Square function estimates and functional calculi

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To the memory of Nigel Kalton (1946–2010)

Abstract.

Contents

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1. Introduction

Square functions and square function estimates are a classical topic and a central tool in harmonic analysis, in particular in the so-called Littlewood–Paley theory. Their history can be traced back to almost a century ago, see [?] for a historical account and [?, ?, ?] for the development from the 1960s on. One of the classical instances of a square function is

$$(S_\phi f)(x) := \Big(\int_0^\infty \big|(\phi_t * f)(x)\big|^2 \, \tfrac{\mathrm{d}t}{t} \Big)^{1/2} \tag{1.1}$$

where $\phi \in \mathrm{L}_2(\mathbb{R}^d)$ decays reasonably fast at infinity and $\phi_t(x) = t^{-d}\phi(x/t)$ for $x \in \mathbb{R}^d$ and $t > 0$. A "square function estimate" then reads

$$\|S_\phi f\|_{\mathrm{L}_p} = \Big\| \Big(\int_0^\infty \big|(\phi_t * f)(x)\big|^2 \, \tfrac{\mathrm{d}t}{t} \Big)^{1/2} \Big\|_{\mathrm{L}_p} \lesssim \|f\|_{\mathrm{L}_p} . \tag{1.2}$$

In many situations, ϕ is radial. Then its Fourier transform is radial, too, and can be written as $\widehat{\phi}(\xi) = \psi(|\xi|)$ for $\xi \in \mathbb{R}^d$. Hence,

$$\phi_t * f = \mathcal{F}^{-1}(\widehat{\phi}(t\xi) \cdot \widehat{f}\,(\xi)) = \mathcal{F}^{-1}(\widehat{\psi}(|t\xi|) \cdot \widehat{f}\,(\xi)) = \psi(t\sqrt{-\Delta})\, f,$$

where we employ the functional calculus for the Laplace, or better, the Poisson operator. Hence, the abstract form of (1.2) is

$$\Big\| \Big(\int_0^\infty |\psi(tA)f|^2 \, \tfrac{\mathrm{d}t}{t} \Big)^{1/2} \Big\|_{\mathrm{L}_p} \lesssim \|f\|_{\mathrm{L}_p} , \tag{1.3}$$

where $A := \sqrt{-\Delta}$; and taking $\psi(z) := z\mathrm{e}^{-z}$ we recover the classical Littlewood-Paley g-function.

From the mid 1980's on, the theory of functional calculus for sectorial operators was developed by several people. Building on the seminal works [?] and [?] and inspired by [?], Cowling, Doust, McIntosh and Yagi in [?] established a strong link between the boundedness of the H^∞-calculus for sectorial operators A on (closed subspaces of) L_p-spaces and square functions of the form (1.3). Kalton and Weis in an unpublished and unfortunately never finalized manuscript [?] then showed how one could pass from L_p-spaces to general Banach spaces. Their manuscript subsequently circulated and inspired a considerable amount of research.

The main novelty in Kalton and Weis' approach from [?] was to employ the class of so-called *γ-radonifying operators* in order to define square functions. This step is motivated by two observations. On the one hand,

$$\Big(\int_0^\infty |\psi(tA)f|^2 \,\tfrac{\mathrm{d}t}{t}\Big)^{1/2} = \Big(\sum_{k=1}^\infty |(Tf)e_n|^2\Big)^{1/2}$$

where $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $H := \mathrm{L}_2(\mathbb{R}_+; {}^{\mathrm{d}t}\!/_t)$ and $Tf : H \to X := \mathrm{L}_p(\mathbb{R}^d)$ is the operator defined by

$$(Tf)\, h := \int_0^\infty h(t)\psi(tA)f \,\tfrac{\mathrm{d}t}{t} \qquad (h \in H). \tag{1.4}$$

(This works in every Banach lattice X, see Appendix **??**.) The second, more decisive step, is based on the norm equivalence

$$\Big\| \Big(\sum\nolimits_k |x_k|^2\Big)^{1/2} \Big\|_X \sim \Big(\mathbb{E}\Big\|\sum\nolimits_k \gamma_k \otimes x_k\Big\|_X^2\Big)^{1/2}$$

where $(\gamma_k)_k$ is an independent sequence of standard Gaussian random variables. (This equivalence holds true in every Banach lattice X of finite cotype, see Theorem **??**.) Hence, the square function estimate (1.3) can be reformulated as

$$\Big(\mathbb{E}\Big\|\sum\nolimits_k \gamma_k \otimes (Tf)e_k\Big\|_X^2\Big)^{1/2} \lesssim \|f\|_X \tag{1.5}$$

with Tf being as above. But this means that the operator Tf is γ-radonifying and its γ-norm satisfies $\|Tf\|_\gamma \lesssim \|f\|_X$. (See Appendix B for the definition and basic properties of the space $\gamma(H;X)$ of γ-radonifying operators.)

The decisive feature of this new formulation of the square function estimate is that the lattice structure of $X = \mathrm{L}_p$ does not appear any more. With it, a door is opened to define square function estimates over general Banach spaces X. Hence, the following definition.

Definition 1.1. Let X, Y be Banach spaces. Then an (abstract) (X,Y)**-square function** is a linear operator

$$Q : \mathrm{dom}(Q) \to \gamma(H;Y), \quad \mathrm{dom}(Q) \subseteq X$$

for some Hilbert space H. A **dual** (X,Y)**-square function** is a linear operator

$$Q^d : \mathrm{dom}(Q^d) \to \gamma(H;Y)' \cong \gamma'(H';Y'), \quad \mathrm{dom}(Q^d) \subseteq X'$$

for some Hilbert space H.

A **square function estimate** or a **quadratic estimate** for the (X,Y)-square function Q is any inequality of the form

$$\|Qx\|_\gamma \le C\,\|x\| \qquad \text{for all } x \in \mathrm{dom}(Q) \tag{1.6}$$

for some constant $C \geq 0$. If Q is densely defined, such a square function estimate holds true if and only if Q extends to a bounded operator $Q : X \to \gamma(H; Y)$. Note that a closed and densely defined square function satisfies a square function estimate if and only if it is fully defined.

Similarly, an estimate of the form

$$\left\| Q^d x' \right\|_{\gamma'} \leq C \left\| x' \right\| \qquad (x' \in \mathrm{dom}(Q^d))$$

is called a **dual square function (quadratic) estimate**. The usual examples of dual square functions are not densely, but only weakly*-densely defined, and hence in general a dual square function estimate does not automatically lead to a bounded operator $X' \to \gamma'(H'; Y')$.

Note that one always arrives at a (X, Y)-square function by starting with an operator

$$A : \mathrm{dom}(A) \to \mathcal{L}(H; Y), \qquad \mathrm{dom}(A) \subseteq X$$

and taking its part in $\gamma(H; X)$, i.e., $A_\gamma : \mathrm{dom}(A_\gamma) \to \gamma(H; Y)$ with

$$\mathrm{dom}(A_\gamma) = \{x \in \mathrm{dom}(A) \mid Ax \in \gamma(H; Y)\}, \quad A_\gamma x := Ax.$$

It is easy to see that A_γ is a closed square function if A is closed. (Obviously, a similar construction is possible to obtain dual square functions.)

If one takes H the one-dimensional Hilbert space, then $\mathcal{L}(H; Y) \cong Y$, and hence every (bounded) operator can be trivially viewed as a square function (estimate).

A Functional Calculus with Square Functions.
In this paper, more precisely in Chapter 2, we build on the above definition and present a novel and systematic account of square function estimates related to functional calculus. For the sake of readiblity, this will be carried out in the context of H^∞-calculus only. However, in Chapter **??** we sketch how notions and results can be generalized to other types of functional calculi.

One main feat is that we cover square functions associated with expressions of the general form

$$\psi(t, A),$$

where the common square functions for sectorial or strip-type operators usually work with expressions of the form

$$\psi(tA) \quad \text{or} \quad \psi(t + A),$$

repectively. What is more, we do not deal with square functions individually but develop a whole *calculus of square functions* in the spirit of general functional calculus philosophy. (That is: working with functions instead of working with operators.)

Let us illustrate this idea for the case of sectorial operators. (For convenience we have included a brief introduction to the functional calculi of strip-type and sectorial operators in Chapter **??**.)

Given a sectorial operator A of angle ω_0 on a Banach space X and a function $\psi \in \mathrm{H}_0^\infty(\mathrm{S}_\omega)$ with $\omega \in (\omega_0, \pi)$ one considers — for fixed $x \in X$ — the vector-valued function

$$(0,\infty) \longrightarrow X, \qquad t \mapsto \psi(tA)x.$$

Following Kalton and Weis [**?**] one should interpret this function as an operator

$$T_\psi x : \mathrm{L}_2(\mathbb{R}_+; {}^{\mathrm{d}t}\!/\!_t) \longrightarrow X$$

via (Pettis) integration[1], i.e.

$$(T_\psi x)\, h := \int_0^\infty h(t)\psi(tA)x \,\frac{\mathrm{d}t}{t}$$

for $h \in H := \mathrm{L}_2(\mathbb{R}_+; {}^{\mathrm{d}t}\!/\!_t)$, cp. (1.4). One then asks whether the operator $T_\psi x$ is γ-radonifying and obeys an estimate of the form

$$\left\| T_\psi x \right\|_{\gamma(H;X)} \lesssim \left\| x \right\| . \tag{1.7}$$

However, for $x \in \mathrm{dom}(A) \cap \mathrm{ran}(A)$ one can employ the definition of the functional calculus by Cauchy integrals to obtain

$$\begin{aligned}(T_\psi x)h &= \int_0^\infty h(t)\psi(tA)x \,\tfrac{\mathrm{d}t}{t} = \int_0^\infty h(t)\frac{1}{2\pi\mathrm{i}}\int_\Gamma \psi(tz)R(z,A)x \,\mathrm{d}z \,\tfrac{\mathrm{d}t}{t}\\ &= \frac{1}{2\pi\mathrm{i}}\int_\Gamma \Big(\int_0^\infty h(t)\psi(tz)\,\tfrac{\mathrm{d}t}{t}\Big) R(z,A)x\,\mathrm{d}z\\ &= \Big(\int_0^\infty h(t)\psi(tz)\,\tfrac{\mathrm{d}t}{t}\Big)(A)x.\end{aligned}$$

This step indicates an important change in perspective: We interpret $\psi(tz)$ as a function of $z \in \mathrm{S}_\omega$ with values in H and write

$$\Psi : \mathrm{S}_\omega \longrightarrow H, \qquad \Psi(z)(t) := \psi(tz).$$

Then the mapping

$$z \mapsto (h \circ \Psi)(z) := \int_0^\infty h(t)\psi(tz)\,\tfrac{\mathrm{d}t}{t} = \langle h, \Psi(z)\rangle$$

[1] cf. Appendix **??**

is a scalar H^∞-function into which A can be inserted by the functional calculus. Finally, the resulting operator can be applied to $x \in \mathrm{dom}(A) \cap \mathrm{ran}(A)$ resulting in

$$(h \circ \Psi)(A)x = \Big(\int_0^\infty h(t)\psi(tz) \,\tfrac{\mathrm{d}t}{t} \Big)(A)x.$$

Now, for fixed such x this yields the operator

$$\Psi(A)x : H \to X, \qquad (\Psi(A)x)h := (h \circ \Psi)(A)x.$$

(Note that $\Psi(A)x = T_\psi x$, by the computation above.) This operator turns out to be γ-radonifying (see Section 7.4), so we end up with a square function

$$\Psi(A) : \mathrm{dom}(A) \cap \mathrm{ran}(A) \to \gamma(H; X).$$

In this way, the problem of a square function estimate (1.7) has been tranformed into the problem of the boundedness of the operator $\Psi(A)$ on X.

Evolving this idea one realizes that the mapping

$$\Psi \mapsto \Psi(A),$$

which associates an (in general unbounded) square function with an H-valued H^∞-function Ψ, has many properties of a functional calculus (Lemma 2.9, Lemma 2.10). The only difference now is that the functions we are considering have to be H-valued instead of scalar-valued, and this calculus is a module rather than an algebra homomorphism. We call this calculus the **square functional calculus** or, following Le Merdyin [**?**], the *vectorial functional calculus*.

As a result of this new perspective, the problem of a square function estimate is recognised as just another instance of the central problem of functional calculus, namely whether applying an unbounded functional calculus to a certain function leads to a bounded operator or not.

Achievements.
The main objectives of our paper are on the one hand to devise abstract theorems that govern the calculus of square functions and on the other hand to reduce known concrete results (for sectorial and strip type operators) to them. This is achieved in Chapters 3 and 4 where we identify *three basic principles*:

Subordination: The two square functions are connected via a bounded operator between the underlying Hilbert spaces. (This principle is often applied to the Fourier transform in the case of strip-type and the Mellin transform in the case of sectorial operators.)

Integral representations: Here square function and dual square function estimates are combined with a deep result from the theory of γ-radonifying operators. Integral representations are the key to results that assert bounded (vectorial) H^∞-functional calculus from the boundedness of certain carefully chosen square functions (Theorem 3.12).

ℓ_1-Frame-boundedness: This is a (natural, but still rather enigmatic) boundedness concept for subsets of Hilbert spaces. It lies at the heart of all known results inferring the boundedness of certain square functions from a bounded scalar H^∞-calculus. Basically, the main abstract theorem here asserts that if the H-valued function Ψ has ℓ_1-frame-bounded range, then the associated square function $\Psi(A)$ is bounded (Theorem 4.3).

These abstract results are then, in Chapters 7 and 8 applied to operators of strip type and to sectorial operators. In particular, we discuss all the classical integral representations used to infer bounded H^∞-calculus from bounded square functions, namely:

- Cauchy–Gauß Representation (Section 8.1)
- Poisson Representation (Section 8.2)
- CDMcY-Representation (Section 8.3)
- Laplace Transform Representation (Section 8.4)
- Franks-McIntosh Representation (Section 8.5
- Singular Cauchy Representation (Section 8.6).

(The first one of these is actually new, and has not been used so far in the literature.)

One consequence of Theorem 3.12 on integral representations is that it allows to infer not just the boundedness of a *scalar* H^∞-calculus but of the *vectorial* one. Hence one of the main results, Theorem 8.1, states that a densely defined operator (on a Banach space with finite cotype) with a bounded scalar H^∞-calculus on a strip has a bounded H^∞-square functional calculus on each larger strip. (This result has been obtained independently of us by Le Merdy in [**?**].)

Relation with Work by Others.

Was sollen wir hier schreiben?

Notation and Terminology.
Banach spaces are denoted by X, Y, Z and understood to be *complex* unless otherwise noted. The duality between a Banach space X and its dual space X' is denoted by $\langle \cdot, \cdot \rangle$ or $\langle \cdot, \cdot \rangle_{X,X'}$.

For a closed linear operator A on a complex Banach space X we denote by $\operatorname{dom}(A)$, $\operatorname{ran}(A)$, $\ker(A)$, $\sigma(A)$ and $\varrho(A)$ the *domain*, the *range*, the *kernel*, the *spectrum* and the *resolvent set* of A, respectively. The norm-closure of the range is written as $\overline{\operatorname{ran}}(A)$. The space of bounded linear operators on X is denoted by $\mathcal{L}(X)$. For two possibly unbounded linear operators A, B on X their *product* AB is defined on its natural domain $\operatorname{dom}(AB) := \{x \in \operatorname{dom}(B) \mid Bx \in \operatorname{dom}(A)\}$. An inclusion $A \subseteq B$ denotes inclusion of graphs, i.e., it means that B extends A.

The inner product of two elements u, v of a Hilbert space H is generically written as $[u, v]$ or $[u, v]_H$. We usually do *not* identify a Hilbert space H with its dual space H'. Rather, we write

$$\overline{u} := [\,\cdot\,, u] \in H',$$

for $u \in H$, i.e., the mapping

$$H \longrightarrow H', \qquad u \longmapsto \overline{u} = [\,\cdot\,, u]$$

is the canonical (conjugate-linear) bijection of H onto its dual H'. The definition

$$[\overline{u}, \overline{v}]_{H'} := [v, u]_H \qquad (u,\ v \in H) \tag{1.8}$$

turns H' canonically into a Hilbert space, and a short computation yields $\overline{\overline{u}} = u$ under the canonical identification $H = H''$. Moreover, (1.8) becomes

$$[\overline{x}, \overline{y}]_H = [y, x]_{H'} \qquad (x, y \in H'). \tag{1.9}$$

If $H = \mathrm{L}_2(\Omega) = \mathrm{L}_2(\Omega; \mathbb{K})$ for some measure space (Ω, Σ, μ), we can identify $H' = \mathrm{L}_2(\Omega)$ via the duality

$$H \times H \longrightarrow \mathbb{K}, \qquad (u, v) \longmapsto \langle u, v \rangle := \int_\Omega u\, v \,\mathrm{d}\mu \qquad (u,\ v \in \mathrm{L}_2(\Omega)). \tag{1.10}$$

Under this identification, the conjugate $\overline{u}$ of $u \in H$ as defined above coincides with the usual complex conjugate of u as a function on Ω.

For an open subset $O \subseteq \mathbb{C}$ of the complex plane we let $\mathrm{H}^\infty(O)$ be the algebra of bounded holomorphic functions on O with norm $\|f\|_{\mathrm{H}^\infty} := \sup\{|f(z)| \mid z \in O\}$.

Unless explicitly noted otherwise, the real line $\mathbb{R}$ carries the Lebesgue measure $\mathrm{d}t$ and the set $(0, \infty)$ of positive reals carries the measure $\mathrm{d}t/t$.

We abbreviate

$$\mathrm{L}_p^*(0,\infty) := \mathrm{L}_p((0,\infty); {}^{\mathrm{d}t}\!/\!_t) \qquad (0 < p \le \infty).$$

The *Fourier transform* of a function $f \in \mathrm{L}_1(\mathbb{R})$ is

$$\mathcal{F}(f)(t) = \widehat{f}(t) = \int_{\mathbb{R}} f(s)\mathrm{e}^{-\mathrm{i}st}\,\mathrm{d}s \qquad (t \in \mathbb{R}).$$

The *inverse Fourier transform* is then given by the formula

$$(\mathcal{F}^{-1}g)(s) = g^{\vee}(s) = \frac{1}{2\pi}\int_{\mathbb{R}} g(t)\mathrm{e}^{\mathrm{i}st}\,\mathrm{d}t \qquad (s \in \mathbb{R})$$

for $g \in \mathrm{L}_1(\mathbb{R})$.

Part I. Theory

2. Square Functions Associated with a Functional Calculus

In this chapter we shall associate square functions with a given (holomorphic) functional calculus. The definition of a general square function, the outline of our approach and a justification for it have been presented in the Introduction (page 3) so that we do not repeat them here. We start with a short introduction on abstract functional calculus and then pass to the "vectorial" case, where square functions appear.

2.1. Scalar Functional Calculus.

Let $\mathcal{F}$ be a commutative algebra with a unit element $\mathbf{1}$ and X a Banach space, and let furthermore a mapping

$$\Phi : \mathcal{F} \to \{\text{closed single-valued operators on } X\}$$

be given. For each $f \in \mathcal{F}$ the set of its **regularisers** is

$$\mathrm{Reg}_\Phi(f) := \{e \in \mathcal{F} \mid \Phi(e),\ \Phi(ef) \in \mathcal{L}(X)\}.$$

A subset $\mathcal{M} \subseteq \mathrm{Reg}_\Phi(f)$ is called **determining** for $\Phi(f)$ if one has

$$\Phi(f)x = y \iff \forall e \in \mathcal{M} : \Phi(ef)x = \Phi(e)y \tag{2.1}$$

for all $x, y \in X$. Since $\Phi(f)$ is a (single-valued) operator it follows that if $\mathcal{M}$ is determining for $\Phi(f)$ then

$$\bigcap_{e \in \mathcal{M}} \ker(\Phi(e)) = \{0\}.$$

Now, the pair $(\mathcal{F}, \Phi)$ is called an **(unbounded) $\mathcal{F}$-calculus** on X if the following axioms hold:

1) $\Phi(\mathbf{1}) = \mathrm{I}$
2) $\Phi(f) + \Phi(g) \subseteq \Phi(f+g)$.
3) $\Phi(f)\Phi(g) \subseteq \Phi(fg)$ and
$$\mathrm{dom}(\Phi(f)\Phi(g)) = \mathrm{dom}(\Phi(g)) \cap \mathrm{dom}(\Phi(fg)).$$
4) For each $f \in \mathcal{F}$ its set of regularisers $\mathrm{Reg}_\Phi(f)$ is determining for $\Phi(f)$.

Each element $e \in \bigcap_{f\in\mathcal{F}} \mathrm{Reg}_\Phi(f)$ is called a **universal regularizer**. A subset $\mathcal{M}$ of universal regularisers is called **(universally) determining** if it is determining for each $f \in \mathcal{F}$. (Universally determining sets need not exist.)

Remark 2.1. It follows easily from 2) and 3) that $\Phi(f+g) = \Phi(f) + \Phi(g)$ and $\Phi(fg) = \Phi(f)\Phi(g)$ if $\Phi(g) \in \mathcal{L}(X)$. From 3) it follows that if $e \in \mathrm{Reg}_\Phi(f)$ and $y = \Phi(f)x$, then
$$\Phi(e)y = \Phi(e)\Phi(f)x = \Phi(ef)x$$
and hence the implication "$\Longrightarrow$" in (2.1) already follows from 3).

If $e \in \mathrm{Reg}_\Phi(e)$ is injective, then it follows from 4) that
$$\Phi(f) = \Phi(e)^{-1}\Phi(ef).$$
In [**?**, Chapter 1] it has been described how this formula can be used to extend a so-called "elementary" functional calculus and this is the way most functional calculi are obtained. In [**?**] it has been described how the more general requirement 4) can be made the basis of a similar extension procedure. In any case, for the purpose of this paper, it is not relevant how to arrive at a given unbounded calculus. The only thing matters are its formal properties.

From now on, we shall almost exclusively consider the case $\mathcal{F} = \mathrm{H}^\infty(O)$, the algebra of bounded holomorphic functions on some open set $O \subseteq \mathbb{C}$ (or $\mathbb{C}^d$). Then we speak of a (possibly unbounded) H^∞-calculus on O. In this situation, if $\Phi(f) \in \mathcal{L}(X)$ for each $f \in \mathrm{H}^\infty(O)$, then
$$\Phi : \mathrm{H}^\infty(O) \to \mathcal{L}(X)$$
is an algebra homomorphism. (Conversely, each such algebra homomorphism constitutes a functional calculus, i.e., satisfies the axioms from above.) If, in addition, there is $C \geq 0$ such that
$$\|\Phi(f)\| \leq C\,\|f\|_{\mathrm{H}^\infty} \qquad \text{for all } f \in \mathrm{H}^\infty(O),$$
then we speak of Φ as a **bounded** H^∞**-calculus** on O. (By the closed graph theorem, in practically all interesting cases a $\mathrm{H}^\infty(O)$-calculus

satisfying $\Phi(f) \in \mathcal{L}(X)$ for all $f \in \mathrm{H}^\infty(O)$ will be a bounded H^∞-calculus.)

Remark 2.2. If $O \subseteq \mathbb{C}$ is open, then by Liouville's theorem the algebra $\mathrm{H}^\infty(\Omega)$ is only interesting if $\emptyset \neq O \neq \mathbb{C}$.

Now suppose that $(\mathrm{H}^\infty(O), \Phi)$ is a functional calculus on X and suppose in addition that $\mathbb{C} \setminus O$ has nonempty interior U, say. For each $\lambda \in U$ the function $r_\lambda(z) := (\lambda - z)^{-1}$ is holomorphic and bounded on O. If we suppose in addition that $R_\lambda := \Phi(r_\lambda) \in \mathcal{L}(X)$, then this yields a pseudo-resolvent on U. Hence by [**?**, Proposition A.2.4] there is a unique operator A with $R(\lambda, A) = R_\lambda$ for all $\lambda \in U$. (This operator is single-valued if and only if one/each R_λ is injective.) It is common to call Φ a functional calculus *for* A and write $f(A) := \Phi_A(f) := \Phi(f)$ for $f \in \mathrm{H}^\infty(O)$.

Note that if $f_n \in \mathrm{H}^\infty(O)$ is a uniformly bounded sequence which converges pointwise (= locally uniformly) on O to a function f, then also $f \in \mathrm{H}^\infty(O)$. In this case we say that $f_n \to f$ **pointwise and boundedly**, and call this **bp-convergence**.

At times we shall need that the functional calculus $(\mathrm{H}^\infty(O), \Phi)$ in question is in some sense continuous with respect to bp-convergence. In order to make this more precise, let us call a point $x \in X$ **bp-good** if $x \in \mathrm{dom}(\Phi(f))$ for all $f \in \mathrm{H}^\infty(O)$, and if $(f_n)_n$ is a sequence in $\mathrm{H}^\infty(O)$ that bp-converges on O to $f \in \mathrm{H}^\infty(O)$, then

$$\Phi(f_n)x \to \Phi(f)x$$

in X. The following is a useful fact about bp-good points. Its proof is straightforward.

Lemma 2.3. *Let* $(\mathrm{H}^\infty(O), \Phi)$ *be a functional calculus. Then the set of bp-good points is a subspace of* X*, invariant under each* $\Phi(f)$*,* $f \in \mathrm{H}^\infty(O)$*.*

A universal regulariser $e \in \mathrm{H}^\infty(O)$ is called **bp-good** whenever $(f_n)_n$ is a sequence in $\mathrm{H}^\infty(O)$ that bp-converges on O to $f \in \mathrm{H}^\infty(O)$, then

$$\Phi(ef_n) \to \Phi(ef)$$

strongly on X. In other words, e is bp-good if each $x \in \mathrm{ran}(\Phi(e))$ is bp-good.

Our first continuity property of the functional calculus shall be expressed in terms of the set

$$\mathcal{C}_\Phi := \{e \in \mathrm{H}^\infty(O) \mid e \text{ is bp-good}\}.$$

A functional calculus $(\mathrm{H}^\infty(O), \Phi)$ is called **standard** if the set $\mathcal{C}_\Phi$ of bp-good universal regularisers is universally determining.

Lemma 2.4. *Let* $(\mathrm{H}^\infty(O), \Phi)$ *be a standard functional calculus. Then the following holds: If* $f_n \to f$ *pointwise and boundedly, and if* $x, y \in X$ *are such that* $x \in \mathrm{dom}(\Phi(f_n))$ *for all* $n \in \mathbb{N}$ *and*

$$\Phi(f_n)x \to y,$$

then $x \in \mathrm{dom}(\Phi(f))$ *and* $\Phi(f)x = y$.

We say that the **convergence lemma** holds for a functional calculus $(\mathrm{H}^\infty(O), \Phi)$ if the following is true: whenever $f_n \to f$ pointwise and boundedly on O and $\Phi(f_n) \in \mathcal{L}(X)$ for all $n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} \|\Phi(f_n)\| < \infty$, then $\Phi(f) \in \mathcal{L}(X)$ and $\Phi(f_n) \to \Phi(f)$ strongly as $n \to \infty$.

Lemma 2.5. *Let* $(\mathrm{H}^\infty(O), \Phi)$ *be a functional calculus such that the space of bp-good points of* X *is dense in* X*. Then the convergence lemma holds.*

Corollary 2.6. *Let* $(\mathrm{H}^\infty(O), \Phi)$ *be a functional calculus such that each operator* $\Phi(f)$*,* $f \in \mathrm{H}^\infty(O)$*, is bounded. If the functional calculus is standard or the convergence lemma holds, then* $\Phi : \mathrm{H}^\infty(O) \to \mathcal{L}(X)$ *is bounded.*

The functional calculi most relevant in the remainder of the paper are the ones for sectorial and strip type operators. Although their construction can be found at several places in the literature, e.g. in [**?**, Chapters 2 and 4], the standard presentations suffer from an unnatural asymmetry in view of the exp / log-correspondence of sectors and strips, cf. Remark 7.11 below. Therefore, we have taken the opportunity to give a slightly modified account that avoids that shortcoming and in fact appears to be the most natural and the most general at the same time. See Chapter 7 for details.

The functional calculus for a strip type operator A is standard, and the bp-good points form a dense subspace if A is densely defined, see Lemma 7.2. Analogously, the functional calculus for an injective sectorial operator A is standard, and the bp-good points form a dense subspace if A has dense domain and dense range, see **??**.

2.2. The Square Functional Calculus.

We shall now associate square functions with a given functional calculus. For the necessary definitions and results about γ-radonifying operators, the reader is referred to Appendix B. Let us fix (once and for all) a functional calculus $(\mathrm{H}^\infty(O), \Phi)$ over some open set $O \subseteq \mathbb{C}$ (or $\mathbb{C}^d$) on some Banach space X.

Let H be a Hilbert space with (Banach space) dual H', where the duality is denoted by

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{H,H'} : H \times H' \to \mathbb{C}, \qquad \langle h, h' \rangle = h'(h)$$

for $h \in H$, $h' \in H'$. We identify H with H'' via the canonical mapping, i.e., elements $h \in H$ are regarded also as functionals on H'. Hence, for a H'-valued function $f : O \to H'$ we can form the scalar function

$$h \circ f : O \to \mathbb{C}, \qquad (h \circ f)(z) = \langle h, f(z) \rangle_{H,H'} \quad (z \in O,\, h \in H).$$

If $f \in \mathrm{H}^\infty(O; H')$ then $h \circ f \in \mathrm{H}^\infty(O)$ and hence $\Phi(h \circ f)$ is defined as a closed operator on X. We then define the operator

$$\Phi(f) : \mathrm{dom}(\Phi(f)) \to \mathcal{L}(H; X)$$

by

$$[\Phi(f)x]h := \Phi(h \circ f)x,$$

on its natural domain $\mathrm{dom}(\Phi(f))$ given by

$$x \in \mathrm{dom}(\Phi(f)) \overset{\text{def}}{\iff} x \in \mathrm{dom}(\Phi(h \circ f)) \text{ for all } h \in H \text{ and } \\ (h \mapsto \Phi(h \circ f)x \in \mathcal{L}(H; X).$$

This definition/notation is consistent with the original notation under the identification $\mathrm{H}^\infty(O; H') = \mathrm{H}^\infty(O)$ in the case that $H = \mathbb{C}$ is one-dimensional. The following lemma is straightforward.

Lemma 2.7. *Let* $(\mathrm{H}^\infty(O), \Phi)$ *ba functional calculus and* $f \in \mathrm{H}^\infty(O; H')$. *Then* $\mathrm{dom}(\Phi(f))$ *contains all bp-good points.*

In the next step we take the part of $\Phi(f)$ in $\gamma(H; X)$ to arrive at the square function

$$\Phi_\gamma(f) : \mathrm{dom}(\Phi_\gamma(f)) \to \gamma(H; X), \qquad \Phi_\gamma(f)x := \Phi(f)x, \quad \text{where}$$

$$\mathrm{dom}(\Phi_\gamma(f)) = \{x \in \mathrm{dom}(\Phi(f)) \mid \Phi(f)x \in \gamma(H; X)\}.$$

We call the square function $\Phi_\gamma(f)$ **bounded** if $\mathrm{dom}(\Phi_\gamma(f)) = X$ and

$$\Phi_\gamma(f) : X \to \gamma(H; X)$$

is a bounded operator.

In the same way one obtains the operator $\Phi_{\gamma_\infty}(f)$ by taking the part of $\Phi(f)$ in $\gamma_\infty(H; X)$. If X does not contain a copy of c_0, $\Phi_\gamma(f) = \Phi_{\gamma_\infty}(f)$.

If the functional calculus is standard, the following lemma sometimes facilitates computations.

Lemma 2.8. *Let* $(\mathrm{H}^\infty(O), \Phi)$ *be a standard functional calculus on some Banach space* X, *let* $f \in \mathrm{H}^\infty(O; H')$ *and* $x \in X$. *Define*

$$D(f, x) := \{h \in H \mid x \in \mathrm{dom}(\Phi(h\circ f))\} \subseteq H.$$

Then the following assertions hold:

a) *One has* $x \in \mathrm{dom}(\Phi(f))$ *if and only if there is a dense subspace* D *of* H *with* $D \subseteq D(f, x)$ *and the operator* $D \to X$, $h \mapsto \Phi(h\circ f)x$ *is bounded.*

b) *One has* $x \in \mathrm{dom}(\Phi_{\gamma_\infty}(f))$ *if and only if there is a dense subspace* D *of* H *with* $D \subseteq D(f, x)$ *and there is* $c \geq 0$ *such that*

$$\mathbb{E}\Big\|\sum_{e\in F} \gamma_e \Phi(e\circ f)x\Big\|_X^2 \leq c^2$$

for all finite orthonormal systems $F \subseteq D$. *In this case,*

$$\|\Phi_{\gamma_\infty}(f)x\|_\gamma \leq c.$$

c) *Suppose that* X *does not contain a copy of* c_0 *and* $(e_\alpha)_{\alpha\in I}$ *is a fixed orthonormal basis of* H. *Then* $\Phi_\gamma(f)$ *is bounded if and only if* $\Phi(e_\alpha\circ f) \in \mathcal{L}(X)$ *for all* α *and there is a constant* $c \geq 0$ *such that*

$$\mathbb{E}\big\|\textstyle\sum_{\alpha\in F} \gamma_\alpha \Phi(e_\alpha\circ f)x\big\|^2 \leq c^2 \|x\|^2$$

for all $x \in X$. *In this case,* $\|\Phi_\gamma(f)\| \leq c$.

