Vector-Valued Holomorphic Functions and Abstract Fubini-Type Theorems

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Abstract. Let $f = f(z,t)$ be a function holomorphic in $z \in O \subseteq \mathbb{C}^d$ for fixed $t \in \Omega$ and measurable in t for fixed z and such that $z \mapsto f(z,\cdot)$ is bounded with values in $E := \mathrm{L}_p(\Omega)$, $1 \le p \le \infty$. It is proved (among other things) that

$$\langle t \mapsto \varphi(f(\cdot,t)), \mu\rangle = \varphi(z \mapsto \langle f(z,\cdot), \mu\rangle)$$

whenever $\mu \in E'$ and φ is a bp-continuous linear functional on $\mathrm{H}^\infty(O)$.

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1. Introduction

Let O and Ω be non-empty sets and let

$$f : O \times \Omega \to \mathbb{C}, \qquad (z,t) \mapsto f(z,t)$$

be a function from which we define two functions,

$$F(z) := f(z,\cdot) \in \mathbb{C}^\Omega \qquad (z \in O) \qquad \text{and} \qquad f_t := f(\cdot,t) \in \mathbb{C}^O \qquad (t \in \Omega).$$

Given linear functionals μ on $\mathbb{C}^\Omega$ and φ on $\mathbb{C}^O$, we might ask whether their application commutes, i.e. whether one has

$$\langle t \mapsto \varphi(f_t), \mu\rangle = \varphi(z \mapsto \langle F(z), \mu\rangle). \tag{1.1}$$

We shall call a theorem stating the validity of (1.1) under certain conditions an *abstract Fubini-type theorem*, for obvious reasons.

L_2-valued bounded holomorphic functions play a prominent role in our work on functional calculus [12, 13], and we came naturally across some particular instances of abstract Fubini-type theorems in that context. It is the purpose of this note to present these results independently of their relevance for functional calculus theory, because in our view they are interesting in their own right.

The abstract Fubini-type theorems we are aiming at involve holomorphy in the first and measurability in the second variable. To wit, we suppose that $O \subseteq \mathbb{C}^d$ is open, Ω is a measure space and $f : O \times \Omega \to \mathbb{C}$ is such that f_t is holomorphic for each $t \in \Omega$ and $F(z)$ is measurable for each $z \in O$ (plus additional hypotheses). Thus, contrary to the classical Fubini theorem, there is a built-in asymmetry motivated by the aim to eventually regard F as a bounded holomorphic function with values in some Banach space E of (equivalence classes of) measurable functions.

The paper is organized as follows. In Section 2 we recall classical results for vector-valued holomorphic functions. Next, in Section 3, we state a corollary of Mujica's linearization theorem from [19], but with a new proof. In Section 4, we consider first the case of a closed subspace E of the space $\mathrm{BM}(\Omega, \Sigma)$ of bounded measurable functions. The main result is then Theorem 4.3. It treats the case $E = \mathrm{L}_p(\Omega)$ for $1 \le p < \infty$. It is complemented by Theorem 4.4, which deals with $E = \mathrm{L}_\infty(\Omega)$.

Theorem 4.3 extends results of Mattner [18], who only treats the case d=1 and p=1. In the concluding Section 5 we discuss Mattner's theorem and its relation to our work. In particular, we present an alternative proof of Theorem 4.3 for the case p=1 using Mattner's results as a starting point.

As it is often the case with papers in functional analysis, the thrust of the presentation is not just about results (which are only partially new) but also about proofs. In particular, we strive to keep the presentation self-contained, at least for readers with a background in functional analysis.

Terminology and Notation. Generically, E, F denote complex Banach spaces, and E', F' are their respective duals; the canonical duality $E \times E' \to \mathbb{C}$ is denoted by $(x, x') \mapsto \langle x, x' \rangle$.

For any set Ω we write $\ell^\infty(\Omega; E)$ for the space of all bounded E-valued functions on Ω, endowed with the supremum norm; moreover, we abbreviate $\ell^\infty(\Omega) := \ell^\infty(\Omega; \mathbb{C})$.

We say that a sequence $(f_n)_n$ in $\ell^\infty(\Omega; E)$ *bp-converges* to $f : \Omega \to E$ if $f_n \to f$ pointwise on Ω and $\sup_n \|f_n\|_\infty < \infty$.

A subset $M \subseteq \ell^\infty(\Omega; E)$ is called *bp-closed* if it is closed under taking bp-limits of sequences in M.

Let $F \subseteq \ell^\infty(\Omega; E)$ be a bp-closed subspace. A linear mapping $\varphi : F \to X$ (where X is any topological vector space) is called *bp-continuous* if

$$f_n \to f \quad (\mathrm{bp}) \qquad \Rightarrow \quad Tf_n \to Tf.$$

We agree that it would be more accurate to speak of "sequentially bp-closed" sets and "sequentially bp-continuous" mappings. However, we decided to drop the word "sequentially" for the sake of brevity.

