\mathrm{d}s + \int_0^t e^{-(t-s)\mathcal{A}(t)}f(s)\,\mathrm{d}s.$$

This gives (2.4) by observing that $v(t) = u(t)$ and $v(0) = e^{-t\,\mathcal{A}(t)}u_0$. $\square$

Lemma 2.4. *Suppose (1.3). Then for almost all $t \in [0, \tau]$*

$$\mathcal{A}(t)u(t) = (Q\mathcal{A}(\cdot)u(\cdot))(t) + (Lf)(t) + (Ru_0)(t)$$

in V', where

$$(Qg)(t) := \int_0^t \mathcal{A}(t)e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t) - \mathcal{A}(s))\mathcal{A}(s)^{-1}g(s)\,\mathrm{d}s \qquad (2.5)$$

and

$$(Lf)(t) := \mathcal{A}(t)\int_0^t e^{-(t-s)\,\mathcal{A}(t)}f(s)\,\mathrm{d}s \quad and \quad (Ru_0)(t) := \mathcal{A}(t)e^{-t\,\mathcal{A}(t)}u_0. \qquad (2.6)$$

Proof. As in the proof of the previous lemma, we fix $t \in (0, \tau)$ such that $V = \mathcal{D}(\mathcal{A}(t))$. It is enough to prove that each term in the sum (2.4) is in V. Observe that by analyticity, $e^{-t\,\mathcal{A}(t)}u_0 \in \mathcal{D}(\mathcal{A}(t)) = V$. In passing we also note that since $u_0 \in H$, $\mathcal{A}(t)e^{-t\,\mathcal{A}(t)}u_0 = A(t)e^{-t\,A(t)}u_0$.
Concerning the first term, we recall that $u(s) \in V$ for almost all s (note that $u \in L_2(0, \tau; V)$). Therefore, $e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t) - \mathcal{A}(s))u(s) \in V$ for almost every $s < t$ by the analyticity of the semigroup generated by $-\mathcal{A}(t)$. In addition,

$$\|\mathcal{A}(t)e^{(t-s)\mathcal{A}(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s)\|_{V'} \leq \tfrac{C}{t-s}\|(\mathcal{A}(t)-\mathcal{A}(s))u(s)\|_{V'}$$

$$= \tfrac{C}{t-s}\sup_{\|v\|_V=1}|\mathfrak{a}(t; u(s), v) - \mathfrak{a}(s; u(s), v)|$$

$$\leq \tfrac{C}{t-s}\omega(t-s)\|u(s)\|_V.$$

Note that the operator

$$\mathcal{H}:\ h \mapsto \int_0^t \tfrac{\omega(t-s)}{t-s}\,h(s)\,\mathrm{d}s \qquad (2.7)$$

is bounded on $L_p(0, \tau; \mathbb{R})$ for all $p \in [1, \infty]$. The reason is that the associated kernel $(t, s) \mapsto \mathbb{1}_{[0,t]}(s)\tfrac{\omega(t-s)}{t-s}$ is integrable with respect to each variable with a uniform bound with respect to the other variable as can be seen easily from (1.3). Recall again that $\|u(\cdot)\|_V \in L_2(0, \tau; \mathbb{R})$. Hence

$$s \mapsto \mathbb{1}_{[0,t]}(s) \cdot \mathcal{A}(t)e^{-(t-s)\,\mathcal{A}(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s)$$

is in $L_1(0, \tau; V')$. Therefore, for every $\varepsilon > 0$

$$\int_0^{t-\varepsilon} e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s)\,\mathrm{d}s \in \mathcal{D}(\mathcal{A}(t))$$

and the fact that $\mathcal{A}(t)$ is a closed operator gives

$$\mathcal{A}(t)\int_0^{t-\varepsilon} e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s)\,\mathrm{d}s = \int_0^{t-\varepsilon} \mathcal{A}(t)e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s)\,\mathrm{d}s.$$

Since

$$s \mapsto \mathbb{1}_{[0,t]}(s) \cdot \mathcal{A}(t)e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s) \in L_1(0, \tau; V'),$$

we may let $\varepsilon \to 0$ and obtain $\int_0^t e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s)\,\mathrm{d}s \in \mathcal{D}(\mathcal{A}(t))$ and

$$\mathcal{A}(t)\int_0^t e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s)\,\mathrm{d}s = \int_0^t \mathcal{A}(t)e^{-(t-s)\mathcal{A}(t)}(\mathcal{A}(t)-\mathcal{A}(s))u(s)\,\mathrm{d}s.$$

Finally, the equality (2.4) and the fact that $u(t) \in V$ for almost all t yields

$$\int_0^t e^{-(t-s)\mathcal{A}(t)}f(s)\,\mathrm{d}s \in \mathcal{D}(\mathcal{A}(t)).$$

This proves the lemma. $\qquad\square$

Recall the definition of the operator L,

$$Lf(t) = \mathcal{A}(t)\int_0^t e^{-(t-s)A(t)}f(s)\,\mathrm{d}s.$$

Let $f \in C_c^\infty(0, \tau; H)$ and denote by f_0 its extension to the whole of $\mathbb{R}$ by 0 outside $(0, \tau)$. Observe that f_0 is then in the Schwarz class $\mathcal{S}(\mathbb{R}; H)$. We denote for Fourier transform of f_0 by $\mathcal{F}f_0$ or $\widehat{f_0}$. Clearly,

$$\int_{-\infty}^{t} e^{-(t-s)A(t)} f_0(s) \, \mathrm{d}s = \tfrac{1}{2\pi} \int_{-\infty}^{t} e^{-(t-s)A(t)} \int_{\mathbb{R}} e^{\mathrm{i}s\xi} \widehat{f_0}(\xi) \, \mathrm{d}\xi \, \mathrm{d}s$$

Now, exponential stability of $(e^{-s\,A(t)})_{s \geq 0}$ and the fact that $f_0 \in \mathcal{S}(\mathbb{R}; H)$ allows us to use Fubini's theorem, giving

$$\int_{-\infty}^{t} e^{-(t-s)A(t)} \int_{\mathbb{R}} e^{\mathrm{i}s\xi} \widehat{f_0}(\xi) \, \mathrm{d}\xi \, \mathrm{d}s = \int_{\mathbb{R}} \Big(\int_{-\infty}^{t} e^{-(t-s)(\mathrm{i}\xi + A(t))} \, \mathrm{d}s \Big) \widehat{f_0}(\xi) e^{\mathrm{i}t\xi} \, \mathrm{d}\xi$$

$$= \int_{\mathbb{R}} (\mathrm{i}\xi + A(t))^{-1} \widehat{f_0}(\xi) e^{\mathrm{i}t\xi} \, \mathrm{d}\xi.$$

It follows that

$$\int_{-\infty}^{t} e^{-(t-s)A(t)} f_0(s) \, \mathrm{d}s = \tfrac{1}{2\pi} \int_{\mathbb{R}} (\mathrm{i}\xi + A(t))^{-1} \widehat{f_0}(\xi) e^{\mathrm{i}t\xi} \, \mathrm{d}\xi. \tag{2.8}$$