Proof. a) One implication is clear. For the converse, suppose that $D \subseteq D(f, x)$ is dense in H and $c \geq 0$ is such that

$$\|\Phi(h\circ f)x\| \leq c\,\|h\| \tag{2.2}$$

for all $h \in D$. If $h \in H$ is arbitrary, there is a sequence $(h_n)_n$ in D with $h_n \to h$. Then $h_n\circ f$ bp-converges to $h\circ f$ and $(\Phi(h_n\circ f)x)_n$ is a Cauchy sequence in X. Since the functional calculus is standard, by Lemma 2.4 it follows that $h \in D(f, x)$ and the norm estimate (2.2) holds. Hence $x \in \mathrm{dom}(\Phi(f))$ as claimed.

b) Again, one implication is trivial. For the converse we note that it follows from the assumption that $\|\Phi(h\circ f)\| \leq \sqrt{2}c\,\|h\|$ for every $h \in D$. By a), $x \in \mathrm{dom}(\Phi(f))$. It is left to show that $\Phi(f) \in \gamma_\infty(H; X)$. If $(e_1, \ldots, e_d)$ is any finite orthonormal system in H we can find, by density and the Gram–Schmidt procedure, a sequence $(e_{1,n}, \ldots, e_{d,n})$ of orthonormal systems in D such that $e_{j,n} \to e_j$ for each $1 \leq j \leq d$. As the estimate

$$\mathbb{E}\Big\|\sum_{j=1}^{d} \gamma_e [\Phi(f)x] e_{j,n}\Big\|_X^2 \leq c^2$$

holds for each $n \in \mathbb{N}$, it holds also in the limit.

c) As before, one implication is clear. For the converse note that the hypothesis implies that the dense subspace $D := \operatorname{span}\{e_\alpha \mid \alpha \in I\}$ is contained in $D(f, x)$ for each $x \in X$. Moreover, if F is a finite orthonormal system in D, each vector in F is a finite linear combination of the e_α. From the hypothesis it follows by virtue of the contraction principle that

$$\mathbb{E}\Big\|\sum_{e\in F} \gamma_e \Phi(e\circ f)x\Big\|^2 \le c^2 \, \|x\|^2 .$$

Hence, applying b) concludes the proof. □

In the following lemma we collect some properties of the so-obtained square functions. Note that $\mathrm{H}^\infty(O; H')$ is an $\mathrm{H}^\infty(O)$-module with respect to pointwise multiplication.

Lemma 2.9. *In the situation just described, the following assertions hold for each $f \in \mathrm{H}^\infty(O; H')$:*

a) *The operators $\Phi(f)$ and $\Phi_\gamma(f)$ are closed.*

b) *For each $g \in \mathrm{H}^\infty(O; H')$*

$$\Phi_\gamma(f) + \Phi_\gamma(g) \subseteq \Phi_\gamma(f + g).$$

c) *For each $g \in \mathrm{H}^\infty(O)$*

$$\Phi_\gamma(f)\Phi(g) \subseteq \Phi_\gamma(f \cdot g)$$

with $\operatorname{dom}(\Phi_\gamma(f)\Phi(g)) = \operatorname{dom}(\Phi(g)) \cap \operatorname{dom}(\Phi_\gamma(f \cdot g))$.

d) *For each $g \in \mathrm{H}^\infty(O)$*

$$\Phi(g) \circ \Phi_\gamma(f) \subseteq \Phi_\gamma(f \cdot g)$$

e) *For each $g \in \mathrm{H}^\infty(O)$ with $\Phi(g) \in \mathcal{L}(X)$*

$$\Phi(g) \circ \Phi_\gamma(f) \subseteq \Phi_\gamma(f \cdot g) = \Phi_\gamma(f)\Phi(g)$$

In particular, $\operatorname{dom}(\Phi_\gamma(f))$ *is invariant under* $\Phi(g)$.

The assertion d) means: if $x \in \operatorname{dom}(\Phi_\gamma(f))$ and $\Phi(g)[\Phi_\gamma(f)x] \in \gamma(H; X)$, then $x \in \operatorname{dom}(\Phi_\gamma(f \cdot g))$ and $\Phi(g)[\Phi_\gamma(f)x] = \Phi_\gamma(f \cdot g)x$.

Proof. The assertions in b) and c) follow more or less directly from the corresponding statements about the scalar calculus $(\mathrm{H}^\infty(O), \Phi)$. Assertion d) is straightforward, and e) is a consequence of c) and d). (Note that by the ideal property of $\gamma(H; X)$, $\operatorname{dom}(\Phi(g) \circ \Phi_\gamma(f)) = \operatorname{dom}(\Phi_\gamma(f))$. □

From Lemma 2.9 we see that the mapping $f \mapsto \Phi_\gamma(f)$ behaves like a functional calculus, so we call it the **vectorial** H^∞-calculus on O. It is **bounded** if $\Phi_\gamma(f)$ is a bounded square function for each $f \in \mathrm{H}^\infty(O; H')$ and there is a constant $C \ge 0$ such that

$$\|\Phi_\gamma(f)x\|_\gamma \le C \, \|f\|_{\mathrm{H}^\infty(O)} \, \|x\| \quad (x \in X, \; f \in \mathrm{H}^\infty(O; H')).$$

This amounts to saying that the mapping

$$\mathrm{H}^\infty(O; H') \longrightarrow \mathcal{L}(X; \gamma(H; X)), \qquad f \mapsto \Phi_\gamma(f)$$

is a bounded operator.

Clearly, if the vectorial H^∞-calculus is bounded, then the underlying scalar H^∞-calculus is bounded. We shall prove that, essentially, the converse holds for sectorial/strip type operators, when one allows for opening up the sector/strip (Theorem 8.1).

Lemma 2.10 (Convergence Lemma for Square Functions)**.**
Let $(\mathrm{H}^\infty(O), \Phi)$ *be a functional calculus on a Banach space* X *such that the scalar convergence lemma holds. Then the* **vectorial convergence lemma** *holds. More precisely: Let* $(f_n)_n$ *be a sequence in* $\mathrm{H}^\infty(O; H')$ *satisfying*

1) $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$,
2) $f_n(z) \to f(z)$ *weakly for all* $z \in O$,
3) $\Phi_\gamma(f_n) \in \mathcal{L}(X; \gamma(H; X))$ *for all* $n \in \mathbb{N}$ *and*
4) $\sup_{n\in\mathbb{N}} \|\Phi_\gamma(f_n)\|_{\mathcal{L}(X;\gamma(H;X))} < \infty$.

Then $\Phi_{\gamma_\infty}(f) \in \mathcal{L}(X; \gamma_\infty(H; X))$ *and* $\Phi_\gamma(f_n)x \to \Phi_{\gamma_\infty}(f)x$ *strongly in* $\mathcal{L}(H; X)$ *as* $n \to \infty$*, for each* $x \in X$.

Proof. Fix $h \in H$. Then $\sup_n \|h \circ f_n\|_\infty \le \|h\| \sup_n \|f_n\|_\infty < \infty$ and $h \circ f_n \to h \circ f$ pointwise on O. Moreover, $\Phi(h \circ f_n) \in \mathcal{L}(X)$ and

$$\begin{aligned} \|\Phi(h\circ f_n)x\|_X = \|[\Phi_\gamma(f_n)x]h\|_X &\le \|h\| \, \|\Phi_\gamma(f_n)x\|_{\mathcal{L}(H;X)} \\ &\le \|h\| \, \|\Phi_\gamma(f_n)x\|_{\gamma(H;X)} \end{aligned}$$

for all $n \in \mathbb{N}$. This yields

$$\sup_{n\in\mathbb{N}} \|\Phi(h\circ f_n)\|_{\mathcal{L}} \le \|h\| \sup_n \|\Phi_\gamma(f_n)\|_{\mathcal{L}(X;\gamma(H;X))} .$$

By the scalar convergence lemma, $\Phi(h\circ f) \in \mathcal{L}(X)$ and $\Phi(h\circ f_n) \to \Phi(h\circ f)$ strongly on X. That is, $\Phi_\gamma(f_n)x \to \Phi(f)x$ strongly in $\mathcal{L}(H; X)$ for every $x \in X$. By the γ-Fatou Lemma B.5, $\Phi(f)x \in \gamma_\infty(H; X)$. $\square$

2.3. Dual Square Functions.
Let again $(\mathrm{H}^\infty(O), \Phi)$ be a functional calculus as above. For a Hilbert space H and a function $f : O \to H$ we have

$$h'\circ f : O \to \mathbb{C}, \qquad (h'\circ f)(z) = \langle f(z), h' \rangle_{H,H'} \quad (z \in O,\ h' \in H').$$

If $f \in \mathrm{H}^\infty(O; H)$ then $h'\circ f \in \mathrm{H}^\infty(O)$ for each $h' \in H'$. Hence we can define the operator

$$\Phi^d(f) : \mathrm{dom}(\Phi^d(f)) \to \mathcal{L}(H'; X')$$

by

$$[\Phi^d(f)x']\, h' := \Phi(h'\circ f)'x'$$

on its natural domain $\mathrm{dom}(\Phi^d(f)) \subseteq X'$ given by

$$x' \in \mathrm{dom}(\Phi^d(f)) \overset{\text{def}}{\iff} x' \in \mathrm{dom}(\Phi(h'\circ f)') \text{ for all } h' \in H' \text{ and } (h' \mapsto \Phi(h'\circ f)'x') \in \mathcal{L}(H'; X').$$

Then we pass to the associated dual square function

$$\Phi_{\gamma'}(f) : \mathrm{dom}(\Phi_{\gamma'}(f)) \to \gamma'(H'; X'), \qquad \Phi_{\gamma'}(f)x' := \Phi^d(f)x'$$
$$\mathrm{dom}(\Phi_{\gamma'}(f)) = \{x' \in \mathrm{dom}(\Phi^d(f) \mid \Phi^d(f)x' \in \gamma(H'; X')\} \subseteq X'.$$

Of course, this is only meaningful if $\Phi(h'\circ f)'$ is single-valued, i.e., if $\Phi(h'\circ f)$ is densely defined for each $h' \in H'$. We therefore make the following

Standing assumption for dual square functions: *Whenever we speak of a dual square function associated with a function $f \in \mathrm{H}^\infty(O; H)$ we require that for each $h' \in H'$ the operator $\Phi(h'\circ f)$ is densely defined.*

Remark 2.11. In order to talk about square functions for *each* $f \in \mathrm{H}^\infty(O; H)$ we would need that $\Phi(f)$ is densely defined for each scalar-valued $f \in \mathrm{H}^\infty(O)$. For a strip-type operator this means that it must be densely defined, whereas for a sectorial operator this means that it must have dense domain and range.

The following lemma is the analogue of Lemma 2.9.

Lemma 2.12. *In the situation just described, the following assertions hold for $f \in \mathrm{H}^\infty(O; H)$:*

a) *The operator $\Phi_{\gamma'}(f)$ is weak*-to-weak* closed.*
b) *If $g \in \mathrm{H}^\infty(O)$ such that $\Phi(g) \in \mathcal{L}(X)$ and if $x' \in \mathrm{dom}(\Phi_{\gamma'}(f \cdot g))$, then $\Phi(g)'x' \in \mathrm{dom}(\Phi_{\gamma'}(f))$ and*

$$\Phi_{\gamma'}(f)\Phi(g)'x' = \Phi_{\gamma'}(f \cdot g)x'.$$

Proof. a) is left to the reader. For the proof of b) we fix $h' \in H'$ and note first that since $\Phi(g)$ is bounded we have

$$\Phi(h'\circ(f \cdot g))' = \Phi((h'\circ f)g)' \subseteq \big(\Phi(g)\Phi(h'\circ f)\big)' = \Phi(h'\circ f)'\Phi(g)'$$

by [**?**, A.4.2 and 1.2.2]. The claim now follows easily. $\square$

The following theorem yields a useful characterisation of "dual square function estimates". (Recall our notation $[u, v]$ for the scalar product on H. For an H-valued function f and $v \in H$ we also write $[f, v]$ for the function $z \mapsto [f(z), v]$.)

Theorem 2.13. *Let $(e_\alpha)_{\alpha\in I}$ be a fixed orthonormal basis of H. The following assertions are equivalent for $f \in \mathrm{H}^\infty(O; H)$:*

(i) *$\Phi_{\gamma'}(f)$ is a bounded operator $\Phi_{\gamma'}(f) : X' \to \gamma'(H'; X')$.*

(ii) *The assignment*

$$T(h' \otimes x) := \Phi(h'\circ f)x, \qquad h' \in H',\ x \in \mathrm{dom}(\Phi(h'\circ f))$$

extends to a bounded operator $T : \gamma(H; X) \to X$.

(iii) *There is a constant $c \geq 0$ such that*

$$\Big\| \sum_{\alpha\in F} \Phi([f, e_\alpha])x_\alpha \Big\|_X^2 \leq c\, \mathbb{E}\Big\| \sum_{\alpha\in F} \gamma_\alpha x_\alpha \Big\|^2 \tag{2.3}$$

for all finite subsets $F \subseteq I$ and $x_\alpha \in \mathrm{dom}(\Phi([f, e_\alpha]))$ for $\alpha \in F$.

In this case $T = \Phi_{\gamma'}(f)'\big|_{\gamma(H;X)}$ is the pre-adjoint of $\Phi_{\gamma'}(f)$ under the identification $\gamma'(H'; X') \cong \gamma(H; X)'$, and (2.3) holds with $c = \|T\| = \|\Phi_{\gamma'}(f)\|$.

Furthermore, if $g \in \mathrm{H}^\infty(O)$ is such that $\Phi(g) \in \mathcal{L}(X)$, then

$$\Phi_{\gamma'}(f)'(\Phi(g) \circ S) = \Phi(g)\big(\Phi_{\gamma'}(f)'S\big) \quad \text{for all } S \in \gamma(H; X). \tag{2.4}$$

Proof. (i)⇒(ii): By hypothesis, $\Phi_{\gamma'}(f)' : \gamma(H; X)'' \to X''$ is bounded. Fix $x' \in X'$, $h' \in H'$ and $x \in \mathrm{dom}(\Phi(h'\circ f))$. Then

$$\begin{aligned}\langle \Phi_{\gamma'}(f)'(h' \otimes x), x' \rangle_{X'',X'} &= \langle h' \otimes x, \Phi_{\gamma'}(f)x' \rangle \\ &= \mathrm{tr}\,\big((\Phi_{\gamma'}(f)x')'(h' \otimes x)\big) \\ &= \langle x, [\Phi_{\gamma'}(f)x']\, h' \rangle = \langle x, \Phi(h'\circ f)'x' \rangle \\ &= \langle \Phi(h'\circ f)x, x' \rangle\,. \end{aligned}$$

Consequently,

$$\Phi_{\gamma'}(f)'(h' \otimes x) = \Phi(h'\circ f)x = T(h' \otimes x) \in X.$$

Since $\operatorname{dom}(\Phi(h'\circ f))$ is dense in X, the linear span of such elements $h' \otimes x$ is dense in $\gamma(H;X)$ and the claim follows.
(ii)⇔(iii): This follows since $T(\sum_{\alpha\in F} \overline{e_\alpha} \otimes x_\alpha) = \sum_{\alpha \in F} \Phi([f, e_\alpha])x_\alpha$.
(ii)⇒(i): It suffices to show that $\Phi_{\gamma'}(f) = T' : X' \to \gamma(H;X)' \cong \gamma'(H';X')$. Fix $x' \in X'$. Then

$$\begin{aligned}\langle x, (T'x')(h')\rangle_{X,X'} &= \langle h' \otimes x, T'x'\rangle_{\gamma,\gamma'} = \langle T(h'\otimes x), x'\rangle_{X,X'}\\ &= \langle \Phi(h'\circ f)x, x'\rangle\end{aligned}$$

for all $h' \in H'$ and $x \in \operatorname{dom}(\Phi(h'\circ f))$. Hence $x' \in \operatorname{dom}(\Phi(h'\circ f)')$ and

$$[\Phi_{\gamma'}(f)x']\, h' = \Phi(h'\circ f)'x' = (T'x')h' \qquad \text{for all } h' \in H'.$$

That is, $\Phi_{\gamma'}(f) = T'$.
For the remaining statement let again $h' \in H'$ and $x \in \operatorname{dom}(\Phi(h'\circ f))$. Then, with $S := h' \otimes x$,

$$\begin{aligned}\Phi(g)(T(S)) &= \Phi(g)\Phi(h'\circ f)x = \Phi(h'\circ f)\Phi(g)x = T(h'\otimes \Phi(g)x)\\ &= T(\Phi(g)\circ S).\end{aligned}$$

Since the linear span of such operators S is a dense subset of $\gamma(H;X)$, the claim follows from the ideal property of $\gamma(H;X)$. □

2.4. Square Functions over $\mathbf{L_2}$-Spaces.

Up to now we worked with a general Hilbert space H. If one is in the special situation $H = \mathrm{L}_2(\Omega) = H'$ for some measure space (Ω,μ), it is natural to consider functions of *two variables* $f = f(t,z)$ in the construction of square functions.

Lemma 2.14. *Let $O \subseteq \mathbb{C}$ be an open subset of the complex plane, let $f : \Omega \times O \to \mathbb{C}$ be measurable and suppose in addition that*

1) $f(t,\cdot) \in \mathrm{H}^\infty(O)$ *for μ-almost all $t \in \Omega$ and*
2) $\sup_{z\in O} \int_\Omega |f(t,z)|^2 \, \mu(\mathrm{d}t) < \infty$.

Then $(z \mapsto f(\cdot,z)) \in \mathrm{H}^\infty(O;\mathrm{L}_2(\Omega))$.

Proof. Let $g \in \mathrm{L}_2(\Omega)$. It remains to show that the function

$$F(z) := \int_\Omega g(t) f(t,z)\, \mu(\mathrm{d}t)$$

is holomorphic. To this end, let B be any open ball such that $\overline{B} \subseteq O$. Then for $a \in B$

$$f(t,a) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B} \frac{f(t,z)\mathrm{d}z}{z-a}$$

for almost all $t \in \Omega$, by the Cauchy formula. Fubini's theorem yields

$$F(a) = \int_\Omega g(t)f(t,a)\,\mu(\mathrm{d}t) = \frac{1}{2\pi\mathrm{i}}\int_{\partial B}\frac{F(z)\mathrm{d}z}{z-a}$$

for all $a \in B$. By a standard result in complex function theory [**?**, Theorem 10.7], F is holomorphic. □

Remark 2.15. Suppose in addition that $(\mathrm{H}^\infty(O), \Phi)$ is a functional calculus for the (possibly multivalued) operator A, cf. Remark 2.2. For f as in Lemma 2.14 we then have

$$[\Phi(f)x]\,h = \Big(\int_\Omega h(t)f(t,z)\,\mu(\mathrm{d}t)\Big)(A)x$$

if $x \in \mathrm{dom}(\Phi(f))$ and $h \in H = \mathrm{L}_2(\Omega)$. In many situations one has

$$[\Phi(f)x]\,h = \Big(\int_\Omega h(t)f(t,z)\,\mathrm{d}t\Big)(A)x = \int_\Omega h(t)f(t,A)x\,\mu(\mathrm{d}t)$$

at least for vectors x from a large subspace of X. We therefore use the symbol $f(\cdot, A)x$ or $f(t,A)x$ as a convenient alternative notation — as a *façon de parler* — for the operator $\Phi(f)x$. So, whenever expressions of the form

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$$\|f(t,A)x\|_\gamma$$

appear, this is just a suggestive notation. It is *by no means* implied that "$f(t,A)x$" has to make sense literally, which would mean that $x \in \mathrm{dom}(f(t,A))$ for almost all $t \in \Omega$ and $[\Phi(f)x]h = \int_\Omega h(t)f(t,A)x\,\mathrm{d}t$ as an integral of an X-valued function as in Appendix **??**. It is actually one of the advantages of our approach to square functions that one does not have to worry about vector-valued integration too much.

3. Square Function Estimates: New from Old

In this chapter we discuss certain general principles that allow to generate new (dual) square function estimates from known ones. A fairly trivial instance of such a principle is given by subordination.

3.1. Subordination.

Let us, for the moment, return to a general setting, where X, Y are Banach and H, K are Hilbert spaces. A square function $Q : \mathrm{dom}(Q) \to \gamma(H;Y)$ is called **subordinate** to a square function $R : \mathrm{dom}(R) \to \gamma(K;Y)$, in symbols:

$$Q \precsim R,$$

if $\mathrm{dom}(Q) \supseteq \mathrm{dom}(R)$ and there is a bounded operator $T : H \to K$ such that

$$Qx = Rx \circ T \quad \text{for all } x \in \mathrm{dom}(R).$$

The square functions Q and R are called **strongly equivalent**, in symbols:

$$Q \approx R,$$

if $Q \precsim R$ and $R \precsim Q$. Note that if $Q \precsim R$ then, by the ideal property, there is a constant $c \geq 0$ such that

$$\|Qx\|_\gamma \leq c\,\|Rx\|_\gamma \quad \text{for all } x \in \operatorname{dom}(R).$$

Analogously, a dual square function $Q^d : \operatorname{dom}(Q^d) \to \gamma'(H'; Y')$ is **subordinate** to a dual square function $R^d : \operatorname{dom}(R^d) \to \gamma'(H'; Y')$ if $\operatorname{dom}(Q^d) \supseteq \operatorname{dom}(R^d)$ and there is a bounded operator $T : H' \to K'$ such that

$$Q^d x' = R^d x' \circ T \quad \text{for all } x' \in \operatorname{dom}(R^d).$$

It is evident that any (dual) square function subordinate to a bounded (dual) square function is itself bounded. Subordination is a (rather trivial) way to generate new square function estimates from known ones.

Now let us return to square functions associated with a functional calculus $(\mathrm{H}^\infty(O), \Phi)$ over a set O.

Theorem 3.1 (Subordination)**.** *Let K be another Hilbert space and $T : K \to H$ a bounded linear operator.*

a) *If $g \in \mathrm{H}^\infty(O; H')$ then $\operatorname{dom}(\Phi_\gamma(g)) \subseteq \operatorname{dom}(\Phi_\gamma(T' \circ g))$ and*

$$\Phi_\gamma(T' \circ g)x = \Phi_\gamma(g)x \circ T \qquad \text{for all } x \in \operatorname{dom}(\Phi_\gamma(g)).$$

In particular, $\Phi_\gamma(T' \circ g) \precsim \Phi_\gamma(g)$. If, moreover, T is surjective, then $\Phi_\gamma(T' \circ g) \approx \Phi_\gamma(g)$.

b) *If $f \in \mathrm{H}^\infty(O; K)$ then $\operatorname{dom}(\Phi_{\gamma'}(f)) \subseteq \operatorname{dom}(\Phi_{\gamma'}(T \circ f))$ and*

$$\Phi_{\gamma'}(T \circ f)x' = \Phi_{\gamma'}(f)x' \circ T' \qquad \text{for all } x' \in \operatorname{dom}(\Phi_\gamma(f)).$$

In particular, $\Phi_{\gamma'}(T \circ f) \precsim \Phi_{\gamma'}(f)$. If, moreover, T is injective with closed range, then $\Phi_{\gamma'}(T \circ f) \approx \Phi_{\gamma'}(f)$.

Proof. The first part of a) is an easy exercise. If T is surjective then there is a bounded operator $S : H \to K$ with $TS = \mathrm{I}_H$. By the first part, $\Phi_\gamma(g) = \Phi_\gamma(S' \circ T' \circ g) \precsim \Phi_\gamma(T' \circ g) \precsim \Phi_\gamma(g)$. The proof of b) is similar. □

Whenever it is convenient, we shall abbreviate $\Phi_\gamma(f) \precsim \Phi_\gamma(g)$ and $\Phi_\gamma(f) \approx \Phi_\gamma(g)$ simply by

$$f \precsim g \quad \text{and} \quad f \approx g,$$

respectively. The same abbreviation is used in the case of dual square functions. For applications of the subordination principle see Chapter 7 below.

3.2. Direct Sums.

Suppose that $H = H_1 \oplus H_2$ is the direct sum of Hilbert spaces H_1, H_2. Then H' can be canonically identified with

$$H' = (H_1 \oplus H_2)' \cong H_1' \oplus H_2'$$

and we shall do so in the following. Given $f_j \in \mathrm{H}^\infty(O; H_j')$ for $j = 1, 2$, we write

$$\Phi_\gamma(f_1) \oplus \Phi_\gamma(f_2)$$

for the operator $X \to \gamma(H_1 \oplus H_2; X)$ with domain

$$\operatorname{dom}\bigl(\Phi_\gamma(f_1) \oplus \Phi_\gamma(f_2)\bigr) := \operatorname{dom}(\Phi_\gamma(f_1)) \cap \operatorname{dom}(\Phi_\gamma(f_2))$$

and mapping x from that domain to the operator

$$H_1 \oplus H_2 \to X, \quad (h_1, h_2) \mapsto [\Phi_\gamma(f_1)x]h_1 + [\Phi_\gamma(f_2)x]h_2.$$

On the other hand, we may consider the function

$$f_1 \oplus f_2 := (f_1, f_2) \in \mathrm{H}^\infty(O; H_1' \oplus H_2'), \qquad (f_1 \oplus f_2)(z) := (f_1(z), f_2(z))$$

and the associated square function $\Phi_\gamma(f_1 \oplus f_2)$.

Corollary 3.2. *Let H_1, H_2 be Hilbert spaces, and $f_j \in \mathrm{H}^\infty(O; H_j')$ for $j = 1, 2$. Then*

$$\Phi_\gamma(f_1) \oplus \Phi_\gamma(f_2) = \Phi_\gamma(f_1 \oplus f_2).$$

Proof. Abbreviate $H := H_1 \oplus H_2$ and $f := f_1 \oplus f_2$. We consider for $j = 1, 2$ the projection operators

$$P_j : H \to H_j, \qquad P_j(h_1, h_2) = h_j$$

and the injections

$$J_1(h_1) = (h_1, 0), \quad J_2(h_2) = (0, h_2)$$

for $h_j \in H_j$, $j = 1, 2$. Note that the adjoints P_j' and J_j' are simply the corresponding injection and projection operators, respectively, for the direct sum $H_1' \oplus H_2'$.

Now clearly $f = P_1' \circ f_1 + P_2' \circ f_2$, hence by a basic rule from Lemma 2.9

$$\Phi_\gamma(P_1' \circ f_1) + \Phi_\gamma(P_2' \circ f_2) \subseteq \Phi_\gamma(f).$$

Since P_j is surjective, Theorem 3.1 yields

$$\Phi_\gamma(P_j' \circ f_j) \approx \Phi_\gamma(f_j).$$

In particular, they have equal domains. This leads to

$$\Phi_\gamma(P_1' \circ f_1) + \Phi_\gamma(P_2' \circ f_2) = \Phi_\gamma(f_1) \oplus \Phi_\gamma(f_2)$$

by our definition from above.

Next, observe that $f_j = J_j' \circ f$ and hence

$$\Phi_\gamma(f_j) \precsim \Phi_\gamma(f)$$

for $j = 1, 2$. In particular,

$$\mathrm{dom}(\Phi_\gamma(f)) \subseteq \mathrm{dom}(\Phi_\gamma(f_1)) \cap \mathrm{dom}(\Phi_\gamma(f_2))$$

and this is what was missing to establish the claim. □

Direct sums of square functions appear, for example, in Example 7.9.

3.3. Tensor Products and Property $(\boldsymbol{\alpha}^+)$.

Let H, K be Hilbert spaces and $f \in \mathrm{H}^\infty(O; H')$ and $g \in \mathrm{H}^\infty(O; K')$. Then one can consider the function

$$f \otimes g : O \longrightarrow H' \otimes_a K' \;\subseteq\; (H \otimes K)' \qquad (f \otimes g)(z) := f(z) \otimes g(z).$$

It is then clear that $(f \otimes g) \in \mathrm{H}^\infty(O; (H \otimes K)')$.

The following result yields information about the square function $\Phi_\gamma(f \otimes g)$ in terms of the square functions $\Phi_\gamma(f)$ and $\Phi_\gamma(g)$. (We refer to Appendix B.6 for background on tensor products of Hilbert spaces, property (α^+), and the associated constant C^+.)

Theorem 3.3. *Let $(\mathrm{H}^\infty(O); \Phi)$ be a standard functional calculus on a space X with property (α^+), and let $f \in \mathrm{H}^\infty(O; H)$ and $g \in \mathrm{H}^\infty(O; K)$, where H and K are Hilbert spaces. Suppose that the square function $\Phi_\gamma(g)$ is bounded and let $x \in \mathrm{dom}(\Phi_\gamma(f))$. Then $x \in \mathrm{dom}(\Phi_\gamma(f \otimes g))$ and*

$$\|\Phi_\gamma(f \otimes g)x\|_\gamma \le C^+ \|\Phi_\gamma(g)\| \, \|\Phi_\gamma(f)x\|_\gamma .$$

Proof. By the ideal property, composition with the operator $\Phi_\gamma(g) : X \to \gamma(K; X)$ yields a bounded operator

$$\Phi_\gamma(g)^\otimes : \gamma(H; X) \to \gamma(H; \gamma(K; X)), \qquad \Phi_\gamma(g)^\otimes T := \Phi_\gamma(g) \circ T,$$

the "tensor extension". As such, its norm is bounded by $\|\Phi_\gamma(g)\|$. Since X has property (α^+) the natural mapping $H' \otimes_a (K' \otimes_a X) \to (H \otimes_a K)' \otimes_a X$ extends to a bounded operator

$$J^+ : \gamma(H; \gamma(K; X)) \to \gamma(H \otimes K; X)$$

with norm $C^+ = C^+(X)$. Hence $\Phi_\gamma(g)^\otimes$ as an operator $\gamma(H; X) \to \gamma(H \otimes K; X)$ with norm $\le C^+ \|\Phi_\gamma(g)\|$.

Fix $x \in \mathrm{dom}(\Phi_\gamma(f)$ and note that for $h \in H$ and $k \in K$ we have

$$(h \otimes k) \circ (f \otimes g) = (h \circ f)\,(k \circ g).$$

Since, by hypothesis, $\Phi_\gamma(g)$ is bounded, $\Phi(k\circ g) \in \mathcal{L}(X)$ and hence x lies in the domain of $\Phi((h \otimes k)\circ(f \otimes g))$ with

$$\begin{aligned}\Phi((h \otimes k)\circ(f \otimes g))x &= \Phi((h\circ f)(k\circ g))x \\ &= \Phi(k\circ g)\Phi(h\circ f)x = \Big[\Phi_\gamma(g)\big(\Phi(h\circ f)x\big)\Big]k \\ &= \Big[\Phi_\gamma(g)\big([\Phi_\gamma(f)x]h\big)\Big]k = \Big[\big[\Phi_\gamma(g)^{\otimes}(\Phi_\gamma(f)x)\big]h\Big]k \\ &= \Big[J^{+}\big[\Phi_\gamma(g)^{\otimes}(\Phi_\gamma(f)x)\big]\Big](h \otimes k).\end{aligned}$$

So the space $D(f \otimes g, x)$ (see Lemma 2.8) contains the dense subspace $D := H \otimes_a K$ of $H \otimes K$, and

$$\Phi(f \otimes g)x = J^{+}\big[\Phi_\gamma(g)^{\otimes}(\Phi_\gamma(f)x)\big] \tag{3.1}$$

as operators $D \to X$. As the functional calculus is standard, Lemma 2.8 applies and yields that $x \in \mathrm{dom}(\Phi(f \otimes g))$. Moreover, (3.1) holds as operators $H \otimes K \to X$. Since the right side lies in $\gamma(H \otimes K; X)$, it follows that $x \in \mathrm{dom}(\Phi_\gamma(f \otimes g))$ and

$$\Phi_\gamma(f \otimes g)x = J^{+}\big[\Phi_\gamma(g)^{\otimes}(\Phi_\gamma(f)x)\big]$$

The claimed norm estimate follows. $\square$

Remark 3.4. The proof of Theorem 3.3 reveals that one can omit the assumption that the functional calculus is standard when one requires in addition that $x \in \mathrm{dom}(\Phi(f \otimes g))$.

Corollary 3.5. *Let H, K be two Hilbert spaces and X be a Banach space with property (α^{+}). Suppose further that for a standard functional calculus the square functions*

$$\Phi_\gamma(f) : \ X \longrightarrow \gamma(H; X) \quad \text{and} \quad \Phi_\gamma(g) : \ X \longrightarrow \gamma(K; X)$$

are bounded. Then the tensor square function

$$\Phi_\gamma(f \otimes g) : \ X \longrightarrow \gamma(H \otimes K; X)$$

is bounded, too, with

$$\|\Phi_\gamma(f \otimes g)\| \le C^{+} \ \|\Phi_\gamma(f)\| \ \|\Phi_\gamma(g)\|$$

For an application of this lemma, see Lemma 8.10 below.

3.4. Lower Square Function Estimates I.

A **lower square function estimate** is an estimate of the form

$$\|x\|_X \le C\ \|\Phi_\gamma(g)x\|_\gamma \qquad (x \in \mathrm{dom}(\Phi_\gamma(g))).$$

In certain situations one can combine a lower square function estimate, a usual square function estimate and a subordination to show the boundedness of an operator $\Phi(f)$ by means of the following result.

Lemma 3.6. *Let H, K be Hilbert spaces and let $g \in \mathrm{H}^\infty(O; K')$ and $\tilde{g} \in \mathrm{H}^\infty(O; H')$. Suppose that one has a lower square function estimate for $\Phi_\gamma(g)$*

$$\|x\|_X \le C\ \|\Phi_\gamma(g)x\|_\gamma \qquad \big(x \in \mathrm{dom}(\Phi_\gamma(g))\big).$$

Suppose further that the scalar-valued function $f \in \mathrm{H}^\infty(O)$ is such that there is $T_f \in \mathcal{L}(H; K)$ with

$$f \cdot g = T_f' \circ \tilde{g}.$$

Then

$$\|\Phi(f)x\| \le C\, \|T_f\|\, \|\Phi_\gamma(\tilde{g})x\|_\gamma$$

for all $x \in \mathrm{dom}(\Phi(f)) \cap \mathrm{dom}(\Phi_\gamma(\tilde{g})))$.

Proof. By Lemma 2.9.c),

$$\Phi_\gamma(g)\Phi(f) \subseteq \Phi_\gamma(f \cdot g) = \Phi_\gamma(T_f' \circ \tilde{g})$$

If $x \in \mathrm{dom}(\Phi_\gamma(\tilde{g}))$, then, by subordination $x \in \mathrm{dom}(\Phi_\gamma(T_f' \circ \tilde{g}))$ as well. If, in addition $x \in \mathrm{dom}(\Phi(f))$ then, still by Lemma 2.9.c), $\Phi(f)x \in \mathrm{dom}(\Phi_\gamma(g))$. Hence,

$$\begin{aligned}\|\Phi(f)x\|_X &\le C\, \|\Phi_\gamma(g)\Phi(f)x\|_\gamma = C\, \|\Phi_\gamma(\tilde{g})x \circ T_f\|_\gamma \\ &\le C\, \|T_f\|\, \|\Phi_\gamma(\tilde{g})x\|_\gamma .\end{aligned}$$

□

Lemma 3.6 is an abstract version of the "pushing the operator through the square function"-technique used by Kalton and Weis in [**?**] (see also [**?**, Theorem 10.9]) to show that a norm equivalence

$$\|R(\pm \mathrm{i}\omega + \cdot, A)x\|_{\gamma(\mathrm{L}_2(\mathbb{R}); X)} \sim \|x\|_X$$

for a strip type operator A implies the boundedness of the H^∞-calculus on a strip, see Section 8.6 below for details.