For any measurable space (Ω, Σ) we let $\mathrm{BM}(\Omega) := \mathrm{BM}(\Omega, \Sigma)$ be the Banach space of all bounded and measurable $\mathbb{C}$-valued functions, endowed with the supremum norm. This is a bp-closed subspace of $\ell^\infty(\Omega)$.

2. Vector-Valued Multivariate Holomorphic Functions

Here and in the following, $O \subseteq \mathbb{C}^d$ is a fixed open and not-empty set and E is a complex Banach space. A function $f : O \to E$ is **holomorphic** if it is totally differentiable (= Fréchet-differentiable) with $\mathbb{C}$-linear derivative at each point of O [14, Def. 147, p.68]. The following theorem is a useful characterization of holomorphy. It extends well-known results for multi-variable scalar-valued and one-variable vector-valued functions.

Theorem 2.1. *Let E be a complex Banach space, and $N \subseteq E'$ an E-norming subset of E'. For a mapping $f : O \to E$ the following assertions are equivalent:*

(i) *f is holomorphic.*

(ii) *f is continuous and separately holomorphic.*

(iii) *f is locally bounded and $x' \circ f$ is separately holomorphic for all $x' \in N$.*

In this case for each $a = (a_j)_{j=1}^d \in O$ and each $r > 0$ with $\prod_{j=1}^d \mathrm{Ball}[a_j; r] \subseteq O$ one has the **Cauchy formula**

$$f(z) = \frac{1}{(2\pi \mathrm{i})^d} \int_{|w_d - a_d| = r} \cdots \int_{|w_1 - a_d| = r} \frac{f(w)\,\mathrm{d}w_1 \dots \mathrm{d}w_d}{(w_1 - z_1) \cdots (w_d - z_d)}$$

for all $z \in \mathbb{C}^d$ with $|z - a|_\infty < r$.

Here, "separately holomorphic" means partially complex differentiable in each coordinate direction. For the proof of Theorem 2.1 we recall the following simple consequence of Schwarz' lemma.

Lemma 2.2. *Let $a \in \mathbb{C}, r > 0$ and $f : \mathrm{Ball}(a; r) \to \mathbb{C}$ holomorphic and bounded. Then*

$$|f(z) - f(a)| \le \frac{2}{r} \|f\|_\infty |z - a| \qquad (|z - a| < r).$$

Proof. Fix $M > \|f\|_\infty$ and define $\varphi : \mathrm{Ball}(0; 1) \to \mathbb{C}$ by $\varphi(z) := \frac{1}{2M}(f(a + rz) - f(a))$. Then φ is holomorphic and bounded by 1 on $\mathrm{Ball}(0; 1)$ with $\varphi(0) = 0$. By Schwarz' Lemma, $|\varphi(z)| \le |z|$ for $|z| < 1$. Replacing z by $\frac{1}{r}(z - a)$ yields the claim. □

Proof of Theorem 2.1. Clearly, (i) implies (ii) and (ii) implies (iii). Now suppose (iii) and let $D := \prod_j \mathrm{Ball}(a_j; r)$ be any polydisc with $\overline{D} \subseteq O$. Note that, by hypothesis, f is bounded on $\overline{D}$.

We first show as in [16, proof of 1.3] that f is continuous at a. Let $z = (z_j)_j \in D$ and write

$$f(z) - f(a) = \sum_{j=1}^d f(z_1, \dots, z_j, a_{j+1}, \dots, a_d) - f(z_1, \dots, z_{j-1}, a_j, \dots, a_d).$$

Composing with $x' \in N$ and applying Lemma 2.2 to each summand we obtain

$$|x'(f(z) - f(a))| \le \frac{2}{r}\|f\|_\infty \sum_{j=1}^{d} |z_j - a_j|.$$

Taking the supremum over $x' \in N$ yields

$$\|f(z) - f(a)\| \le \frac{2}{r}\|f\|_\infty \sum_{j=1}^{d} |z_j - a_j| \quad \text{whenever} \quad |z - a|_\infty < r.$$

In particular, f is continuous at a. Since $a \in O$ was arbitrary, f is continuous.

Continuity of f implies that the function

$$g(z) := \frac{1}{(2\pi \mathrm{i})^d} \int_{|w_d - a_d| = r} \cdots \int_{|w_1 - a_d| = r} \frac{f(w)\, \mathrm{d}w_1 \ldots \mathrm{d}w_d}{(w_1 - z_1) \cdots (w_d - z_d)} \qquad (z \in D)$$

is a well-defined E-valued holomorphic function on D. Indeed, g is certainly holomorphic in each variable separately and each partial derivative is continuous. Hence one can apply [17, XIII, Thm. 7.1].

Composing with $x' \in N$ yields, by the scalar Cauchy formula in each variable separately, the identity

$$x'(g(z)) = x'(f(z)) \quad (z \in D,\ x' \in N).$$

Since N is norming (in particular: separating), it follows that $g = f$ on D. Hence, f is holomorphic on D. This implies (i). □

Remarks 2.3. 1) For univariate functions, Theorem 2.1 is well-known, see [3, Appendix A]. For scalar-valued functions, the equivalence (i) ⇔ (ii) is called Osgood's lemma; the (a priori) stronger equivalence (i) ⇔ (iii) is [16, Thm. 1.3].