Observe that the right hand side of (2.8) converges in norm (as a Bochner integral) and that the same holds for

$$\int_{\mathbb{R}} A(t)(\mathrm{i}\xi + A(t))^{-1} \widehat{f_0}(\xi) e^{\mathrm{i}t\xi} \, \mathrm{d}\xi$$

since $\widehat{f_0} \in \mathcal{S}(\mathbb{R}; H)$. Thus, both terms in (2.8) take values in $\mathcal{D}(A(t))$. This shows that for $f \in C_c^\infty(0, \tau; H)$, $(Lf)(t)$ is a well-defined function taking pointwise values in H. Hille's theorem (see e.g. [11, II.2, Theorem 6]) then allows us to take the closed operator $A(t)$ inside the integral which finally gives the representation formula

$$Lf(t) = \mathcal{F}^{-1}\big(\xi \mapsto \sigma(t, \xi)\widehat{f_0}(\xi) \big)(t), \tag{2.9}$$

that allows us to see L as a pseudo-differential operator with operator-valued symbol

$$\sigma(t, \xi) = \begin{cases} A(0)(\mathrm{i}\xi + A(0))^{-1} & \text{if} \quad t < 0 \\ A(t)(\mathrm{i}\xi + A(t))^{-1} & \text{if} \quad 0 \leq t \leq \tau \\ A(\tau)(\mathrm{i}\xi + A(\tau))^{-1} & \text{if} \quad t > \tau \end{cases} \tag{2.10}$$

Lemma 2.5. *Suppose that in addition to our standing assumptions [H1]- [H3] that (1.2) holds with $\omega : [0, \tau] \to [0, \infty)$ a non-decreasing function such that*

$$\int_0^\tau \tfrac{\omega(t)^2}{t} \, \mathrm{d}t < \infty. \tag{2.11}$$

Then L is a bounded operator on $L_2(0, \tau; H)$.

Proof. We prove the Lemma by verifying the conditions of Theorem 4.1 below. Let $\sigma(t, \xi)$ be given by (2.10). We need to prove that

$$\big\| \partial_\xi^k \, \sigma(t, \xi) \big\|_{\mathcal{B}(H)} \leq c \tfrac{1}{(1+\xi^2)^{k/2}}, \tag{2.12}$$

$$\big\| \partial_\xi^k \, \sigma(t, \xi) - \partial_\xi^k \, \sigma(s, \xi) \big\|_{\mathcal{B}(H)} \leq c \tfrac{\omega(t-s)}{(1+\xi^2)^{k/2}}, \tag{2.13}$$

for $k = 0, 1, 2$. For $k = 0$, (2.12) is just the sectoriality of $A(t)$, see Proposition 2.1 whereas (2.13) is precisely Corollary 2.2. Observe that a holomorphic function that satisfies

$$\|f(z)\| \leq C \, \tfrac{1}{|z|}$$

on the complement of a sector of angle θ will automatically satisfy

$$\|f^{(k)}(z)\| \leq C_{\theta,k} \, C \, \tfrac{1}{|z|^{k+1}}$$

on the complement of strictly larger sectors, simply by Cauchy's integral formula for derivatives. Conditions (2.12) and (2.13) follow therefore for all $k \geq 1$. $\qquad\square$

Next we prove that the operator L extends to a bounded operator on $L_p(0, \tau; H)$.

Lemma 2.6. *Under the assumptions of the previous lemma the operator L is bounded on $L_p(0, \tau; H)$ for all $p \in (1, \infty)$.*

Proof. The operator L is a singular integral operator with operator-valued kernel

$$K(t,s) = \mathbb{1}_{\{0 \le s \le t \le \tau\}} A(t) e^{-(t-s)A(t)},$$

where $\mathbb{1}$ denotes the indicator function. We prove that both L and L^* are of weak type $(1,1)$ operators and we conclude by the Marcinkiewicz interpolation theorem together with the previous lemma that L is bounded on $L_p(0,\tau;H)$ for all $p \in (1,\infty)$. It is known (see, e.g. [24, Theorems III.1.2 and III.1.3]) that L (respectively L^*) is of weak type $(1,1)$ if the corresponding kernel $K(t,s)$ satisfies the Hörmander integral condition. This means that we have to verify

$$\int_{|t-s| \ge 2|s'-s|} \|K(t,s) - K(t,s')\|_{\mathcal{B}(H)} \, \mathrm{d}t \le C \tag{2.14}$$

and

$$\int_{|t-s| \ge 2|s'-s|} \|K(s,t) - K(s',t)\|_{\mathcal{B}(H)} \, \mathrm{d}t \le C \tag{2.15}$$

for some constant C independent of $s, s' \in (0,\tau)$. Note that the above mentioned theorems in [24] are formulated for integral operators on $L_p(\mathbb{R};H)$ instead of $L_p(0,\tau;H)$; however it is known that Hörmander's integral condition works on any space satisfying the volume doubling condition, see [24, page 15].

First consider the integral in (2.14). When $s \le s'$ and $2|s'-s| > \tau$ the integral vanishes. When $0 \le s \le s' \le t \le \tau$ and $2s'-s \le \tau$, using that the semigroup $(e^{-s\,A(t)})_{s \ge 0}$ generated by $-A(t)$ is bounded holomorphic, with a norm bound independent of t, we have for some constant C

$$\int_{|t-s| \ge 2|s'-s|} \|K(t,s) - K(t,s')\|_{\mathcal{B}(H)} \, \mathrm{d}t$$

$$= \int_{2s'-s}^{\tau} \|A(t) e^{-(t-s)A(t)} - A(t) e^{-(t-s')A(t)}\|_{\mathcal{B}(H)} \, \mathrm{d}t$$

$$= \int_{2s'-s}^{\tau} \int_{s}^{s'} \|A(t)^2 e^{-(t-r)A(t)} \, \mathrm{d}r\|_{\mathcal{B}(H)} \, \mathrm{d}t$$

$$\le C \int_{2s'-s}^{\tau} \int_{s}^{s'} \frac{1}{(t-r)^2} \, \mathrm{d}r \, \mathrm{d}t = C \int_{2s'-s}^{\tau} \left[\frac{1}{t-s'} - \frac{1}{t-s} \right] \mathrm{d}t$$

$$= C \left[\log \frac{t-s'}{t-s} \right]_{t=2s'-s}^{t=\tau} \le C \log 2.$$

When $s' < s$ and $3s - 2s' > \tau$, the integral (2.14) vanishes. When $s' < s$ and $3s - 2s' < \tau$, a similar calculation to the above shows that the integral is bounded by $C \log(3/2)$.

We now consider (2.15). When $s \le s'$, as above, we may assume that $3s-2s' > 0$, since otherwise the integral in (2.15) vanishes. We have

$$\int_{|t-s| \ge 2|s'-s|} \|K(s,t) - K(s',t)\|_{\mathcal{B}(H)} \mathrm{d}t$$

$$= \int_{0}^{3s-2s'} \|A(s) e^{-(s-t)A(s)} - A(s') e^{-(s'-t)A(s')}\|_{\mathcal{B}(H)} \, \mathrm{d}t$$

$$\le \int_{0}^{3s-2s'} \|A(s) e^{-(s-t)A(s)} - A(s) e^{-(s'-t)A(s)}\|_{\mathcal{B}(H)} \, \mathrm{d}t$$

$$+ \int_{0}^{3s-2s'} \|A(s) e^{-(s'-t)A(s)} - A(s') e^{-(s'-t)A(s')}\|_{\mathcal{B}(H)} \, \mathrm{d}t =: I_1 + I_2.$$