3.5. Lower Square Function Estimates II.

We now present some methods to *establish* lower square function estimates. These, however, require slightly stronger assumptions about the underlying functional calculus. Indeed, eventually we shall work with a *standard* functional calculus $(\mathrm{H}^\infty(O), \Phi)$. Recall from Section 2.1 that this means that the set $\mathcal{C}_\Phi$ of bp-good universal regularisers is universally determining.

Remark 3.7. The following considerations are motivated by McIntosh's approximation formula

$$x = \int_0^\infty \varphi(tA)\psi(tA)x \, \frac{\mathrm{d}t}{t}$$

for $x \in \overline{\mathrm{dom}}(A) \cap \overline{\mathrm{ran}}(A)$, sectorial operators A and appropriate functions φ, ψ, see [**?**] and [**?**, Sec. 5.2].

For $f \in \mathrm{H}^\infty(O; H)$ and $g \in \mathrm{H}^\infty(O; H')$ let $f \diamond g \in \mathrm{H}^\infty(O)$ be defined by

$$(f \diamond g)(z) := \langle f(z), g(z)\rangle_{H,H'} \qquad (z \in O).$$

Then, if we regard elements $h \in H$ and $h' \in H'$ as constant mappings from O to H and H', respectively, we have

$$h \diamond g = h \circ g \quad \text{and} \quad f \diamond h' = h' \circ f.$$

Clearly, we expect the formula

$$\langle \Phi(f \diamond g)x, x'\rangle_{X,X'} = \langle \Phi_\gamma(g)x, \Phi_{\gamma'}(f)x'\rangle_{\gamma,\gamma'} \tag{3.2}$$

to hold. The following result gives some conditions for this.

Lemma 3.8. *Suppose that $f \in \mathrm{H}^\infty(O; H)$ and $g \in \mathrm{H}^\infty(O; H')$, and that $x \in \mathrm{dom}(\Phi_\gamma(g))$ is bp-good. Then*

$$\langle \Phi(f \diamond g)x, x'\rangle = \langle \Phi_\gamma(g)x, \Phi_{\gamma'}(f)x'\rangle$$

for all $x' \in \mathrm{dom}(\Phi_{\gamma'}(f))$.

Proof. For the proof of the claim we let $(e_\alpha)_{\alpha\in I}$ be an orthonormal basis of H and denote

$$g_\alpha(z) := \langle e_\alpha, g(z)\rangle = \big[e_\alpha, \overline{g(z)}\big]_{H'}, \quad f_\alpha(z) := \langle f(z), \overline{e_\alpha}\rangle = \big[f(z), e_\alpha\big]_H$$

for $\alpha \in I$. Then by general Hilbert space theory

$$(f \diamond g)(z) = \sum\nolimits_\alpha f_\alpha(z) \cdot g_\alpha(z)$$

for each $z \in O$, and the partial sums are uniformly bounded. (Note that $f_\alpha(z) \cdot g_\alpha(z) \neq 0$ for at most countably many α since $\{\overline{g(z)}, f(z) \mid z \in O\}$ is separable.) Hence, for $x' \in \operatorname{dom}(\Phi_{\gamma'}(f))$ we can compute

$$\begin{aligned}\langle \Phi_\gamma(g)x, \Phi_{\gamma'}(f)x' \rangle_{\gamma,\gamma'} &\overset{1)}{=} \sum_\alpha \langle \Phi(g_\alpha)x, \Phi(f_\alpha)'x' \rangle_{X,X'} \\ &\overset{2)}{=} \sum_\alpha \langle \Phi(f_\alpha)\Phi(g_\alpha)x, x' \rangle_{X,X'} \\ &\overset{3)}{=} \sum_\alpha \langle \Phi(f_\alpha g_\alpha)x, x' \rangle_{X,X'} = \langle \Phi(f \diamond g)x, x' \rangle_{X,X'}\end{aligned}$$

Here, 1) follows from c) of Theorem B.17 and 3) since x is bp-good. Since x being bp-good implies in particular that $\Phi(g_\alpha)x \in \operatorname{dom}(\Phi(f_\alpha))$, also 2) is justified. □

The following is the main result of the present section.

Theorem 3.9. *Let* $(\mathrm{H}^\infty(O), \Phi)$ *be a standard functional calculus, and let* $f \in \mathrm{H}^\infty(O; H)$ *and* $g \in \mathrm{H}^\infty(O; H')$. *If* $\Phi_{\gamma'}(f)$ *is a bounded operator, then*

$$\Phi_{\gamma'}(f)'\Phi_\gamma(g) \subseteq \Phi(f \diamond g).$$

In other words, $\operatorname{dom}(\Phi_\gamma(g)) \subseteq \operatorname{dom}(\Phi(f \diamond g))$ *and*

$$\langle \Phi(f \diamond g)x, x' \rangle = \langle \Phi_\gamma(g)x, \Phi_{\gamma'}(f)x' \rangle$$

for all $x \in \operatorname{dom}(\Phi_\gamma(g))$ *and all* $x' \in X'$. *In particular, one has the lower estimate*

$$\|\Phi(f \diamond g)x\|_X \lesssim \|\Phi_\gamma(g)x\|_\gamma \quad \text{for all } x \in \operatorname{dom}(\Phi_\gamma(g)).$$

Proof. We let $y := \Phi_{\gamma'}(f)'[\Phi_\gamma(g)x] \in X$ by Theorem 2.13. Take $e \in \mathcal{C}_\Phi$, i.e. a bp-good universal regulariser. Then by Lemma 3.8, for each $x' \in X'$ we have

$$\begin{aligned}\langle \Phi(e(f \diamond g))x, x' \rangle &= \langle \Phi(f \diamond g)\Phi(e)x, x' \rangle = \langle \Phi_\gamma(g)\Phi(e)x, \Phi_{\gamma'}(f)x' \rangle \\ &= \langle \Phi(e) \circ [\Phi_\gamma(g)x], \Phi_{\gamma'}(f)x' \rangle \\ &= \big\langle \Phi_{\gamma'}(f)'\big(\Phi(e) \circ [\Phi_\gamma(g)x]\big), x' \big\rangle \\ &= \langle \Phi(e)(\Phi_{\gamma'}(f)'[\Phi_\gamma(g)x]), x' \rangle = \langle \Phi(e)y, x' \rangle,\end{aligned}$$

where we used Lemma 2.9.e) and (2.4). It follows that

$$\Phi(e(f \diamond g))x = \Phi(e)y.$$

Since, by hypothesis, the set $\mathcal{C}_\Phi$ is determining, it follows that $x \in \operatorname{dom}(\Phi(f \diamond g))$ and $\Phi(f \diamond g)x = y$. The remaining assertions follow easily. □

Corollary 3.10. *Let* $(\mathrm{H}^\infty(O), \Phi)$ *be a standard functional calculus. Suppose that* $f \in \mathrm{H}^\infty(O; H)$ *and* $g \in \mathrm{H}^\infty(O; H')$ *are such that* $\Phi_\gamma(g)$ *and* $\Phi_{\gamma'}(f)$ *are bounded operators. Then* $\Phi(f \diamond g)$ *is a bounded operator and*

$$\langle \Phi(f \diamond g)x, x' \rangle = \langle \Phi_\gamma(g)x, \Phi_{\gamma'}(f)x' \rangle \quad \text{for all } x \in X \text{ and } x' \in X'.$$

In particular, if $f \diamond g = \mathbf{1}$ *then one has the norm equivalence*

$$\|x\|_X \simeq \|\Phi_\gamma(g)x\|_\gamma \quad \text{for all } x \in X.$$

The problem whether to a given function $f \in \mathrm{H}^\infty(O; H)$ there exists a function $g \in \mathrm{H}^\infty(O; H')$ with $f \diamond g = \mathbf{1}$ is known as the *Corona problem*. By a result of Tolokonnikov [**?**] and Uchiyama [**?**], for the case $O = \mathbb{D}$ such a function g exists provided $\inf_{z\in\mathbb{D}} \|f(z)\|_H > 0$, see also [**?**, Appendix 3]. By a conformal mapping this result extends to O immediately (recall $\emptyset \subsetneq O \subsetneq \mathbb{C}$).

Corollary 3.11. *Let* $(\mathrm{H}^\infty(O), \Phi)$ *be a standard functional calculus over a simply connected domain* $O \subseteq \mathbb{C}$ *such that* $\Phi_{\gamma'}(f)$ *is a bounded operator for all* $f \in \mathrm{H}^\infty(O; H)$. *Then there is a constant* $C \geq 0$ *with the following property: whenever* $g \in \mathrm{H}^\infty(O; H')$ *is such that* $\delta := \inf_{z\in O} \|g(z)\|_{H'} > 0$, *one has*

$$\|x\| \leq C\, c(\delta)\, \|\Phi_\gamma(g)\, x\|_\gamma \qquad \text{for all } x \in \mathrm{dom}(\Phi_\gamma(g))$$

with $c(\delta) \leq \delta^{-2} \ln(1 + \frac{1}{\delta})^{3/2}$.

Proof. The closed graph theorem yields a constant C_1 with $\|\Phi_{\gamma'}(f)\| \leq C_1 \|f\|_\infty$ for all $f \in \mathrm{H}^\infty(O; H)$. And the Tolokonnikov–Uchiyama lemma yields for given g a function $f \in \mathrm{H}^\infty(O)$ with $f \diamond g = \mathbf{1}$ and $\|f\|_\infty \leq C_2 \delta^{-2} \ln(1 + \frac{1}{\delta})^{3/2}$. Now the claim follows from Theorem 3.9 with $C = C_1 C_2$. ☐

Sollte man hierzu nicht eine Anwendung beschreiben?

3.6. Integral Representations.

In this section we describe a method for obtaining new square function estimates from known ones via integral representations. We build on the previous results and hence work again with a *standard* functional calculus $(\mathrm{H}^\infty(O), \Phi)$ on a Banach space X. The following is the main result.

Dieser Abschnitt muss nochmal ueberarbeitet werden

Theorem 3.12. *Let* $(\mathrm{H}^\infty(O), \Phi)$ *be a standard functional calculus on the Banach space* X. *Let* H, K *be Hilbert spaces, where* $K := \mathrm{L}_2(\Omega)$ *for some measure space* (Ω, μ). *Suppose further that* f, $g \in \mathrm{H}^\infty(O; K)$ *such that the dual square function associated with* f,

$$\Phi_{\gamma'}(f) : X' \to \gamma'(K; X'),$$

is bounded. Consider, for $m \in \mathrm{L}_\infty(\Omega; H')$, the function $u \in \mathrm{H}^\infty(O; H')$ defined by

$$u(z) := \int_\Omega m(t) \cdot f(t,z)\, g(t,z)\, \mu(\mathrm{d}t) \;\in\; H' \qquad (z \in O). \tag{3.3}$$

If H has finite dimension or X has finite cotype then

$$\mathrm{dom}(\Phi_\gamma(g)) \subseteq \mathrm{dom}(\Phi_\gamma(u))$$

and for each $x \in \mathrm{dom}(\Phi_\gamma(u))$,

$$\|\Phi_\gamma(u)x\|_\gamma \le C\; \|m\|_{\mathrm{L}_\infty(\Omega;H')}\; \|\Phi_{\gamma'}(f)\|\, \|\Phi_\gamma(g)x\|_\gamma\,, \tag{3.4}$$

where C depends on $\dim(H)$ or the cotype (constant) of X, respectively.

The following proof shows that the constant C appearing here satisfies

$$C \le \min(\sqrt{2\dim(H)},\; c(q, \mathrm{c}_q(X)))$$

where q is the cotype of X and $c(q, \mathrm{c}_q(\gamma(K;X))$ is the constant appearing in Theorem B.21.

Proof. Fix $x \in \mathrm{dom}(\Phi_\gamma(g))$. We claim that $\Phi(u)x$ is defined and factorizes as

$$\Phi(u)x \;:\; H \xrightarrow{S_m} \mathrm{L}_\infty(\Omega) \xhookrightarrow{M} \mathcal{L}(K) \xrightarrow{\lambda_x} \gamma(K;X) \xrightarrow{\pi} X, \tag{3.5}$$

where

1) $\pi = \Phi_{\gamma'}(f)' : \gamma(K;X) \to X$, cf. Theorem 2.13;
2) $\lambda_x = (R \mapsto \Phi_\gamma(g)x \circ R) : \mathcal{L}(K) \to \gamma(K;X)$, which is well-defined by the ideal property of $\gamma(K;X)$;
3) $M : \mathrm{L}_\infty(\Omega) \to \mathcal{L}(K)$ is the representation of $\mathrm{L}_\infty(\Omega)$ as multiplication operators on $K = \mathrm{L}_2(\Omega)$;
4) $S_m : H \to \mathrm{L}_\infty(\Omega)$ maps $h \in H$ to $S_m h := h \circ m$.

Suppose that this factorisation holds. If $\dim(H) < \infty$ then $\gamma(H;X) = \mathcal{L}(H;X)$ with estimate

$$\|T\|_\gamma \le \sqrt{2\dim(H)}\, \|T\| \quad \text{for each } T \in \mathcal{L}(H;X).$$

Hence the factorization (3.5) shows that $\Phi(u)x$ is in $\gamma(H;X)$ and one has the norm estimate (3.7) with $C \le \sqrt{2\dim(H)}$.

If X has cotype $q < \infty$, so does $\gamma(K;X)$ (Lemma B.20); hence, by Theorem B.21 and the ideal property,

$$\|\Phi(u)x\|_{\gamma(H;X)} \le c(q, \mathrm{c}_q(\gamma(K;X))\; \|S_m\|\; \|\lambda_x\|\; \|\pi\|\,,$$

from which the norm estimate follows.

To establish the factorization (3.5), fix $h \in H$. Then

$$\lambda_x M S_m(h) = \lambda_x M(h\circ m) = \lambda_x(M_{h\circ m}) = \big(\Phi_\gamma(g)x\big)\, M_{h\circ m}.$$

Now for any $k \in K$,

$$\begin{aligned}[\lambda_x M S_m(h)]k &= \big[\Phi_\gamma(g)x\big]((h\circ m)k) = \Phi\big(\,\langle g(\cdot),(h\circ m)k\rangle\,\big)x\\ &= \Phi\big(\,\langle (h\circ m)g(\cdot),k\rangle\,\big)x.\end{aligned}$$

Hence $x \in \operatorname{dom}(\Phi_\gamma((h\circ m)g)$ and

$$\lambda_x M S_m(h) = \Phi_\gamma((h\circ m)g)x.$$

Since, for $z \in O$,

$$\begin{aligned}\big(f \diamond (h\circ m)g\big)(z) &= \int_\Omega \langle h, m(\cdot)\rangle\, f(z)g(z)\\ &= \Big\langle h, \int_\Omega m f(z)g(z)\Big\rangle = (h\circ u)(z),\end{aligned}$$

we may apply Theorem 3.9, and obtain for any $h \in H$

$$\begin{aligned}\pi\lambda_x M S_m(h) &= \Phi_{\gamma'}(f)'\Phi_\gamma((h\circ m)g)x = \Phi(f\diamond (h\circ m)g)x\\ &= \Phi(h\circ u)x = [\Phi(u)x]h,\end{aligned}$$

which shows $x \in \operatorname{dom}(\Phi(u))$ and proves the desired factorization. □

Corollary 3.13. *Let* $(\mathrm{H}^\infty(O),\Phi)$ *be a standard functional calculus on the Banach space* X*. Let* H*,* K *be Hilbert spaces, where* $K := \mathrm{L}_2(\Omega)$ *for some measure space* (Ω,μ)*. Suppose further that* f*,* $g \in \mathrm{H}^\infty(O;K)$ *and* $m \in \mathrm{L}_\infty(\Omega;H')$ *such that*

1) $\Phi_\gamma(g) : X \to \gamma(K;X)$ *is bounded and*
2) $\Phi_{\gamma'}(f) : X' \to \gamma'(K;X')$ *is bounded.*

Consider the function $u \in \mathrm{H}^\infty(O;H')$ *defined by*

$$u(z) := \int_\Omega m(t)\cdot f(t,z)\, g(t,z)\,\mu(\mathrm{d}t) \;\in\; H' \qquad (z\in O). \tag{3.6}$$

If H *has finite dimension or* X *has finite cotype then the operator* $\Phi_\gamma(u) : X \to \gamma(H;X)$ *is bounded, too, with*

$$\|\Phi_\gamma(u)\| \le c\,\|m\|_{\mathrm{L}_\infty(\Omega;H')}\,\|\Phi_{\gamma'}(f)\|\,\|\Phi_\gamma(g)\|, \tag{3.7}$$

where c *depends on* $\dim(H)$ *or the cotype (constant) of* X*, respectively.*

Suppose that Φ is a standard functional calculus for the operator A, $H = \mathrm{L}_2(\Omega')$, and

$$u(s,z) = \int_\Omega m(s,t)f(t,z)g(t,z)\,\mathrm{d}t.$$

Then Theorem 3.12 says the following: if the square and dual square functions associated with $g(\cdot, A)$ and $f(\cdot, A)$, respectively, are bounded, then also the square function associated with $u(\cdot, A)$ is bounded. (Note our convention from Section 2.4.) For $H = \mathbb{C}$ this theorem is the main tool to infer bounded H^∞-calculus from square and dual square function estimates. Examples are given in Chapter 8 below.

Remark 3.14. We do not know of a proper dual analogue of Theorem 3.12. However, under certain conditions one can use it to obtain bounded square functions for the dual functional calculus and by the inclusion $\gamma(H'; X') \subseteq \gamma'(H'; X')$ this yields a bounded dual square function for the original calculus.

4. Franks–McIntosh Representations and ℓ_1-Conditions

Quadratic estimates in relation to H^∞-calculus use bounded holomorphic H-valued functions f defined on some open subset $O \subseteq \mathbb{C}$. After introducing an orthonormal basis in H we may identify H with the space ℓ_2. In this chapter we consider functions f with stronger properties, namely bounded holomorphic functions f defined and taking values in ℓ_1. Writing $f(z) = (f_n(z))_{n\in\mathbb{N}}$ we thus have: each $f_n \in \mathrm{H}^\infty(O)$ and

$$\|f\|_{\mathrm{H}^\infty(O;\ell_1)} = \sup_{z\in O} \sum_{n=1}^{\infty} |f_n(z)| < \infty. \tag{4.1}$$

Given a bounded H^∞-functional calculus $\Phi : \mathrm{H}^\infty(O) \to \mathcal{L}(X)$, such functions have remarkable properties.

Theorem 4.1. *Let $O \subseteq \mathbb{C}$ be an open subset of the complex plane, let X be Banach space, and let $\Phi : \mathrm{H}^\infty(O) \to \mathcal{L}(X)$ be a bounded algebra homomorphism with norm $\|\Phi\|$. For a function $f = (f_n)_{n\in\mathbb{N}} \in \mathrm{H}^\infty(O; \ell_1)$, the following assertions hold:*

a) *The set of operators $\{\Phi(f_n) \mid n \in \mathbb{N}\}$ is R-bounded in $\mathcal{L}(X)$ and*

$$\sup_{N\in\mathbb{N},\, \varepsilon\in\{0,1\}^N} \Big\| \sum_{n=1}^{N} \varepsilon_n \Phi(f_n) \Big\| < \infty. \tag{4.2}$$

b) *The dual square function associated with f (considered as a function into ℓ_2) is bounded. More precisely, $\Phi_{\gamma'}(f) \in \mathcal{L}(X'; \gamma'(\ell_2; X))$ with*

$$\|\Phi_{\gamma'}(f)x'\|_{\gamma'} \le \tfrac{\sqrt{\pi}}{2}\, \|\Phi\|\, \|f\|_{\mathrm{H}^\infty(O;\ell_1)}\, \|x'\|_{X'} \qquad (x' \in X').$$

c) *If X has finite cotype $q < \infty$, then the square function $\Phi_\gamma(f)$ associated with f is bounded. More precisely: $\Phi_\gamma(f) \in \mathcal{L}(X; \gamma(\ell_2; X))$ and*

$$\|\Phi_\gamma(f)x\|_\gamma \le \sqrt{2}\mathrm{c}_q(X)m_q \, \|\Phi\| \, \|f\|_{\mathrm{H}^\infty(O;\ell_1)} \, \|x\|_X \qquad (x \in X),$$

where $\mathrm{c}_q(X)$ is the cotype-q constant of X and m_q is the q-th absolute moment of the normal distribution.

Proof. a) Let $N \in \mathbb{N}$ and $\varepsilon \in \{0,1\}^N$. Then

$$\Big\|\sum_{n=1}^N \varepsilon_n \Phi(f_n)\Big\| = \Big\|\Phi\Big(\sum_{n=1}^N \varepsilon_n f_n\Big)\Big\| \le \|\Phi\| \sup_z \Big|\sum_{n=1}^N \varepsilon_n f_n(z)\Big| \le \|\Phi\| \, \|f\|_{\mathrm{H}^\infty(O;\ell_1)} \, .$$

This establishes (4.2). For the proof of the R-boundedness, let $x_1, \dots, x_N \in X$ and $r_1, r_1', \dots, r_N, r_N'$ be independent real Rademacher variables,

$$\begin{aligned}
\mathbb{E}\Big\|\sum_{n=1}^N r_n \Phi(f_n) x_n\Big\| &= \mathbb{E}\Big\|\mathbb{E}'\sum_{n=1}^N r_n' r_n \Phi(f_n) \sum_{k=1}^N r_k' x_k\Big\| \\
&\le \mathbb{E}\mathbb{E}'\Big\|\Phi\Big(\sum_{n=1}^N r_n f_n\Big)\Big(\sum_{k=1}^N r_k' x_k\Big)\Big\| \\
&\le \|\Phi\| \, \Big(\mathbb{E}\sup_z \Big|\sum_{n=1}^N r_n f_n(z)\Big|\Big) \cdot \mathbb{E}'\Big\|\sum_{k=1}^N r_k' x_k\Big\| \\
&\le \|\Phi\| \, \|f\|_{\mathrm{H}^\infty(O;\ell_1)} \, \mathbb{E}'\Big\|\sum_{k=1}^N r_k' x_k\Big\|.
\end{aligned}$$

b) Fix $N \in \mathbb{N}$, $x_1, \dots, x_N \in X$ and independent (complex) Rademacher variables $r_1, \dots, r_N$. Then, since $\mathbb{E}(\overline{r_n} r_k) = 2\delta_{kn}$,

$$\begin{aligned}
\Big\|\sum_{n=1}^N \Phi(f_n) x_n\Big\|^2 &= \tfrac{1}{4}\Big\|\mathbb{E}\sum_{n=1}^N \overline{r_n}\Phi(f_n)\sum_{k=1}^N r_k x_k\Big\|^2 \\
&\le \tfrac{1}{4}\mathbb{E}\,\Big\|\Phi\Big(\sum_{n=1}^N \overline{r_n} f_n\Big)\,\Big(\sum_{k=1}^N r_k x_k\Big)\Big\|^2 \\
&\le \tfrac{1}{4}\,\|\Phi\|^2\Big(\mathbb{E}\,\sup_z \Big|\sum_{n=1}^N \overline{r_n} f_n(z)\Big|\Big)^2 \,\Big\|\sum_{k=1}^N r_k x_k\Big\|^2 \\
&\le \tfrac{1}{2}\,\|\Phi\|^2\,\|f\|^2_{\mathrm{H}^\infty(O;\ell_1)}\;\mathbb{E}\Big\|\sum_{k=1}^N r_k x_k\Big\|^2 \\
&\le \tfrac{\pi}{4}\,\|\Phi\|^2\,\|f\|^2_{\mathrm{H}^\infty(O;\ell_1)}\;\mathbb{E}\Big\|\sum_{k=1}^N \gamma_k x_k\Big\|^2,
\end{aligned}$$

where in the last step we appplied (B.4) with $q = 2$. By Theorem 2.13, the dual square function $\Phi_{\gamma'}(f)$ is bounded, and its norm satisfies $\|\Phi_{\gamma'}(f)\| \le \frac{\sqrt{\pi}}{2}\,\|\Phi\|\,\|f\|_{\mathrm{H}^\infty(O;\ell_1)}$.

c) By Theorem B.19,

$$\mathbb{E}\,\Big\|\sum_{n=1}^N \gamma_n \Phi(f_n) x\Big\|^2 \le \; \mathrm{c}_q(X)^2 m_q^2\,\mathbb{E}\,\Big\|\sum_{n=1}^N r_n \Phi(f_n) x\Big\|^2$$

$$
\begin{aligned}
&= \mathrm{c}_q(X)^2\, m_q^2\, \mathbb{E} \left\| \Phi\Big(\sum\nolimits_{n=1}^N r_n f_n\Big) x \right\|^2 \\
&\le \big(\mathrm{c}_q(X) m_q \, \|\Phi\| \, \|x\|\big)^2 \, \mathbb{E} \sup_{z\in O} \Big| \sum\nolimits_n r_n f_n(z) \Big|^2 \\
&\le \big(\mathrm{c}_q(X) m_q \, \|\Phi\| \, \|x\|\big)^2 \, 2 \, \|f\|_{\mathrm{H}^\infty(O;\ell_1)} .
\end{aligned}
$$

This shows that $\Phi(f)x \in \gamma_\infty(\ell_2; X)$ and

$$
\|\Phi(f)x\|_\gamma \le \sqrt{2}\mathrm{c}_q(X) m_q \, \|\Phi\| \, \|f\|_{\mathrm{H}^\infty(O;\ell_1)} \, \|x\| .
$$

Finally, since X has finite cotype it cannot contain a copy of c_0 and hence $\Phi(f)x \in \gamma(H; X)$ by the Hoffmann-Jørgensen–Kwapień theorems, cf. Appendix B.2. □

Remark 4.2. It is clear from the proof that assertions b) and c) remain true if instead of $f \in \mathrm{H}^\infty(O; \ell_1)$ one has

$$
\mathbb{E} \sup_{z\in O,\, N\in\mathbb{N}} \Big| \sum\nolimits_{n=1}^N r_n f_n(z) \Big| < \infty
$$

where $(r_n)_n$ is any sequence of independent complex Rademachers.

4.1. Functions with ℓ_1-Frame-Bounded Range.

The condition (4.1) means that f—considered as a function into ℓ_2—has its image in a multiple of the ℓ_1-unit ball. However, in many concrete cases, this condition is too restrictive. Rather, the following more general notion is often helpful, see Example **??** below.

Let H be a complex Hilbert space. The ℓ_1**-frame-bound** of a subset $M \subseteq H$ is defined as

$$
|M|_1 := \inf_{(L,R)} \|L\| \sup_{x\in M} \sum\nolimits_{\alpha\in I} |\langle Rx, e_\alpha\rangle| , \tag{4.3}
$$

where the infimum is taken over all pairs (L, R) of bounded linear operators

$$
R : H \to \ell_2(I), \qquad L : \ell_2(I) \to H
$$

(I being any (sufficiently large) index set) such that

$$
LR = \mathrm{I}_H.
$$

And M is called ℓ_1**-frame-bounded** if $|M|_1 < \infty$.

This notion is, to the best of our knowledge, new and presumably interesting in its own right. Its name derives from the fact that for a pair (R, L) as in the definition, the family $(R^* e_\alpha)_{\alpha\in I}$ is a frame for H.

Note that in the presence of operators (R, L) as above, for a function $f \in \mathrm{H}^\infty(O; H)$ the square functions associated to f and to $R \circ f$ are mutually subordinate, and hence equivalent, i.e.,

$$f \approx R \circ f$$

in the terminology of Section 3.2. To say that f has ℓ_1-frame-bounded range means that there is at least one pair (R, L) as above such that

$$\sup_{z \in O} \sum_\alpha |\langle Rf(z), e_\alpha \rangle| < \infty$$

Since O is separable, the sum here involves only countably many α, and hence one is in the situation of Theorem 4.1 above. As a result, we obtain the following theorem.

Theorem 4.3. *Let $O \subseteq \mathbb{C}$ be an open set, let X be a Banach space, and let $\Phi : \mathrm{H}^\infty(O) \to \mathcal{L}(X)$ be a bounded algebra homomorphism with norm $\|\Phi\|$. Furthermore, let $f \in \mathrm{H}^\infty(O; H)$ and $g \in \mathrm{H}^\infty(O; H')$. Then the following assertions hold.*

a) *If f has ℓ_1-frame-bounded image in H, then the dual square function associated with f is bounded, i.e., $\Phi_{\gamma'}(f) \in \mathcal{L}(X'; \gamma'(H'; X'))$. Moreover,*

$$\|\Phi_{\gamma'}(f)x'\|_{\gamma'} \le \sqrt{\tfrac{\pi}{2}}\ \|\Phi\|\ |f(O)|_1 \cdot \|x'\|_{X'} \qquad (x' \in X').$$

b) *If g has ℓ_1-frame-bounded image in H' and X has cotype $q < \infty$, then the square function associated with f is bounded, i.e., $\Phi_\gamma(g) \in \mathcal{L}(X; \gamma(H; X))$. Moreover,*

$$\|\Phi_\gamma(g)x\|_\gamma \le \sqrt{2}\mathrm{c}_q(X) m_q\ \|\Phi\|\ |g(O)|_1 \cdot \|x\|_X \qquad (x \in X),$$

where $\mathrm{c}_q(X)$ is the cotype-q constant of X and m_q is the q-th absolute moment of the normal distribution.

The most prominent examples of H^∞-functions with ℓ^1–frame-bounded range are the shift-type functions

$$f(t, z) = \psi(t + z)$$

defined for $t \in \mathbb{R}$ and z from a horizontal strip, where ψ is an elementary function of the functional calculus of strip type, see Theorem 7.6 below. By the exp / log-correspondence, this result transfers to sectors and one obtains that also the dilation type functions

$$f(s, z) = \varphi(sz)$$

defined for $s > 0$ and z from a sector have ℓ^1-frame-bounded range, if φ is an elementary function for the sectorial calculus, see **??**.

4.2. Franks-McIntosh Representations.

Suppose that $\emptyset \neq O \subseteq O'$ are open subsets of the complex plane. A **Franks-McIntosh representation** for the pair (O, O') on a Banach space X consists of a pair of functions $f, g \in \mathrm{H}^\infty(O'; \ell_1)$ and a bounded linear operator

$$a : \mathrm{H}^\infty(O'; X) \to \ell_\infty(X)$$

such that

$$\varphi(z) = \sum_{n=1}^{\infty} a_n(\varphi) f_n(z) g_n(z) \quad \text{for } z \in O$$

holds for all $\varphi \in \mathrm{H}^\infty(O', X)$.

Note that by composing with the restriction map

$$\mathrm{H}^\infty(O') \to \mathrm{H}^\infty(O), \qquad f \mapsto f\big|_O$$

each $\mathrm{H}^\infty(O)$-functional calculus Φ induces an $\mathrm{H}^\infty(O')$-functional calculus, which by abuse of notation we denote again by Φ. (If O' is connected, then a holomorphic function on O' is uniquely determined by its restriction to O, so that the restriction mapping is injective.)

From a combination of the integral representation Theorem 3.12 and Theorem 4.1 we obtain the following result.

Theorem 4.4. *Let $\emptyset \neq O \subseteq O'$ be open subsets of the complex plane. Let X be a Banach space of finite cotype and let $\Phi : \mathrm{H}^\infty(O) \to \mathcal{L}(X)$ be a bounded standard functional calculus. Let H be an infinite-dimensional Hilbert space and suppose that there exists a Franks-McIntosh representation (f, g, a) for the pair (O, O') on H'. Then the associated vectorial $\mathrm{H}^\infty(O', H')$-calculus ist bounded. More precisely, one has*

$$\|\Phi_\gamma(\varphi)x\|_{\gamma(H;X)} \leq C \, \|\Phi\|^2 \, \|\varphi\|_{\mathrm{H}^\infty(O';H')} \, \|x\|_X$$

for all $x \in X$ and $\varphi \in \mathrm{H}^\infty(O', H')$, where

$$C = c \, \|a\| \, \|f\|_{\mathrm{H}^\infty(O';\ell_1)} \, \|g\|_{\mathrm{H}^\infty(O',\ell_1)}$$

and c depends only on the cotype and the cotype constant of X.

Proof. Take $\Omega := \mathbb{N}$ with the counting measure, so that $K := \mathrm{L}_2(\Omega) = \ell_2$ and regard f and g as mappings $O \to K = K'$. By Theorem 4.1 b) and c) the dual square function $\Phi_{\gamma'}(f)$ and the square function $\Phi_\gamma(g)$ are both bounded. (Recall that X has finite cotype.)

The Franks-McIntosh representation

$$\varphi(z) = \sum_{n=1}^{\infty} a_n(\varphi) f_n(z) g_n(z)$$

on O is an instance of (3.6), with $m = (a_n(\varphi))_n$. Applying Theorem 3.12 yields the claim. □

The Franks-McIntosh representations carry their name after the paper [**?**], where Franks and McIntosh constructed such representations for the case $(O, O') = (\mathrm{S}_\alpha, \mathrm{S}_\omega)$, where $0 < \alpha < \omega < \pi$. See **??** below.

5. Square Function Estimates and γ-Boundedness

In this chapter we examine the relationship between the boundedness of certain square functions and the γ-boundedness of certain sets of operators. Basically, two questions are interesting:

1) What can γ-boundedness do for the boundedness of square functions?
2) What can the boundedness of square functions do for obtaining γ-bounded sets of operators?

We shall address these two questions in this order. As with the theory of γ-radonifying operators, we refer to Appendix B.7 and the literature listed there for the definition and basic properties of γ-bounded sets of operators.

5.1. The Multiplier Theorem.

The Multiplier Theorem is one (and in fact up to now the only one) answer to the first question from above. We work, again, with a functional calculus $(\mathrm{H}^\infty(O), \Phi)$ over an open set $O \subseteq \mathbb{C}$ (or $\mathbb{C}^d$). Suppose that

$$M : O \to \mathcal{L}(H'; K')$$

is an operator-valued function on O. For an H'-valued function $f : O \to H'$ one can then form the K'-valued function

$$Mf : O \to K', \qquad (Mf)(z) = M(z)f(z)$$

and ask about a relation of $\Phi_\gamma(f)$ and $\Phi_\gamma(Mf)$. If $M(z) \equiv T$ is constant, this is just subordination as treated in Section 3.2. It becomes interesting when M is not constant.

Of course, one first has to deal with the question whether $\Phi_\gamma(Mf)$ is meaningful. It is reasonable here to require at least

Condition 1: *For all $k \in K$ and $h' \in H'$ the function*

$$M_{k,h'} : O \to \mathbb{C}, \qquad M_{k,h'}(z) := \langle k, M(z)h' \rangle$$

is a member of H^∞.