2) Many books start from a different definition of holomorphy than ours and do not even mention Fréchet-differentiability. A noteworthy exception is [14]. See also [14, Theorem 160] for further characterizations of holomorphy.

3) Assertion (iii) involving a norming subset is due to Grothendieck [11], see also [15, p.139]. It implies (via the Hahn–Banach theorem) the following equivalences:

a) A function $f : O \to E$ is holomorphic if and only if it is weakly holomorphic (Dunford's theorem, see [14, Thm. 148, p.68]).

b) A function $T : O \to \mathcal{L}(E; F)$ is holomorphic if and only if it is strongly holomorphic, if and only if for each $x \in E$ and each $x' \in N$ from a norming subset $N \subseteq F'$ the function $\langle F(\cdot)x, x' \rangle$ is holomorphic.

4) Theorem 2.1 still holds if N is merely a *separating* subset of E'. This follows from the analogous result for univariate functions, due to Grosse-Erdmann [10]. Different proofs have been given by Arendt and Nikolski [4, Thm. 3.1] (see also [2]) and Grosse-Erdmann [9, Thm. 1]. For more general results in this direction see Frerick, Jordá and Wengeroth [8], in particular their Theorem 3.2.

5) One may replace (ii) by the weaker assertion

(ii)' f is weakly separately holomorphic.

This follows from Hartogs' theorem, which says that a scalar multi-variable function is already holomorphic if it is merely separately holomorphic [14, Thm. 153, p.69].

It follows easily from the Cauchy integral formula that each partial derivative $\frac{\partial}{\partial z_j} f$ of a holomorphic function $f : O \to E$ is again holomorphic. Iterating this yields for $\alpha \in \mathbb{N}_0^d$ the holomorphic function

$$D^\alpha f := \prod_{j=1}^{d} \frac{\partial^{\alpha_j}}{\partial z_j^{\alpha_j}} f,$$

and one has the well-known Cauchy integral formula for derivatives. From there, the following lemma is straightforward.

Lemma 2.4. *Let $f : O \to E$ holomorphic and $T : E \to F$ bounded and linear. Then $T \circ f$ is holomorphic and $D^\alpha(T \circ f) = T \circ D^\alpha f$.*

In the case that $f = f(z,t) : O \times \Omega \to \mathbb{C}$ depends on an additional parameter $t \in \Omega$, we shall write

$$D_z^\alpha f(a,t) \mapsto D^\alpha(f(\cdot,t))(a) \qquad (a,t) \in O \times \Omega.$$

We write $\mathrm{H}^\infty(O;E)$ for the space of all bounded holomorphic E-valued functions on O. If we endow O with the Borel σ-algebra, $\mathrm{H}^\infty(O;E)$ becomes a (sequentially) bp-closed subspace of $\mathrm{BM}(O;E)$. From the Cauchy integral formula it follows that for fixed $a \in O$ and $\alpha \in \mathbb{N}_0^d$ the mapping

$$\mathrm{H}^\infty(O;E) \to E, \qquad f \mapsto D^\alpha f(a)$$

is bp-continuous.

3. The Linearization Theorem

The following theorem is a corollary of Mujica's linearization theorem [19, Thm. 2.1], however with a different proof (see Remarks 3.2 below).

Theorem 3.1 (Linearization). *Let $O \subseteq \mathbb{C}^d$ be open, E a Banach space and $f \in \mathrm{H}^\infty(O;E)$. Then for each bp-continuous functional $\varphi \in \mathrm{H}^\infty(O)'$ there is a unique element $\varphi_f \in E$ such that*

$$\langle \varphi_f, x' \rangle = \langle x' \circ f, \varphi \rangle \qquad (x' \in E').$$

Moreover, $\varphi_f \in \overline{\mathrm{span}} f(O)$.

Proof. Uniqueness and the second assertion follow from the Hahn–Banach theorem. For existence we may suppose without loss of generality that $E = \overline{\mathrm{span}} f(O)$. As f is continuous and O is separable, so is E. Define the linear functional $\varphi_f : E' \to \mathbb{C}$ by

$$\varphi_f(x') := \varphi(z \mapsto (x' \circ f)(z)) \qquad (x' \in E').$$

We need to prove that $\varphi_f \in E$ under the natural embedding $E \hookrightarrow E''$. Since E is separable, the weak*-topology on the unit ball of E' is metrizable. Since φ is bp-continuous, φ_f is weakly* sequentially continuous on the unit ball of E', and hence weakly* continuous there. Then, by a well-known theorem of Banach (see [21, Lemma 1.2] for an elegant proof), it follows that $\varphi_f \in E$ as claimed. □

Remarks 3.2. 1) As it stands, Theorem 3.1 is a consequence of Mujica's linearization theorem [19, Theorem 2.1]. There, Theorem 3.1 is stated for open $O \subseteq F$, where F is any Banach space, and $\varphi \in \mathrm{G}^\infty(O)$. Here, $\mathrm{G}^\infty(O)$ is the space of all linear functionals which on bounded subsets of $\mathrm{H}^\infty(O)$ are continuous with respect to τ_c, the topology of uniform convergence on compacts.