The first term I_1 is handled exactly as in the proof of (2.14). For the second term I_2, we write by the functional calculus

$$A(s) e^{-(s'-t)A(s)} - A(s') e^{-(s'-t)A(s')} = \frac{1}{2\pi i} \int_{\Gamma} z e^{-tz} \big[R(z, A(s)) - R(z, A(s')) \big] \, \mathrm{d}z$$

where Γ is the boundary of an appropriate sector S_θ. We apply Corollary 2.2 to deduce

$$\|A(s) e^{-(s'-t)A(s)} - A(s') e^{-(s'-t)A(s')}\|_{\mathcal{B}(H)} \le C \int_{0}^{\infty} r e^{-(s'-t)r \cos\theta} \frac{\omega(s'-s)}{r} \, \mathrm{d}r$$

$$\leq C \, \tfrac{\omega(s'-s)}{s'-t}.$$

Therefore,

$$\int_0^{3s-2s'} \|A(s)e^{-(s'-t)A(s)} - A(s')e^{-(s'-t)A(s')}\|_{\mathcal{B}(H)} \, \mathrm{d}t$$

$$\leq C \int_0^{3s-2s'} \tfrac{\omega(s'-s)}{s'-t} \, \mathrm{d}t \leq \; C \int_0^{\tau} \omega(r) \, \tfrac{\mathrm{d}r}{r} = C',$$

where we used the fact that ω is non-decreasing and $s'-s \leq s'-t$ to write the second inequality. Finally, the integral (2.15) in the case $s' < s$ is treated similarly. Remark: A similar reasoning for the weak type $(1,1)$ estimate for L and L^* appears in [13, p. 1051]. $\qquad\square$

Now we study the operator R.

Lemma 2.7. *Assume (1.4). Then there exists $C > 0$ such that for every $u_0 \in (H, \mathcal{D}(A(0)))_{1-1/p,p}$:*

$$\|Ru_0\|_{L_p(0,\tau;H)} \leq C\|u_0\|_{(H,\mathcal{D}(A(0)))_{1-1/p,p}}.$$

Proof. Recall that the operator R is given by $(Rg)(t) = A(t)e^{-t\,A(t)}g$ for $g \in H$. Let

$$(R_0 g)(t) := A(0)e^{-t\,A(0)}g.$$

We aim to estimate the difference $(R - R_0)g$. Let $\Gamma = \partial S_\theta$ with $\theta \in (\omega_0, \pi/2)$ and ω_0 is as in the proof of Proposition 2.1. Then, for $v \in V$, the functional calculus for the sectorial operators $A(t)$ and $A(0)$ gives

$$\left\langle A(t)e^{-t\,A(t)}g - A(0)e^{-t\,A(0)}g, v \right\rangle$$

$$= \tfrac{1}{2\pi i} \int_\Gamma \left\langle ze^{-tz}\big[R(z,A(t)) - R(z,A(0))\big]g, v \right\rangle \, \mathrm{d}z$$

$$= \tfrac{1}{2\pi i} \int_\Gamma \left\langle ze^{-tz}R(z,\mathcal{A}(t))\big[\mathcal{A}(0) - \mathcal{A}(t)\big]R(z,A(0))g, v \right\rangle \, \mathrm{d}z$$

$$= \tfrac{1}{2\pi i} \int_\Gamma \left\langle ze^{-tz}\big[\mathcal{A}(0) - \mathcal{A}(t)\big]R(z,A(0))g, R(z,A(t))^*v \right\rangle \, \mathrm{d}z$$

$$= \tfrac{1}{2\pi i} \int_\Gamma ze^{-tz}\, \mathfrak{a}(0; R(z,A(0))g, R(z,A(t))^*v) - \mathfrak{a}(t; R(z,A(0))g, R(z,A(t))^*v) \, \mathrm{d}z.$$

Now, taking the absolute value it follows from Proposition 2.1(d) that

$$|\langle (Rg - R_0 g)(t), v\rangle| \leq \tfrac{C_\alpha}{2\pi} \int_\Gamma \omega(t)|z|e^{-t\,\mathrm{Re}(z)}\|R(z,A(0))g\|_V \|R(z,A(t))^*v\|_V \, |\mathrm{d}z|$$

$$\leq \tfrac{C_{\alpha,\theta}}{2\pi}\omega(t)\|g\|_H\|v\|_H \int_\Gamma e^{-t\,\mathrm{Re}z} \, |\mathrm{d}z|$$

$$\leq C' \tfrac{\omega(t)}{t}\|g\|_H\|v\|_H.$$

Since this true for all $v \in H$ we conclude that

$$\|(Ru_0)(t) - (R_0 u_0)(t)\|_H \leq C' \tfrac{\omega(t)}{t}\|u_0\|_H. \tag{2.16}$$

From the hypothesis (1.4) it follows that $Ru_0 - R_0 u_0 \in L_p(0,\tau;H)$. On the other hand, since $A(0)$ is invertible, it is well-known that $A(0)e^{-t\,A(0)}u_0 \in L_p(0,\tau;H)$ if and only if $u_0 \in (H,\mathcal{D}(A(0)))_{1-1/p,p}$ (see Triebel [26, Theorem 1.14]). Therefore, $Ru_0 \in L_p(0,\tau;H)$ and the lemma is proved. $\qquad\square$

3. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1.2. Assume first that $u_0 = 0$ and let $f \in C_c^\infty(0,\tau;H)$. From Lemma 2.4 it is clear that

$$(I - Q)A(\cdot)u(\cdot) = Lf(\cdot). \tag{3.1}$$

Recall that L is bounded on $L_p(0, \tau; H)$ by Lemma 2.6. We shall now prove that Q is bounded on $L_p(0, \tau; H)$. Let $g \in C_c^\infty(0, \tau; H)$. By Proposition 2.1 we have

$$\|(Qg)(t)\|_H \leq \int_0^t \tfrac{C}{t-s} \|e^{-(t-s)A(t)/2}(\mathcal{A}(t) - \mathcal{A}(s))\mathcal{A}(s)^{-1}g(s)\|_H \, ds$$

$$\leq \int_0^t \tfrac{C'}{(t-s)^{3/2}} \|(\mathcal{A}(t) - \mathcal{A}(s))\mathcal{A}(s)^{-1}g(s)\|_{V'} \, ds.$$

Since $\|\mathcal{A}(t)x\|_{V'} = \sup_{\|v\|_V = 1} |\mathfrak{a}(t; x, v)|$, we use the regularity assumption (1.2) to bound Qg further by

$$\|(Qg)(t)\|_H \leq \int_0^t \tfrac{C'}{(t-s)^{3/2}} \, \omega(t-s) \, \|\mathcal{A}(s)^{-1}g(s)\|_V \, ds. \qquad (3.2)$$

Now we estimate $\|\mathcal{A}(s)^{-1}g(s)\|_V$. By coercivity

$$\alpha\|\mathcal{A}(s)^{-1}g(s)\|_V^2 \leq \operatorname{Re} \mathfrak{a}(s; \mathcal{A}(s)^{-1}g(s), \mathcal{A}(s)^{-1}g(s))$$

$$= \operatorname{Re}\langle \mathcal{A}(s)\mathcal{A}(s)^{-1}g(s), \mathcal{A}(s)^{-1}g(s)\rangle$$

$$= \operatorname{Re}\big[g(s) \,|\, \mathcal{A}(s)^{-1}g(s)\big]_H$$

$$\leq \|g(s)\|_H^2 \|\mathcal{A}(s)^{-1}\|_{\mathcal{B}(H)}.$$

We obtain from (3.2) that

$$\|(Qg)(t)\|_H \leq \int_0^t \tfrac{C'}{(t-s)^{3/2}} \, \omega(t-s) \, \|\mathcal{A}(s)^{-1}\|_{\mathcal{B}(H)}^{1/2} \|g(s)\|_H \, ds. \qquad (3.3)$$