By some standard results on vector-valued holomorphic functions, Condition 1 is actually equivalent to the (formally stronger) statement that

$$M \in \mathrm{H}^\infty(O; \mathcal{L}(H', K')).$$

In particular, the pointwise product Mf satisfies $Mf \in \mathrm{H}^\infty(O; K')$.

Whereas Condition 1 is more or less natural, the next requirement imposes a serious restriction on the function M:

Condition 2: *One has $H = K$ and the $M(z)$ are pairwise commuting and normal operators on H'.*

Note that Condition 2 implies by the Fuglede–Putnam-Rosenblum theorem [**?**, 12.16] that the range of M generates a commutative C^*-subalgebra of $\mathcal{L}(H')$. Hence, by the spectral theorem, Condition 2 is (up to a fixed subordinating unitary operator) equivalent to saying that $H = H' = \mathrm{L}_2(\Omega)$ for some measure space Ω and $\{M(z) \mid z \in O\}$ is a family of bounded scalar multiplication operators thereon.

Condition 1 allows to consider, for each pair $(h, h') \in H \oplus H'$ the operator

$$\Phi(M_{h,h'})$$

on X. In our third condition we shall require the following strong form of uniform boundedness:

Condition 3: *Each operator $\Phi(M_{h,h'})$ is bounded and the set of operators*

$$\{\Phi(M_{h,h'}) \mid \|h\|, \|h'\| \le 1\} \subseteq \mathcal{L}(X)$$

is γ-bounded.

Before we state and prove the main theorem, let us look more closely at this third requirement.

Remark 5.1. Suppose that $H = H' = \mathrm{L}_2(\Omega)$ for some measure space $(\Omega, \mathrm{d}t)$ and that $M(z)$ is given by multiplication with the function $m(z) \in \mathrm{L}_\infty(\Omega)$. Let us define for $\psi \in \mathrm{L}_1(\Omega)$

$$m_\psi(z) := \int_\Omega m(z)\, \psi \qquad (z \in O).$$

Then

$$M_{h,h'}(z) = \int_\Omega m(z)\, hh' = m_{hh'}(z),$$

and since

$$\{hh' \mid \|h\|_2, \|h'\|_2 \le 1\} = \{\psi \mid \|\psi\|_1 \le 1\},$$

Condition 3 simply says that the set

$$\{\Phi(m_\psi) \mid \|\psi\|_1 \le 1\}$$

is γ-bounded. This can be established, for instance, under the following hypotheses:

Ist das $\gamma[1]$-boundedness in Krieglerspeak?

1) The function m can be regarded as a function of two variables $(t,z) \in \Omega \times O$ and, in some (if very weak) sense, one can write

$$\Phi(m_\psi) = \int_\Omega \Phi(m(t,\cdot))\,\psi(t)\,\mathrm{d}t.$$

2) The set

$$\{\Phi(m(t,\cdot)) \mid t \in \Omega\}$$

is γ-bounded.

See [**?**]

Reference to examples!

We now come to the main result.

Theorem 5.2 (Multiplier theorem)**.** *Let* $(\mathrm{H}^\infty(O), \Phi)$ *be a functional calculus over an open set* $O \subseteq \mathbb{C}^d$*. Let* H *be a Hilbert space and* $(M(z))_{z\in O}$ *a family of operators in* $\mathcal{L}(H')$ *satisfying Conditions 1–3 from above.*

Let, furthermore, $f \in \mathrm{H}^\infty(O; H')$ *and* $x \in \mathrm{dom}(\Phi_\gamma(f))$*. If the point* x *is bp-good, then* $x \in \mathrm{dom}(\Phi_{\gamma_\infty}(Mf))$ *and*

$$\|\Phi_\gamma(Mf)x\|_\gamma \le C\,\|\Phi_\gamma(f)x\|_\gamma\,,$$

where $C = [\![\, \Phi(M_{h,h'}) \mid \|h\|, \|h'\| \le 1 \,]\!]^\gamma$ *is the* γ*-bound of the set considered in Condition 3.*

Proof. First step. We consider the case that f has values in a finite-dimensional subspace of H' that has an orthonormal basis consisting of eigenvectors of each $M(z)$. That is to say, there is a finite orthonormal system $(e_j)_{j=1}^n$ in H such that $f(z) = \sum_{j=1}^n f_j(z)\overline{e_j}$ and $M(z)\overline{e_j} = \lambda_j(z)\overline{e_j}$ for all j and all $z \in O$. Since $f \in \mathrm{H}^\infty(O; H')$ and by Condition 1 it follows that all the scalar-valued functions f_j and λ_j are contained in H^∞. Then $(Mf)(z) = \sum_{j=1}^n \lambda_j(z) f_j(z)\overline{e_j} \in \mathrm{H}^\infty(O; H')$. For $h \in H$:

$$h\circ(Mf) = \sum_{j=1}^n [h, e_j]\,\lambda_j f_j.$$

If $x \in \mathrm{dom}(\Phi_\gamma(f))$ then, in particular $x \in \mathrm{dom}(\Phi(f_j))$ for each j. Since, by Condition 3, each operator $\Phi(\lambda_j)$ is bounded, $x \in \mathrm{dom}(\Phi(\lambda_j f_j))$ and

$$\Phi(\lambda_j f_j)x = \Phi(\lambda_j)\Phi(f_j)x.$$

Hence, $x \in \operatorname{dom}\big(\Phi(h\circ(Mf))\big)$ with

$$[\Phi(Mf)x]h = \sum_{j=1}^{n} [h, e_j]\, \Phi(\lambda_j)\Phi(f_j)x.$$

Now let $(h_k)_{k=1}^m$ be any orthonormal system in H and $\gamma_1, \gamma_2 \dots$ a sequence of independent standard Gaussians on some probability space. Then

$$\begin{aligned}
\mathbb{E}\Big\| \sum_{k=1}^{m} \gamma_k [\Phi(Mf)x]h_k \Big\|_X^2 &= \mathbb{E}\Big\| \sum_{k=1}^{m}\sum_{j=1}^{n} \gamma_k \,\langle h_k, e_j\rangle\, \Phi(\lambda_j)\Phi(f_j)x \Big\|_X^2 \\
\leq \mathbb{E}\Big\|\sum_{j=1}^{n} \gamma_j \Phi(\lambda_j)\Phi(f_j)x\Big\|_X^2 &\leq C^2\, \mathbb{E}\Big\|\sum_{j=1}^{n} \gamma_j \Phi(f_j)x\Big\|_X^2 \\
= C^2\, \mathbb{E}\Big\|\sum_{j=1}^{n} \gamma_j [\Phi(f)x]e_j\Big\|_X^2 &\leq C^2\, \|\Phi_\gamma(f)x\|_\gamma^2 .
\end{aligned}$$

Here, the first inequality is due to the contraction principle (Theorem B.1) and the second one to Condition 3, which implies that the γ-bound of the set $\{\Phi(\lambda_j) \mid j = 1, \dots, n\}$ is less than C. It follows that

$$\|\Phi_\gamma(Mf)x\|_\gamma \leq C\, \|\Phi_\gamma(f)x\|_\gamma$$

as desired.

Second step. We employ the spectral theorem and may suppose that $H = H' = \mathrm{L}_2(\Omega, \mu)$, where Ω is a locally compact Hausdorff space endowed with a Radon measure μ and $M(z)$ is multiplication with a function $m(z) = m(z, t)$ which is bounded and continuous on Ω.

This representation of M and H allows to construct conditional expectation operators as follows. A *partition* of Ω is a finite family $\alpha = (A_1, \dots, A_n)$ of μ-essentially disjoint measurable subsets of Ω with $0 < \mu(A_j)$ for each $j = 1, \dots, n$. The set of all partitions is denoted by $\mathcal{A}$. It is directed in a natural way by

$$\alpha = (A_1, \dots, A_n) \prec \beta = (B_1, \dots, B_m)$$

whenever for each k there is j such that $B_k \subseteq A_j$ (μ-essentially). To each partition α we form the associated conditional expectation operator

$$P_\alpha \varphi := {\sum_j}' \Big(\frac{1}{\mu(A_j)} \int_{A_j} \varphi \Big)\, \mathbf{1}_{A_j},$$

where the primed sum $\sum'$ indicates that the sum ranges only over indices j with $0 < \mu(A_j) < \infty$. This definition applies, of course, to each function φ on Ω such that $\varphi \mathbf{1}_{A_j}$ is integrable whenever $\mu(A_j) < \infty$.

It is a simple exercise to show that

$$P_\beta P_\alpha = \begin{cases} P_\beta & \text{if } \beta \prec \alpha \text{ and} \\ P_\alpha & \text{if } \alpha \prec \beta. \end{cases}$$

Moreover, if $P_\alpha \varphi$ and $P_\alpha \psi$ are defined, then so is $P_\alpha(\varphi P_\alpha \psi)$ and one has

$$P_\alpha(\varphi P_\alpha \psi) = (P_\alpha \varphi)\,(P_\alpha \psi).$$

The operators P_α are positive orthogonal projections on $\mathrm{L}_2(\Omega)$, and $P'_\alpha = P_\alpha$ under the natural identification of $\mathrm{L}_2(\Omega)$ with its dual. Moreover, $P_\alpha \to \mathrm{I}$ strongly on $\mathrm{L}_2(\Omega)$ as $\alpha \to \infty$ with respect to $\prec$.

Third step. Recall that $f \in \mathrm{H}^\infty(O; \mathrm{L}_2(\Omega))$. We write f_α to denote the L_2-valued function

$$f_\alpha(z) = (P_\alpha \circ f)(z) = P_\alpha(f(z))$$

(and likewise for m and $m_\alpha(z)$). Now fix $x \in \mathrm{dom}(\Phi_\gamma(f))$. Then, by subordination, $x \in \mathrm{dom}(\Phi_\gamma(f_\alpha))$ and

$$\|\Phi_\gamma(f_\alpha)x\|_\gamma \le \|\Phi_\gamma(f)x\|_\gamma .$$

Moreover, the range of f_α lies in the finite-dimensional subspace

$$H_\alpha = \mathrm{span}\{\mathbf{1}_{A_j} \mid 0 < \mu(A_j) < \infty\}$$

that has an orthonormal basis, namely the functions

$$e_j := \frac{1}{\sqrt{\mu(A_j)}} \mathbf{1}_{A_j}$$

of eigenvectors of the multiplication operators associated with the functions $m_\alpha = P_\alpha \circ m$.

In order to apply the first step, we need to assure that Condition 3 is satisfied for m_α. If $\varphi \in \mathrm{L}_1(\Omega)$ with $\|\varphi\|_1 \le 1$ then for each $z \in O$

$$\int_\Omega P_\alpha(m(z))\,\varphi = \int_\Omega m(z)\,(P_\alpha \varphi)$$

and by Condition 3, the Φ-images of these functions of z are γ-bounded with bound C. So we indeed can apply the first step and obtain that $x \in \mathrm{dom}(\Phi_\gamma(m_\alpha f_\alpha))$ with

$$\|\Phi(m_\alpha f_\alpha)x\|_\gamma \le C\,\|\Phi_\gamma(f_\alpha)x\|_\gamma \le C\,\|\Phi_\gamma(f)x\|_\gamma . \tag{5.1}$$

Fourth step. We now let $\alpha \to \infty$ for the direction of the set of finite partitions of Ω. Actually, we shall show that for x satisfying the hypothesis of the theorem, the operator $\Phi(mf)x$ lies in the strong closure of the operator family $\{\Phi(m_\alpha f_\alpha)x \mid \alpha\}$. To this aim, note that f and mf both have separable range. Fix any separable subspace K of H

that contains the ranges of f and mf. Then, by Lemma A.1, there is a sequence $(\alpha_n)_n$ such that $P_{\alpha_n}k \to k$ for all $k \in K$. It follows that for each $k \in K$ one has

$$k\circ(m_{\alpha_n} f_{\alpha_n}) = k\circ(mf_{\alpha_n})_{\alpha_n} = (P_{\alpha_n}k)\circ(mf_{\alpha_n}) \to k\circ(mf)$$

pointwise and boundedly on O. Since x is supposed to be bp-good, it follows that

$$\Phi(m_{\alpha_n} f_{\alpha_n})x \to \Phi(mf)x$$

strongly on K. This establishes the claim that $\Phi(mf)x$ is in the strong operator closure of the operators $\Phi(m_\alpha f_\alpha)x$.

Fifth step. Now we can apply the γ-Fatou Lemma B.5 to conclude that $\Phi(mf)x \in \gamma_\infty(H;X)$ and

$$\|\Phi_{\gamma_\infty}(mf)x\|_\gamma \le \sup_\alpha \|\Phi(m_\alpha f_\alpha)x\|_\gamma \le C\,\|\Phi_\gamma(f)\|_\gamma$$

as claimed. □

Corollary 5.3. *Let* $(\mathrm{H}^\infty(O), \Phi)$ *be a functional calculus on a Banach space* X *over an open set* $O \subseteq \mathbb{C}^d$ *such that the set of bp-good points is dense in* X*. Let* H *be a Hilbert space and* $(M(z))_{z\in O}$ *a family of operators in* $\mathcal{L}(H')$ *satisfying Conditions 1–3 from above.*

Let, furthermore, $f \in \mathrm{H}^\infty(O;H')$ *such that* $\Phi_\gamma(f)$ *is bounded. Then* $\Phi_\gamma(Mf)$ *is bounded and*

$$\|\Phi_\gamma(Mf)\| \le C\,\|\Phi_\gamma(f)\|\,,$$

where $C = [\![\, \Phi(M_{h,h'}) \mid \|h\|, \|h'\| \le 1 \,]\!]^\gamma$ *is the* γ*-bound of the set considered in Condition 3.*

5.2. How to Obtain γ-Bounded Subsets.

In this section we look at the second question from above: *What can the boundedness of square functions do for obtaining* γ*-bounded sets of operators?* A first answer to this question rests on the following result, which is a slight generalization of a result of Haak and Kunstmann from [**?**, Theorem 3.18].

Theorem 5.4 (Haak and Kunstmann)**.** *Let* X *be a Banach space with property* (α) *and let* H*,* K *be Hilbert spaces. For each* $R \in \mathcal{L}(H;K)$ *consider the operator "subordination by* R*"*

$$S_R : \gamma(K;X) \to \gamma(H;X), \qquad S_R(T) := TR$$

Then the set

$$\{S_R \mid R \in \mathcal{L}(H;K),\ \|R\| \le 1\} \ \subseteq\ \mathcal{L}\big(\gamma(K;X);\gamma(H;X)\big)$$

is γ-bounded by C^-C^+, the condition number of the isomorphism

$$\gamma(\ell_2 \otimes \ell_2; X) \cong \gamma(\ell_2; \gamma(\ell_2; X)),$$

Proof. Let $(R_j)_{j=1}^J$ be a finite sequence of bounded operators $H \to K$, with $\|R_j\| \le 1$ for all j and let $T_j \in \gamma(K; X)$ for each j. Furthermore, let $(e_n)_{n=1}^N$ be an orthonormal system in H and let $(f_l)_{l=1}^L$ be an orthonormal basis for the space

$$\operatorname{span}\{R_j e_n \mid 1 \le j \le J,\, 1 \le n \le N\} \subseteq K.$$

Then

$$\begin{aligned}
\mathbb{E}\mathbb{E}'\Big\| \sum_n \gamma_n' \Big(\sum_j \gamma_j T_j R_j \Big) e_n \Big\|_X^2 &= \mathbb{E}\mathbb{E}' \Big\| \sum_{j,n} \gamma_j \gamma_n' T_j R_j e_n \Big\|^2 \\
&= \Big\| \sum_{j,n} e_j \otimes e_n' \otimes \big(T_j R_j e_n\big) \Big\|^2_{\gamma(\ell_2^J; \gamma(\ell_2^N; X))} \\
&\le (C^-)^2 \Big\| \sum_{j,n} e_j \otimes e_n' \otimes \big(T_j R_j e_n\big) \Big\|^2_{\gamma(\ell_2^J \otimes \ell_2^N; X)} \\
&= (C^-)^2\, \mathbb{E} \Big\| \sum_{(j,n)} \gamma_{j,n} \sum_l \left[R_j e_n, f_l\right]_K T_j f_l \Big\|_X^2 =: \Lambda^2.
\end{aligned}$$

Here, C^- is just the norm of the identification map

$$\gamma(\ell_2 \otimes \ell_2; X) \to \gamma(\ell_2; \gamma(\ell_2; X)),$$

We now use a little trick in writing

$$\sum_l \left[R_j e_n, f_l\right]_K T_j f_l = \sum_{(l,m)} \left[R_j e_n, f_l\right]_K \delta_{jm}\, T_m f_l,$$

where m also ranges over $1, \dots, J$. Now we can apply the contraction principle for Gaussian sums (Theorem B.1) and obtain

$$\Lambda^2 \le C\, \|A\|^2\, \mathbb{E} \Big\| \sum_{(l,m)} \gamma_{l,m} T_m f_l \Big\|_X^2,$$

where A is the operator

$$A : \ell_2^L \otimes \ell_2^J \to \ell_2^N \otimes \ell_2^J$$

given by the matrix

$$A \sim \big(\left[R_j e_n, f_l\right]_K \delta_{jm} \big)_{(l,m),(n,j)}.$$

To estimate this norm let $u = (u_{(l,m)})_{(l,m)}$ and $v = (v_{(n,j)})_{(n,j)}$ be unit vectors in $\ell_2^L \otimes \ell_2^J$ and $\ell_2^N \otimes \ell_2^J$, respectively. Then

$$\begin{aligned}|[A\overline{u}, \overline{v}]|^2 &= \Big| \sum_{(n,j),(l,m)} [R_j e_n, f_l]_K \, \delta_{jm} \overline{u_{(l,m)}} v_{(j,n)} \Big|^2 \\ &= \Big| \sum_{n,j,l} [R_j e_n, f_l]_K \, \overline{u_{(l,j)}} v_{(j,n)} \Big|^2 \\ &= \Big| \sum_j \Big[R_j \sum_n v_{(n,j)} e_n, \sum_l u_{(l,j)} f_l \Big]_K \Big|^2 \\ &\le \Big(\sum_j \sqrt{\sum_n \big| v_{(n,j)} \big|^2} \sqrt{\sum_l \big| u_{(l,j)} \big|^2} \Big)^2 \le \|v\|^2 \|u\|^2 \le 1.\end{aligned}$$

Hence $\|A\| \le 1$. We therefore can continue with

$$\begin{aligned}\Lambda^2 &\le (C^-)^2 \, \mathbb{E} \Big\| \sum_{(l,m)} \gamma_{l,m} T_m f_l \Big\|_X^2 \\ &\le (C^-)^2 \, (C^+)^2 \, \mathbb{E}\mathbb{E}' \Big\| \sum_l \sum_m \gamma_l \gamma'_m T_m f_l \Big\|_X^2 \\ &= (C^- C^+)^2 \, \mathbb{E}'\mathbb{E} \Big\| \sum_l \gamma_l \Big(\sum_m \gamma'_m T_m \Big) f_l \Big\|_X^2 \\ &\le (C^- C^+)^2 \, \mathbb{E}' \Big\| \sum_m \gamma'_m T_m \Big\|_{\gamma(K;X)}^2.\end{aligned}$$

Here, C^+ is the norm of the identification mapping

$$\gamma(\ell_2; \gamma(\ell_2; X)) \to \gamma(\ell_2 \otimes \ell_2; X),$$

hence $C^+ C^-$ is the condition number of this map.

Now let $F := \operatorname{span}\{e_n \mid n = 1, \ldots, N\}$ and $P_F := \sum_n \overline{e_n} \otimes e_n$ the orthogonal projection onto F. Then

$$\mathbb{E}\mathbb{E}' \Big\| \sum_n \gamma'_n \Big(\sum_j \gamma_j T_j R_j \Big) e_n \Big\|_X^2 = \mathbb{E} \Big\| \sum_j \gamma_j T_j R_j P_F \Big\|_{\gamma(H,X)}^2.$$

If we let $F \nearrow H$ with respect to the natural direction, then $T_j R_j P_F \to T_j R_j$ in the norm of $\gamma(H; X)$, by Theorem B.8. Therefore

$$\sum_j \gamma_j T_j R_j P_F \longrightarrow \sum_j \gamma_j T_j R_j$$

as $\gamma(H; X)$-valued functions uniformly on each set

$$\big[\max_j |\gamma_j| \le c \big], \qquad c > 0.$$

Since one has the square-integrable majorant

$$\sum_j |\gamma_j| \, \|T_j R_j\|_{\gamma(H;X)},$$

the net-version of the dominated convergence theorem (Lemma A.2) yields convergence in L_2 as $F \to H$, and we obtain

$$\mathbb{E}\Big\|\sum_j \gamma_j T_j R_j\Big\|^2_{\gamma(H;X)} \le C\,C'\,\mathbb{E}'\Big\|\sum_m \gamma'_m T_m\Big\|^2_{\gamma(K;X)}.$$

And this is what we intended to prove. □

Remark 5.5. *If Theorem 5.4 both Hilbert spaces H and K are infinite dimensional, then the hypothesis that X has property (α) is necessary.*

Proof. Suppose that the set $\{S_R \mid R \in \mathcal{L}(H;K), \|R\| \le 1\}$ is γ-bounded with bound $C \ge 0$. Then, let N, $J \ge 1$ be entire numbers and for $1 \le n \le N$ and $1 \le j \le K$ let $x_{n,j} \in X$ and $\alpha_{n,j} \in \mathbb{C}$ with $|\alpha_{n,j}| \le 1$ be given.

Choose any orthogonal systems $(e_n)_{n\le N}$ and $(f_j)_{j\le J}$ in H and K, respectively, and define the operators

$$T_n := \sum_{j=1}^{J} \overline{f_j} \otimes x_{n,j} \in \gamma(K;X) \quad \text{and}$$

$$R_n := \sum_{j=1}^{J} \overline{e_j} \otimes \alpha_{n,j} f_j \in \mathcal{L}(H;K).$$

A simple computation using the fact that $|\alpha_{n,j}| \le 1$ yields $\|R_n\| \le 1$ for each $n \in \mathbb{N}$.

By hypothesis, the set $\{S_{R_n} \mid 1 \le n \le N\}$ is γ-bounded with bound C. In particular, one has the estimate

$$\mathbb{E}\Big\|\sum_{n=1}^{N} \gamma_n T_n R_n\Big\|_{\gamma(H;X)} \le C\,\mathbb{E}\Big\|\sum_{n=1}^{N} \gamma_n T_n\Big\|_{\gamma(K;X)}.$$

Since

$$T_n R_n = \sum_{j=1}^{J} \overline{e_j} \otimes (\alpha_{n,j} x_{n,j}),$$

writing out both γ-norms yields

$$\mathbb{E}\mathbb{E}'\Big\|\sum_{n=1}^{N}\sum_{j=1}^{J} \alpha_{n,j}\gamma'_j\gamma_n x_{n,j}\Big\|^2_X \le C^2\,\mathbb{E}\mathbb{E}'\Big\|\sum_{n=1}^{N}\sum_{j=1}^{J} \gamma_n\gamma'_j x_{n,j}\Big\|^2_X.$$

This shows that X has Pisier's contraction property (in its Gaussian form) which is equivalent to property (α), see Appendix B.6. □

Theorem 5.4 has the following interesting consequence. Recall the situation of Theorem 3.12 about integral representations in its "scalar form": So one is given a standard functional calculus $(\mathrm{H}^\infty(O); \Phi)$ a Hilbert space K of the form $K = \mathrm{L}_2(\Omega, \mu)$ for some measure space (Ω, μ), and functions f, $g \in \mathrm{H}^\infty(O; K)$ such that $\Phi_{\gamma'}(f)$ and $\Phi_\gamma(g)$ are bounded. For each $m \in \mathrm{L}_\infty(\Omega, \mu)$ the function $u_m \in \mathrm{H}^\infty(O)$ is given by

$$u_m(z) := \int_\Omega m \cdot f(z) g(z) \,\mathrm{d}\mu \qquad (z \in O).$$

In Corollary 3.13 it was proved that $\Phi(u_m) \in \mathcal{L}(X)$ with $\|\Phi(u_m)\| \lesssim \|m\|_\infty$. If X has property (α) we can assert more.

Theorem 5.6. *Suppose that X has property (α). Then, under the hypotheses of Corollary 3.13 (scalar case, i.e., with $H = \mathbb{C}$) the set*

$$\{\Phi(u_m) \mid m \in \mathrm{L}_\infty(\Omega, \mu),\ \|m\|_\infty \le 1\}$$

is γ-bounded, with bound

$$[\![\, \Phi(u_m) \mid \ \|m\|_\infty \le 1 \,]\!]^\gamma \le C \,\|\Phi_\gamma(g)\|_\gamma \,\|\Phi_{\gamma'}(f)\|_{\gamma'}\,,$$

where C is as in Theorem 5.4.

Proof. Recall from the proof of 3.12 the factorisation

$$\Phi(u_m)x = \Phi_{\gamma'}(f)'(\Phi_\gamma(x) \circ M_m)$$

where M_m is the multiplication operator on $K = \mathrm{L}_2(\Omega, \mu)$ with the function $m \in \mathrm{L}_\infty(\Omega, \mu)$. In other words,

$$\Phi(u_m) = \Phi_{\gamma'}(f)' \circ S_{M_m} \circ \Phi_\gamma(g).$$

Since X has property (α), Theorem 3.12 yields that the set of operators

$$\{S_{M_m} \mid \ \|m\|_\infty \le 1\}$$

is γ-bounded with bound C. Hence, also the set $\{\Phi(u_m) \mid \ \|m\|_\infty \le 1\}$ is γ-bounded with bound

$$[\![\, \Phi(u_m) \mid \ \|m\|_\infty \le 1 \,]\!]^\gamma \le C \,\|\Phi_{\gamma'}(f)\| \,\|\Phi_\gamma(g)\|\,,$$

as claimed. □

Theorem 5.6 is the abstract version of many concrete results that assert "γ-bounded H^∞-calculus" on spaces with property (α). For sectorial operators, such a result was first obtained by Kalton and Weis in [**?**, Theorem 5.3], see also **??** below. In **??** we treat a different example.

Remark 5.7. It must remain open at this point whether the vectorial form of the integral representation theorem allows for a similar conclusion regarding γ-boundedness.

5.3. The Multiplier Theorem (II).

The multiplier theorem (Corollary **??**) states up to some technical subtelties that if a multiplier M satisfies a certain γ-boundedness condition and if f yields a bounded square function, then Mf yields a bounded square function as well.

Now suppose that we vary f within a set of bounded square functions and M within a set of multipliers? Under which conditions can we guarantee that the emerging set of square functions Mf is γ-bounded? The following theorem gives an answer:

Theorem 5.8 (Multiplier Theorem, γ-version)**.** *Let the following hypotheses be satisfied:*

1) *X is a Banach space with property (α) and H is a Hilbert space.*
2) *$(\mathrm{H}^\infty(O), \Phi)$ is a functional calculus on X over an open set $O \subseteq \mathbb{C}^d$ such that the set of bp-good points is dense in X.*
3) *$\mathcal{F} \subseteq \mathrm{H}^\infty(O; H')$ is a set of vector-valued functions such that the corresponding set of square functions*
$$\Phi_\gamma(\mathcal{F}) := \{\Phi_\gamma(f) \mid f \in \mathcal{F}\} \;\subseteq\; \mathcal{L}(X; \gamma(H; X))$$
is γ-bounded.
4) *$\mathcal{M} \subseteq \mathrm{H}^\infty(O; \mathcal{L}(H))$ is a set of multipliers satisfying the following conditions:*
   - 4.1) *The operators $\{M(z) \mid z \in O,\ M \in \mathcal{M}\}$ are normal and pairwise commuting.*
   - 4.2) *The set of operators*
   $$\{\Phi(M_{h,h'}) \mid \ \|h\|, \|h'\| \le 1,\ M \in \mathcal{M}\}$$
   is γ-bounded.

Then the set
$$\Phi_\gamma(\mathcal{M}\,\mathcal{F}) := \{\Phi_\gamma(Mf) \mid M \in \mathcal{M},\ f \in \mathcal{F}\} \subseteq \mathcal{L}(X; \gamma(H; X))$$
is γ-bounded, with
$$[\![\, \Phi_\gamma(\mathcal{M}\,\mathcal{F}) \,]\!]^\gamma \le C\ [\![\, \mathcal{M} \,]\!]^\gamma\ [\![\, \mathcal{F} \,]\!]^\gamma,$$
where
$$[\![\, \mathcal{M} \,]\!]^\gamma := [\![\, M_{h,h'} \mid \ \|h\|, \|h'\| \le 1,\ M \in \mathcal{M} \,]\!]^\gamma.$$

Note that a space with property (α) has finite cotype and hence does not contain a copy of c_0. In particular, $\gamma_\infty(H;X) = \gamma(H;X)$.

Proof. □

5.4. Averaged γ-Boundedness and Related Concepts.

Let X, Y, Z be Banach spaces. A linear operator $T : X \to \mathcal{L}(Y;Z)$ is called γ**-strict**, if the set

$$\{Tx \mid \|x\| \le 1\} \subseteq \mathcal{L}(Y;Z)$$

is γ-bounded. The corresponding γ-bound

$$|\!|\!| T |\!|\!|_\gamma := [\![\, Tx \mid \|x\| \le 1 \,]\!]^\gamma$$

is called the γ-strict norm of T. It is a simple exercise to verify that the set

$$\Gamma(X;\mathcal{L}(Y;Z)) := \{T : X \to \mathcal{L}(Y;Z) \mid T \text{ is } \gamma\text{-strict}\}$$

of γ-strict operators is a linear subspace of $\mathcal{L}(X;\mathcal{L}(Y;Z)$ and the mapping $|\!|\!|\cdot|\!|\!|_\gamma$ is a complete norm on it. Moreover, if $S_1 : X_1 \to X$, $S_2 : Y_1 \to Y$ and $S_3 : Z \to Z_1$ are bounded linear mappings and $T : X \to \mathcal{L}(X;\mathcal{L}(Y;Z))$ is γ-strict, then so is the mapping

$$S_3(TS_1)S_2 : X_1 \to \mathcal{L}(Y_1;Z_1), \qquad x_1 \to S_3(TS_1x)S_2$$

and one has

$$|\!|\!| S_3(TS_1)S_2 |\!|\!|_\gamma \le \|S_1\| \, \|S_2\| \, \|S_3\| \, |\!|\!| T |\!|\!|_\gamma.$$

Using our new terminology, we can rephrase the Haak–Kunstmann Theorem 5.4 as follows: *If X has property (α) then for any pair of Hilbert spaces H, K the mapping*

$$\mathcal{L}(H;K) \to \mathcal{L}(\gamma(K;X);\gamma(H;X)), \qquad R \mapsto S_R$$

is γ-strict.

Now, examining the proof of this theorem, we observe that if either H or K is one-dimensional, one does not need the full power of property (α) to obtain the result.

For example, let us suppose first that $\dim H = 1$, i.e., $H = \mathbb{C}$. The operators $R : \mathbb{C} \to K$ simply correspond to elements $k \in K$, and subordination with such an operator is nothing else than evaluation at k. We hence arrive at the following result, where for convenience we use the letter H instead of K for the generic Hilbert space. Recall that C^+ is our generic notation for the "property (α^+)-constant" of a space, cf. Appendix B.6.

Theorem 5.9. *Let H be Hilbert space and let X be Banach space with property (α^+). For each $h \in H$ let*

$$\delta_h : \gamma(H;X) \to X, \qquad \delta_h(T) := T(h)$$

be the evaluation mapping at h. Then the set

$$\{\delta_h \mid \|h\| \le 1\} \subseteq \mathcal{L}(\gamma(H;X);X)$$

is γ-bounded by C^+. In other words: the operator

$$\Delta : H \to \mathcal{L}(\gamma(H;X);X), \qquad h \mapsto \delta_h$$

is γ-strict and $\|\Delta\|_\gamma \le C^+$.

Proof. As said above, the proof is the same as for Theorem 5.4 with the further assumption that $H = \mathbb{C}$ there. With the notation of that proof, since $N = 1$, property (α^-) is not needed there.

For convenience, here is also a direct proof. Let $T_1, \dots, T_n \in X$ and $h_1, \dots, h_n \in H$ with $\|h_k\| \le 1$ for all k. We have to prove

$$\mathbb{E}\Big\|\sum_k \gamma_k T_k h_k\Big\|_X^2 \le (C^+)^2 \mathbb{E}\Big\|\sum_j \gamma_j T_j\Big\|_{\gamma(H;X)}^2$$

By the Kahane contraction principle, we may suppose without loss of generality that $\|h_k\| = 1$ for all k. Now let us take $K = \ell^2$ with canonical basis $(e_k)_k$. By property (α^+) one has a bounded operator

$$\gamma(K;\gamma(H;X)) \to \gamma(K \otimes H; X)$$

with norm $\le C^+$ (see Lemma B.25).

Note that the vectors $(e_k \otimes h_k)_{1 \le k \le n}$ form an orthonormal sequence in $K \otimes H$. Hence, we obtain

$$\begin{aligned}\mathbb{E}\Big\| \sum_k \gamma_k T_k h_k\Big\|_X^2 &= \mathbb{E}\Big\|\sum_k \gamma_k \big[\big(\sum_j \overline{e_j} \otimes T_j\big)\big](e_k \otimes h_k)\Big\|_X^2 \\ &\le \Big\|\big(\sum_j \overline{e_j} \otimes T_j\big)\Big\|_{\gamma(K\otimes H;X))}^2 \le (C^+)^2 \,\Big\|\sum_j \overline{e_j} \otimes T_j\Big\|_{\gamma(K;\gamma(H;X))}^2 \\ &= (C^+)^2\, \mathbb{E}\Big\|\sum_j \gamma_j T_j\Big\|_{\gamma(H;X)}^2,\end{aligned}$$

as desired. $\square$

As a corollary we obtain a result that has been first obtained by Le Merdy in [?] for L_p-spaces and later proved in the present generality by Van Neerven and Weis in [?, Theorem 5.4].

Corollary 5.10. *Let X, Y be Banach spaces and $T : X \to \gamma(H; Y)$ a bounded linear operator (i.e., a bounded square function). If Y has property (α^+), then the set of operators*

$$\big\{[T\,\cdot]h \mid h \in H,\ \|h\| \le 1\big\} \subseteq \mathcal{L}(X; Y)$$

is γ-bounded with γ-bound $\le C^+ \|T\|$.