Now, since any open set $O \subseteq \mathbb{C}^d$ is locally compact and σ-compact, the topology τ_c is metrizable, and hence on bounded subsets of $\mathrm{H}^\infty(O)$, bp-continuity and τ_c-continuity of a functional coincide.

2) Theorem 3.1 remains true (with essentially the same proof) under the following weaker hypotheses: F is any Banach space, $O \subseteq F$ open, and $f \in \mathrm{H}^\infty(O; E)$ is separably-valued.

Under these hypotheses, a bp-continuous functional may not be τ_c-continuous on bounded sets, and hence the result may then not be covered by Mujica's theorem. (Unfortunately, we do not know of an example showing that this is indeed the case.)

3) Our proof of Theorem 3.1 is more elementary than Mujica's from [19], as it avoids the bipolar theorem and is built just on Banach's theorem (which has a much more elementary proof). Moreover, a slight modification of our argument also works in the more general setting of Mujica's theorem: Namely, if $(x'_\alpha)_\alpha$ is a norm-bounded net and weakly*-convergent to $x' \in E'$, then the net $(x'_\alpha \circ f)_\alpha$ is τ_c-convergent to $x' \circ f$, by equicontinuity.

Using this argument leads to a simplification of the proof of Mujica's theorem. However, it should be noted that the linearization result is only one (although essential) part of Mujica's original theorem from [19].

The proof of Theorem 3.1 is primarily functional-analytic. It may be interesting to see that one can alternatively employ concepts from measure theory. Here, we suppose in addition that φ is integration with respect to a measure. (Cf., however, Remark 3.3 below.)

Second Proof of Theorem 3.1. Let the functional $\varphi : \mathrm{H}^\infty(O) \to \mathbb{C}$ be given by integration against a complex Borel measure ν. Since f is holomorphic, it is weakly holomorphic and has values in a separable subspace. Hence, by Pettis' measurability theorem, f is strongly $|\nu|$-measurable. Since f is bounded, it is Bochner integrable with respect to $|\nu|$, and hence the Bochner integral

$$h := \int_O f \, \mathrm{d}\nu \in E$$

exists. Applying $x' \in E'$ yields

$$\langle h, x' \rangle = \int_O \langle f(\cdot), x' \rangle \, \mathrm{d}\nu = \varphi(z \mapsto \langle f(z), x' \rangle). \qquad \square$$

Remark 3.3. The assumption that the bp-continuous functional $\varphi \in \mathrm{H}^\infty(O)'$ is integration with respect to a measure is only virtually restrictive. Indeed, it turns out that each bp-continuous functional on $\mathrm{H}^\infty(O)$ is already given by integration against some finite measure, and even one that has a density with respect to Lebesgue measure. This follows from identifying the space of bp-continuous functionals on $\mathrm{H}^\infty(O)$ with the dual of $\mathrm{H}^\infty(O)$ with respect to the so-called *mixed topology*, see [6, Chap. V.1], in particular part 5) of Proposition 1.1 and Proposition 1.2. (Actually, Cooper [6] only treats the case $d = 1$, but we expect the same result for $d > 1$.)

4. Holomorphic Families of Measurable and Integrable Functions

The Linearization Theorem 3.1 takes the form of an abstract Fubini-type theorem if E is a space of functions on a set Ω. We exploit this idea first for Banach spaces of bounded functions. Note that every space $E = \mathrm{BM}(\Omega, \Sigma)$ for some σ-algebra Σ on a set Ω is a closed subspace of $\ell^\infty(\Omega)$.

Theorem 4.1. *Let Ω be a set and E a closed subspace of $\ell^\infty(\Omega)$, let $O \subseteq \mathbb{C}^d$ be open and $f : O \times \Omega \to \mathbb{C}$ a bounded function with the following properties:*

- *$F(z) = f(z, \cdot) \in E$ for each $z \in O$ and*
- *$f_t = f(\cdot, t)$ is holomorphic for each $t \in \Omega$.*

Finally, let $\varphi : \mathrm{H}^\infty(O) \to \mathbb{C}$ be a bp-continuous functional. Then the following assertions hold:

a) $F \in \mathrm{H}^\infty(O; E)$.

b) *For each multi-index $\alpha \in \mathbb{N}_0^d$ and each $a \in O$ one has*

$$(D^\alpha F)(a) = t \mapsto D_z^\alpha f(z,t)\Big|_{z=a} \in E$$

and, for all $\mu \in E'$, $\quad D_z^\alpha \langle t \mapsto f(z,t), \mu \rangle\Big|_{z=a} = \langle t \mapsto D_z^\alpha f(z,t)\Big|_{z=a}, \mu \rangle.$

c) *The function $t \mapsto \varphi(f_t)$ is contained in $\overline{\mathrm{span}}(F(O)) \subseteq E$.*

d) *For all $\mu \in E'$:* $\quad \langle t \mapsto \varphi(f_t), \mu \rangle = \varphi(z \mapsto \langle F(z), \mu \rangle)$.