Now, once we replace $A(t)$ by $A(t) + \mu$, (3.2) is valid with a constant independent of $\mu \geq 0$ by Proposition 2.1(e). Using the estimate

$$\|(\mathcal{A}(s) + \mu)^{-1}\|_{\mathcal{B}(H)} \leq \tfrac{1}{\mu},$$

in (3.3) for $A(s) + \mu$ we see that

$$\|(Qg)(t)\|_H \leq \tfrac{C'}{\sqrt{\mu}} \int_0^t \tfrac{\omega(t-s)}{(t-s)^{3/2}} \, \|g(s)\|_H \, ds.$$

It remains to see that the operator S defined by

$$Sh(t) := \int_0^t \tfrac{\omega(t-s)}{(t-s)^{3/2}} h(s) \, ds$$

is bounded on $L_p(0, \tau; \mathbb{R})$. Observe that S is an integral operator with kernel function $(t, s) \mapsto \mathbb{1}_{[0,t]}(s)\tfrac{\omega(t-s)}{(t-s)^{3/2}}$. Hence by assumption (1.3) it is integrable with respect to each of the two variables with uniform bound with respect to the other variable. This implies that S is bounded on $L_1(0, \tau; H)$ and on $L_\infty(0, \tau; H)$ and hence bounded on $L_p(0, \tau; H)$.

It follows that Q is bounded on $L_p(0, \tau; H)$ with norm of at most $\tfrac{C''}{\sqrt{\mu}}$ for some constant C''. Taking then μ large enough makes Q strictly contractive such that $(I - Q)^{-1}$ is bounded by the Neumann series. Then, for $f \in C_c^\infty(0, \tau; H)$, (3.1) can be rewritten as

$$A(\cdot)u(\cdot) = (I - Q)^{-1}Lf(\cdot).$$

For general $u_0 \in (H, \mathcal{D}(A(0)))_{1 - 1/p, p}$ we suppose in addition to (1.3) that (1.4) holds. Lemma 2.7 shows that $Ru_0 \in L_p(0, \tau; H)$. As previously we conclude that

$$A(\cdot)u(\cdot) = (I - Q)^{-1}(Lf + Ru_0),$$

whenever $f \in C_c^\infty(0, \tau; H)$. Thus taking the L_p norms we have

$$\|A(\cdot)u(\cdot)\|_{L_p(0, \tau; H)} \leq C\|(Lf + Ru_0)\|_{L_p(0, \tau; H)}.$$

We use again the previous estimates on L and R to obtain

$$\|A(\cdot)u(\cdot)\|_{L_p(0, \tau; H)} \leq C' \left[\|f\|_{L_p(0, \tau; H)} + \|u_0\|_{(H, \mathcal{D}(A(0)))_{1 - 1/p, p}}\right].$$

Using the equation (P) we obtain a similar estimate for u' and so

$$\|u'(\cdot)\|_{L_p(0,\tau;H)} + \|A(\cdot)u(\cdot)\|_{L_p(0,\tau;H)} \leq C'' \left[\|f\|_{L_p(0,\tau;H)} + \|u_0\|_{(H,\mathcal{D}(A(0)))_{1-1/p,p}} \right].$$

We write $u(t) = A(t)^{-1}A(t)u(t)$ and use one again the fact that the norms of $A(t)^{-1}$ on H are uniformly bounded we obtain

$$\|u(t)\|_{L_p(0,\tau;H)} \leq C_1 \|A(\cdot)u(\cdot)\|_{L_p(0,\tau;H)} \leq C_2 \left[\|f\|_{L_p(0,\tau;H)} + \|u_0\|_{(H,\mathcal{D}(A(0)))_{1-1/p,p}} \right].$$

We conclude therefore that the following a priori estimate holds

$$\begin{aligned}
\|u\|_{L_p(0,\tau;H)} &+ \|u'\|_{L_p(0,\tau;H)} + \|A(\cdot)u(\cdot)\|_{L_p(0,\tau;H)} \\
&\leq C \left[\|f\|_{L_p(0,\tau;H)} + \|u_0\|_{(H,\mathcal{D}(A(0)))_{1-1/p,p}} \right],
\end{aligned} \tag{3.4}$$

where the constant C does not depend on $f \in C_c^\infty(0,\tau;H)$.

Now let $f \in L_p(0,\tau;H)$ and $(f_n) \subset C_c^\infty(0,\tau;H)$ be an approximating sequence that converges in L_p and pointwise almost everywhere. For each n, denote by u_n the solution of (P) with right hand side f_n. We apply (3.4) to $u_n - u_m$ and we see that there exists $u \in W_p^1(0,\tau;H)$ and $v \in L_p(0,\tau;H)$ such that

$$u_n \xrightarrow{L_p} u \qquad u_n' \xrightarrow{L_p} u' \quad \text{and} \quad A(\cdot)u_n(\cdot) \xrightarrow{L_p} v \tag{3.5}$$

By extracting a subsequence, we may assume that these limits hold in the pointwise a.e. sense as well. For a fixed t, the operator $A(t)$ is closed and so $v(\cdot) = A(\cdot)u(\cdot)$. Passing to the limit in the equation

$$u_n'(t) + A(t)\,u_n(t) = f_n(t)$$

shows that

$$u'(t) + A(t)u(t) = f(t)$$

for a.e. $t \in (0,\tau)$. On the other hand, by Sobolev embedding, (u_n) is bounded in $C([0,\tau];H)$ and hence $u(0) = u_0$ since $u_n(0) = u_0$ by the definition of u_n. We conclude that u satisfies

$$u'(t) + A(t)\,u(t) = f(t) \qquad u(0) = u_0$$

in the L_p sense. This means that u is a solution to (P). Moreover, (3.4) transfers from u_n to u. The uniqueness of the solution u follows from the a priori estimate (3.4) as well. □

Proof of Corollary 1.3. The result follows from Theorem 1.2 and the observation that

$$(H,\mathcal{D}(A(0)))_{1/2,2} = [(H,\mathcal{D}(A(0))]_{1/2} = \mathcal{D}((\delta + A(0))^{1/2}),$$

see e.g. [17, Corollaries 4.37 and 4.30]. □

Proof of Corollary 1.4. By the definition of maximal regularity, one may modify the operators $A(t)$, $0 \leq t \leq \tau$, on a set of Lebesgue measure zero. Therefore, we may assume without loss of generality that the mapping $t \mapsto \mathfrak{a}(t;u,v)$ is right continuous. We may assume again that the operators $A(\cdot)$ are all invertible. We apply Corollary 1.3 to the evolution equation

$$\begin{cases} u_j'(t) + A(t)u_j(t) = f(t) & t \in (t_j, t_{j+1}) \\ \quad u_j(t_j) = u_{j-1}(t_j), \end{cases}$$

since it is obvious that the assumed α-Hölder continuity for some $\alpha > 1/2$ implies both (1.3) and (1.5). The solution u_j is in $W_2^1(t_j, t_{j+1};H)$ provided the initial data satisfies

$$u_j(t_j) := u_{j-1}(t_j) \in \mathcal{D}(A(t_j)^{1/2}).$$

Note that the endpoint $u_{j-1}(t_j)$ is well defined since $u_j \in C([t_j, t_{j+1}];H)$ by [9, XVIII Chapter 3, p. 513]. In order to obtain a solution $u \in W_2^1(0,\tau;H)$, we glue the solutions u_j. That is, we set $u(t) = u_j(t)$ for $t \in [t_j, t_{j+1}]$. What remains then to prove is that $u(t_j) \in \mathcal{D}(A(t_j)^{1/2})$, where $u \in W_2^1(0,\tau;V')$ is the solution in V' given by Lions' theorem.