Proof. Since $[T\cdot]h = \delta_h \circ T$, the assertion follows from Theorem 5.9. □

Remark 5.11. The concept of γ-strict operators is motivated by the notion of "E-averaged R-boundedness" coined by Kriegler and Weis in [**?**, Definition 5.1]. For the special case $E = \mathrm{L}_2(\Omega)$ this amounts to the following: If $N = N(t)$ is a weakly square-integrable operator-valued function defined on a measure space Ω then N is L^2-averaged R-bounded, in short: $R[L^2]$-bounded, if the operators

$$x \mapsto \int_\Omega h(t)N(t)x\,\mathrm{d}t, \qquad \|h\|_{\mathrm{L}_2} \le 1$$

form an R-bounded set. In Example 5.3 c) of their paper, Kroegler and Weis mention that if N defines a bounded square function on a space with property (α), then N is $R[L^2]$-bounded. Indeed, this is true by an application of Corollary 5.10 (even under the weaker assumption of property (α^+).

Let us briefly comment on the other simplification of Theorem 5.4, the case that $K = \mathbb{C}$. Identifying as before $\gamma(\mathbb{C}; X) \cong X$, a short analysis yields that we are considering the family of operators

$$T_h : X \to \gamma(H; X), \qquad T_h(x) := \overline{h} \otimes x$$

for $h \in H$. Similarly as before, the proof of Theorem 5.4 will work for this special case if X has property (α^-). However, the result holds under even weaker conditions.

Theorem 5.12. *Let X be a space of finite cotype q and let H be any Hilbert space. Then the set of operators*

$$\{T_h \mid h \in H,\ \|h\| \le 1\} \subseteq \mathcal{L}(X; \gamma(H; X))$$

is γ-bounded by $C = C(q, \mathrm{c}_q(X))$.

(The constant $C = C(q, \mathrm{c}_q(X))$ is the same as in Theorem B.21.)

Proof. Let $h_1 \dots, h_N \in H$ and $x_1, \dots, x_n \in X$. Let $e_1, \dots, e_J$ be an orthonormal basis for the space $\operatorname{span}\{h_1, \dots, h_N\}$. Then

$$\mathbb{E}\Big\|\sum_n \gamma_n \left(\overline{h_n} \otimes x_n\right)\Big\|_\gamma^2 = \mathbb{E}\mathbb{E}'\| \sum_{n,j} \gamma_n \gamma_j' \left[e_j, h_n\right] x_n\Big\|_X^2$$

The Kaiser–Weis Lemma (Corollary B.22) yields a constant $C \geq 0$ such that the latter random sum can be estimated by

$$\dots \leq C^2 \sup_n \|([e_j, h_n])_j\|_{\ell_2^J}^2 \, \mathbb{E}\| \sum_n \gamma_n x_n\Big\|_X^2$$

and the claim follows since

$$\|([e_j, h_n])_j\|_{\ell_2^J}^2 = \sum_j |[e_j, h_n]|^2 = \|h_n\|_H^2$$

for each $n \in \{1, \dots, N\}$. □

Remark 5.13. It is easily seen that Theorem 5.12 is actually equivalent to the Kaiser–Weis Lemma employed in its proof.

A closer examination of Theorem 5.9 reveals that its statement is equivalent to the following inequality which bears a faint resemblance to the Kaiser–Weis Lemma:

$$\mathbb{E}\Big\|\sum_n \gamma_n \sum_j \alpha_{n,j} x_{n,j}\Big\|_X^2 \leq C^2 \sup_n \|(\alpha_{n,j})_j\|_{\ell_2}^2 \, \mathbb{E}\mathbb{E}'\Big\|\sum_{n,j} \gamma_n \gamma_j' x_{n,j}\Big\|_X^2.$$

We know that this holds if X has property (α^+), but it is reasonable to conjecture that it holds under weaker conditions. The question must remain open for now.

6. Alternative Types of Functional Calculi

The square functions considered so far were defined in terms of an (possibly unbounded) H$^\infty$-functional calculus. This has basically two reasons: (1) many readers are well acquainted with this type of functional calculus and (2) this is the case for which we, the authors, know the applications. It seemed unreasonable to strive for greater generality unless one has some serious applications in mind. On the other hand, applications might well appear as soon as one understands what is needed for generalizing the concepts and results from the previous chapters.

Therefore, in this chapter we shall review the basic concepts and sketch how to develop a theory which leaves the framework of H$^\infty$-calculus towards other functional calculi.

...

To be completed

Part II. Applications

7. Sectorial, Strip type and Ritt Operators

In this chapter we review the standard examples of functional calculi for sectorial, strip type and Ritt operators and associated square functions. We start with the strip case and postpone the discussion of sectorial and Ritt operators to **??** and **??** below.

Although the construction of the functional calculi for sectorial and strip type operators can be found at several places in the literature, the usual accounts suffer from an unnatural asymmetry in view of the exp / log-correspondence of sectors and strips, cf. Remark 7.11 below. Therefore, we give a slightly modified account that avoids that shortcoming and in fact appears to be the most natural and the most general at the same time.

7.1. Function Theory on Strips.

When we speak of a *strip* in the following we mean a horizontal strip in the complex plane, symmetric about the real axis. Formally,

$$\mathrm{St}_\omega := \{z \in \mathbb{C} \mid \ |\operatorname{Im} z| < \omega\}$$

for $\omega > 0$ and (the degenerate strip) $\mathrm{St}_0 := \mathbb{R}$.

The algebra of **elementary functions** on St_ω, $\omega > 0$, is

$$\mathcal{E}(\mathrm{St}_\omega) := \Big\{ f \in \mathrm{H}^\infty(\mathrm{St}_\omega) \mid \int_{-\infty}^{\infty} |f(r + \mathrm{i}\alpha)| \,\mathrm{d}r < \infty \text{ for all } |\alpha| < \omega \Big\}.$$

It is common in the literature to use explicit growth conditions to define elementary functions. However, it will appear from our discussion that this is not necessary. See also Remark 7.11 below.

It is clear that $f \in \mathcal{E}(\mathrm{St}_\omega)$ if and only if $f(\cdot + r) \in \mathcal{E}(\mathrm{St}_\omega)$ for some/each $r \in \mathbb{R}$. Moreover, by Cauchy's theorem, the following formulae hold for any elementary function $f \in \mathcal{E}(\mathrm{St}_\omega)$:

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{St}_{\omega'}} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta \tag{7.1}$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{St}_{\omega'}} f(\zeta) \frac{\mathrm{e}^{-(\zeta - z)^2}}{\zeta - z} \,\mathrm{d}\zeta \qquad (z \in \mathrm{St}_{\omega'},\ 0 < \omega' < \omega). \tag{7.2}$$

Note that for $z \in \mathbb{C}$ and $\zeta \in \mathrm{St}_\omega$,

$$\begin{aligned} \left|\mathrm{e}^{-(\zeta - z)^2}\right| &= \mathrm{e}^{-\operatorname{Re}(\zeta - z)^2} = \mathrm{e}^{-(\operatorname{Re}\zeta - \operatorname{Re} z)^2} \cdot \mathrm{e}^{(\operatorname{Im}\zeta - \operatorname{Im} z)^2} \\ &\le \mathrm{e}^{-(\operatorname{Re}\zeta - \operatorname{Re} z)^2} \mathrm{e}^{(\omega + |\operatorname{Im} z|)^2}. \end{aligned}$$

Consequently, for fixed $z \in \mathbb{C}$ the function $\zeta \mapsto \mathrm{e}^{-(\zeta-z)^2}$ is an elementary function on St_ω. It follows that the representation formula (7.2) actually holds for *all* $f \in \mathrm{H}^\infty(\mathrm{St}_\omega)$.

Lemma 7.1. *Let $0 < \alpha < \omega$ and $f \in \mathcal{E}(\mathrm{St}_\omega)$. Then the following assertions hold:*

a) $\sup_{|s|\le\alpha} \int_{-\infty}^{\infty} |f(r + \mathrm{i}s)| \,\mathrm{d}r < \infty$.

b) $f \in \mathcal{E}(\mathrm{St}_\alpha) \cap \mathrm{C}_0(\overline{\mathrm{St}_\alpha})$.

c) $\int_{\partial \mathrm{St}_\alpha} f(z) \,\mathrm{d}z = 0$.

d) $f' \in \mathcal{E}(\mathrm{St}_\alpha)$.

Proof. For the proof of a) fix $\alpha < \omega' < \omega$. Then for $0 \le s \le \alpha$,

$$\begin{aligned}\int_{\partial \mathrm{St}_s} |f(z)| \, |\mathrm{d}z| &\le \frac{1}{2\pi} \int_{\partial \mathrm{St}_s} \int_{\partial \mathrm{St}_{\omega'}} \Big| f(\zeta) \frac{\mathrm{e}^{-(\zeta-z)^2}}{\zeta - z} \Big| \, |\mathrm{d}\zeta| \, |\mathrm{d}z| \\ &= \frac{1}{2\pi} \int_{\partial \mathrm{St}_{\omega'}} |f(\zeta)| \int_{\partial \mathrm{St}_s} \Big| \frac{\mathrm{e}^{-(\zeta-z)^2}}{\zeta - z} \Big| \, |\mathrm{d}z| \, |\mathrm{d}\zeta| \\ &\le \frac{1}{2\pi} \, \|f\|_{\mathrm{L}_1(\partial \mathrm{St}_{\omega'})} \int_{\partial \mathrm{St}_s} \frac{\mathrm{e}^{-(\operatorname{Re} z)^2} \mathrm{e}^{(\omega'+\alpha)^2}}{\omega' - \alpha} \, |\mathrm{d}z| \\ &= \frac{\mathrm{e}^{(\omega'+\alpha)^2}}{\sqrt{\pi}(\omega' - \alpha)} \, \|f\|_{\mathrm{L}_1(\partial \mathrm{St}_{\omega'})} .\end{aligned}$$

b) To see that $|f(z)| \to 0$ as $|\operatorname{Re} z| \to \infty, |\operatorname{Im} z| \le \alpha$ one uses the representation formula (7.1) or (7.2) and the dominated convergence theorem.

c) By Cauchy's formula one has $0 = \int_{R_n} f(z) \,\mathrm{d}z$ where R_n is the rectangle with corners at $\pm n \pm \mathrm{i}\alpha$, $n \in \mathbb{N}$. When letting $n \to \infty$ the upper and the lower side of the rectangle approach $\partial \mathrm{St}_\omega$ and the integrals over the left and right side vanish since $f \in \mathrm{C}_0(\overline{\mathrm{St}_{\omega'}})$ by b).

d) Let $\alpha < \omega' < \omega$. Then by Cauchy's integral formula,

$$f'(z) = \frac{1}{2\pi\mathrm{i}} \int_{\partial \mathrm{St}_{\omega'}} \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - z)^2} \quad (|\operatorname{Im} z| < \omega').$$

In particular,

$$\int\limits_{|\operatorname{Im} z| = \alpha} |f(z)| \, |\mathrm{d}z| \le \Big(\int\limits_{\partial \mathrm{St}_{\omega'}} \frac{|f(\zeta)| \, |\mathrm{d}\zeta|}{2\pi} \Big) \Big(\max_{\zeta = \pm \mathrm{i}\omega'} \int\limits_{|\operatorname{Im} z| = \alpha} \frac{|\mathrm{d}z|}{|\zeta - z|^2} \Big) < \infty.$$

$\square$

After having introduced the relevant function class, we now turn to operators.

7.2. Operators of Strip Type and their Functional Calculus. A closed operator A on a Banach space X is called of **strip type** $\alpha \geq 0$, if $\sigma(A) \subseteq \overline{\mathrm{St}_\alpha}$ and for all $\beta > \alpha$ the resolvent $R(\cdot, A)$ is uniformly bounded on $\mathbb{C} \setminus \mathrm{St}_\beta$. If for each $\beta > \alpha$ we have an estimate $\|R(\lambda, A)\| \lesssim (|\mathrm{Im}\,\lambda| - \beta)^{-1}$ on $\mathbb{C} \setminus \overline{\mathrm{St}_\beta}$, A is called of **strong strip type** α.

For an operator A of strip type $\alpha \geq 0$, $\omega > \alpha$ and an elementary function $f \in \mathcal{E}(\mathrm{St}_\omega)$ there is a natural definition of the operator $f(A) \in \mathcal{L}(X)$ by

$$f(A) := \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{St}_{\omega'}} f(z) R(z, A)\, \mathrm{d}z, \tag{7.3}$$

which is independent of $\omega' \in (\alpha, \omega)$ by Cauchy's theorem. The mapping $\mathcal{E}(\mathrm{St}_\omega) \to \mathcal{L}(X)$ given by $f \mapsto f(A)$ is called the elementary calculus for A. It is rather routine to show by virtue of the resolvent identity, the residue theorem and contour deformation arguments that this is a homomorphism of algebras with

$$\Big(\frac{f(z)}{\lambda - z}\Big)(A) = f(A) R(\lambda, A) \quad \text{and}$$
$$\Big(\frac{1}{(\lambda - z)(\mu - z)}\Big)(A) = R(\lambda, A) R(\mu, A)$$

for all $\lambda, \mu \in \mathbb{C} \setminus \overline{\mathrm{St}_\omega}$, cf. [**?**, Chapter 2] or [**?**].

The elementary functional calculus from above can be extended to a functional calculus

$$\Phi : \mathrm{H}^\infty(\mathrm{St}_\omega) \to \{\text{closed unbounded operators on } E\}$$

by virtue of the definition

$$\Phi(f) := \Phi(e)^{-1} \Phi(ef)$$

where $f \in \mathrm{H}^\infty(\mathrm{St}_\omega)$ and $e \in \mathcal{E}(\mathrm{St}_\omega)$ is such that $e(A)$ is injective. (One can take $e(z) = (\lambda - z)^{-2}$ for any $\lambda \in \mathbb{C}$ with $|\mathrm{Im}\,\lambda| > \omega$.) The following lemma summarizes the most important properties.

Lemma 7.2. *Let A be a strip type operator of type $\alpha \geq 0$ on a Banach space E, let $\omega > 0$. Then the associated functional calculus* $(\Phi, \mathrm{H}^\infty(\mathrm{St}_\omega)$ *has the following properties:*

a) *Each elementary function is a bp-good universal regulariser. In other words: $\mathcal{E}(\mathrm{St}_\omega) \subseteq \mathcal{C}_\Phi$.*
b) *The functional calculus* $(\Phi, \mathrm{H}^\infty(\mathrm{St}_\omega)$ *is standard.*

c) *If A is densely defined, the set of bp-good points is dense in E and the convergence lemma holds.*

Proof. a) If $e \in \mathcal{E}(\mathrm{St}_\omega)$ and $f \in \mathrm{H}^\infty(\mathrm{St}_\omega)$, then $ef \in \mathcal{E}(\mathrm{St}_\omega)$ again. Hence, each elementary function is a universal regulariser. Moreover, if $(f_n)_n$ is a sequence in $\mathrm{H}^\infty(\mathrm{St}_\omega)$ bp-converging to f, then $ef_n \to ef$ converges pointwise and dominated (by a constant times $|e|$) on each vertical line, so $\Phi(ef_n) = (ef_n)(A) \to (ef)(A) = \Phi(ef)$ in operator norm.

b) Clearly $\mathcal{E}(\mathrm{St}_\omega)$ is universally determining. In fact: whenever $e \in \mathcal{E}(\mathrm{St}_\omega)$ is such that $e(A)$ is in injective, then $\{e\}$ is universally determining.

c) Let $\operatorname{Im}\lambda > \omega$ and $e(z) := (\lambda - z)^{-2}$. Then e is elementary and $\operatorname{ran}(e(A)) = \operatorname{dom}(A^2)$, which is dense if $\operatorname{dom}(A)$ is. Since by a), all points in $\operatorname{ran}(e(A))$ are bp-good, the first assertion in c) is proved. The second follows from Lemma 2.5. □

An operator A of strip type $\alpha \in [0, \omega)$ on a Banach space X **has a bounded** $\mathrm{H}^\infty(\mathrm{St}_\omega)$**-calculus** if there is a constant $C \geq 0$ such that

$$\|f(A)\| \leq C \|f\|_{\mathrm{H}^\infty(\mathrm{St}_\omega)} \quad \text{for all } f \in \mathcal{E}(\mathrm{St}_\omega).$$

The following lemma gives a characterization of this notion in the case that A is densely defined.

Lemma 7.3. *Let $0 \leq \alpha < \omega$, and let A be a densely defined operator A of strip type $\alpha \in [0, \omega)$ on a Banach space X. Then A has a bounded $\mathrm{H}^\infty(\mathrm{St}_\omega)$-calculus if and only if the elementary calculus has an extension to a bounded algebra homomorphism $\Phi : \mathrm{H}^\infty(\mathrm{St}_\omega) \to \mathcal{L}(X)$. In this case, such an extension is unique and $\|\Phi(f)\| \leq C \|f\|_\infty$ holds for every $f \in \mathrm{H}^\infty(\mathrm{St}_\omega)$ if it holds for every $f \in \mathcal{E}(\mathrm{St}_\omega)$.*

Proof. To prove uniqueness, suppose first that the bounded algebra homomorphism $\Phi : \mathrm{H}^\infty(\mathrm{St}_\omega) \to \mathcal{L}(X)$ extends the elementary calculus. If $f \in \mathrm{H}^\infty(\mathrm{St}_\omega)$ and $e \in \mathcal{E}(\mathrm{St}_\omega)$ such that $ef \in \mathcal{E}(\mathrm{St}_\omega)$ and $e(A)$ is injective, then $(ef)(A) = \Phi(ef) = \Phi(e)\Phi(f) = e(A)\Phi(f)$. Hence

$$\Phi(f) = e(A)^{-1}(ef)(A)$$

with the natural domain. Since each function $e(z) = (\lambda - z)^{-2}$ with $|\operatorname{Im}\lambda| > \omega$ is an instance, this shows uniqueness.

Now suppose that $\|f(A)\| \leq C \|f\|_\infty$ for each $f \in \mathcal{E}(\mathrm{St}_\omega)$. We consider the extension Φ to all of $\mathrm{H}^\infty(\mathrm{St}_\omega)$ by regularisation. Let $f \in \mathrm{H}^\infty(\mathrm{St}_\omega)$. Then for each $n \in \mathbb{N}$ the function $e_n(z) := \mathrm{e}^{-(1/n)z^2}$ is elementary, hence also fe_n is, and thus

$$\|(e_n f)(A)\| \le C\,\|e_n f\|_\infty \le C\,\|e_n\|_\infty\,\|f\|_\infty \le C\mathrm{e}^{-\frac{1}{n}\omega^2}\,\|f\|_\infty\,.$$

On the other hand, it is easy to see that for x in the dense subspace $\mathrm{dom}(A^2)$ of X, one has $x \in \mathrm{dom}(f(A))$ and $(e_n f)(A)x \to \Phi(f)x$. Hence $\Phi(f)$ is bounded by $C\,\|f\|_\infty$, and since it is closed and densely defined, $\Phi(f) \in \mathcal{L}(X)$ and $\|\Phi(f)\| \le C\,\|f\|_\infty$. □

7.3. Square Functions for Strip Type Operators.

From the scalar functional calculus constructed in the previous section, we now pass to the vectorial version, i.e., to square functions. Again we suppose that A is an operator of strip type ω_0 on a Banach space X, and consider the functional calculus $(\mathcal{E}(\mathrm{St}_\omega), \mathrm{H}^\infty(\mathrm{St}_\omega), \Phi)$.

Since the product eg of an elementary function e with a general H^∞-function g is again elementary, it follows from the definition of $g(A)$ that $\mathrm{ran}(e(A)) \subseteq \mathrm{dom}(g(A))$. The following result tells that the same is true for square functions.

Lemma 7.4. *In the described situation the following assertions hold:*

a) *Let* $g \in \mathrm{H}^\infty(\mathrm{St}_\omega; H')$ *and let* $\Phi_\gamma(g)$ *be the associated square function*

$$\Phi_\gamma(g) : \mathrm{dom}(\Phi_\gamma(g)) \to \gamma(H;X), \quad \Phi_\gamma(g)x = \Big(h \mapsto (h\circ g)(A)x\Big).$$

Then $\mathrm{dom}(\Phi_\gamma(g))$ *contains* $\mathrm{ran}(e(A))$ *for each* $e \in \mathcal{E}(\mathrm{St}_\omega)$.

b) *Suppose that* A *is densely defined. Then each operator* $f(A)$, $f \in \mathrm{H}^\infty(\mathrm{St}_\omega)$, *is densely defined and dual square functions are well defined.*

Let $f \in \mathrm{H}^\infty(\mathrm{St}_\omega; H)$ *and let* $\Phi_{\gamma'}(f)$ *be the associated dual square function*

$$\Phi_{\gamma'}(f) : \mathrm{dom}(\Phi_{\gamma'}(f)) \to \gamma(H;X), \quad \Phi_{\gamma'}(f)x' = \Big(h' \mapsto (h'\circ f)(A)'x'\Big)$$

Then $\mathrm{dom}(\Phi_{\gamma'}(f))$ *contains* $\mathrm{ran}(e(A)')$ *for each* $e \in \mathcal{E}(\mathrm{St}_\omega)$.

Proof. a) Let $e \in \mathcal{E}(\mathrm{St}_\omega)$, let $x \in X$ and $h \in H$. Then

$$\begin{aligned}[\Phi(f)e(A)x]\,h &= (h\circ f)(A)e(A)x = ((h\circ f)e)(A)x \\ &= \frac{1}{2\pi\mathrm{i}}\int_{\partial\mathrm{St}_{\omega'}} \langle h, f(z)\rangle\, e(z)R(z,A)x\,\mathrm{d}z\end{aligned}$$

with $\omega' \in (\omega_0, \omega)$. This shows that $e(A)x \in \mathrm{dom}(\Phi(f))$ and that

$$\Phi(f)e(A)x = \frac{1}{2\pi\mathrm{i}} \int_{\partial \mathrm{St}_{\omega'}} f(z) \otimes e(z)R(z,A)x \,\mathrm{d}z$$

is in $\gamma(H;X)$ by Lemma B.12. The proof of b) is similar. □

In the following, we shall discuss several instances of square functions for strip type operators. We recall our standing assumption that whenever we speak of dual square functions the dual calculus is supposed to be well defined, cf. Section 2.3. We begin with the square functions "of shift type".

Example 7.5 (Shift type square functions)**.** Let $\omega' > \omega$ and $\psi \in \mathcal{E}(\mathrm{St}_{\omega'})$, and define

$$g : \mathbb{R} \times \mathrm{St}_\omega \to \mathbb{C}, \qquad g(t,z) := \psi(t+z) \quad (t \in \mathbb{R},\ z \in \mathrm{St}_\omega).$$

Then g satisfies the hypotheses of Lemma 2.14, i.e., it can be regarded as a H^∞- function

$$g : \mathrm{St}_\omega \to \mathrm{L}_2(\mathbb{R}).$$

This gives rise to the square and dual square function

$$[\Phi_\gamma(g)x]\, h = \Big(\int_{\mathbb{R}} h(t)\psi(t+z)\,\mathrm{d}t\Big)(A)x \qquad (x \in \mathrm{dom}(\Phi_\gamma(g)),$$

$$[\Phi_{\gamma'}(g)x']\, h = \Big(\int_{\mathbb{R}} h(t)\psi(t+z)\,\mathrm{d}t\Big)(A)'x' \qquad (x' \in \mathrm{dom}(\Phi_{\gamma'}(g))$$

for $h \in \mathrm{L}_2(\mathbb{R})$.

For $e \in \mathcal{E}(\mathrm{St}_\omega)$ and $x \in \mathrm{ran}(e(A))$, Lemma 7.4 and a simple Fubini argument show that $\Phi_\gamma(g)x$ is integration against the vector-valued function $t \mapsto \psi(t+A)x$. This why we use the symbol "$\psi(t+A)$" as an abbreviation for $\Phi_\gamma(g)$, cf. Remark 2.15.

If X is itself a Hilbert space, one has

$$\|\Phi_\gamma(f)x\|_\gamma = \Big(\int_{\mathbb{R}} \|\psi(t{+}A)x\|^2 \,\mathrm{d}t\Big)^{1/2},$$

and hence a square function estimate for this square function takes the form

$$\int_{\mathbb{R}} \|\psi(t{+}A)x\|^2 \,\mathrm{d}t \le C\, \|x\|^2 .$$

Similar remarks hold true for the dual square functions of shift type.

The next result explains why the shift type square functions are of such great importance.

Theorem 7.6. *Let $\omega' > \omega > 0$, let $\psi \in \mathcal{E}(\mathrm{St}_{\omega'})$ be an elementary function on the strip $\mathrm{St}_{\omega'}$. Then the function*

$$g : \mathrm{St}_\omega \to \mathrm{L}_2(\mathbb{R}), \quad g(z) = (t \mapsto \psi(t+z))$$

has ℓ_1-frame-bounded range.

Proof. We first note that

$$\sup_{z\in\mathrm{St}_\omega} \int_{\mathbb{R}} |\psi(t+z)| \, \mathrm{d}t < \infty$$

by Lemma 7.1.a). Moreover, $\psi', \psi'' \in \mathcal{E}(\mathrm{St}_\alpha)$ for each $\alpha \in (\omega, \omega')$ by Lemma 7.1.d). As before, this implies that

$$\sup_{z\in\mathrm{St}_\omega} \|\psi(\cdot+z)\|_{\mathrm{W}^2_1(\mathbb{R})}$$
$$= \sup_{z\in\mathrm{St}_\omega} \int_{\mathbb{R}} |\psi(t+z)| + |\psi'(t+z)| + |\psi''(t+z)| \, \mathrm{d}t < \infty.$$

By Lemma C.6 the claim is proved. □

Theorem 4.3 yields the following important result.

Corollary 7.7. *Suppose that the $\mathrm{H}^\infty(\mathrm{St}_\omega)$-calculus for A is bounded and $\psi \in \mathcal{E}(\mathrm{St}_{\omega'})$ for some $\omega' > \omega$. Then the dual square function associated with $\psi(t+z)$ is bounded. If X has finite cotype, then also the square function associated with $\psi(t+z)$ is bounded.*

Let us pass to other square functions for strip type operators. Since the Fourier transform is an isomorphism on $\mathrm{L}_2(\mathbb{R})$, we may subordinate the shift type square functions from above via the Fourier transform. This yields the "weighted group orbits", to be discussed next.

Example 7.8 (Weighted group orbits)**.** Let as before $\omega' > \omega$ and $\psi \in \mathcal{E}(\mathrm{St}_{\omega'})$. Taking the inverse Fourier transform with respect to the variable t in the $\mathrm{L}_2(\mathbb{R})$-valued function $\psi(t+z)$ yields the function

$$\psi^\vee(s)\mathrm{e}^{-\mathrm{i}sz} = \mathcal{F}_t^{-1}(\psi(t+z))(s).$$

Hence, the (dual) square functions associated with $\psi(t+z)$ and its Fourier transform $\psi^\vee(s)\mathrm{e}^{-\mathrm{i}sz}$ are strongly equivalent. In particular,

$$\text{if} \quad \psi(z) = \frac{\pi/\omega}{\cosh((\pi/2\omega)z)} \quad \text{then} \quad \psi^\vee(s) = \frac{1}{\cosh \omega s}$$

(cf. Remark 8.4 below), hence

$$\frac{\pi/\omega}{\cosh((\pi/2\omega)(t+z))} \quad \approx \quad \frac{\mathrm{e}^{-\mathrm{i}sz}}{\cosh(\omega s)}.$$

Hence, by Theorem 7.6 the latter function also has ℓ_1-frame-bounded range in $\mathrm{L}_2(\mathbb{R})$. Moreover, Corollary 7.7 holds mutatis mutandis for this square function.

By refining the subordination technique, we are led to yet another type of square functions.

Example 7.9 (Resolvents on horizontal lines)**.** Let $m > 0$ be a function on $\mathbb{R}$ such that m, $m^{-1} \in \mathrm{L}_\infty(\mathbb{R})$. Then Multiplying with m an isomorphism on $\mathrm{L}_2(\mathbb{R})$ and hence suitable for generating (by subordination) equivalent square functions. Applying this observation to the weighted group orbits from above and then use the Fourier transform again, we obtain

$$\begin{aligned}\frac{\pi/\omega}{\cosh((\pi/2\omega)(t+z))} &\approx \frac{\mathrm{e}^{-\mathrm{i}sz}}{\cosh(\omega s)} \approx \mathrm{e}^{-\omega|s|}\mathrm{e}^{-\mathrm{i}sz} \\ &\approx \big(\mathbf{1}_{\mathbb{R}_+}(s)\mathrm{e}^{-\omega s}\mathrm{e}^{-isz},\, \mathbf{1}_{\mathbb{R}_+}(s)\mathrm{e}^{-\omega s}\mathrm{e}^{isz}\big) \\ &\approx \big(\pm\mathrm{i}\omega + t - z\big)^{-1}.\end{aligned}$$

This means, in particular, that the weighted group orbit square function on $\mathrm{L}_2(\mathbb{R})$

$$f(s,z) = \frac{\mathrm{e}^{-\mathrm{i}sz}}{\cosh(\omega s)}$$

is strongly equivalent to the *direct sum* of square functions given by the "resolvent functions"

$$f_1(t,z) = \frac{1}{\mathrm{i}\omega + t - z} \quad \text{and} \quad f_2(t,z) = \frac{1}{-\mathrm{i}\omega + t - z}$$

on $\mathrm{L}_2(\mathbb{R}) \oplus \mathrm{L}_2(\mathbb{R})$, cf. Corollary 3.2. Consequently, we may (informally!) write

$$\|\cosh(\omega s)^{-1}\mathrm{e}^{-\mathrm{i}sA}x\|_\gamma \approx \|R(\pm\mathrm{i}\omega + t, A)x\|_\gamma\,.$$

Such square functions were considered in [**?**, Theorem 6.2], see Section 8.6 below.

7.4. Sectorial Operators.

When we speak of a *sector* we usually mean a sector in the complex plane with vertex at the origin and which is symmetric about the positive real axis. Formally,

$$\mathrm{S}_\omega := \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \omega\},$$

for $0 < \omega \le \pi$ and (the degenerate sector) $\mathrm{S}_0 := (0, \infty)$. The transformations

$$w = \log z \quad \text{and} \quad z = \exp(w)$$

form a pair of mutually inverse holomorphic mappings from the sector S_ω to the strip St_ω and vice versa. Consequently, the function theory for St_ω and S_ω are equivalent, and we obtain the following analogue of Lemma 7.1.

Corollary 7.10. *Let $0 < \alpha < \omega \le \pi$ and $f \in \mathcal{E}(\mathrm{S}_\omega)$. Then the following assertions hold:*

a) $\displaystyle \sup_{|s|\le\alpha} \int_0^\infty \left|f(re^{\mathrm{i}s})\right| \frac{\mathrm{d}r}{r} < \infty.$

b) $f \in \mathcal{E}(\mathrm{S}_\alpha) \cap \mathrm{C}_0(\overline{\mathrm{S}_\alpha} \setminus \{0\})$.

c) $\displaystyle \int_{\partial \mathrm{S}_\alpha} f(z) \frac{\mathrm{d}z}{z} = 0.$

d) $zf'(z) \in \mathcal{E}(\mathrm{S}_\alpha)$.

The algebra of **elementary functions** on S_ω, $0 < \omega \le \pi$, is

$$\mathcal{E}(\mathrm{S}_\omega) := \Big\{ f \in \mathrm{H}^\infty(\mathrm{S}_\omega) \mid \int_0^\infty \left|f(re^{\mathrm{i}\alpha})\right| \frac{\mathrm{d}r}{r} < \infty \text{ for all } |\alpha| < \omega \Big\}.$$

Then $\mathcal{E}(\mathrm{S}_\omega) = \{ f \circ \log \mid f \in \mathcal{E}(\mathrm{St}_\omega)\}$.

Remark 7.11. It is common in the literature to use a class of elementary functions defined via explicit growth conditions instead of integrability. In this approach, the class

$$\mathrm{H}_0^\infty(\mathrm{S}_\omega) = \{ f \in \mathrm{H}^\infty(\mathrm{S}_\omega) \mid \exists s, C > 0 : |f(z)| \le C \min(|z|^s, |z|^{-s})\}$$

features prominently. However, integrability conditions are more natural and do work as well. Moreover, the common growth conditions for elmentary functions on sectors and strips are not compatible with the exp / log-correspondence, whereas our definition is.

A closed operator A with dense domain and dense range on a Banach space X is called **sectorial of angle** $\alpha \in [0, \pi)$, if $\sigma(A) \subseteq \overline{\mathrm{S}_\alpha}$ and for all $\beta \in (\alpha, \pi)$ the mapping $z \mapsto zR(z, A)$ is uniformly bounded on $\mathbb{C} \setminus \mathrm{S}_\beta$.

One can set up a functional calculus for sectorial operators on sectors analogously to the strip case. Namely, $f(A)$ is defined for an elementary function $f \in \mathcal{E}(\mathrm{S}_\omega)$ by means of (7.3) with $\partial \mathrm{St}_{\omega'}$ replaced by $\partial \mathrm{S}_{\omega'}$. For a general $f \in \mathrm{H}^\infty(\mathrm{S}_\omega)$, $f(A)$ is defined by regularisation as described above. Then the sectorial analogue of Lemma 7.3 holds.

It turns out [**?**, Proposition 3.5.2] that each sectorial operator A of angle α has a logarithm $\log(A)$, which is of (strong) strip type α. The functional calculi of these operators are linked via the exp/log-correspondence, i.e., $f(\log A) = (f \circ \log)(A)$ for all $f \in \mathrm{H}^\infty(\mathrm{St}_\omega)$, see

[?, Theorem 4.2.4]. It is not true in general that every (strong) strip type operator is the logarithm of a sectorial one [?, Example 4.4.1]. However, as long as one confines oneself to operators with bounded H^∞-calculus, the correspondence is perfect [?, Proposition 5.5.3], and hence it suffices to consider in detail only one of these cases.

Finally, we turn to square functions for sectorial operators. The results of the previous section have their natural analogues for sectorial operators via the exp / log-correspondence. Of course, one has to use the Hilbert space $\mathrm{L}_2^*(0,\infty)$ and the "shift type" square functions become "dilation type" square functions of the form $\psi(tz)$. The analogue of the Fourier transform is the Mellin transform, and the square functions of "weighted group orbit"-type translate into square functions of the form $\psi(s)z^{-\mathrm{i}s}$, i.e., the group of imaginary powers emerges here.