Note that the second part of b) is of the type "differentiation under the integral" when one is inclined to interpret the application of μ as some kind of integral.

Proof of Theorem 4.1. Observe that the set of Dirac (= point evaluation) functionals $\{\delta_t \mid t \in \Omega\}$ is norming for E. Since F is bounded and (by hypothesis) the functions

$$z \mapsto \langle F(z), \delta_t \rangle = f(z,t) \qquad (t \in \Omega)$$

are all holomorphic, F is holomorphic by Theorem 2.1. It follows that $D^\alpha F$ takes values in E and

$$\langle (D^\alpha F)(z), \mu\rangle = D_z^\alpha \langle F(z), \mu\rangle$$

as scalar functions on O for each $\mu \in E'$ (Lemma 2.4). Specializing $\mu = \delta_t$ for $t \in \Omega$ yields $(D^\alpha F)(z)(t) = D_z^\alpha f(z,t)$, and hence b) is proved.

Next, we apply Theorem 3.1 to the mapping $F \in \mathrm{H}^\infty(O; E)$ and obtain, for any bp-continuous functional $\varphi \in \mathrm{H}^\infty(O)'$ an element $\varphi_F \in \overline{\operatorname{span}}(F(O))$ with

$$\langle \varphi_F, \mu\rangle = \varphi(z \mapsto \langle F(z), \mu\rangle) \qquad (\mu \in E').$$

By specializing $\mu = \delta_t$ for $t \in \Omega$, we find $\varphi_F = (t \mapsto \varphi(f_t))$. This concludes the proof of c) and d). $\square$

Next, we replace the *set* Ω by a *measure space* (Ω, Σ, μ). The corresponding L_p-space is denoted by $\mathrm{L}_p(\Omega)$, and p, q are always dual exponents, i.e., $1 \le p, q \le \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Before we turn to the main results, let us fix some auxiliary information.

Lemma 4.2. *Let (Ω, Σ) be a measurable space, $O \subseteq \mathbb{C}^d$ an open subset of $\mathbb{C}^d$ and*

$$f : O \times \Omega \to \mathbb{C}$$

a function with the following properties:

- *$F(z) := f(z, \cdot)$ is measurable for each $z \in O$ and*
- *$f_t := f(\cdot, t)$ is bounded and holomorphic for each $t \in \Omega$.*

Then the following assertions hold.

a) *For each $\alpha \in \mathbb{N}_0^d$ and $a \in O$, the function $D_z^\alpha f(a, \cdot)$ is measurable.*

b) *The function $t \mapsto \|f_t\|_\infty = \sup_{z \in O} |f(z,t)|$ is measurable.*

c) *For each bp-continuous functional $\varphi : \mathrm{H}^\infty(O) \to \mathbb{C}$ the function $t \mapsto \varphi(f_t)$ is measurable.*

Proof. a) By induction, it suffices to prove the statement for the case $|\alpha| = 1$. As a partial derivative is a limit of a sequence of difference quotients, the claim follows.

b) Note that, by hypothesis, $f_t \in \mathrm{H}^\infty(O)$, i.e., $\|f_t\|_\infty < \infty$ for each $t \in \Omega$. Let $D \subseteq O$ be a countable dense set. Then, since each f_t is continuous,

$$c(t) := \|f_t\|_\infty = \sup_{z \in O} |f(z,t)| = \sup_{z \in D} |f(z,t)| \qquad (t \in \Omega).$$

Hence $c : \Omega \to \mathbb{R}_+$ is a pointwise supremum of countably many measurable functions, and hence measurable.

c) Define

$$m(z,t) := \frac{f(z,t)}{1 + c(t)} \qquad (z \in O,\ t \in \Omega).$$

Then $|m(z,t)| \le 1$ for all $(z,t) \in O \times \Omega$. Hence, m meets the conditions of Theorem 4.1 with $E = \mathrm{BM}(\Omega,\Sigma)$, the space of bounded Σ-measurable functions on Ω. Given a bp-continuous functional φ on $\mathrm{H}^\infty(O)$ it follows that

$$t \mapsto \varphi(m_t) = \frac{\varphi(f_t)}{1+c(t)}$$

is measurable. As $1+c$ is measurable, so is $t \mapsto \varphi(f_t)$. □

The next is our main theorem. Observe that assertion d) is an abstract Fubini-type result as in (1.1).