Fix one of the discontinuity points t_j and consider the autonomous equation

$$v'(s) + A(t_j^-)v(s) = f(s), \qquad v(0) = 0.$$

By maximal regularity of $A(t_j^-)$, its solution $v(s) = \int_0^s e^{-(s-r)A(t_j^-)} f(r)\,\mathrm{d}r$ satisfies $v(s) \in \mathcal{D}(A(t_j^-))$ for almost all s. Choose a sequence (s_n) converging to t_j from the left such that $v(s_n) \in \mathcal{D}(A(t_j^-))$. Since $A(t_j^-)$ is an accretive and sectorial operator it has a bounded H^∞-calculus of some angle $< \pi/2$. Hence $A(t_j^-)$ and its adjoint admit square-function estimates of the form:

$$\int_0^\infty \big\| A(t_j^-)^{1/2} e^{-r\,A(t_j^-)} x \big\|_H^2 \, \mathrm{d}r \le C \|x\|_H^2 \text{ for all } x \in H,$$

see e.g. [18, Section 8]. It follows that

$$
\begin{aligned}
&\int_0^{s_n} \big\| A(t_j^-)^{1/2} e^{-(t_j-r)A(t_j^-)} f(r) \big\|_H \, \mathrm{d}r \\
&= \int_0^{s_n} \sup_{\|h\|_H \le 1} \Big[f(r) \,\big|\, A(t_j^-)^{*1/2} e^{-(t_j-r)A(t_j^-)^*} h \Big]_H \, \mathrm{d}r \\
&\le \|f\|_{L_2(0,\tau;H)} \sup_{\|h\|_H \le 1} \Big(\int_0^{s_n} \big\| A(t_j^-)^{*1/2} e^{-(t_j-r)A(t_j^-)^*} h \big\|_H^2 \, \mathrm{d}r \Big)^{1/2} \\
&\le C \|f\|_{L_2(0,\tau;H)}.
\end{aligned}
\tag{3.6}
$$

Thus, (3.6) implies that the sequence $(v(s_n))_{n \ge 0}$ is bounded in the Hilbert space $\mathcal{D}(A(t_j^-)^{1/2})$. It has a weakly convergent subsequence. By extracting a subsequence, we may assume that $(v(s_n))_{n \ge 0}$ converges weakly to some v in $\mathcal{D}(A(t_j^-)^{1/2})$. But the continuity of the solution $v(\cdot)$ implies that $v(s_n)$ tends also to $v(t_j)$ in H. Therefore,

$$v(t_j) = \int_0^{t_j} e^{-(t_j-r)A(t_j^-)} f(r)\,\mathrm{d}r \in \mathcal{D}(A(t_j^-)^{1/2}).$$

In particular,

$$\int_{t_{j-1}}^{t_j} e^{-(t_j-r)A(t_j^-)} f(r)\,\mathrm{d}r \in \mathcal{D}(A(t_j^-)^{1/2}). \tag{3.7}$$

On the other hand, as in the proof of Lemma 2.3, we have for all $t > t_{j-1}$

$$
\begin{aligned}
&u(t) - e^{-(t-t_{j-1})\mathcal{A}(t)} u(t_{j-1}) \\
&= \int_{t_{j-1}}^t e^{-(t-s)\mathcal{A}(t)} (\mathcal{A}(t) - \mathcal{A}(s)) u(s)\,\mathrm{d}s + \int_{t_{j-1}}^t e^{-(t-s)\mathcal{A}(t)} f(s)\,\mathrm{d}s.
\end{aligned}
\tag{3.8}
$$

By analyticity of the semigroup $e^{-s\,\mathcal{A}(t_j^-)}$ it follows that $e^{-(t_j-t_{j-1})\,\mathcal{A}(t_j^-)} u(t_{j-1}) \in \mathcal{D}(A(t_j^-)^{1/2})$. Now we prove that $\int_{t_{j-1}}^{t_j} e^{-(t_j-s)\,\mathcal{A}(t_j^-)} (\mathcal{A}(t_j^-) - \mathcal{A}(s)) u(s)\,\mathrm{d}s \in \mathcal{D}(A(t_j^-)^{1/2})$. It is enough to prove that

$$\int_{t_{j-1}}^{t_j} A(t_j^-)^{1/2} e^{-(t_j-s)\mathcal{A}(t_j^-)} (\mathcal{A}(t_j^-) - \mathcal{A}(s)) u(s)\,\mathrm{d}s \in H. \tag{3.9}$$

To this end, let $h \in H$ be such that $\|h\|_H \le 1$. By Proposition 2.1, (c), we have

$$\big\| A(t_j^-)^{*1/2} e^{-(t_j-s)A(t_j^-)^*} h \big\|_V \le C |t_j - s|^{-1} \tag{3.10}$$

Thus, since the form is C^α on (t_{j-1}, t_j), we have for every small $\epsilon > 0$

$$
\begin{aligned}
&\Big| \big\langle \int_{t_{j-1}}^{t_j-\epsilon} A(t_j^-)^{1/2} e^{-(t_j-s)\mathcal{A}(t_j^-)} (\mathcal{A}(t_j^-) - \mathcal{A}(s)) u(s)\,\mathrm{d}s, h \big\rangle \Big| \\
&= \Big| \int_{t_{j-1}}^{t_j-\epsilon} \mathfrak{a}(t_j; u(s), A(t_j^-)^{*1/2} e^{-(t_j-s)A(t_j^-)^*} h) - \mathfrak{a}(s; u(s), A(t_j^-)^{*1/2} e^{-(t_j-s)A(t_j^-)^*} h)\,\mathrm{d}s \Big| \\
&\le C \int_{t_{j-1}}^{t_j-\epsilon} |t_j - s|^\alpha \|u(s)\|_V \|A(t_j^-)^{*1/2} e^{-(t_j-s)A(t_j^-)^*} h\|_V \, \mathrm{d}s \\
&\le C' \int_{t_{j-1}}^{t_j} |t_j - s|^{\alpha-1} \|u(s)\|_V \, \mathrm{d}s.
\end{aligned}
$$

Taking the supremum over all $h \in H$ of norm we obtain

$$\| \int_{t_{j-1}}^{t_j - \epsilon} A(t_j^-)^{1/2} e^{-(t_j - s)\mathcal{A}(t_j^-)} (\mathcal{A}(t_j^-) - \mathcal{A}(s)) u(s) \, \mathrm{d}s \|_H \leq C'' \|u\|_{L_2(0,\tau;V)}.$$

Since this is true for all $\epsilon > 0$ we obtain (3.9). We conclude from this, (3.7) and (3.8) that $u_{j-1}(t_j) = u(t_j) \in \mathcal{D}(A(t_j^-)^{1/2})$. Finally, the latter space coincides with $\mathcal{D}(A(t_j^-)^{1/2}) = \mathcal{D}(A(t_j^+)^{1/2})$ by the assumptions of the corollary. $\qquad\square$

4. OPERATOR-VALUED PSEUDO-DIFFERENTIAL OPERATORS

Given a Hilbert space H, our aim in this section is to prove results on boundedness on $L_2(\mathbb{R}^n; H)$ for pseudo-differential operators with minimal smoothness assumption on the symbol. The main results we will show here were proved in [19] in the scalar case (i.e. $H=\mathbb{C}$), see also [4]. The operator-valued version follows the lines in [19] and we give the details here for the sake of completeness. Let us mention the paper [14] where results on L_p–boundedness of pseudo-differential operators with operator-valued symbols are proved even when H is not a Hilbert space. We do not appeal to the results from [14] in order to avoid assuming continuity and concavity assumptions on the function ω in Theorem 1.2.