7.5. Ritt Operators.

A bounded operator on a Banach space is a **Ritt** operator if

$$\sum\nolimits_{k\ge 1} k \left\| T^{k-1}(\mathrm{I}-T)\right\| < \infty.$$

The semigroup $\{T^n \mid n \ge 0\}$ is the discrete analogue of an analytic semigroup, see [?]. The spectrum of a Ritt operator is contained in a *Stolz domain* and one has a natural functional calculus there, see [?, ?, ?, ?]. In the recent article [?], LeMerdy considers square functions associated with the ℓ_2-valued H^∞-mappings

$$f_m(k,z) := k^{m-1/2} z^{k-1}(1-z)^m \qquad (k\in\mathbb{N})$$

which are the discrete analogues of the $\mathrm{L}_2^*(0,\infty)$-valued mappings

$$g_m(t,z) := (tz)^m \mathrm{e}^{-tz}$$

of dilation type. To some extent, the theory of bounded H^∞-calculus and square function estimates on Stolz domains is equivalent to the strip or the sector case, by conformal equivalence of the underlying complex domains.

Should we say more here?

8. Applications

In this chapter we present several applications of the integral representation Theorem 3.12. In each case one starts from very specific bounded square and dual square functions and concludes the boundedness of an H^∞-calculus or even, in the case that the Banach space has finite cotype, the boundedness of a vectorial H^∞-calculus. However, one usually has to pay a price in the form that the domain set for the holomorphic functions represented by the integral formula has to be larger than the domain set used for the square functions.

8.1. Cauchy–Gauß Representation.

Our first instance uses the variant of the usual Cauchy integral formula with an additional Gaussian factor.

Let $0 < \omega < \omega'$, and let $\Gamma := \partial \mathrm{St}_\omega$ with arc length (=Lebesgue) measure. Then it is simple complex analysis to show that

$$u(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma u(w) \tfrac{\mathrm{e}^{-(w-z)^2}}{w-z} \, \mathrm{d}w \qquad (|\operatorname{Im} z| < \omega)$$

whenever $u \in \mathrm{H}^\infty(\mathrm{St}_{\omega'}; H)$, cf. Formula (7.2). To interpret it in the light of Theorem 3.12 we let $K := \mathrm{L}_2(\Gamma)$ and define

$$m(w) := u(w), \quad f(w,z) := \tfrac{\mathrm{e}^{-\frac{1}{2}(w-z)^2}}{w-z}, \quad \text{and} \quad g(w,z) := \mathrm{e}^{-\frac{1}{2}(w-z)^2}$$

for $w \in \Gamma$ and $z \in \mathrm{St}_\omega$. Then $f, g \in \mathrm{H}^\infty(\mathrm{St}_\alpha; K)$ for each $\alpha \in (0, \omega)$. Consequently, if for an operator A of strip type $\omega_0 < \omega$ on a Banach space X the square and dual square functions associated respectively with f and g are bounded, then A has a bounded $\mathrm{H}^\infty(\mathrm{St}_{\omega'})$-calculus. And if X has finite cotype, then A has a bounded $\mathrm{H}^\infty(\mathrm{St}_{\omega'})$-*square* functional calculus.

Actually, one can say more here. Theorem 3.12 yields a constant $C \geq 0$ such that

$$\|f(A)\|_\gamma \leq C \, \|f\|_{\mathrm{H}^\infty(\mathrm{St}_\omega)} \quad \text{for all} \quad f \in \bigcup\nolimits_{\omega' > \omega} \mathrm{H}^\infty(\mathrm{St}_{\omega'}).$$

If the operator A is densely defined, then by the scalar/vectorial convergence lemma one obtains a bounded (vectorial) $\mathrm{H}^\infty(\mathrm{St}_\omega)$-calculus.

Combining these results with Theorem 4.3, or rather with Corollary 7.7, we arrive at the following central result.

Theorem 8.1. *Let $\alpha > 0$ and let $\Phi : \mathrm{H}^\infty(\mathrm{St}_\alpha) \to \mathcal{L}(X)$ be a bounded H^∞-calculus over the strip St_α on a Banach space X of finite cotype. Further, let $\beta > \alpha$ and H be an arbitrary Hilbert space. Then, for each $u \in \mathrm{H}^\infty(\mathrm{St}_\beta; H')$ the square function $\Phi_\gamma(u) : X \to \gamma(H; X)$ is a bounded operator and there is a constant $C \geq 0$ such that*

$$\|\Phi_\gamma(u)x\|_\gamma \leq C \, \|u\|_{\mathrm{H}^\infty(\mathrm{St}_\beta)} \, \|x\|_X \quad \textit{for all} \quad u \in \mathrm{H}^\infty(\mathrm{St}_\beta; H'), \, x \in X.$$

Proof. Fix $\omega \in (\alpha, \beta)$. Then, as in Example 7.5,

$$\sup_{z \in \mathrm{St}_\alpha} \|g(z)\|_{\mathrm{W}_1^2(\Gamma)} + \|f(z)\|_{\mathrm{W}_1^2(\Gamma)} < \infty$$

and hence $g, f : \mathrm{St}_\alpha \to K$ have ℓ_1-frame-bounded range. By Theorem 4.3, the associated square and dual square functions are bounded. As explained above, the claim now follows from Theorem 3.12. □

Clearly, Theorem 8.1 has a straightforward analogue for sectorial operators. Note that the vectorial calculus in Theorem 8.1 "lives" on a slightly larger strip. Consequently, in the sectorial version one needs to enlarge the sector.

Remark 8.2. While we were working on the present manuscript, Christian Le Merdy independently found the equivalent result of Theorem 8.1 for sectorial operators [**?**, Theorem 6.3]. (His "quadratic" H^∞-calculus is essentially what we call a "bounded vectorial" H^∞-calculus.) Le Merdy's proof, which rests implicitly on an ℓ_1-frame-boundedness argument, is based on the Franks–McIntosh decomposition, to be treated below in Section 8.5.

8.2. Poisson Representation.
Our next example uses a variant of the Poisson formula for the strip.

Lemma 8.3. *Let* $0 < \omega < \omega'$ *and* $u \in \mathrm{H}^\infty(\mathrm{St}_{\omega'}; H)$. *Then*

$$u(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\pi/2\omega}{\cosh(\pi/2\omega(z+s))} \big(u(\mathrm{i}\omega - s) + u(-\mathrm{i}\omega - s)\big)\, \mathrm{d}s \tag{8.1}$$

whenever $|\mathrm{Im}\, z| < \omega$.

Proof. Fix $0 < \alpha \le \pi/2\omega$, then for $|\mathrm{Im}\, z| < \omega$ the function

$$f(w) = \frac{\alpha(z-w)}{\sinh(\alpha(w-z))}\, u(w)$$

is analytic in a strip larger than St_ω. (Note that $w = z$ is a removable singularity.) Hence, by Cauchy's integral formula.

$$u(z) = f(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\alpha}{\sinh(\alpha(w-z))}\, u(w)\, \mathrm{d}w$$

where $\Gamma := \partial \mathrm{St}_\omega$ with the natural orientation. Now write out the parametrisation, specialise $\alpha = \pi/2\omega$ and use that $\sinh(a \pm \mathrm{i}\pi/2) = \pm \mathrm{i} \cosh(a)$. □

Remark 8.4. Specialising $u(z) = \mathrm{e}^{\mathrm{i}zx}$ and $z = 0$ in (8.1) one obtains again the formula

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{\pi/2\omega}{\cosh(\pi/2\omega\, s)} \mathrm{e}^{\mathrm{i}st}\, \mathrm{d}s = \frac{1}{\cosh(\omega t)} \qquad (t \in \mathbb{R})$$

used in Example 7.8.

In order to apply Theorem 3.12, we need to factorise the integral kernel in (8.1). A possibility is

$$\frac{\pi/_{2\omega}}{\cosh(\pi/_{2\omega} z)} = \Big[\frac{\alpha}{\omega}\frac{\cosh(\pi/_{2\alpha} z)}{\cosh(\pi/_{2\omega} z)}\Big] \cdot \frac{\pi/_{2\omega}}{\cosh(\pi/_{2\alpha} z)} = f(z)\cdot g(z), \tag{8.2}$$

for $\alpha > \omega$. With $m_u(s) := \big(u(\mathrm{i}\omega - s) + u(-\mathrm{i}\omega - s)\big)$, Formula (8.1) then becomes

$$u(z) = \frac{1}{2\pi}\int_{\mathbb{R}} m_u(s) f(z-s) g(z-s)\,\mathrm{d}s. \tag{8.3}$$

This is an instance of (3.6), hence Theorem 3.12 can be applied. The function f still looks a little unwieldy, but turns out to be strongly equivalent to g, since

$$\frac{\alpha}{\omega}\frac{\cosh(\pi/_{2\alpha}(z+s))}{\cosh(\pi/_{2\omega}(z+s))} \approx \frac{2\alpha}{\pi}\cos\Big(\frac{\pi\omega}{2\alpha}\Big)\frac{\cosh(\omega t)}{\cos(\pi\omega/_{\alpha}) + \cosh(2\omega t)}\,\mathrm{e}^{-\mathrm{i}tz} \approx \frac{\mathrm{e}^{-\mathrm{i}tz}}{\cosh(\omega t)}.$$

Here, the first equivalence comes from taking the inverse Fourier transform, and the second holds by multiplying by L_∞-functions.

Theorem 8.5. *Let A be a densely defined operator of strip type $\omega_0 \geq 0$ on a Banach space X (of finite cotype). Let $\omega > \omega_0$ and suppose that the square and dual square functions associated with the weighted group orbit $\mathrm{e}^{-itz}/\cosh(\omega t)$ are bounded, i.e.,*

$$\Big\|\frac{\mathrm{e}^{-\mathrm{i}tA}x}{\cosh(\omega t)}\Big\|_\gamma \lesssim \|x\| \quad \textit{and} \quad \Big\|\frac{\mathrm{e}^{-\mathrm{i}tA'}x'}{\cosh(\omega t)}\Big\|_{\gamma'} \lesssim \|x'\|.$$

Then A has a bounded (vectorial) H^∞-calculus on St_ω.

Proof. We apply the preceding remarks to obtain

$$\|\Phi_\gamma(f)\|_\gamma \lesssim \|f\|_{\mathrm{H}^\infty(\mathrm{St}_\omega)}$$

for $f \in \bigcup_{\omega'>\omega}\mathrm{H}^\infty(\mathrm{St}_{\omega'})$. The remaining step to a full vectorial $\mathrm{H}^\infty(\mathrm{St}_\omega)$-calculus is made via the vectorial convergence lemma (Lemma 2.10). □

Remark 8.6. The factorisation 8.2 has been used in [**?**] to prove the transference principle for groups. A close inspection reveals that Formula (8.3) is — after taking a Fourier transform — just the transference identity in disguise. Using the arguments in the proof of [**?**, Theorem 3.2] leads to an alternative proof of Lemma 8.3, see the following section.

8.3. CDMcY-Representation.

A variant of the Poisson type representation in the previous section was used by Cowling, Doust, McIntosh and Yagi in their influential paper [?]. To motivate it we first sketch an

Alternative proof of Lemma 8.3: Suppose first that $f = \widehat{g}$ is the Fourier transform of a function g on $\mathbb{R}$ with $\int_{\mathbb{R}} \cosh(\omega t) \, |g(t)| \, \mathrm{d}t < \infty$. We abbreviate $g_\omega(t) := \cosh(\omega t) g(t)$. Then

$$\begin{aligned} f(z) &= \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}tz} g(t) \, \mathrm{d}t = \int_{\mathbb{R}} \frac{\mathrm{e}^{-\mathrm{i}tz}}{\cosh(\omega t)} \cosh(\omega t) g(t) \, \mathrm{d}t = \int_{\mathbb{R}} \frac{\mathrm{e}^{-\mathrm{i}tz}}{\cosh(\omega t)} g_\omega(t) \, \mathrm{d}t \\ &= \int_{\mathbb{R}} \frac{\pi/\omega}{\cosh(\pi/2\omega (z+s))} \mathcal{F}^{-1}(g_\omega)(s) \, \mathrm{d}s \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}^{-1}(g_\omega)(s) &= \frac{1}{2\pi} \int_{\mathbb{R}} g(t) \cosh(\omega t) \mathrm{e}^{\mathrm{i}ts} \, \mathrm{d}t \\ &= \frac{1}{4\pi} \int_{\mathbb{R}} g(t) \big(\mathrm{e}^{-\mathrm{i}((\mathrm{i}\omega - s)t} + \mathrm{e}^{-\mathrm{i}(-\mathrm{i}\omega - s)t} \big) \, \mathrm{d}s \\ &= \frac{1}{4\pi} \big(f(\mathrm{i}\omega - s) + f(-\mathrm{i}\omega - s) \big). \end{aligned}$$

Hence, (8.1) is valid for such functions f, and the general case is proved by approximation. □

The idea behind the CDMcY-representation is to sneak in an additional factor in the previous argument and compute formally[2]

$$\begin{aligned} f(z) &= \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}tz} g(t) \, \mathrm{d}t = \int_{\mathbb{R}} \psi^\vee(t) \mathrm{e}^{-\mathrm{i}tz} \frac{g_\omega(t)}{\psi^\vee(t) \cosh(\omega t)} \, \mathrm{d}t \\ &= \int_{\mathbb{R}} \psi(z+s) \, \mathcal{F}^{-1} \Big[\frac{g_\omega(t)}{\psi^\vee(t) \cosh(\omega t)} \Big](s) \, \mathrm{d}s \\ &= \int_{\mathbb{R}} \psi(z+s) \, \Big[\mathcal{F}^{-1} \Big(\frac{1}{\psi^\vee(t) \cosh(\omega t)} \Big) * \mathcal{F}^{-1}(g_\omega) \Big](s) \, \mathrm{d}s. \end{aligned}$$

To make this work, the authors require that

$$\frac{1}{\psi^\vee(t) \cosh(\nu t)} \in \mathrm{L}_\infty(\mathbb{R}) \quad \text{for some } \nu < \omega. \tag{8.4}$$

[2] In order to keep our own presentation consistent, we deviate inessentially from [?] in that we use inverse Fourier transforms in place of Fourier transforms, and work on strips in place of sectors.

In order to obtain an L_∞-bound on

$$m_f(t) := \mathcal{F}^{-1}\Big(\frac{1}{\psi^\vee(t)\cosh(\omega t)}\Big) * \mathcal{F}^{-1}(g_\omega)$$

in terms of the H^∞-norm of f it remains to ensure that the first factor in the convolution is in $\mathrm{L}_1(\mathbb{R})$. Hence, by the well known Carlson–Bernstein criterion and under the hypothesis (8.4), it suffices to have

$$\frac{(\psi^\vee)'}{\psi^\vee}\frac{\cosh(\nu t)}{\cosh(\omega t)} \in \mathrm{L}_2(\mathbb{R}).$$

Under the additional assumption (made in [**?**]) that $\psi(z) = \varphi(\mathrm{e}^z)$, and $\varphi \in \mathrm{H}^\infty_0$ on a sector, this is the case, see [**?**, p. 67].

Remark 8.7. The authors of [**?**] used this representation to infer bounded H^∞-calculus from "weak quadratic estimates" of the form

$$\int_{\mathbb{R}} |\langle \psi(t+A)x, x'\rangle| \,\mathrm{d}t \lesssim \|x\| \, \|x'\| \,.$$

This notion is not covered so far in our approach (which avoids computing with X-valued functions). However, when it comes to square function estimates, it is not clear whether there is really a surplus compared with Theorem 8.5. The reason is that requirement (8.4) implies that

$$\frac{\mathrm{e}^{-itz}}{\cosh(\nu t)} \;\lesssim\; \mathrm{e}^{-\mathrm{i}tz}\psi^\vee(t) \;\approx\; \psi(z{+}s)$$

and hence the boundedness of the shift-type square function associated with ψ implies the boundedness of the "weighted group orbit"-square functions considered in Theorem 8.5. (Even more, the CDMcY-choice of ψ implies also that $\psi^\vee/\cosh(\omega'\cdot) \in \mathrm{L}_\infty(\mathbb{R})$ for some ω' and hence square function estimates for ψ are basically equivalent with square function estimates for weighted group orbits.)

8.4. Laplace (Transform) Representation.

In this section we work with a sectorial operator A of angle $\theta < \pi/2$, i.e., $-A$ generates a (sectorially) bounded holomorphic semigroup $(e^{-tA})_{t>0}$. Ubiquitous square functions in this context are dilation type square functions $\psi(tz)$ with $H = \mathrm{L}_2^*(0,\infty)$, in particular for the choice $\psi = \psi_\alpha$, where

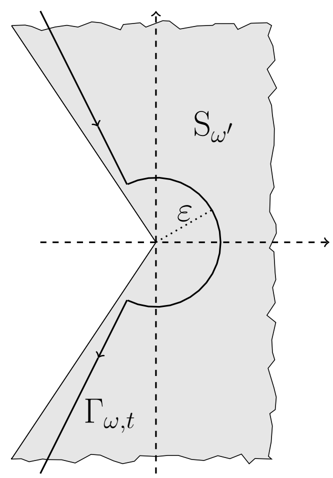
$$\psi_\alpha(z) = z^\alpha \mathrm{e}^{-z} \qquad (\alpha > 0)$$

and z is from a sufficiently large sector. Aiming at an application of Theorem 3.12 we look for a representation

$$u(z) = \int_0^\infty m_u(t)\psi_\alpha(tz)\psi_\beta(tz)\,\tfrac{\mathrm{d}t}{t} = z^{\alpha+\beta}\int_0^\infty m_u(t)t^{\alpha+\beta-1}\mathrm{e}^{-2tz}\,\mathrm{d}t$$

$$= \frac{1}{2^{\alpha+\beta}} z^{\alpha+\beta} \int_0^\infty m_u(t/2) t^{\alpha+\beta-1} \mathrm{e}^{-tz}\, \mathrm{d}t$$

with $m_u \in \mathrm{L}_\infty(0,\infty)$. This means that $\frac{1}{2^{\alpha+\beta}} m_u(t/2) t^{\alpha+\beta-1}$ is the inverse Laplace transform of $u(z)/z^{\alpha+\beta}$. Now let us suppose that $u \in \mathrm{H}^\infty(\mathrm{S}_{\omega'})$ for some $\omega' > \pi/2$. Then one can use the complex inversion formula to compute



$$\frac{m_u(t/2)}{2^{\alpha+\beta}} t^{\alpha+\beta-1} = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\omega,t}} \frac{u(z)}{z^{\alpha+\beta}} \mathrm{e}^{tz}\, \mathrm{d}z$$

Here, $\pi/2 < \omega < \omega'$ and the contour $\Gamma_{\omega,t}$ is the boundary of the region $\mathrm{S}_\omega \setminus \{|z| \le t\}$. Hence, with a change of variable,

$$\begin{aligned} m(t/2) &= \frac{2^{\alpha+\beta}}{2\pi \mathrm{i}} \int_{\Gamma_{\omega,t}} \frac{u(z) t^{\alpha+\beta-1}}{z^{\alpha+\beta}} \mathrm{e}^{tz}\, \mathrm{d}z \\ &= \frac{2^{\alpha+\beta}}{2\pi \mathrm{i}} \int_{\Gamma_{\omega,1}} \frac{u(z/t)}{z^{\alpha+\beta}} \mathrm{e}^{z}\, \mathrm{d}z, \end{aligned}$$

and this yields an estimate

$$\|m_u\|_{\mathrm{L}_\infty(0,\infty)} \lesssim \Big(\int_{\Gamma_{\omega,t}} \frac{\mathrm{e}^{\operatorname{Re} z}}{|z|^{\alpha+\beta}} |\mathrm{d}z| \Big) \ \|u\|_{\mathrm{H}^\infty(\mathrm{S}_\omega)} .$$

Combining these consideration with Theorem 3.12 we obtain the following result.

Theorem 8.8. *Let A be a sectorial operator, with dense domain and range, of angle $\theta < \pi/2$ on a Banach space X (of finite cotype). Let $\alpha, \beta > 0$ and suppose that the square function associated with $\varphi_\alpha(tz) = (zt)^\alpha \mathrm{e}^{-tz}$ and the dual square function associated with $\varphi_\beta(tz) = (zt)^\beta \mathrm{e}^{-tz}$ are bounded operators. Then A has a bounded (vectorial) H^∞-calculus on each sector $\mathrm{S}_{\omega'}$ with $\omega' > \pi/2$.*

Remark 8.9. If $\alpha + \beta > 1$, then one can choose $\omega = \pi/2$ in the complex inversion formula. Hence one obtains an estimate $\|m_u\|_{\mathrm{L}_\infty} \lesssim \|u\|_{\mathrm{H}^\infty(\mathrm{S}_{\pi/2})}$ and then, by the convergence lemma, a bounded $\mathrm{H}^\infty(\mathrm{S}_{\pi/2})$-calculus.

It is an intriguing question under which conditions one can actually push the "H^∞-angle" (that is, the angle ω such that A has a bounded (vectorial) $\mathrm{H}(\mathrm{St}_\omega)$-calculus) down below $\pi/2$. To the best of our knowledge, this requires using the concept of R-boundedness and the multiplier theorem for γ-spaces. Recently [?], Christian Le Merdy has shown that if X has Pisier's property (α), then boundedness of the (dual) square function associated with $\varphi_{1/2}(tz) = (tz)^{1/2}\mathrm{e}^{-tz}$ already suffices. Apart from a result by Kalton and Weis involving R-boundedness, Le Merdy needed to "improve the exponent", i.e., to pass from $\varphi_{1/2}$ to φ_1 and even to $\varphi_{3/2}$. His clever argument, carried out for X being an L_p-space, can be covered by our abstract theory.

Lemma 8.10 (Le Merdy). *Suppose that X is a Banach space with property (α^+), and let A be a sectorial operator of angle $\theta < \pi/2$, with sectorial functional calculus Φ. Suppose that for given $\alpha, \beta > 0$ the square functions $\Phi_\gamma(\varphi_\alpha)$ and $\Phi_\gamma(\varphi_\beta)$ are bounded operators. Then $\Phi_\gamma(\varphi_{\alpha+\beta})$ is bounded, too.*

Proof. The proof relies on the tensor product square function and subordination. We abbreviate $H = \mathrm{L}_2(\mathbb{R}_+)$. Since X has property (α), Lemma **??** shows that the function

$$(\varphi_\alpha \otimes \varphi_\beta)(s,t,z) = s^\alpha t^\beta z^{\alpha+\beta}\mathrm{e}^{-(t+s)z}$$

yields a bounded square function on $\mathrm{L}_2^*(0,\infty)\otimes\mathrm{L}_2^*(0,\infty)$. Equivalently, the function

$$(f_\alpha \otimes f_\beta)(s,t,z) = s^{\alpha-1/2} t^{\beta-1/2} z^{\alpha+\beta}\mathrm{e}^{-(t+s)z}$$

yields a bounded square function on $H\otimes H$, where we have put $f_\alpha(t,z) := t^{\alpha-1/2}z^\alpha\mathrm{e}^{-tz}$.

Next, observe that $T : H \to H \otimes H$ defined by

$$(Tf)(s,t) = (t+s)^{-1/2} f(t+s)$$

is isometric. Indeed,

$$\int_0^\infty\int_0^\infty \tfrac{|f(t+s)|^2}{t+s}\,\mathrm{d}t\mathrm{d}s = \int_0^\infty\int_s^\infty \tfrac{|f(t)|^2}{t}\,\mathrm{d}t\mathrm{d}s = \int_0^\infty |f(t)|^2\Big(\tfrac{1}{t}\int_0^t \mathrm{d}s\Big)\mathrm{d}t.$$

Therefore, $T^*T = \mathrm{Id}_H$. As a consequence, $\Phi_\gamma(f) \in \mathcal{L}(X;\gamma(H;X))$ if and only if $\Phi_\gamma(T\circ f) \in \mathcal{L}(X;\gamma(H\otimes H;X))$. Now,

$$\begin{aligned} T^*(f_\alpha\otimes f_\beta)(t,s,z) &= \tfrac{1}{\sqrt{t}}\int_0^t f_\alpha(t-s,z)f_\beta(s,z)\,ds \\ &= c_{\alpha,\beta}\, t^{\alpha+\beta-1/2}z^{\alpha+\beta}\mathrm{e}^{-tz}, \end{aligned}$$

and this concludes the proof. □

Remark 8.11. Passing from $\mathrm{L}_2^*(0,\infty)$ to $\mathrm{L}_2(\mathbb{R}_+)$ and then to $\mathrm{L}_2(\mathbb{R})$ via the Fourier transform, one has

$$\varphi_{1/2}(tz) = (tz)^{1/2}\mathrm{e}^{-tz} \quad \text{on } \mathrm{L}_2^*(0,\infty) \quad \approx \quad \frac{z^{1/2}}{z+\mathrm{i}s} \quad \text{on } \mathrm{L}_2(\mathbb{R}).$$

These square functions — in the form $A^{1/2}R(\mathrm{i}s, A)x$ and $(A')^{1/2}R(\mathrm{i}s, A')x'$ — were considered by Kalton and Weis in [**?**, Theorem 7.2].

8.5. Franks–McIntosh Representation.
In [**?**] Franks and McIntosh prove the following result: Given $\theta \in (0, \pi)$ there exist sequences $(f_n)_n, (g_n)_n$ in $\mathrm{H}^\infty(\mathrm{S}_\theta)$ such that

a) $\sup_{z\in\mathrm{S}_\theta} \sum_n |f_n(z)| + |g_n(z)| \le C$,

b) Any $\phi \in \mathrm{H}^\infty(\mathrm{S}_\theta; X)$ decomposes as $\phi(z) = \sum_n a_n f_n(z) g_n(z)$ with coefficients $a_n \in X$ satisfying $\|a_n\| \lesssim \|\phi\|_\infty$.

The decomposition b) is an instance of our representation formula (3.6) for $K = \ell_2$. Condition a) tells — in our terminology — that the ℓ_2-valued H^∞-functions $F(z) = (f_n(z))_n$ and $G(z) = (g_n(z))_n$ have ℓ_1-frame-bounded range.

In [**?**] Le Merdy employs this representation to prove that on a space X of finite cotype each sectorial operator with a bounded H^∞-calculus on a sector has bounded vectorial ("quadratic") H^∞-calculus on each larger sector, i.e., the sectorial equivalent to our Theorem 8.1, cf. Remark 8.2.

Theorem 8.12. *Let X be a Banach space and $0 < \omega < \alpha$. Then there exist functions f, $g \in H^\infty(\mathrm{St}_\alpha, \ell_1)$ such that each $F \in H^\infty(\mathrm{St}_\omega; X)$, decomposes uniquely as*

$$F(z) = \sum_j a_j f_j(z) g_j(z)$$

where the vector-valued sequence (a_j) is bounded and satisfies

$$\|(a_j)\|_{\ell^\infty(X)} \le C \|F\|_{H^\infty(\mathrm{St}_\omega)}.$$

Proof. Let $\tau = \frac{\alpha-\omega}{4}$ and $\beta = \frac{\alpha-\omega}{2}$. Further, let $I_k^\pm$ be the translated interval $I_k^\pm = [k\tau, (k+1)\tau] \pm i\beta$. Notice that, as a function in $H^\infty(\mathrm{St}_\omega)$,

$$F(z) = \tfrac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{St}_\beta} k(\zeta, z) F(\zeta)\, \mathrm{d}\zeta \quad \text{where} \quad k(\zeta, z) := \frac{\exp(-(\zeta - z)^2)}{\zeta - z}.$$

Let $e_{j,k}^\pm$ be the "Legendre polynomials" on $I_k^\pm$, i.e. the orthonormal sequence obtained by Gram-Schmidt starting from the functions z^j on $I_k^\pm$. Further, let $\phi_{j,k}^\pm = \int_{\partial\mathrm{St}_\beta} k(\zeta, z) e_{j,k}^\pm(\zeta) \mathrm{d}\zeta$.

Observe that for $\zeta \in I_k^\pm$ and $\mathrm{Re}(z) \in [n\tau, (n+1)\tau)$, $|k(\zeta, z)| \le Ce^{-(n-k)^2}$ where $C = (\beta-\omega)^{-1} e^{\beta-\omega}$. Let $B_k^\pm = B(c_k^\pm, \tau)$ denote the ball of radius τ centered around $I_k^\pm$. Then

$$\sup_{\zeta\in B_k^\pm} |k(\zeta,z)| \leq C' e^{-(n-k)^2},$$

and so the normalised power series expansion of $k(\zeta,z) = \sum_j b_{j,k}^\pm \left(\frac{\zeta - c_k^\pm}{\tau}\right)^j$ in $B_k^\pm$ has coefficients $b_{j,k}^\pm$ satisfying

$$|b_{j,k}^\pm| \leq \left\|(b_{j,k}^\pm)_j\right\|_{\ell_2} = \|k\|_{H^2(B_k^\pm)} \leq Ce^{-(n-k)^2}$$

Finally, using the orthogonality of z^l and $e_{j,k}^\pm$ for $l < j$ and the fact that $|\zeta - c_k^\pm| \leq \tau/2$ for $\zeta \in I_k^\pm$ yields

$$|\phi_{j,k}^\pm| \leq \tfrac{1}{2\pi} \int_{I_k^\pm} \sum_{l=j}^{\infty} e_{j,k}^\pm b_{l,k}^\pm \left(\tfrac{\zeta - c_k^\pm}{\tau}\right)^l \leq C'' e^{-(n-k)^2} 2^{-j}.$$

This decay rate allows to conclude

$$\begin{aligned} F(z) &= \lim_{n\to\infty} \sum_{k=-n}^{n} \int_{I_k^\pm} F(\zeta)k(\zeta,z)\,\mathrm{d}\zeta \\ &= \lim_{n\to\infty} \sum_{k=-n}^{n} \int_{I_k^\pm} \sum_j \underbrace{\langle F, e_{j,k}^\pm\rangle}_{=a_{j,k}^\pm} e_{j,k}^\pm(\zeta) k(\zeta,z)\,\mathrm{d}\zeta \\ &= \sum_k \sum_j a_{j,k}^\pm \phi_{j,k}^\pm, \end{aligned}$$

and a suitable factorisation $\phi_{j,k}^\pm = f_{j,k}^\pm g_{j,k}^\pm$ finishes the proof. □

8.6. Singular Cauchy Representation.
All the results in this chapter so far were applications of Theorem 3.12, that is, they infer a bounded (vectorial) H^∞-calculus from bounded square and dual square functions. In the present section, however, we shall treat an application of Lemma 3.6. That is, we want to infer bounded H^∞-calculus from upper and lower square function estimates. We discuss an example due to Kalton and Weis [**?**], see also [**?**, Theorem 10.9].

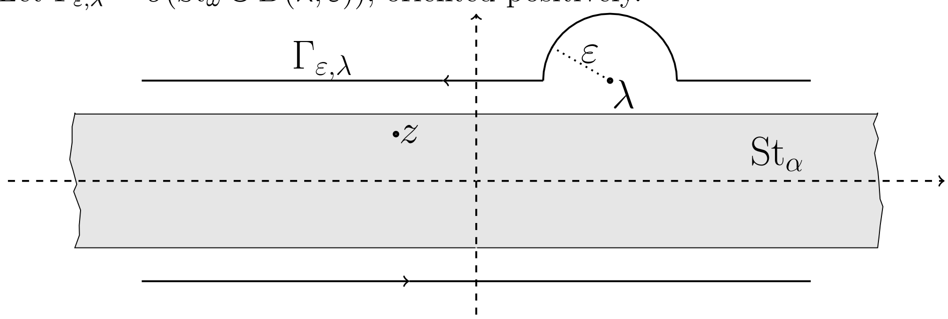
Let A be a densely defined operator of strip type $\omega_0 \geq 0$ on a Banach space X. We fix $\omega > \omega_0$ and let $\Gamma_\omega = \partial\mathrm{St}_\omega = (\mathrm{i}\omega + \mathbb{R}) \cup (-\mathrm{i}\omega + \mathbb{R})$ with arc length measure, let $H := \mathrm{L}_2(\Gamma_\omega)$ and consider the H-valued function $g(\lambda, z) := \frac{1}{\lambda - z}$. Under the canonical isomorphism $H \cong \mathrm{L}_2(\mathbb{R}) \oplus \mathrm{L}_2(\mathbb{R})$, the function g is strongly equivalent with the *pair* $(\pm\mathrm{i}\omega + s - z)^{-1}$ of shift-type square functions, which — as demonstrated in Section 7.5 — is again strongly equivalent with the weighted group orbit square function associated with $\mathrm{e}^{-\mathrm{i}sz}/\cosh(\omega s)$. Our aim is to prove the following remarkable result of Kalton and Weis [**?**, Theorem 6.2].

Theorem 8.13. *Let A be a densely defined operator of strip type ω_0 and let $\omega > \omega_0$. Suppose that*

$$\|R(\lambda, A)x\|_{\gamma(\mathrm{L}_2(\Gamma_\omega);X)} \quad \simeq \quad \|x\| \qquad (x \in X).$$

Then A has a bounded $\mathrm{H}^\infty(\mathrm{St}_\omega)$-calculus.

Proof. Let $0 \le \omega_0 < \alpha < \omega < \omega'$ and let $f \in \mathrm{H}^\infty(\mathrm{St}_{\omega'})$ and $\lambda \in \Gamma_\omega$. Let $\Gamma_{\varepsilon,\lambda} = \partial(\mathrm{St}_\omega \cup B(\lambda, \varepsilon))$, oriented positively.