Theorem 4.3. *Let (Ω,Σ,μ) be a measure space, $1 \le p < \infty$, $O \subseteq \mathbb{C}^d$ an open subset of $\mathbb{C}^d$ and*

$$f : O \times \Omega \to \mathbb{C}$$

a function with the following properties:

- $\forall z \in O:\ F(z) := f(z,\cdot)$ *is measurable and* $\sup_{z\in O} \int_\Omega |f(z,t)|^p \, \mu(\mathrm{d}t) < \infty$;
- $\forall t \in \Omega:\ f_t := f(\cdot,t)$ *is bounded and holomorphic.*

Fix $\alpha \in \mathbb{N}_0^d$, $a \in O$, and a bp-continuous linear functional $\varphi : \mathrm{H}^\infty(O) \to \mathbb{C}$. Then the following assertions hold:

a) $F \in \mathrm{H}^\infty(O;\mathrm{L}_p(\Omega))$ *and* $D^\alpha F(a) = D_z^\alpha f(a,\cdot)$ *μ-almost everywhere.*

b) *For each $h \in \mathrm{L}_q(\Omega)$ the function* $z \mapsto \int_\Omega f(z,t)h(t)\,\mu(\mathrm{d}t)$ *is holomorphic and*

$$D^\alpha\Big(z \mapsto \int_\Omega f(z,t)h(t)\,\mu(\mathrm{d}t)\Big) = z \mapsto \int_\Omega D_z^\alpha f(z,t)h(t)\,\mu(\mathrm{d}t).$$

c) *The measurable function $(t \mapsto \varphi(f_t)) : \Omega \to \mathbb{C}$ is p-integrable with*

$$\Big(\int_\Omega |\varphi(f_t)|^p \, \mu(\mathrm{d}t)\Big)^{\frac{1}{p}} \le \|\varphi\| \sup_{z\in O} \|F(z)\|_p,$$

and it is contained in the subspace $\overline{\mathrm{span}} F(O) \subseteq \mathrm{L}_p(\Omega)$.

d) $\int_\Omega \varphi(f_t)h(t)\,\mu(\mathrm{d}t) = \varphi\Big(z \mapsto \int_\Omega f(z,t)h(t)\,\mu(\mathrm{d}t)\Big)$ *for all* $h \in \mathrm{L}_q(\Omega)$.

Proof. (1) Observe that f meets the conditions of Lemma 4.2 and hence that the functions

$$t \mapsto D_z^\alpha f(a,t), \quad t \mapsto \|f_t\|_\infty \quad \text{and} \quad t \mapsto \varphi(f_t)$$

are measurable.

(2) Let $D \subseteq O$ be a countable dense set. Then, since each f_t is continuous,

$$\bigcup_{z\in O} [\,F(z) \neq 0\,] = \bigcup_{z\in D} [\,F(z) \neq 0\,],$$

where $[\,F(z) \neq 0\,] = \{t \in \Omega \mid f(z,t) \neq 0\}$. The right-hand side is a σ-finite subset of Ω. Hence, we may suppose that μ is σ-finite. Accordingly, we fix measurable subsets $\Omega_n \subseteq \Omega$ with $\mu(\Omega_n) < \infty$ and $\Omega_n \nearrow \Omega$.

(3) For each $n \in \mathbb{N}$ let

$$f_n : O \times \Omega \to \mathbb{C}, \quad f_n(z,t) := \frac{n}{n + \|f_t\|_\infty} f(z,t) \cdot \mathbf{1}_{\Omega_n}(t) \qquad (z \in O,\, t \in \Omega).$$

Then f_n is bounded, measurable in the second and holomorphic in the first variable. Applying Theorem 4.1 with $E = E_n := \{g \in \mathrm{BM}(\Omega) \mid g\mathbf{1}_{\Omega_n^c} = 0\}$ we conclude that the function

$$F_n : O \to E_n, \qquad F_n(z) := f_n(z, \cdot)$$

is bounded and holomorphic and

$$D^\alpha F_n(a)(t) = D_z^\alpha f_n(a,t) = \frac{n}{n + \|f_t\|_\infty} D_z^\alpha f(a,t) \cdot \mathbf{1}_{\Omega_n}(t) \qquad (t \in \Omega).$$

Since $\mu(\Omega_n) < \infty$, $E_n \subseteq \mathrm{L}_p(\Omega)$ continuously and hence $F_n \in \mathrm{H}^\infty(O; \mathrm{L}_p(\Omega))$. Clearly, the sequence $(F_n)_n$ bp-converges to F, and hence $F \in \mathrm{H}^\infty(O; \mathrm{L}_p(\Omega))$. Moreover, for each $a \in O$

$$D^\alpha F_n(a) \to D^\alpha F(a) \quad \text{in } \mathrm{L}_p \text{ and } \quad D_z^\alpha f_n(a,\cdot) \to D_z^\alpha f(a,\cdot) \quad \text{pointwise.}$$

It follows that $D^\alpha F(a) = D_z^\alpha f(a,\cdot)$ almost everywhere and the proof of a) is complete.

(4) Assertion b) follows from a) and Lemma 2.4 on noting that integration against $h \in \mathrm{L}_q(\Omega)$ is a bounded linear functional on $\mathrm{L}_p(\Omega)$.