Let H be a Hilbert space on $\mathbb{C}$, with scalar product $[\cdot \, | \, \cdot]_H$ and associated norm $\| \cdot \|_H$.

$$\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{B}(H)$$

be bounded measurable. We define for f in the Schwartz space $\mathcal{S}(\mathbb{R}^n; H)$

$$T_\sigma f(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \sigma(x, \xi) \widehat{f}(\xi) e^{ix\xi} \, \mathrm{d}\xi.$$

where we write $\widehat{f}$ for the Fourier transform of f. We shall also use the notation $|\xi|$ for the Euclidean distance in $\mathbb{R}^n$ and write henceforth $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$. For the rest of this section, we will ignore the normalisation constant in the definition of the Fourier transform.

Theorem 4.1. *Suppose that there exists a non-decreasing function $\omega : [0, \infty) \to [0, \infty)$ such that*

$$\|\partial_\xi^\alpha \sigma(x, \xi)\|_{\mathcal{B}(H)} \leq C_\alpha \langle \xi \rangle^{-|\alpha|}$$

and

$$\|\partial_\xi^\alpha \sigma(x, \xi) - \partial_\xi^\alpha \sigma(x', \xi)\|_{\mathcal{B}(H)} \leq C_\alpha \langle \xi \rangle^{-|\alpha|} \omega(|x - x'|)$$

for all $|\alpha| \leq [n/2] + 2$ and some positive constant C_α. Suppose in addition that

$$\int_0^1 \omega(t)^2 \, \frac{\mathrm{d}t}{t} < \infty,$$

then T_σ is a bounded operator on $L_2(\mathbb{R}; H)$.[†]

Proof. Let $\varphi \in C^\infty(\mathbb{R}^n)$ be a non-negative function with support in the unit ball such that $\int_{\mathbb{R}^n} \varphi(x) \, \mathrm{d}x = 1$. Fix a constant $\delta \in (0, 1)$ and define the symbols

$$\sigma_1(x, \xi) := \int_{\mathbb{R}^n} \varphi(y) \, \sigma(x - \frac{y}{\langle \xi \rangle^\delta}, \xi) \, \mathrm{d}y$$

and

$$\sigma_2(x, \xi) := \sigma(x, \xi) - \sigma_1(x, \xi).$$

It is clear that

$$\sigma_1(x, \xi) = \int_{\mathbb{R}^n} \varphi(\langle \xi \rangle^\delta (x - y)) \, \sigma(y, \xi) \langle \xi \rangle^{n\delta} \, \mathrm{d}y$$

and one checks that

$$\|\partial_x^\beta \partial_\xi^\alpha \sigma_1(x, \xi)\|_{\mathcal{B}(H)} \leq c_{\alpha,\beta} \langle \xi \rangle^{-|\alpha| + \delta|\beta|} \leq c_{\alpha,\beta} \langle \xi \rangle^{-\delta(|\alpha| - |\beta|)} \tag{4.1}$$

and

$$\|\partial_\xi^\alpha \sigma_2(x, \xi)\|_{\mathcal{B}(H)} \leq c_\alpha \omega(\langle \xi \rangle^{-\delta}) \langle \xi \rangle^{-|\alpha|} \tag{4.2}$$

[†]In [13], $L_2(\mathbb{R}; H)$–boundedness of T_σ is claimed for symbols $\sigma : \mathbb{R} \times \mathbb{R} \to \mathcal{B}(H)$ that admit a bounded holomorphic extension to a double sector of $\mathbb{C}$ in the variable ξ, without any kind of regularity in the variable x.

for $|\alpha| \leq [\frac{n}{2}] + 2$ and all β. Using (4.2) we conclude by the next theorem that T_{σ_2} is bounded on $L_2(\mathbb{R}^n; H)$. The boundedness of T_{σ_1} on $L_2(\mathbb{R}^n; H)$ follows from (4.1) and Theorem 1 in [20]. Note that it is assumed there that the symbol is C^∞ but the estimate needed in Theorem 1 is exactly (4.1) with $|\alpha| \leq [\frac{n}{2}] + 2$. $\qquad\square$

Theorem 4.2. *Let $\delta \in (0,1)$ and $w : [0,1] \to \mathbb{R}_+$ be a non-decreasing measurable function satisfying*

$$\int_0^1 \omega(t)^2 \, \tfrac{\mathrm{d}t}{t} < \infty.$$

If a bounded strongly measurable symbol $\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{B}(H)$ satisfies

$$\|\partial_\xi^\alpha \sigma(x,\xi)\|_{\mathcal{B}(H)} \leq C_\alpha \langle\xi\rangle^{-\alpha} \, \omega(\langle\xi\rangle^{-\delta}) \tag{4.3}$$

for $|\alpha| \leq \kappa := [n/2] + 1$, then T_σ is bounded on $L_2(\mathbb{R}^n; H)$.

Proof. Let φ be a non-negative C_c^∞ function satisfying $\varphi(\xi) = 1$ for $|\xi| \leq 2$ and $\varphi(\xi) = 0$ for $|\xi| > 3$. Then we may rewrite

$$\sigma(x,\xi) = \varphi(\xi)\sigma(x,\xi) + (1-\varphi(\xi))\sigma(x,\xi) = \sigma_1(x,\xi) + \sigma_2(x,\xi)$$

and treat both parts separately. For the first part, let $f \in \mathcal{S}(\mathbb{R}^n; H)$ and $h \in H$. Then

$$[(T_{\sigma_1}f)(x) \,|\, h]_H = \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left[\sigma_1(x,\xi)\widehat{f}(\xi) \,|\, h\right]_H \mathrm{d}\xi$$

$$= \int_{\mathbb{R}^n} \left[f(y) \,|\, \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\sigma_1(x,\xi)^*h \, \mathrm{d}\xi\right]_H \mathrm{d}y$$

$$=: \int_{\mathbb{R}^n} [f(y) \,|\, K(x, x-y)]_H \, \mathrm{d}y.$$

By Plancherel's theorem,

$$\int_{\mathbb{R}^n} \|\langle z\rangle^{2\kappa} K(x,z)\|^2 \, \mathrm{d}z \leq \sum_{|\alpha|\leq 2\kappa} c_\alpha \int_{\mathbb{R}^n} \|z^\alpha K(x,z)\|^2 \, \mathrm{d}z$$

$$= \sum_{|\alpha|\leq 2\kappa} c_\alpha \int_{\mathbb{R}^n} \|\partial_\xi^\alpha \sigma_1(x,\xi)^*h\|^2 \, \mathrm{d}z =: C_1\|h\|^2,$$

where C_1 is finite due to the support of σ_1. By the Cauchy-Schwarz inequality,

$$\|T_{\sigma_1}f\|_{L_2(\mathbb{R}^n;H)}^2$$

$$= \int_{\mathbb{R}^n} \sup_{h\in H, \|h\|\leq 1} \left|\int_{\mathbb{R}^n} [f(y) \,|\, K(x, x-y)]_H \, \mathrm{d}y\right|^2 \mathrm{d}x$$

$$\leq \int_{\mathbb{R}^n} \sup_{h\in H, \|h\|\leq 1} \left(\int_{\mathbb{R}^n} \langle x-y\rangle^{-2\kappa} \|f(y)\|_H^2 \, \mathrm{d}y\right)\left(\int_{\mathbb{R}^n} \|\langle x-y\rangle^{2\kappa} K(x, x-y)\|_H^2 \, \mathrm{d}y\right) \mathrm{d}x$$

$$\leq C_1 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle x-y\rangle^{-2\kappa} \, \mathrm{d}x \, \|f(y)\|_H^2 \, \mathrm{d}y = C_1 C_2 \|f\|_{L_2(\mathbb{R}^n;H)}^2.$$

Thus, T_{σ_1} is bounded on $L_2(\mathbb{R}^n; H)$.