Then for $z \in \mathrm{St}_\alpha$

$$\frac{f(w)}{(w-z)(\lambda-z)} = \frac{f(w)}{(w-\lambda)(\lambda-z)} - \frac{f(w)}{(w-\lambda)(w-z)}.$$

Integrating this with respect to w over $\{w \in \Gamma_{\varepsilon,\lambda}, |w| \le r\}$ and letting $r \to \infty$ yields

$$\frac{f(z)}{\lambda - z} = \frac{f(\lambda)}{\lambda - z} - \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_{\varepsilon,\lambda}} \frac{f(w)}{(w-\lambda)(w-z)}\, \mathrm{d}w.$$

By the fractional Cauchy theorem, the limit as $\varepsilon \to 0$ of the integral over the half circle avoiding $\lambda \in \Gamma_\omega$ at distance ε is

$$\frac{1}{2\pi\mathrm{i}} \cdot (\mathrm{i}\pi) \frac{f(\lambda)}{\lambda - z} = \frac{1}{2} \frac{f(\lambda)}{\lambda - z}.$$

Hence, as $\varepsilon \to 0$ we obtain

$$\begin{aligned} \frac{f(z)}{\lambda - z} &= \frac{f(\lambda)}{\lambda - z} - \frac{1}{2\pi\mathrm{i}} \mathrm{p.v.}\!\int_{\Gamma_\omega} \frac{f(z)}{(w-\lambda)(w-z)}\, \mathrm{d}w - \frac{f(\lambda)}{2(\lambda - z)} \\ &= \frac{f(\lambda)}{2(\lambda - z)} + \frac{1}{2\pi\mathrm{i}} \mathrm{p.v.}\!\int_{\Gamma_\omega} \frac{f(w)}{(\lambda - w)(w-z)}\, \mathrm{d}w \end{aligned}$$

Let $T_f : \mathrm{L}_2(\Gamma_\omega) \to \mathrm{L}_2(\Gamma_\omega)$ be defined by

$$(T_f h)(\lambda) := \frac{f(\lambda)}{2} h(\lambda) + \frac{1}{2\pi\mathrm{i}} \mathrm{p.v.}\!\int_{\Gamma_\omega} \frac{f(w)}{(\lambda - w)} h(w) \mathrm{d}w \qquad (\lambda \in \Gamma_\omega).$$

Note that T_f is bounded by a constant times $\|f\|_{\mathrm{H}^\infty(\mathrm{St}_\omega)}$: the first summand is simply multiplication by $\frac{1}{2}f$ and the second is multiplication with f composed with convolution with $1/w$.
Now, by the computations above, we have

$$f(z)g(\lambda, z) = \big(T_f(g(\cdot, z))\big)(\lambda) \qquad (z \in \mathrm{St}_\alpha, \lambda \in \Gamma_\omega).$$

Viewing g as a function in $\mathrm{H}^\infty(\mathrm{St}_\alpha; \mathrm{L}_2(\Gamma_\omega))$ we hence have

$$f \cdot g = T_f \circ g$$

as in the hypotheses of Lemma 3.6. (Note that we as usual identify $\mathrm{L}_2(\Gamma_\omega) = \mathrm{L}_2(\Gamma_\omega)'$ here.) We hence obtain a constant $C \geq 0$ independent of $\omega' > \omega$ such that

$$\|f(A)\| \leq C \, \|f\|_{\mathrm{H}^\infty(\mathrm{St}_\omega)} \quad \text{for all } f \in \mathrm{H}^\infty(\mathrm{St}_{\omega'}).$$

The claim now follows from the scalar convergence lemma [**?**, Section 5.1]. □

Appendix

Appendix A. Some Analytical Lemmas

In this appendix we collect some auxiliary results from analysis for the sake of easy referencing. We include sketches of proofs for the convenience of the reader.

Lemma A.1. *Let $(J, \leq)$ be a directed set, let $(T_\alpha)_\alpha$ be a J-net of bounded linear operators on a Banach space X such that $T_\alpha \to 0$ strongly and $\sup_\alpha \|T_\alpha\| < \infty$. Let, furthermore, Y be a separable subspace of X. Then there is an increasing sequence $(\alpha_n)_{n \in N}$ in J such that $\lim_{n\to\infty} T_{\alpha_n} x = 0$ for all $x \in Y$.*

Proof. Let $\{x_n \mid n \in \mathbb{N}\} \subseteq Y$ be dense subset of Y. The sequence $(\alpha_n)_n$ is constructed recursively with the property that

$$\|T_{\alpha_n} x_k\| \leq \frac{1}{n} \qquad (k = 1, \dots, n).$$

(It is obvious that this can be done.) Clearly, for fixed k we have $T_{\alpha_n} x_k \to 0$ as $n \to \infty$. And by the uniform boundedness of the operators T_{α_n}, the claim follows. □

The next result states a net-version of the dominated convergence theorem

Lemma A.2 (Dominated Convergence)**.** *Let X be a Banach space, (Ω, Σ, μ) a measure space, $1 \le p < \infty$, and $(f_\alpha)_\alpha$ a net in $\mathrm{L}_p(\Omega, \Sigma, \mu; X)$ that satisfies the following conditions:*

1) *There is $g \in \mathrm{L}_p(\Omega, \Sigma, \mu)$ with $\|f_\alpha(\cdot)\|_X \le g$ almost everywhere for each index α.*
2) *There is a sequence $(A_n)_{n\in\mathbb{N}}$ in Σ of finite measure with $\bigcup_{n\in\mathbb{N}} A_n = [g > 0]$ and such that $\|f_\alpha \mathbf{1}_{A_n}\|_{\mathrm{L}_\infty} \to 0$ as $\alpha \to \infty$ for each $n \in \mathbb{N}$.*

Then $\|f_\alpha\|_p \to 0$ as $\alpha \to \infty$.

Proof. This follows from the estimate

$$\int_\Omega \|f_\alpha\|_X^p \le \int_{A_n} \|f_\alpha\|_X^p + \int_{A_n^c} \|f_\alpha\|_X^p \le \mu(A_n)\, \|f_\alpha \mathbf{1}_{A_n}\|_{\mathrm{L}_\infty}^p + \int_{A_n^c} |g|^p$$

which holds for all $n \in \mathbb{N}$ and all indices α. □

Corollary A.3. *Let X be a Banach space, (Ω, Σ, μ) a measure space, $1 \le p < \infty$, and $\gamma_1, \ldots, \gamma_N \in \mathrm{L}_p(\Omega, \Sigma\mu)$. Furthermore, let for each $j = 1, \ldots, N$ a bounded and convergent net $(x_j^\alpha)_\alpha$ in X be given. Denote by $x_j := \lim_\alpha x_j^\alpha$ the respective limit. Then*

$$\sum_{j=1}^N \gamma_j x_j^\alpha \to \sum_{j=1}^N \gamma_j x_j \qquad \text{as } \alpha \to \infty$$

in $\mathrm{L}_p(\Omega, \Sigma, \mu; X)$.

Proof. Let $C := \sup_{j,\alpha} \left\|x_j^\alpha\right\|$. Then the assertion follows from Lemma A.2 by setting $f_\alpha := \sum_{j=1}^N \gamma_j(x_j^\alpha - x_j)$, $g := 2C \sum_{j=1}^N |\gamma_j|$ and $A_n := \bigcap_{j=1}^N [\frac{1}{n} \le |\gamma_j| \le n]$. □

The next lemma tells something about L_p-norms of the sums of independent vector-valued random variables.

Lemma A.4. *Let X be a Banach space and let ξ and η be independent X-valued random variables on a probability space Ω. If η is symmetric then*

$$\mathbb{E}\, \|\xi\|_X^p \le \mathbb{E}\, \|\xi + \eta\|_X^p$$

for each $1 \le p < \infty$.

Proof. Since η is symmetric and ξ and η are independent, both summands on the right hand side of

$$\xi = \frac{1}{2}(\xi + \eta) + \frac{1}{2}(\xi - \eta)$$

have the same distribution and hence the same L_p-norms. □

Appendix B. γ-Radonifying Operators

In this appendix we review and develop the theory of γ-summing and γ-radonifying operators to an extent that serves our purposes. The presented results in this chapter are essentially from or closely inspired by the breakthrough paper [?] of Kalton and Weis, cf. Chapter 1 above.

The draft character of Kalton and Weis' original preprint stimulated us and various other people to elaborate the theory or to detail and streamline the proofs. Traces of these activities can be found in many published papers, for example in Jan van Neerven's excellent survey [?] that contains also historical remarks and an extensive bibliography on the topic. Hytonen, Van Neerven, Veraar and Weis are currently preparing a multi-volume monograph on "Analysis in Banach Spaces" on these and other topics. The second volume [?], at present only available in preprint form, is particularly relevant for us.

However, for the convenience of the reader and in order to keep this paper as self-contained as possible, we shall present our own account of the theory of γ-radonifying operators. As this account dates back to times when no-one but the authors had any notice of [?], there are some (mostly inessential) differences, which will be pointed out when they occur. When possible and convenient, we shall point to existing proofs in the literature. However, some of the results presented here have no direct counterpart in [?] (so far, one should say).

As this chapter is intended as a reference for notation, definitions and results, we shall be brief with proofs and refer to [?] and [?] whenever it is convenient.

One of the main differences of our presentation to both the original Kalton-Weis paper and van Neerven's survey is that those works deal exclusively with *real* Banach spaces, whereas we develop the theory for *complex* ones. The reason is that functional calculus questions, where complex contour integrals are ubiquitous, call for rather a complex than a real setting.

For the theory we need the notion of a **complex standard Gaussian** random variable, by which we mean a random variable γ of the form

$$\gamma = \gamma_r + \mathrm{i}\,\gamma_i$$

where γ_r and γ_i are independent standard *real* Gaussians. Basically, the whole theory for real spaces carries over to complex spaces when real Gaussians are replaced by complex ones.

B.1. The Contraction Principle for Gaussian Sums.
The following result is fundamental when working with Gaussian sums. We work over the scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Theorem B.1 (Contraction Principle). *Let $\gamma_1, \gamma_2, \ldots$ be independent scalar standard Gaussians on some probability space, let X be a Banach space, $x_1, \ldots, x_m \in X$ and let $A = (a_{kj})_{kj}$ be a scalar $n \times m$-matrix. Then*

$$\mathbb{E} \left\| \sum\nolimits_{k=1}^{n} \sum\nolimits_{j=1}^{m} \gamma_k a_{kj} x_j \right\|_X^2 \leq \|A\|^2 \, \mathbb{E} \left\| \sum\nolimits_{j=1}^{m} \gamma_j x_j \right\|_X^2 ,$$

where the matrix A is considered as an operator $A : \ell_2^m \to \ell_2^n$.

The proof, which we include for convenience, proceeds in three steps. In the first step one reduces the problem to the case that $n = m$. If $m > n$ one just extends A to an $m \times m$-matrix by adding 0-rows. If $m < n$ one extends A to an $n \times n$-matrix by adding 0-columns, and defines $x_j := 0$ for $m < j \leq n$.

Now, if $m = n$ after scaling one may suppose that A is a contraction. Then the following lemma reduces the claim to A being an isometry.

Lemma B.2. *Every contraction on the Euclidean space $\mathbb{K}^d$ is a convex combination of at most d isometries.*

Proof. This is well known, but the proof is given here for the convenience of the reader. We may suppose that $\|A\| = 1$. By polar decomposition, $A = U\,|A|$ where $|A| = (A^*A)^{1/2}$, and U is isometric. Hence we may assume that $A = A^*$ is positive semi definite. By the spectral theorem we may even further reduce the problem to A being a diagonal matrix with entries $1 = \lambda_d \geq \cdots \geq \lambda_1 \geq 0$. (Note that 1 has to be an eigenvalue since $\|A\| = 1$.) Now we set $\lambda_0 = 0$ and write

$$\operatorname{diag}(\lambda_1, \ldots, \lambda_d) = \sum\nolimits_{j=1}^{d} (\lambda_j - \lambda_{j-1}) P_j$$

where $P_j(x_1, \ldots, x_d) := (x_1, \ldots, x_j, 0 \ldots, 0)$ is projection onto the first j coordinates. (So $P_d = I$.) This is convex combination of projections. But for any orthogonal projection P on a Hilbert space,

$$P = \frac{1}{2}\mathrm{I} + \frac{1}{2}(2P - \mathrm{I})$$

is a representation as a convex combination of unitaries, since $(2P - I)^*(2P-I) = (2P-I)^2 = 4P^2 - 4P + I = I$. Since in the representation above always the identity I is used, we can collect terms and arrive at a convex combination of at most d terms. ☐

Finally, we have to treat the case that $n = m$ and A is an orthogonal/unitary matrix. Then by the rotation invariance of the n-dimensional, resp. $2n$-dimensional, standard Gaussian measure [**?**, p.239],

$$\mathbb{E}\Big\| \sum_{k=1}^{n}\sum_{j=1}^{n} \gamma_k a_{kj} x_j \Big\|^2 = \mathbb{E}\Big\| \sum_{k=j}^{n}\Big(\sum_{k=1}^{n} a_{kj}\gamma_k\Big) x_j \Big\|^2 = \mathbb{E}\Big\| \sum_{j=1}^{n} \gamma_j x_j \Big\|^2.$$

This concludes the proof of Theorem B.1.

B.2. Definition and the Ideal Property.
Let H be a Hilbert space and X a Banach space over the scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A linear operator $T : H \to X$ is called *γ-summing* if

$$\|T\|_\gamma := \sup_F \mathbb{E}\left(\Big\| \sum\nolimits_{e\in F} \gamma_e \otimes Te \Big\|_X^2 \right)^{1/2} < \infty,$$

where the supremum is taken over all finite orthonormal systems $F \subseteq H$ and $(\gamma_e)_{e\in F}$ is an independent collection of $\mathbb{K}$-valued standard Gaussian random variables on some probability space. We let

$$\gamma_\infty(H; X) := \{T : H \longrightarrow X \mid T \text{ is } \gamma\text{-summing}\}$$

the space of γ-summing operators of H into X. It is clear that each γ-summing operator is bounded with $\|T\| \le \|T\|_\gamma$.

Remark B.3 (Real vs. Complex)**.** In the case $\mathbb{K} = \mathbb{C}$ we can view the complex spaces H, X as real spaces, and we shall indicate this by writing H_r, X_r. Then H_r is a real Hilbert space with respect to the scalar product $[f, g]_r := \mathrm{Re}\,[f, g]$. For $\mathbb{C}$-linear $T : H \to X$ we now have two definitions of $\|T\|_\gamma$, one using $\langle\cdot,\cdot\rangle_r$-orthonormal systems (called $\mathbb{R}$-ons's for short) and real Gaussians, and the other using $\mathbb{C}$-orthonormal systems and complex Gaussians. We claim that both definitions lead to the same quantity. In particular, one has

$$\gamma_\infty(H; X) = \gamma_\infty(H_r; X_r) \cap \mathcal{L}(H; X).$$

In order to see this we note first that if $\{e_1, \dots, e_d\}$ is a $\mathbb{C}$-orthonormal system, then $\{e_1, \dots, e_d, \mathrm{i}e_1, \dots, \mathrm{i}e_d\}$ is an $\mathbb{R}$-ons. Hence, if $\tilde\gamma_j = \gamma_j + \mathrm{i}\gamma_j'$ are independent complex standard Gaussians,

$$\mathbb{E}\left\| \sum\nolimits_j \tilde\gamma_j T e_j \right\|^2 = \mathbb{E}\left\| \sum\nolimits_j \gamma_j T(e_j) + \gamma_j' T(\mathrm{i}e_j) \right\|^2 \le \|T\|_{\gamma,\mathbb{R}}^2$$

with the obvious meaning of $\|T\|_{\gamma,\mathbb{R}}$. This yields $\|T\|_{\gamma,\mathbb{C}} \le \|T\|_{\gamma,\mathbb{R}}$. On the other hand, let $\{f_1, \dots, f_n\}$ be an $\mathbb{R}$-ons and let $\gamma_1, \dots, \gamma_n$ be real standard Gaussians. Pick a $\mathbb{C}$-ons $\{e_1, \dots, e_n\}$ such that $f_k \in$

$\mathrm{span}_{\mathbb{C}}\{e_1, \dots, e_n\}$ for each k. Then we can find $\lambda_{kj} = a_{kj} + \mathrm{i} b_{kj}$ such that

$$f_k = \sum_j (a_{kj} + \mathrm{i} b_{kj}) e_j = \sum_j a_{kj} e_j + b_{kj}(\mathrm{i} e_j) \qquad (1 \le k \le n).$$

Define the real matrices $A := (a_{kj})_{k,j}$, $B := (b_{kj})_{k,j}$ and $C := [A\,B]$, as well as $g_j := e_j$ for $1 \le j \le n$ and $g_j := \mathrm{i} e_j$ for $n < j \le 2n$. Then, by the contraction principle (Theorem B.1),

$$\begin{aligned}
\mathbb{E} \left\| \sum\nolimits_{k=1}^n \gamma_k T f_k \right\|^2 &= \mathbb{E} \left\| \sum\nolimits_{k=1}^n \gamma_k a_{kj} T(f_k) + b_{kj} T(\mathrm{i} f_k) \right\|^2 \\
&= \mathbb{E} \left\| \sum\nolimits_{k=1}^n \sum\nolimits_{j=1}^{2n} \gamma_k c_{kj} T g_j \right\|^2 \le \|C\|^2 \, \mathbb{E} \left\| \sum\nolimits_{j=1}^{2n} \gamma_j T g_j \right\|^2 \\
&= \|C\|^2 \, \mathbb{E} \left\| \sum\nolimits_{j=1}^{n} (\gamma_j + \mathrm{i} \gamma_{n+j}) T e_j \right\|^2 \le \|C\|^2 \, \|T\|_{\gamma,\mathbb{C}}^2 .
\end{aligned}$$

But $c_{kj} = \langle f_k, g_j \rangle_r$ and hence $\|C\| \le 1$. This yields $\|T\|_{\gamma,\mathbb{R}} \le \|T\|_{\gamma,\mathbb{C}}$ and concludes the proof of the claim.

The following lemma yields an equivalent description of the γ-summing norm.

Lemma B.4. *Let $(e_\alpha)_{\alpha \in I}$ be any fixed orthonormal basis of H. Then for any Banach space X and $T \in \mathcal{L}(H; X)$ the net*

$$\mathbb{E} \left\| \sum\nolimits_{\alpha \in F} \gamma_\alpha T e_\alpha \right\|_X^2 , \qquad (F \subseteq I \text{ finite})$$

is increasing (with respect to the natural inclusion order) and

$$\|T\|_\gamma^2 = \sup_F \mathbb{E} \left\| \sum\nolimits_{\alpha \in F} \gamma_\alpha T e_\alpha \right\|_X^2 .$$

Proof. The monotonicity follows from Lemma A.4 since Gaussian random variables are symmetric. For the remaining part let $h_1, \dots, h_N$ be any orthonormal system in H. Then

$$\sum_{j \le N} \gamma_j h_j = \sum_{j \le N} \gamma_j \sum_{\alpha \in I} [h_j, e_\alpha]\, e_\alpha = \sum_{\alpha \in I} \sum_{j \le N} \gamma_j \, [h_j, e_\alpha]\, e_\alpha$$

By Corollary A.3 the latter sum converges in $\mathrm{L}_2(\Omega)$, where Ω is the probability space on which the γ_j are defined. It follows that

$$\begin{aligned}
\mathbb{E} \left\| \sum\nolimits_{j \le N} \gamma_j T h_j \right\|_X^2 &= \lim_F \mathbb{E} \left\| \sum\nolimits_{\alpha \in F} \sum\nolimits_{j \le N} \gamma_j \, [h_j, e_\alpha] \, T e_\alpha \right\|_X^2 \\
&\le \sup_F \mathbb{E} \left\| \sum\nolimits_{\alpha \in F} \gamma_\alpha T e_\alpha \right\|_X^2 .
\end{aligned}$$

Here we have employed the contraction principle (Theorem B.1) applied to the matrix $([h_j, e_\alpha])_{j,\alpha}$. ∎

The following approximation property is [**?**, Proposition 3.18].

Lemma B.5 (γ-Fatou I)**.** *Let $(T_\alpha)_\alpha$ be a bounded net in $\gamma_\infty(H;X)$ such that $T_\alpha \to T \in \mathcal{L}(H;X)$ in the weak operator topology. Then $T \in \gamma_\infty(H;X)$ and*

$$\|T\|_\gamma \le \liminf_{\alpha\to\infty} \|T_\alpha\|_\gamma .$$

It is easy to see that $\gamma_\infty(H;X)$ contains all finite rank operators. The closure in $\gamma_\infty(H;X)$ of the space of finite rank operators is denoted by $\gamma(H;X)$, and its elements $T \in \gamma(H;X)$ are called **γ-radonifying**.

The following property is one of the cornerstones of the theory.

Theorem B.6 (Ideal Property)**.** *Let Y be another Banach space and K another Hilbert space, let $L : X \to Y$ and $R : K \to H$ be bounded linear operators, and let $T \in \gamma_\infty(H;X)$. Then*

$$LTR \in \gamma_\infty(K;Y) \quad \textit{and} \qquad \|LTR\|_\gamma \le \|L\|_{\mathcal{L}(X;Y)} \|T\|_\gamma \|R\|_{\mathcal{L}(K;H)} .$$

If $T \in \gamma(H;X)$, then $LTR \in \gamma(K;Y)$.

Proof. One can handle the left-hand and the right-hand side separately, the first being straightforward. For the latter, pick a finite orthonormal system $\{e_1,\dots,e_n\}$ within K. Then find an orthonormal system $\{f_1,\dots,f_m\}$ with

$$\operatorname{span}\{Re_1,\dots,Re_n\} = \operatorname{span}\{f_1,\dots,f_m\}.$$

Consequently $Re_k = \sum_{j=1}^m a_{kj} f_j$ for some scalar $(n\times m)$-matrix $A = (a_{kj})_{k,j}$. Then, by Theorem B.1 below,

$$\mathbb{E}\left\|\sum\nolimits_{k=1}^n \gamma_k T Re_k\right\|^2 = \mathbb{E}\left\|\sum\nolimits_{k=1}^n \gamma_k T \sum\nolimits_{j=1}^m a_{kj} f_j\right\|^2$$

$$= \mathbb{E}\left\|\sum\nolimits_{k=1}^n \sum\nolimits_{j=1}^m \gamma_k a_{kj} T f_j\right\|^2 \le \|A\|^2\, \mathbb{E}\left\|\sum\nolimits_{j=1}^m \gamma_j T f_j\right\|^2 \le \|A\|^2 \|T\|_\gamma^2 .$$

Since $\|A\|_{\ell_2^m \to \ell_2^n} \le \|R\|_{K\to H}$, the claim is proved. □

See [**?**, Theorem 6.2] for a slightly different proof. Based on the ideal property, we can show that in the case $\mathbb{K} = \mathbb{C}$ a difference between the complex and real approach to $\gamma(H;X)$ is only virtual.

Remark B.7 (Real vs Complex, again)**.** Let H, X be complex spaces. We claim that

$$\gamma(H;X) = \{T \in \gamma(H_r;X_r) \mid T \text{ is } \mathbb{C}\text{-linear}\} = \gamma(H_r;X_r) \cap \mathcal{L}(H;X).$$

The inclusion "$\subseteq$" is trivial, so suppose that $T : H \to X$ is $\mathbb{C}$-linear and in $\gamma(H_r;X_r)$. Then there is a sequence T_n of real-linear finite rank operators such that $\|T_n - T\|_\gamma \to 0$. Define $S_n x := {}^1\!/_2(T_n x - \mathrm{i}T_n(ix))$.

Then each S_n is a $\mathbb{C}$-linear finite rank operator $\|S_n - T\|_\gamma \to 0$. To prove this we note that the operator $M : x \mapsto \mathrm{i}x$ is a linear isometry on H_r commuting with T, hence

$$2\,\|S_n - T\|_\gamma \le \|T_n - T\|_\gamma + \left\|M^{-1}T_nM - T\right\|_\gamma \le \|T_n - T\|_\gamma \to 0$$

by the ideal property. It follows that $T \in \gamma(H;X)$, as claimed.

One might ask whether $\gamma_\infty(H;X)$ can differ from $\gamma(H;X)$. An example from Linde and Pietsch, reproduced in [**?**, Exa. 4.4], shows that this indeed happens if $X = \mathrm{c}_0$. On the other hand, by a theorem of Hoffman-Jørgensen and Kwapień, if X does not contain c_0 then $\gamma(H;X) = \gamma_\infty(H;X)$, see [**?**, Theorem 4.3]. Although this result was obtained for real spaces only, Remark B.7 shows that it continues to hold in the complex case.

For later reference, we quote the following approximation results from [**?**, Corollaries 6.4 and 6.5]. Their proofs are straightforward from the ideal property.

Theorem B.8 (Approximation)**.** *Let H, K be Hilbert and X, Y be Banach spaces, and let $T \in \gamma(H;X)$. Then the following assertions hold:*

a) *If $(L_\alpha)_\alpha \subseteq \mathcal{L}(X;Y)$ is a uniformly bounded net that converges strongly to $L \in \mathcal{L}(X;Y)$, then $L_\alpha T \to LT$ in $\gamma(H;Y)$.*
b) *If $(R_\alpha^*)_\alpha \subseteq \mathcal{L}(H;K)$ is a uniformly bounded net that converges strongly to $R^* \in \mathcal{L}(H;K)$, then $TR_\alpha \to TR$ in $\gamma(K;X)$.*

Note that if $T \in \gamma(H;X)$ the operators LT and TR are again γ-radonifying, by the ideal property.

B.3. Fourier Series and Nuclear Operators.

Recall our notation

$$\overline{g} := [\,\cdot\,, g] \in H'$$

for an element $g \in H$, H any Hilbert space.

Every finite rank operator $T : H \to X$ has the form

$$T = \sum\nolimits_{j=1}^{n} \overline{g_j} \otimes x_j, \tag{B.1}$$

and one can view $\gamma(H;X)$ as a completion of the algebraic tensor product $H' \otimes X$ with respect to the γ-norm.

Note that if $e_1, \dots, e_n$ is an orthonormal system in H, then $\overline{e_1}, \dots, \overline{e_n}$ is an orthonormal system in H', dual to $\{e_1, \dots, e_n\}$ in the sense that

$$\langle e_j, \overline{e_k}\rangle = \langle e_j, \overline{e_k}\rangle_{H,H'} = \delta_{jk} \qquad (j,k = 1, \dots, n).$$

The following shows that a "Gaussian sum" in a Banach space X can be regarded as a γ-norm of a finite rank operator.

Lemma B.9. *Let $g_1, \dots, g_m \in H$ be an orthonormal system in H and $x_1, \dots, x_m \in X$. Then*

$$\left\| \sum\nolimits_{j=1}^{m} \overline{g_j} \otimes x_j \right\|_\gamma^2 = \mathbb{E} \left\| \sum\nolimits_{j=1}^{n} \gamma_j x_j \right\|_X^2 .$$

Proof. Let $e_1, \dots, e_n$ be any finite orthonormal system in H and let T be defined by (B.1). Then

$$\mathbb{E} \left\| \sum\nolimits_{k=1}^{n} \gamma_k T e_k \right\|^2 = \mathbb{E} \left\| \sum\nolimits_{k=1}^{n} \gamma_k \sum\nolimits_{j=1}^{m} [e_k, g_j]\, x_j \right\|^2 \le \mathbb{E} \left\| \sum\nolimits_{j=1}^{m} \gamma_j x_j \right\|^2$$

by Theorem B.1, since the scalar matrix $A := ([e_k, g_j])_{k,j}$ satisfies $\|A\| \le 1$. On the other hand, if we take $n = m$ and $e_k := g_k$, then we obtain equality. ☐

Let $(e_\alpha)_{\alpha \in A}$ be an orthonormal *basis* of H. For a finite set $F \subseteq A$, let

$$P_F := \sum\nolimits_{\alpha \in F} \overline{e_\alpha} \otimes e_\alpha$$

be the orthogonal projection onto $\operatorname{span}\{e_\alpha \mid \alpha \in F\}$. The net $(P_F)_F$ is uniformly bounded and converges strongly to the identity on H. Hence, the following is a consequence of Theorem B.8, part b).

Corollary B.10 (Fourier Series)**.** *If $T \in \gamma(H; X)$ and $(e_\alpha)_\alpha$ is any orthonormal basis of H, then*

$$\sum\nolimits_{\alpha} \overline{e_\alpha} \otimes T e_\alpha = T$$

in the norm of $\gamma(H; X)$.

It follows from Lemma B.9 that

$$\|\overline{g} \otimes x\|_\gamma = \|g\|_H \, \|x\|_X = \|\overline{g}\|_{\overline{H}} \, \|x\|_X$$

for every $g \in H$, $x \in X$, i.e., the γ-norm is a cross-norm.

Recall that $T : H \to X$ is a **nuclear operator** if

$$T = \sum_{n \ge 0} \overline{g_n} \otimes x_n \tag{B.2}$$

for some $g_n \in H, x_n \in X$ with $\sum_{n \ge 0} \|g_n\|_H \, \|x_n\|_X < \infty$. The **nuclear norm** of T is

$$\|T\|_{\text{nu}} := \inf \sum_{n \ge 0} \|g_n\|_H \, \|x_n\|_X ,$$

where the infimum is taken over all representations of the form (B.2). We let $\operatorname{Nu}(H; X)$ be the set of all nuclear operators. If $X = H$ then we write $\mathrm{Nu}(H) := \mathrm{Nu}(H; H)$. Recall that nuclear operators on H are precisely the operators of **trace class**

Corollary B.11. *A nuclear operator $T : H \to X$ is γ-radonifying and $\|T\|_\gamma \le \|T\|_{\mathrm{nu}}$.*

The following application turns out to be quite useful.

Lemma B.12. *Let H, X as before, and let (Ω, Σ, μ) be a measure space. Suppose that $f : \Omega \to H$ and $g : \Omega \to X$ are (strongly) μ-measurable and*

$$\int_\Omega \|f(t)\|_H \, \|g(t)\|_X \; \mu(\mathrm{d}t) < \infty.$$

Then $\overline{f} \otimes g \in \mathrm{L}_1(\Omega; \gamma(H; X))$, and $T := \int_\Omega \overline{f} \otimes g \, \mathrm{d}\mu \in \gamma(H; X)$ satisfies

$$Th = \int_\Omega [h, f(t)] \, g(t) \, \mu(\mathrm{d}t) \qquad (h \in H)$$

and

$$\|T\|_\gamma \le \int_\Omega \|f(t)\|_H \, \|g(t)\|_X \; \mu(\mathrm{d}t).$$

B.4. Trace Duality.

We follow [**?**, **?**], cf. also [**?**, Sec. 9.1.j], and identify the dual of $\gamma(H; X)$ with a subspace of $\mathcal{L}(H'; X')$ via **trace duality**. For a finite rank operator $U : H \to H$ given by

$$U := \sum\nolimits_{j=1}^{n} g'_j \otimes h_j$$

for certain $g'_1, \dots, g'_n \in H'$ and $h_1, \dots, h_n \in H$, its **trace** is

$$\mathrm{tr}(U) = \sum\nolimits_{j=1}^{n} \left\langle h_j, g'_j \right\rangle.$$

This is independent of the representation of U, see [**?**, p. 125]. Now, for $V \in \mathcal{L}(H'; X')$ we define

$$\|V\|_{\gamma'} := \sup \Big\{ |\mathrm{tr}(V'U)| \;\mid\; U \in \mathcal{L}(H; X), \, \|U\|_\gamma \le 1, \, \dim \mathrm{ran}(U) < \infty \Big\},$$

where we regard $V'U : H \to X \subseteq X'' \to H'' = H$, and let

$$\gamma'(H'; X') := \{ V \in \mathcal{L}(H'; X') \mid \|V\|_{\gamma'} < \infty \}.$$

By a short computation, if $U \in \mathcal{L}(H; X)$ has the representation $U = \sum_{j=1}^{n} g'_j \otimes x_j$ and $V \in \mathcal{L}(H'; X')$, then

$$\mathrm{tr}(V'U) = \sum_{j=1}^{n} \left\langle x_j, V g'_j \right\rangle. \tag{B.3}$$

Lemma B.13 (γ'-Fatou)**.** *Let $(V_n)_n$ be a bounded sequence in $\gamma'(H';X')$ and let $V : H' \to X'$ be such that $\langle x, V_n h'\rangle \to \langle x, Vh'\rangle$ for all $x \in X$ and $h' \in H'$. Then $V \in \gamma'(H';X')$ and*

$$\|V\|_{\gamma'} \le \liminf_{n\to\infty} \|V_n\|_{\gamma'}$$

Proof. It follows from (B.3) that $\mathrm{tr}(V_n'U) \to \mathrm{tr}(V'U)$ for every $U : H \to X$ of finite rank. The claim follows. □

We now turn to an alternative description of the γ'-norm. To this end we note the following auxiliary result.

Lemma B.14. *If $T \in \mathrm{Nu}(H)$ then $\mathrm{tr}(TA) = \|T\|_{\mathrm{nu}}$ for some $A \in \mathcal{L}(H)$, $\|A\| \le 1$.*

Proof. By a standard result of Hilbert space operator theory, T has the representation

$$T = \sum\nolimits_{j\in J} s_j \overline{e_j} \otimes f_j$$

where J is either finite or $J = \mathbb{N}$, the e_j as well as the f_j form orthonormal systems, and the numbers $s_j > 0$ are the singular values of T. Define $A := \sum_{j\in J} \overline{f_j} \otimes e_j$, where in case $J = \mathbb{N}$ the series converges strongly. Then $\|A\| \le 1$ and $TA = \sum_{j\in J} s_j \overline{A^* e_j} \otimes f_j$. Hence

$$\mathrm{tr}(TA) = \sum_{j\in J} s_j \,[f_j, A^* e_j] = \sum_{j\in J} s_j = \|T\|_{\mathrm{nu}}\,. \qquad \square$$

As a consequence we arrive at the following characterisation of the γ'-norm.

Corollary B.15. *Let $V \in \mathcal{L}(H';X')$. Then*

$$\|V\|_{\gamma'} = \sup\Big\{ \|V'U\|_{\mathrm{nu}} \mid U \in \mathcal{L}(H;X),\, \|U\|_\gamma \le 1,\, \dim\mathrm{ran}(U) < \infty \Big\}.$$

Proof. Let $U : H \to X$ be of finite rank with $\|U\|_\gamma \le 1$. Then $|\mathrm{tr}(V'U)| \le \|V'U\|_{\mathrm{nu}}$. On the other hand, by applying Lemma B.14 to $T := V'U$ we find $A \in \mathcal{L}(H)$ with $\|A\| \le 1$ and

$$\|V'U\|_{\mathrm{nu}} = \mathrm{tr}(V'UA) \le \|V\|_{\gamma'} \|UA\|_\gamma \le \|V\|_{\gamma'} \|U\|_\gamma \|A\| \le \|V\|_{\gamma'}$$

by the ideal property. □

As a consequence of Corollary B.15 we obtain the ideal property of $\gamma'(H';X')$.

Corollary B.16 (Ideal Property)**.** *Let $R : H \to K$ and $L : Y \to X$ be bounded operators, and $V \in \gamma'(H';X')$. Then $L'VR' \in \gamma'(K';Y')$ with*

$$\|L'VR'\|_{\gamma'} \le \|L\| \, \|V\|_{\gamma'} \, \|R\| \,.$$

Proof. Let $U : K \to Y$ be of finite rank. Then

$$\begin{aligned}\|(L'VR')'U\|_{\mathrm{nu}} = \|RV'(L''U)\|_{\mathrm{nu}} &\le \|R\| \, \|V'(LU)\|_{\mathrm{nu}} \le \|R\| \, \|V'\|_{\gamma'} \, \|LU\|_{\gamma} \\ &\le \|R\| \, \|V'\|_{\gamma'} \, \|L\| \, \|U\|_{\gamma}\end{aligned}$$

by the ideal property of $\mathrm{Nu}(K)$ and $\gamma(K;Y)$. □

With the following results we extend [**?**, Proposition 5.1 and 5.2].