(5) Since Ω_n has finite measure, for each $h \in \mathrm{L}_q(\Omega)$, the map $g \mapsto \int_{\Omega_n} g\, h \,\mathrm{d}\mu$ defines a bounded linear functional on $\mathrm{BM}(\Omega)$. Hence, by d) of Theorem 4.1 applied to f_n,

$$\int_{\Omega_n} \frac{n}{n + \|f_t\|_\infty} \varphi(f_t) h(t)\, \mu(\mathrm{d}t) = \varphi\Big(z \mapsto \int_{\Omega_n} \frac{n}{n + \|f_t\|_\infty} f(z,t) h(t)\, \mu(\mathrm{d}t)\Big).$$

Varying h we arrive at

$$\int_{\Omega_n} \Big(\frac{n}{n + \|f_t\|_\infty}\Big)^p |\varphi(f_t)|^p \,\mu(\mathrm{d}t) \le \|\varphi\|^p \sup_{z \in O} \|F(z)\|_p^p < \infty.$$

When $n \to \infty$ it follows that $(t \mapsto \varphi(f_t)) \in \mathrm{L}_p(\Omega)$ and

$$\|t \mapsto \varphi(f_t)\|_p \le \|\varphi\| \sup_z \|F(z)\|_p.$$

Moreover, by the bp-continuity of φ,

$$\int_\Omega \varphi(f_t) h(t)\, \mu(\mathrm{d}t) = \varphi\Big(z \mapsto \int_\Omega f(z,t) h(t)\, \mu(\mathrm{d}t)\Big),$$

which is d). The remaining part of c) is a consequence of d) and the Hahn–Banach theorem. □

The preceding theorem just covers the case $1 \le p < \infty$. A result for $p = \infty$ needs a special assumption on the measure space. A measure space (Ω, Σ, μ) is called *semi-finite* if for every $B \in \Sigma$ with $\mu(B) = +\infty$ there exists some $A \in \Sigma$, $A \subseteq B$ such that $0 < \mu(A) < \infty$. It is a well-known (and easy-to-prove) fact that the unit ball of $\mathrm{L}_1(\Omega)$ is a norming set for $\mathrm{L}_\infty(\Omega)$ if and only if (Ω, Σ, μ) is semi-finite. Moreover, on a semi-finite measure space, a measurable function f vanishes almost everywhere if and only if for each $g \in \mathrm{L}_1(\Omega)$, the product fg vanishes almost everywhere.

Theorem 4.4. *Let (Ω, Σ, μ) be a semi-finite measure space, $O \subseteq \mathbb{C}^d$ an open subset of $\mathbb{C}^d$ and*

$$f : O \times \Omega \to \mathbb{C}$$

a function with the following properties:

- $\forall z \in O :\ F(z) := f(z, \cdot)$ *is measurable and* $\sup_{z \in O} \operatorname{ess.sup}_{t \in \Omega} |f(z,t)| < \infty$;
- $\forall t \in \Omega :\ f_t := f(\cdot, t)$ *is bounded and holomorphic.*

Fix $\alpha \in \mathbb{N}_0^d$, $a \in O$, and a bp-continuous linear functional $\varphi : \mathrm{H}^\infty(O) \to \mathbb{C}$. Then the following assertions hold:

a) $F \in \mathrm{H}^\infty(O; \mathrm{L}_\infty(\Omega))$ *and* $D^\alpha F(a) = D_z^\alpha f(a, \cdot)$ *μ-almost everywhere.*

b) *For each $g \in \mathrm{L}_1(\Omega)$ the function $z \mapsto \int_\Omega f(z,t) g(t)\, \mu(\mathrm{d}t)$ is holomorphic and*

$$D^\alpha \Big(z \mapsto \int_\Omega f(z,t) g(t)\, \mu(\mathrm{d}t) \Big)(a) = \int_\Omega D_z^\alpha f(a,t) g(t)\, \mu(\mathrm{d}t).$$

c) *The measurable function $t \mapsto \varphi(f_t)$ is essentially bounded with*

$$\operatorname{ess.sup}_{t \in \Omega} |\varphi(f_t)| \leq \|\varphi\| \sup_{z \in O} \big\| F(z) \big\|_{\mathrm{L}_\infty(\Omega)},$$

and it is contained in the subspace $\overline{\operatorname{span} F(O)}^{\sigma(\mathrm{L}_\infty, \mathrm{L}_1)} \subseteq \mathrm{L}_\infty(\Omega)$.

d) $\int_\Omega \varphi(f_t) g(t)\, \mu(\mathrm{d}t) = \varphi\Big(z \mapsto \int_\Omega f(z,t) g(t)\, \mu(\mathrm{d}t) \Big)$ *for all $g \in \mathrm{L}_1(\Omega)$.*

Proof. We start by noticing as in the proof of Theorem 4.3, that by Lemma 4.2 the functions $t \mapsto D_z^\alpha f(a,t), \varphi(f_t)$ are measurable.