Next we show boundedness of T_{σ_2}. Recall that $\mathrm{supp}(\sigma_2) \subset \{(x,\xi) : |\xi| \geq 2\}$. Let $\phi \in C_c^\infty$ such that $\mathrm{supp}(\phi) \subseteq [1,2]$ and $\int_0^\infty |\phi(t)|^2 \, \tfrac{\mathrm{d}t}{t} = 1$. Let $f \in \mathcal{S}(\mathbb{R}^n; H)$ and $h \in H$.

$$[(T_{\sigma_2}f)(x) \,|\, h]_H = \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left[\sigma_2(x,\xi)\widehat{f}(\xi) \,|\, h\right]_H \mathrm{d}\xi$$

$$= \int_0^\infty \int_{\mathbb{R}^n} e^{ix\cdot\xi}\phi(|t\xi|)^2 \left[\widehat{f}(\xi) \,|\, \sigma_2(x,\xi)^*h\right]_H \mathrm{d}\xi \, \tfrac{\mathrm{d}t}{t}$$

$$= \int_0^1 \int_{\mathbb{R}^n} \left[e^{ix\cdot\xi}\phi(|t\xi|)\widehat{f}(\xi) \,|\, \phi(|t\xi|) \, \sigma_2(x,\xi)^*h\right]_H \mathrm{d}\xi \, \tfrac{\mathrm{d}t}{t}$$

$$= \int_0^1 \int_{\mathbb{R}^n} \left[t^{-n}e^{ix/t\cdot\xi}\phi(|\xi|)\widehat{f}(\xi/t) \,|\, \phi(|\xi|) \, \sigma_2(x,\xi/t)^*h\right]_H \mathrm{d}\xi \, \tfrac{\mathrm{d}t}{t}.$$

Recall that $\phi \in \mathcal{S}(\mathbb{R})$, so that by Plancherel's theorem

$$[(T_{\sigma_2}f)(x) \,|\, h]_H = \int_0^1 \int_{\mathbb{R}^n} [K_1(t,x,z) \,|\, K_2(t,x,z)]_H \,\mathrm{d}z \,\tfrac{\mathrm{d}t}{t},$$

where K_1 and K_2 are the respective Fourier transforms

$$K_1(t,x,z) = \int_{\mathbb{R}^n} t^{-n} e^{i(x/t-z)\cdot\xi} \phi(|\xi|) \widehat{f}(\xi/t) \,\mathrm{d}\xi = \int_{\mathbb{R}^n} e^{i(x-tz)\cdot\xi} \phi(|t\xi|) \widehat{f}(\xi) \,\mathrm{d}\xi$$

and

$$K_2(t,x,z) = \int_{\mathbb{R}^n} e^{-iz\cdot\xi} \phi(|\xi|) \sigma_2(x,\xi/t)^* h \,\mathrm{d}\xi.$$

Now, by the Cauchy-Schwarz inequality,

$$\left| [(T_{\sigma_2}f)(x) \,|\, h]_H \right|^2$$

$$= \left| \int_0^1 \int_{\mathbb{R}^n} \left[\langle z \rangle^{-\kappa} K_1(t,x,z) \,|\, \langle z \rangle^{\kappa} K_2(t,x,z) \right]_H \,\mathrm{d}z \,\tfrac{\mathrm{d}t}{t} \right|^2$$

$$\leq \left(\int_0^1 \int_{\mathbb{R}^n} \left\| \langle z \rangle^{-\kappa} K_1(t,x,z) \right\|_H^2 \,\mathrm{d}z \,\tfrac{\mathrm{d}t}{t} \right) \left(\int_0^1 \int_{\mathbb{R}^n} \left\| \langle z \rangle^{\kappa} K_2(t,x,z) \right\|_H^2 \,\mathrm{d}z \,\tfrac{\mathrm{d}t}{t} \right).$$

Observe that $\langle z \rangle^{\kappa} K_2(t,x,z) = \mathcal{F}\big((I-\Delta_\xi)^{\kappa/2} \phi(|\xi|) \sigma_2(x,\xi/t)^* h \big)(z)$. Recall that $\phi \in C_c^\infty(\mathbb{R})$ has its support in $[1,2]$, so that, for $|\xi| \geq 1$, and using the growth assumption (4.3) on derivatives of σ,

$$\left| \partial_\xi^\beta \phi(|\xi|) \right| \leq C \langle \xi \rangle^{-|\beta|} \leq C$$

$$\left\| \partial_\xi^\gamma \sigma_2(x,\xi/t)^* h \right\| \leq C' \, t^{-|\gamma|} \|h\| \left(|\xi/t|^{-|\gamma|} \omega(|\xi/t|^{-\delta}) \right) \leq C'' \|h\| \, \omega(t^{+\delta}).$$

Note that we used here the monotonicity of ω. The Dini type condition on ω then gives

$$\int_0^1 \int_{\mathbb{R}^n} \left\| \langle z \rangle^{\kappa} K_2(t,x,z) \right\|_H^2 \,\mathrm{d}z \,\tfrac{\mathrm{d}t}{t} \leq C'' \|h\|_H^2 \int_0^1 \int_{1 \leq |\xi|^2 \leq 2} \omega(t^{+\delta})^2 \,\mathrm{d}\xi \,\tfrac{\mathrm{d}t}{t} =: C_2 \|h\|_H^2.$$

We conclude by observing that, again by Plancherel,

$$\int_{\mathbb{R}^n} \sup_{h \in H, \|h\| \leq 1} \left| [(T_{\sigma_2}f)(x) \,|\, h]_H \right|^2 \,\mathrm{d}x$$

$$\leq C_2 \int_{\mathbb{R}^n} \int_0^1 \int_{\mathbb{R}^n} \langle z \rangle^{-2\kappa} \left\| K_1(t,x,z) \right\|_H^2 \,\mathrm{d}z \,\tfrac{\mathrm{d}t}{t} \,\mathrm{d}x$$

$$= C_2 \int_0^1 \int_{\mathbb{R}^n} \langle z \rangle^{-2\kappa} \int_{\mathbb{R}^n} \left\| K_1(t,x,z) \right\|_H^2 \,\mathrm{d}x \,\mathrm{d}z \,\tfrac{\mathrm{d}t}{t}$$

$$= C_2' \int_0^1 \int_{\mathbb{R}^n} \langle z \rangle^{-2\kappa} \int_{\mathbb{R}^n} |\phi(|t\xi|)|^2 \|\widehat{f}(\xi)\|_H^2 \,\mathrm{d}\xi \,\mathrm{d}z \,\tfrac{\mathrm{d}t}{t}$$

$$\leq c_n C_2' \int_{\mathbb{R}^n} \|\widehat{f}(\xi)\|_H^2 \int_0^\infty |\phi(|t\xi|)|^2 \,\tfrac{\mathrm{d}t}{t} \,\mathrm{d}\xi$$

$$= c_n C_2' \int_{\mathbb{R}^n} \|\widehat{f}(\xi)\|_H^2 \,\mathrm{d}\xi.$$

Therefore, T_{σ_2} and hence T_σ are bounded on $L_2(\mathbb{R}^n; H)$. $\qquad\square$

5. Examples

In this section we discuss some examples and applications of our results. We shall focus on two simple but relevant linear problems which involve elliptic operators. Note that following [3], we may also consider quasi-linear evolution equations. Our maximal regularity results can be used to improve some results in [3] on existence of solutions to quasi-linear problems in the sense that we assume less regularity with respect to t of the coefficients of the equations. We shall not pursue this direction here.