Theorem B.17. a) *If $U \in \gamma(H;X)$ and $V \in \gamma'(H';X')$, then $V'U \in \mathrm{Nu}(H)$ with $\|V'U\|_{\mathrm{nu}} \le \|V\|_{\gamma'} \|U\|_{\gamma}$. Moreover, the mapping*

$$\gamma'(H';X') \longrightarrow \mathcal{L}\big(\gamma(H;X);\mathrm{Nu}(H)\big), \qquad V \longmapsto (U \longmapsto V'U)$$

is isometric.

b) *The bilinear mapping ("trace duality")*

$$\gamma(H;X) \times \gamma'(H';X') \longrightarrow \mathbb{C}, \qquad (U,V) \longmapsto \langle U, V\rangle := \mathrm{tr}(V'U)$$

establishes an isometric isomorphism $\gamma(H;X)' \cong \gamma'(H';X')$.

c) *Let $(e_\alpha)_\alpha$ be an orthonormal basis of H. Then*

$$\langle U, V\rangle = \mathrm{tr}(V'U) = \sum_\alpha \langle Ue_\alpha, V\overline{e_\alpha}\rangle_{X,X'}$$

for every $U \in \gamma(H;X)$ and $V \in \gamma'(H';X')$.

d) *If $V \in \gamma(H';X')$ then $V \in \gamma'(H';X')$, with $\|V\|_{\gamma'} \le \|V\|_{\gamma}$.*

Proof. a) follows from Corollary B.15 and approximation of a general $U \in \gamma(H;X)$ by finite rank operators.
b) By a) the trace duality is well defined, and it reproduces the norm on $\gamma'(H';X')$ by construction. For surjectivity, let $\Lambda : \gamma(H;X) \to \mathbb{C}$ be a bounded functional and define

$$V : H' \longrightarrow X', \qquad (Vh')(x) := \Lambda(h' \otimes x).$$

A short computation reveals that $\mathrm{tr}(V'U) = \Lambda(U)$ for every rank-one operator $U = h' \otimes x$. Hence $\mathrm{tr}(V'U) = \Lambda(U)$ even for every finite rank-operator $U : H \to X$. But this implies that $V \in \gamma'(H';X')$ and that V induces Λ.
c) By Corollary B.10, $U = \sum_\alpha \overline{e_\alpha} \otimes Ue_\alpha$ and the convergence is in $\|\cdot\|_\gamma$. Hence

$$\langle U, V\rangle = \sum_\alpha \langle \overline{e_\alpha} \otimes Ue_\alpha, V\rangle = \sum_\alpha \langle Ue_\alpha, V\overline{e_\alpha}\rangle_{X,X'}$$

by (B.3).
d) is proved as in [**?**, Theorem 10.9]. □

Remark B.18. It is shown in [**?**, Sec. 10] and [**?**, Thm. 9.43] that the equality $\gamma(H';X') = \gamma'(H';X')$ holds if X is K-convex. By a result of Pisier, a space X is K-convex if and only if it has nontrivial type. See [**?**, Sec. 10] for more about K-convexity in this context.

B.5. Spaces of Finite Cotype.

A **Rademacher** variable is a ±1-valued Bernoulli-($1/2$, $1/2$) random variable. A **complex Rademacher** variable is a random variable of the form

$$r = r_1 + \mathrm{i} r_2$$

where r_1, r_2 are independent real Rademachers on the same probability space. Unless otherwise stated, our Rademacher variables are understood to be complex.[3]

By [**?**, Proposition 2.6] (see also [**?**, Lemma 12.11])

$$\mathbb{E}\Big\|\sum\nolimits_{j=1}^{n} r_j x_j\Big\|_X^q \le (\pi/2)^{q/2}\, \mathbb{E}\Big\|\sum\nolimits_{j=1}^{n} \gamma_j x_j\Big\|_X^q, \tag{B.4}$$

whenever $1 \le q < \infty$, $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$, $r_1, \dots, r_n$ are complex Rademachers and $\gamma_1, \dots, \gamma_n$ are complex Gaussians. (Our reference uses real random variables, but the complex case follows by a straightforward argument, yielding the same constant.)

A converse estimate does not hold in general unless the Banach space has finite cotype. Recall that a Banach space X has *type* $p \in [1, 2]$ if there exists a constant $\mathrm{t}_p(X) \ge 0$ such that for all finite sequences $(x_n)_{n=1}^m$ in X,

$$\Big\|\sum\nolimits_n r_n x_n\Big\|_{\mathrm{L}_2(\Omega;X)} \le \mathrm{t}_p(X)\ \|(x_n)_n\|_{\ell_p(X)},$$

and X has *cotype* $q \in [2, \infty]$ if for some constant $\mathrm{c}_q(X) \ge 0$,

$$\|(x_n)_n\|_{\ell_q(E)} \le \mathrm{c}_q(X)\ \Big\|\sum\nolimits_n r_n x_n\Big\|_{\mathrm{L}_2(\Omega,E)},$$

We refer to [**?**, Chapter 11] for definitions, properties and references on the notions of type and cotype of a Banach space. (Using real in place of complex Rademachers may alter the values of $\mathrm{t}_p(X)$ and $\mathrm{c}_q(X)$ by universal factors, but does not make a qualitative difference.)

Each Banach spaces has cotype ∞ and type 1; therefore, X is said to have *nontrivial type* if it has type p for some $p > 1$, and it said to have *finite cotype* if it has cotype q for some $q < \infty$. Each Banach space of nontrivial type has finite cotype, but the converse is false.

[3] Our definition of complex Rademachers differs from the one given in [**?**], where complex Rademachers are defined as random variables uniformly distributed on the unit circle. The resulting differences are unessential.

It is important for us that if X has finite cotype, then a converse to (B.4) holds. Namely, we have the following deep result from [?, Theorem 12.27].

Theorem B.19. *Let $2 \le q < \infty$ and denote by m_q^q the q-th absolute moment of the normal distribution, i.e.,*

$$m_q := \Big(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |x|^q \, \mathrm{e}^{-\frac{x^2}{2}} \, \mathrm{d}x\Big)^{\frac{1}{q}}.$$

Then

$$\mathbb{E}\Big\|\sum_{j=1}^n \gamma_j x_j\Big\|^2 \le m_q^2 \, \mathrm{c}_q(X)^2 \, \mathbb{E}\Big\|\sum_{j=1}^n r_j x_j\Big\|^2$$

whenever X is a Banach space of cotype q and $x_1, \ldots, x_n \in X$.

Recall that a Banach space X has the same type/cotype as the space $L_2(\Omega, X)$, whenever Ω is a measure space, see e.g. [?, Theorem 11.12]. A similar result holds for the γ-functor.

Lemma B.20. *A Banach space X has the same type and cotype as $\gamma(H; X)$.*

Proof. We show the result only for the case of cotype. For the type case the arguments are similar. Suppose first that X has cotype $q < \infty$, and let $(U_k)_k$ be a finite sequence in $\gamma(H; X)$. Fix an orthonormal basis $(e_\alpha)_\alpha$ of H. Then $U_k = \sum_\alpha \overline{e_\alpha} \otimes U_k e_\alpha$ for each k by Corollary B.10. Hence, with F denoting finite subsets of the index set of the orthonormal basis,

$$\begin{aligned}
\sum_k \|U_k\|_\gamma^q &= \sum_k \lim_F \Big\|\sum_{\alpha\in F} \overline{e_\alpha} \otimes U_k e_\alpha\Big\|_\gamma^q = \lim_F \sum_k \Big\|\sum_{\alpha\in F} \overline{e_\alpha} \otimes U_k e_\alpha\Big\|_\gamma^q \\
&\lesssim \sup_F \sum_k \mathbb{E}'\Big\|\sum_{\alpha\in F} \gamma'_\alpha U_k e_\alpha\Big\|_X^q = \sup_F \mathbb{E}' \sum_k \Big\|\sum_{\alpha\in F} \gamma'_\alpha U_k e_\alpha\Big\|_X^q \\
&\lesssim \sup_F \, \mathrm{c}_q(X)^q \, \mathbb{E}' \, \mathbb{E}\Big\|\sum_k r_k \Big(\sum_{\alpha\in F} \gamma'_\alpha U_k e_\alpha\Big)\Big\|_X^q \\
&\lesssim \sup_F \, \mathrm{c}_q(X)^q \, \mathbb{E} \, \mathbb{E}'\Big\|\sum_{\alpha\in F} \gamma'_\alpha \Big(\sum_k r_k U_k e_\alpha\Big)\Big\|_X^q \\
&\lesssim \mathrm{c}_q(X)^q \, \mathbb{E} \, \Big\|\sum_k r_k U_k\Big\|_X^q,
\end{aligned}$$

where the non-mentioned constants come from the Khinchine–Kahane inequalities. It follows that

$$\|(U_k)_k\|_{\ell_q(\gamma(H;X))} \lesssim \mathrm{c}_q(X) \Big\|\sum_k r_k U_k\Big\|_{\mathrm{L}_2(\Omega;\gamma(H;X))}$$

and this shows that $\mathrm{c}_q(\gamma(H; X)) \lesssim \mathrm{c}_q(X)$.

For the converse suppose that $\gamma(H;X)$ has cotype $q<\infty$. Let $(x_k)_k$ be a finite sequence in X and let $e\in H$ be a unit vector. Abbreviate $E:=\gamma(H;X)$ and $U_k:=\overline{e}\otimes x_k$. Then

$$\Big(\sum\nolimits_k \|x_k\|_X^q\Big)^{1/q} = \Big(\sum\nolimits_k \|U_k\|_E^q\Big)^{1/q} \le \mathrm{c}_q(E)\Big\|\sum\nolimits_k r_k U_k\Big\|_{\mathrm{L}_2(\Omega;E)}.$$

Moreover,

$$\begin{aligned}\Big\|\sum\nolimits_k r_k U_k\Big\|_{\mathrm{L}_2(\Omega;E)}^2 &= \mathbb{E}\Big\|\sum\nolimits_k r_k \overline{e}\otimes x_k\Big\|_E^2 = \mathbb{E}\Big\|\overline{e}\otimes\Big(\sum\nolimits_k r_k x_k\Big)\Big\|_E^2\\ &= \mathbb{E}\Big\|\sum\nolimits_k r_k x_k\Big\|_X^2,\end{aligned}$$

whence it follows that $\mathrm{c}_q(X)\le \mathrm{c}_q(E)$. □

The next result shows the significance of spaces of finite cotype for the theory of γ-radonifying operators.

Theorem B.21. *Let X be a Banach space of finite cotype $q<\infty$. There is a constant $c=c(q,\mathrm{c}_q(X))$ such that the following holds: Whenever K is a compact Hausdorff space, H is a Hilbert space and $T\in\mathcal{L}(H;X)$ is an operator that factorises as $T=UV$ over $\mathrm{C}(K)$, i.e.,*

$$\begin{array}{ccc} H & \xrightarrow{\quad T\quad} & X\\ & {\scriptstyle V}\searrow \quad \nearrow{\scriptstyle U} & \\ & \mathrm{C}(K) & \end{array},$$

then $T\in\gamma(H;K)$ and $\|T\|_{\gamma(H;X)}\le c\,\|U\|\,\|V\|$.

Proof. Let X be of cotype $2\le q<\infty$ and fix $q<p<\infty$. By [**?**, Theorem 11.14] the operator U is p-absolutely summing, and one has $\pi_p(U)\le c\cdot\|U\|$, where c depends on p and $\mathrm{c}_q(X)$. By the ideal property for p-absolutely summing operators, T is p-absolutely summing with $\pi_p(T)\le\pi_p(U)\,\|V\|$. Now, a theorem of Linde and Pietsch [**?**], cf. [**?**, Prop. 12.1], yields that $T\in\gamma(H;X)$ with $\|T\|_\gamma\le\max\{K_{2,p}^\gamma,K_{p,2}^\gamma\}\pi_p(T)$. Here $K_{p,2}^\gamma$ and $K_{2,p}^\gamma$ are the constants in the Khinchine–Kahane inequalities for Gaussians, see [**?**, Proposition 2.7]. By taking the infimum over p we remove the dependence of the constant on p. □

The following consequence is Corollary 3.4 from [**?**].

Corollary B.22 (Kaiser–Weis)**.** *Let X be a Banach space of finite cotype q. Then there is a constant $C=C(q,\mathrm{c}_q(X))$ such that for all finite sequences $x_1,\dots,x_N\in X$ and all complex matrices $\alpha=(\alpha_{n\,j})_{n,j}\in$*

$\mathbb{C}^{N\times J}$ *one has*

$$\mathbb{E}\mathbb{E}'\Big\|\sum_{n=1}^{N}\sum_{j=1}^{J}\gamma_n\gamma'_j\alpha_{n\,j}x_n\Big\|_X^2\leq C^2\,\mathbb{E}\,\Big\|\sum_{n=1}^{N}\gamma_n x_n\|_X^2.$$

Proof. Consider the operators

$$U:\ell_2^J\to\ell_\infty^N,\qquad U(e_j)=(\alpha_{n\,j})_n=\sum_n\alpha_{n\,j}e'_n$$

and

$$V:\ell_\infty^N\to\gamma(\ell_2^N,\qquad V(e'_n)=\overline{e_n}\otimes x_n.$$

Then $\|U\|^2=\sup_n\sum_j|\alpha_{n\,j}|^2$ and $\|V\|^2=\mathbb{E}\,\Big\|\sum_{n=1}^N\gamma_n x_n\|_X^2$, by the contraction principle. Now apply Theorem B.21 to conclude that $VU\in\gamma(\ell_2^J;\gamma(\ell_2^N;X))$ with norm

$$\|VU\|_\gamma\leq C\,\|U\|\,\|V\|\,.$$

Writing out VU and its norm finishes the proof. □

B.6. Spaces with Property ($\boldsymbol{\alpha}^*$).
Let H and K be Hilbert spaces. Their algebraic tensor product, here denoted by $H\otimes_a K$, is a pre-Hilbert space with respect to the (uniquely determined) inner product satisfying

$$[h\otimes k,u\otimes v]=[h,u]\cdot[k,v]$$

for all $h,u\in H$ and $k,v\in K$. We denote by $H\otimes K$ the completion of $H\otimes_a K$ with respect to this inner product. Then we have a natural isometric isomorphism

$$(H\otimes K)'\cong H'\otimes K'.$$

In this section we examine the relation of the spaces $\gamma(H\otimes K;X)$ and $\gamma(H;\gamma(K;X)$.
Note that, algebraically, there is a natural isomorphism

$$H'\otimes_a(K'\otimes_a X)\cong(H'\otimes_a K')\otimes_a X$$

which on the level of elementary tensors is just the "associative law"

$$h'\otimes(k'\otimes x)=(h'\otimes k')\otimes X.$$

Via this natural isomorphism, the algebraic tensor product $H'\otimes_a K'\otimes_a X$ can be viewed as a subset of either space $\gamma(H;\gamma(K;X))$ and $\gamma(H\otimes K;X)$.

Lemma B.23. *The algebraic tensor product $H'\otimes K'\otimes X$ is dense in either space $\gamma(H\otimes K;X)$ and $\gamma(H;\gamma(K;X))$.*

Proof. This follows from Corollary B.10 by noting that if $(e_\alpha)_\alpha$ and $(f_\beta)_\beta$ are orthonormal bases of H and K, respectively, then $(e_\alpha \otimes f_\beta)_{\alpha,\beta}$ is an orthonormal basis of $H \otimes K$. Moreover, for $h \in H$ and $k \in K$ one has $\overline{h \otimes k} = \overline{h} \otimes \overline{k}$ under the canonical identification of $(H \otimes K)'$ with $H' \otimes K'$. □

Now let us look at the first type of relation.

Lemma B.24. *For a Banach space X the following properties are equivalent*

(i) *For each pair H, K of Hilbert spaces, the natural map extends to a bounded operator $\gamma(H \otimes K; X) \to \gamma(H; \gamma(K; X))$.*

(ii) *The natural map extends to a bounded operator*

$$J^- : \gamma(\ell_2 \otimes \ell_2; X) \to \gamma(\ell_2; \gamma(\ell_2; X))$$

(iii) *There is a constant $C^- \geq 0$ such that*

$$\mathbb{E}\mathbb{E}'\Big\|\sum_{j,n} \gamma_j \gamma_n' x_{j,n}\Big\|^2 \leq (C^-)^2 \mathbb{E}\Big\|\sum_{j,n} \gamma_{j,n} x_{j,n}\Big\|^2$$

for each finite array $(x_{j,n})_{j,n}$ in X.

In this case, the best constant C^- in (iii) *equals $\|J^-\|$, and this dominates the norm of the operator in* (i).

Here, $(\gamma_j)_j$ and $(\gamma_n')_n$ are two independent sequences of independent standard Gaussians, and (γ_{ij}) is an array of independent standard Gaussians.

Proof. It is immediately clear that (i) implies (ii) and (ii) implies (iii) with $C^- \leq \|J^-\|$. Suppose that (iii) holds. Then it is clear that for any $T \in H' \otimes_a K' \otimes_a X$ one has

$$\big\|J^-(T)\big\|_{\gamma(H;\gamma(K;X))} \leq C \, \|T\|_{\gamma(H \otimes K;X)} .$$

Now (i) follows from Lemma B.23, and the norm of the considered operator is $\leq C^-$. □

Following Van Neerven and Weis [**?**] we say that a space X having the equivalent properties listed in Lemma B.24 has **(Gaussian) property** (α^-). In Proposition 2.5. of the cited reference, the authors show that the Gaussian property (α^-) implies finite cotype and is equivalent to "Rademacher property (α^-)", by which it is meant that assertion (iii) of Lemma B.24 holds when one replaces Gaussians by Rademachers.

Analogously, a Banach space is said to have **(Gaussian) property** (α^+), if it has the equivalent properties listed in the following lemma.

Lemma B.25. *For a Banach space X the following properties are equivalent*

(i) *For each pair H, K of Hilbert spaces, the natural map extends to a bounded operator $\gamma(H;\gamma(K;X)) \to \gamma(H\otimes K;X)$*

(ii) *The natural map extends to a bounded operator*
$$J^+ : \gamma(\ell_2;\gamma(\ell_2;X)) \to \gamma(\ell_2\otimes\ell_2;X).$$

(iii) *There is a constant $C^+ \geq 0$ such that*
$$\mathbb{E}\mathbb{E}'\Big\|\sum_{j,n}\gamma_j\gamma_n' x_{j,n}\Big\|^2 \leq (C^+)^2\mathbb{E}\Big\|\sum_{j,n}\gamma_{j,n}x_{j,n}\Big\|^2$$
for each finite array $(x_{j,n})_{j,n}$ in X.

In this case, the best constant C^+ in (iii) *equals $\|J^+\|$ and this dominates the norm of the operator in* (i).

Again, by [**?**, Proposition 2.5.], Gaussian property (α^+) is equivalent to Rademacher property (α^+) and implies finite cotype.

A space X is said to have **property** (α) if X has both properties (α^+) and (α^-). It is shown in [**?**, Chapter 13] or [**?**, Thm. 9.72] that this is equivalent to X having **Pisier's contraction property** from [**?**, Definition 2.1]. (The terminology is from [**?**, Thm. 9.72].)

Every Hilbert space has property (α) and each space $\mathrm{L}_p(\Omega;X)$ with $1\leq p<\infty$ inherits this property from X [**?**, Chapter 13].

However, by [**?**, Example 2.4], the p-Schatten classes provide examples of spaces with property (α^-) but not property (α^+) for $p\in(1,2)$ and the other way round for $p\in(2,\infty)$.

B.7. γ-Bounded Sets.

Let X, Y be Banach spaces. A collection $\mathcal{T}\subseteq\mathcal{L}(X;Y)$ is said to be **γ-bounded** if there is a constant $c\geq 0$ such that

$$\text{(B.5)}\qquad \mathbb{E}\left(\Big\|\sum\nolimits_{T\in\mathcal{T}'}\gamma_T T x_T\Big\|_X^2\right)^{1/2} \leq c\ \mathbb{E}\left(\Big\|\sum\nolimits_{T\in\mathcal{T}'}\gamma_T x_T\Big\|_X^2\right)^{1/2}$$

for all finite subsets $\mathcal{T}'\subseteq\mathcal{T}$, $(x_T)_{T\in\mathcal{T}'}\subseteq X$. (As above, $(\gamma_T)_{T\in\mathcal{T}'}$ is an independent collection of standard Gaussian random variables on some probability space.) If $\mathcal{T}$ is γ-bounded, the smallest constant c for which (B.5) holds, is denoted by $[\![\mathcal{T}]\!]^\gamma$ and is called the **γ-bound** of $\mathcal{T}$.

Remark B.26. In order to establish the γ-boundedness of $\mathcal{T}$ with $[\![\mathcal{T}]\!]^\gamma\leq c$, it suffices to check (B.5) only for vectors x_T from a dense subspace of X. This follows from Corollary A.3. Likewise, it suffices to take the operators T from a strongly dense subset of $\mathcal{T}$. Actually, it suffices to take the operators T from a subset $\mathcal{T}'$ such that $\operatorname{absconv}(\mathcal{T}')$ is strongly dense in $\mathcal{T}$. This follows from [**?**].

When one replaces Gaussians by Rademachers in the definition above one obtains the related notion of an R-**bounded** set of operators. If both spaces X and Y have finite cotype, R-boundedness and γ-boundedness are equivelent (on the expense of constants that depend on the cotype and the cotype constants of the spaces).

The notions of γ- and R-bounded sets have been thoroughly studied in the literature, see [**?**, Chapter 7] and the literature listed there. For the following simple fact, however, we could not find any reference in the literature.

Lemma B.27. *Let $(M_\alpha)_\alpha$ be a family of γ-bounded subsets of $\mathcal{L}(X;Y)$ with $C := \sup_\alpha [\![M_\alpha]\!]^\gamma < \infty$. Suppose that $M \subseteq \mathcal{L}(X;Y)$ has the following property:*

$$\forall\, N \in \mathbb{N}\ \forall\, x_1, \dots, x_N \in X\ \forall\, T_1, \dots, T_N \in \mathcal{L}(X;Y)\ \forall\, \varepsilon > 0$$
$$\exists\, \alpha \in I\ \exists\, T_{1\alpha}, \dots, T_{N\alpha} \in M_\alpha : \sum_{j=1}^{N} \|T_{j\alpha} x_j - T_j x_j\| < \varepsilon.$$

Then M is γ-bounded with $[\![M]\!]^\gamma \le C$.

Proof. Fix $x_1, \dots, x_N \in X$ and $T_1, \dots, T_N \in M$. By the hypothesis on M there is a sequence $(\alpha_n)_n$ of indices and respective operators $T_{\alpha_n j} \in M_{\alpha_n}$ with $T_{\alpha_n j} x_j \to T_j x_j$ as $n \to \infty$ for each $j = 1, \dots, N$. By Corollary A.3

$$\mathbb{E} \Big\| \sum\nolimits_j \gamma_j T_{\alpha_n j} x_j \Big\|_X^2 \to \mathbb{E} \Big\| \sum\nolimits_j \gamma_j T_j x_j \Big\|_X^2$$

as $n \to \infty$. The claim follows. □

Remark B.28. Lemma B.27 can be applied, for example, in the situation when one has a subset $M \subseteq \mathcal{L}(X;Y)$ and operator nets $(P_\alpha)_\alpha$ in $\mathcal{L}(X)$ and $(Q_\alpha)_\alpha$ in $\mathcal{L}(Y)$ such that $Q_\alpha T P_\alpha \to T$ strongly for each $T \in M$. Then the sets

$$M_\alpha := \{Q_\alpha T P_\alpha \mid T \in M\}$$

satisfy the technical condition of Lemma **??**. So if one is able to show that $C := \sup_\alpha [\![M_\alpha]\!]^\gamma < \infty$, it follows that M is γ-bounded with $[\![M]\!]^\gamma \le C$.

Appendix C. ℓ_1-Frame-Bounded Sets

Let H be a Hilbert space. A sequence $(f_\alpha)_{\alpha\in I}$ in H is called a **frame** for H if there exist two constants $0 < A < B$ such that

$$A^2 \|h\|_H^2 \le \sum_{\alpha\in I} |\,[h, f_\alpha]\,|^2 \le B^2 \|h\|_H^2 \qquad \text{for all } h \in H. \tag{C.1}$$

Equivalently, a frame is given by a pair of operators (L, R) where $R : H \to \ell_2(I)$ and $L : \ell_2(I) \to H$ such that $L\,R = \mathrm{Id}_H$. Indeed, in that case $f_\alpha := R^* e_\alpha$, $\alpha \in I$, is a frame, where $(e_\alpha)_{\alpha\in I}$ is the canonical basis of $\ell_2(I)$. (One easily obtains (C.1) with $A = \|L\|^{-1}$ and $B = \|R\|$.) Conversely, if $(f_\alpha)_{\alpha\in I}$ is a frame and $R : H \to \ell_2(I)$ is defined by $Rf := ([f, f_\alpha])_{\alpha\in I}$, then R^*R is a selfadjoint, positive and invertible operator, and hence $L := (R^*R)^{-1}R^*$ satisfies $LR = \mathrm{I}_H$.

Let H be a Hilbert space. A subset M of H is called ℓ_1*-frame-bounded* if there exists a frame $(f_\alpha)_{\alpha\in I}$ of H such that

$$\sup_{x\in M} \sum_{\alpha\in I} |[x, f_\alpha]| < \infty,$$

In this case, in view of the discussion above, the ℓ_1**-frame-bound** of a subset $M \subseteq H$ is defined as

$$|M|_1 := \inf\ \|L\| \sup_{x\in M} \sum_{\alpha\in I} |[Rx, e_\alpha]|\} \tag{C.2}$$

where the infimum is taken over all pairs of operators (L, R) with $R : H \to \ell_2(I)$ and $L : \ell_2(I) \to H$ such that $LR = \mathrm{I}_H$.

Let X be a Banach space. An operator $T : X \to H$ is called ℓ_1**-frame-bounded** if T maps the unit ball of X into an ℓ_1-frame-bounded subset of X. In this case,

$$|T|_{\ell_1} := \big|\{Tx \mid \ \|x\|_X \le 1\}\big|_1$$

is called the ℓ_1**-frame-bound** of T.

Remarks C.1. 1) ℓ_1-frame-bounded sets need not be compact.

2) Let X, Y be Banach spaces. If $U : X \to H$ is ℓ_1-frame-bounded and $V : Y \to X$ is bounded, then $UV : Y \to H$ is ℓ_1-frame-bounded and
$$|UV|_{\ell_1} \le |U|_{\ell_1} \|V\|.$$

3) We point out that we do not know yet whether finite unions or translates of ℓ_1-frame-bounded sets are again ℓ_1-frame-bounded, something one would certainly expect to hold for a “good” boundedness concept. Consequently, we do not know whether the set of ℓ_1-frame-bounded operators $X \to H$ form a vector space.

Lemma C.2. *Let H be any Hilbert space and $M \subseteq H$. Then the following assertions hold.*

a) *If M is ℓ_1-frame-bounded, then it is norm-bounded, with*
$$\sup_{x \in M} \|x\| \le |M|_1$$
If $\operatorname{span}(M)$ *is finite-dimensional and M is norm-bounded, then it is ℓ_1-frame-bounded.*

b) *If M is ℓ_1-frame-bounded and $S : H \to K$ is an isomorphism into another Hilbert space, then $S(M)$ is ℓ_1-frame-bounded with*
$$|S(M)|_1 \le \|S\| \; |M|_1$$

c) *If M is ℓ_1-frame-bounded, then* $\overline{\operatorname{absconv}}(M)$ *is ℓ_1-frame-bounded.*

Proof. Parts a) and b) are clear. For the proof of c) it suffices to notice that the closed unit ball of $\ell_1(I)$ is absolutely convex and closed in $\ell_2(I)$. □

Remark C.3. Every ℓ_1-frame-bounded operator $T : X \to H$ factorises through an ℓ_1-space, but the converse is not true in general. Indeed, let $(f_n)_{n \in \mathbb{N}}$ be a countable dense subset of the unit sphere $\{f \in \ell_2 \mid \|f\|_2 = 1\}$ of ℓ_2. Let $T : \ell_1 \to \ell_2$ be the operator defined by $T(x_n)_n := \sum_n x_n f_n$. Then the image under T of the unit ball of ℓ_1 is dense in the unit ball of ℓ_2, and hence T is not ℓ_1-frame-bounded.

For operators between Hilbert spaces, the class of ℓ_1-frame-bounded operators coincides with the class of Hilbert–Schmidt operators.

Lemma C.4. *For an operator $T : K \to H$, K and H Hilbert spaces, the following assertions are equivalent:*

(i) *T is ℓ_1-frame-bounded.*

(ii) *T factorises through an ℓ_1-space.*

(iii) *T is Hilbert-Schmidt.*

Proof. Suppose that (i) holds, i.e., T is ℓ_1-frame-bounded. Let $R : H \to \ell_2(I)$ and $L : \ell_2(I) \to H$ as in (C.2). Then $T = LRT$ factors as $T = VU$ with
$$U : \begin{cases} K & \longrightarrow & \ell_1(I) \\ x & \mapsto & \langle RTx, e_\alpha \rangle \end{cases} \qquad \text{and} \qquad V : \begin{cases} \ell_1(I) & \longrightarrow & H \\ (\lambda_\alpha)_\alpha & \mapsto & \sum_\alpha \lambda_\alpha L e_\alpha \end{cases}$$
and we have (ii). Next, recall that [?, Corollary 4.12] asserts that Hilbert space operators are Hilbert-Schmidt if and only if they factor through an $\mathcal{L}_1$-space. Hence (ii) implies (iii). Finally, if $T : K \to H$ is

Hilbert-Schmidt, the singular value decomposition yields a representation

$$T = \sum_n \tau_n \overline{f_n} \otimes e_n$$

with orthonormal systems $(e_n)_n$ and $(f_n)_n$ and scalars $\tau = (\tau_n)_n \in \ell_2(\mathbb{N})$. We extend $(e_n)_n$ in some way to an orthonormal basis $(e_\alpha)_{\alpha \in I}$ of H. Then

$$\sum_\alpha |\langle Tf, e_\alpha\rangle| = \sum_n |\langle Tf, e_n\rangle| = \sum_n |\tau_n| \, |\langle f, f_n\rangle| \le \|\tau\|_{\ell_2} \|f\|_K$$

by the Cauchy-Schwarz and the Bessel inequalities. Hence T is ℓ_1-frame-bounded with $|T|_{\ell_1} \le \|\tau\|_{\ell_2} = \|T\|_{HS}$. ☐

Let us provide some other examples of ℓ_1-frame-bounded sets/operators.

Examples C.5. 1) The *Wiener algebra* $\mathrm{A}(\mathbb{T})$ is the set of continuous functions on $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ that have absolutely summable Fourier coefficients. Obviously, the embedding $\mathrm{A}(\mathbb{T}) \subseteq \mathrm{L}_2(\mathbb{T})$ is ℓ_1-frame-bounded.

2) As a consequence of the above item, every embedding into $\mathrm{L}_2(\mathbb{T})$ that factors through the Wiener algebra is ℓ_1-frame-bounded. This implies, e.g., that the embedding $\mathrm{C}^s[0,1] \subseteq \mathrm{L}_2[0,1]$ is ℓ_1-frame-bounded for $s > 1/2$

3) The embeddings $\mathrm{B}^s_{pq}[0,1] \subseteq \mathrm{L}_2[0,1]$ and $\mathrm{W}^s_p[0,1] \subseteq \mathrm{L}_2[0,1]$ are ℓ_1-frame-bounded whenever $s > 1/2$.

We do not know whether the continuous analogue of Example 1) is true, namely whether the embedding

$$\mathrm{A}(\mathbb{R}) := \{f \in \mathrm{L}_1(\mathbb{R}) \mid \widehat{f} \in \mathrm{L}_1(\mathbb{R})\} \subseteq \mathrm{L}_2(\mathbb{R})$$

is ℓ_1-frame-bounded. However, we have the following.

Lemma C.6. *The canonical embedding* $\mathrm{W}^2_1(\mathbb{R}) \hookrightarrow \mathrm{L}_2(\mathbb{R})$ *is ℓ_1-frame-bounded.*

Proof. Fix a function $0 \le \eta \in \mathrm{C}^\infty(\mathbb{R})$ with $\operatorname{supp}(\eta) \subseteq (0, 2\pi)$, strictly positive on $[1,2]$ and in such a way that with $\eta_k(t) := \eta(t-k)$ one has

$$1 = \sum_{k \in \mathbb{Z}} \eta_k.$$

Then the double sequence $(f_{n,k})_{(n,k)\in\mathbb{Z}^2}$ given by $f_{n,k} := \eta_k e^{in(\cdot)}$ forms a (Gabor) frame on $\mathrm{L}_2(\mathbb{R})$. (Take $L{=}2\pi$, $\alpha{=}1$ and $\beta{=}(2\pi)^{-1}$ in [**?**, Theorem 6.4.1].) Let $g \in \mathrm{W}_1^2(\mathbb{R})$. For $n = 0$,

$$\sum\nolimits_{k\in\mathbb{Z}} \left| \int_{k-\pi}^{k+\pi} \eta_k(s) g(s) \,\mathrm{d}s \right| \le \|g\|_{\mathrm{L}_1} .$$

For $n \neq 0$, a twofold integration by parts (with vanishing boundary terms) yields

$$\int_{k-\pi}^{k+\pi} \eta_k(s) g(s) \mathrm{e}^{-\mathrm{i}ns} \,\mathrm{d}s = -\tfrac{1}{n^2} \int_{k-\pi}^{k+\pi} [\eta_k(s) g(s)]'' \mathrm{e}^{-\mathrm{i}ns} \,\mathrm{d}s.$$

Since $[\eta_k(s)g(s)]'' = \eta_k(s) g''(s) + 2\eta_k'(s) g'(s) + \eta_k'' g(s)$ and since the η_k's are all translates of the same function,

$$\big| [\eta_k(s) g(s)]'' \big| \lesssim \mathbf{1}_{(k-\pi, k+\pi)}(s) \big(\, |g(s)| + |g'(s)| + |g''(s)| \, \big).$$

Hence

$$\sum_{n\in\mathbb{Z}^*, k\in\mathbb{Z}} |\langle g, f_{n,k} \rangle| \lesssim \|g\|_{\mathrm{W}_1^2} . \qquad \square$$

Using interpolation techniques one can see that $\mathrm{W}_1^\alpha(\mathbb{R}) \subseteq \mathrm{L}_2(\mathbb{R})$ is ℓ_1-frame-bounded for each $\alpha > 1$.

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