5.1. **Elliptic operators.** Define on $H = L_2(\mathbb{R}^d, \mathrm{d}x)$ the sesquilinear forms

$$\mathfrak{a}(t; u, v) = \sum_{k,j=1}^{d} \int_{\mathbb{R}^d} a_{kj}(t,x)\partial_k u \overline{\partial_j v} \, \mathrm{d}x \text{ for } u,v \in W_2^1(\Omega).$$

We assume that $a_{kj} : [0,\tau] \times \mathbb{R}^d \to \mathbb{C}$ such that:

$$a_{kj} \in L_\infty([0,\tau] \times \mathbb{R}^d) \text{ for } 1 \le k, j \le d,$$

and

$$\mathrm{Re} \sum_{k,j=1}^{d} a_{kj}(t,x)\xi_k \overline{\xi_j} \ge \nu|\xi|^2 \text{ for all } \xi \in \mathbb{C}^d \text{ and a.e. } (t,x) \in [0,\tau] \times \mathbb{R}^d.$$

Here $\nu > 0$ is a constant independent of t.

It is easy to check that $\mathfrak{a}(t; \cdot, \cdot)$ is $W_2^1(\mathbb{R}^d)$-bounded and quasi-coercive. The associated operator with $\mathfrak{a}(t; \cdot, \cdot)$ is the elliptic operator given by the formal expression

$$A(t)u = -\sum_{k,j=1}^{d} \partial_j \left(a_{kj}(t,.)\partial_k u\right).$$

In addition to the above assumptions we assume that for some constant M and $\alpha > 1/2$

$$|a_{kj}(t,x) - a_{kj}(s,x)| \le M|t-s|^\alpha \text{ for a.e. } x \in \mathbb{R}^d \text{ and all } t,s \in [0,\tau]. \tag{5.1}$$

By the Kato square root property, it is known that $\mathcal{D}(A(0)^{1/2}) = W_2^1(\mathbb{R}^d)$, see [5]. Therefore, applying Corollary 1.4 we conclude that for every $f \in L_2(0,\tau; H)$ the problem

$$\begin{cases} u'(t) - \sum_{k,j=1}^{d} \partial_j \left(a_{kj}(t,.)\partial_k u(t)\right) &= f(t), \ t \in (0,\tau] \\ u(0) &= u_0 \in W_2^1(\mathbb{R}^d) \end{cases}$$

has a unique solution $u \in W_2^1(0,\tau; H) \cap L_2(0,\tau; W_2^1(\mathbb{R}^d))$.

5.2. **Time-dependent Robin boundary conditions.** We consider here the Laplacian on a domain Ω with a time dependent Robin boundary condition

$$\partial_\nu u(t) + \beta(t,.)u = 0 \text{ on } \Gamma = \partial\Omega, \tag{5.2}$$

for some function $\beta : [0,\tau] \times \Gamma \to \mathbb{R}$. This example is taken from [3]. The difference here is that we assume less regularity on β and also that we can treat maximal L_p–regularity for $p \in (1,\infty)$ whereas the results in [3] are restricted to the case $p=2$.

Let Ω be a bounded domain of $\mathbb{R}^d$ with Lipschitz boundary $\Gamma = \partial\Omega$ and denote by σ the $(d-1)$-dimensional Hausdorff measure on Γ. Let $\beta : [0,\tau] \times \Gamma \to \mathbb{R}$ be a bounded measurable function which is Hölder continuous w.r.t. the first variable, i.e.,

$$|\beta(t,x) - \beta(s,x)| \le M|t-s|^\alpha \tag{5.3}$$

for some constants M, $\alpha > 1/2$ and all $t,s \in [0,\tau]$, $x \in \Gamma$. We consider the symmetric form

$$\mathfrak{a}(t; u, v) = \int_\Omega \nabla u \nabla v \, dx + \int_\Gamma \beta(t,.)uv \, d\sigma, \ u,v \in W_2^1(\Omega). \tag{5.4}$$

The form $\mathfrak{a}(t; \cdot, \cdot)$ is $W_2^1(\Omega)$-bounded and quasi-coercive. The first statement follows from the continuity of the trace operator and the boundedness of β. The second one is a consequence of the inequality

$$\int_\Gamma |u|^2 \, d\sigma \le \varepsilon\|u\|_{W_2^1}^2 + c_\varepsilon\|u\|_{L_2(\Omega)}^2, \tag{5.5}$$

which is valid for all $\varepsilon > 0$ (c_ε is a constant depending on ε). Note that (5.5) is a consequence of compactness of the trace as an operator from $W_2^1(\Omega)$ into $L_2(\Gamma, d\sigma)$, see [21, Chap. 2 § 6, Theorem 6.2].

The operator $A(t)$ associated with $\mathfrak{a}(t; \cdot, \cdot)$ on $H := L_2(\Omega)$ is (minus) the Laplacian with time dependent Robin boundary conditions (5.2). As in [3], we use the following weak definition of the normal derivative. Let $v \in W_2^1(\Omega)$ such that $\Delta v \in L_2(\Omega)$. Let $h \in L_2(\Gamma, d\sigma)$. Then $\partial_\nu v = h$

by definition if $\int_\Omega \nabla v \nabla w + \int_\Omega \Delta v w = \int_\Gamma h w \, d\sigma$ for all $w \in W_2^1(\Omega)$. Based on this definition, the domain of $A(t)$ is the set

$$\mathcal{D}(A(t)) = \{v \in W_2^1(\Omega) : \Delta v \in L_2(\Omega), \partial_\nu v + \beta(t)v|_\Gamma = 0\},$$

and for $v \in \mathcal{D}(A(t))$ the operator is given by $A(t)v = -\Delta v$.

Observe that the form $\mathfrak{a}(t; \cdot, \cdot)$ is symmetric, so that $W_2^1(\Omega) = \mathcal{D}(A(0)^{1/2})$. From Corollary 1.4 it follows that the heat equation

$$\begin{cases} u'(t) - \Delta u(t) = f(t) & \\ \qquad\qquad u(0) = u_0 & u_0 \in W_2^1(\Omega) \\ \partial_\nu u(t) + \beta(t,.)u = 0 & \text{on } \Gamma \end{cases}$$

has a unique solution $u \in W_2^1(0, \tau; L_2(\Omega))$ whenever $f \in L_2(0, \tau; L_2(\Omega))$. This example is also valid for more general elliptic operators than the Laplacian.

Note that in both examples we have assumed α-Hölder continuity in (5.1) and (5.3). We could replace this assumption by piecewise α-Hölder continuity as authorised by Corollary 1.4.

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