For any $g \in \mathrm{L}_1(\Omega)$, the function $(z,t) \mapsto f(z,t) g(t)$ satisfies the assumptions of Theorem 4.3, for the case p=1. Hence b) of that theorem (with $h = \mathbf{1}$) is just the same as b) of the present theorem. Recall that, since (Ω, Σ, μ) is semi-finite, the unit ball of $\mathrm{L}_1(\Omega)$ is norming for $\mathrm{L}_\infty(\Omega)$. Since $F : O \to \mathrm{L}_\infty(\Omega)$ is bounded, it follows from b) and Theorem 2.1 that $F \in \mathrm{H}^\infty(O; \mathrm{L}_\infty(\Omega))$.

From part b) of Theorem 4.3 it follows that $D^\alpha F(a) g = D_z^\alpha f(a, \cdot) g$ μ-almost everywhere. As $g \in \mathrm{L}_1(\Omega)$ is arbitrary here and $D_z^\alpha f(a, \cdot)$ is measurable, this implies that $D^\alpha F(a) = D_z^\alpha f(a, \cdot)$ μ-almost everywhere. Hence, a) is proved.

Part c) of Theorem 4.3 implies that $t \mapsto \varphi(f_t) g(t)$ is integrable with

$$\int_\Omega |\varphi(f_t) g(t)|\, \mu(\mathrm{d}t) \leq \|\varphi\| \sup_z \|F(z) g\|_{\mathrm{L}_1} \leq \|\varphi\| \sup_z \|F(z)\|_{\mathrm{L}_\infty} \cdot \|g\|_{\mathrm{L}_1}.$$

Varying g yields that $t \mapsto \varphi(f_t)$ is essentially bounded with

$$\operatorname{ess.sup}_{t \in \Omega} |\varphi(f_t)| \leq \|\varphi\| \sup_z \|F(z)\|_{\mathrm{L}_\infty},$$

which is part c) halfway. Again by Theorem 4.3 d) applied with $h = \mathbf{1}$, yields d). Finally, it follows from d) and a standard application of the Hahn–Banach theorem that the function $t \mapsto \varphi(f_t)$ (as an element of $\mathrm{L}_\infty(\Omega)$) is contained in the $\sigma(\mathrm{L}_\infty, \mathrm{L}_1)$-closure of span$F(O)$. This concludes the proof. □

Remarks 4.5. 1) For d=1 and p=1, some parts of Theorem 4.3 have been proved by Mattner in [18]. Mattner also has shown in [18, Counterexample 1] that if one replaces the first condition in Theorem 4.3 by the weaker one

$$\int_\Omega |f(z,t)|\, \mu(\mathrm{d}t) < \infty \quad \text{for all } z \in O,$$

then the function $z \mapsto \int_\Omega f(z,t)\, \mu(\mathrm{d}t)$ need not be continuous, let alone holomorphic. In particular, assertion a) may fail.

In Section 5 below, we shall review Mattner's results from [18] and relate them to ours.

2) The following result by Stein tells us that *each* holomorphic L_p-valued function arises in the way considered in Theorem 4.3.

Theorem (Stein). *Let X be a complex Banach space, (Ω, Σ, μ) a σ-finite measure space, $1 \le p < \infty$, and $O \subseteq \mathbb{C}$ an open set. Let $F : O \to \mathrm{L}_p(\Omega; X)$ be a holomorphic function. Then there exists a function $f : O \times \Omega \to X$ such that*

- *f is strongly (product) measurable;*
- *for every $t \in \Omega$, $f(\cdot, t)$ is holomorphic;*
- *for every $z \in O$, $f(z, \cdot) = F(z)$ almost everywhere.*

This theorem goes back to the lemma on page 72 of Stein's book [20] for the special case of orbits of holomorphic semigroups on sectors. Desch and Homan in [7] proved the theorem in full generality and with all the details.

5. Mattner's results

Theorem 4.3 and in particular the Fubini-type result in assertion d) may be surprising on first glance, since the joint measurability of the function f is not assumed (and not needed in the proof). However, as the following result shows, joint measurability is automatic, at least in the case we consider here $(O \subseteq \mathbb{C}^d)$.

Lemma 5.1 (Mattner [18, p.33]**).** *Let O be a second countable topological space, let (Ω, Σ) be a measurable space and let X be a metric space. Furthermore, let*

$$f : O \times \Omega \to X$$

*be a mapping such that $f(\cdot, t) : O \to X$ is continuous for each $t \in \Omega$ and $f(z, \cdot) : \Omega \to X$ is measurable Σ-to-*Borel*. Then f is measurable* (Borel $\otimes\, \Sigma$)*-to-*Borel.

Remark 5.2. Joint measurability results appear to have a long tradition, see e.g. [11, p.42], [5], [1, Lemma 4.51]. However, they are seldom mentioned in courses on measure theory and not widely known. Mattner's version appears to be the strongest so far where measurability in one variable is paired with continuity in the other.

With the help of Lemma 5.1, Mattner in [18] established the following theorem (slightly adapted notationally).

Theorem 5.3 (Mattner [18, p.32]**).** *Let (Ω, Σ, μ) be a measure space, let $\emptyset \neq O \subseteq \mathbb{C}$ be open, and let $f : O \times \Omega \to \mathbb{C}$ be a function subject to the following assumptions:*

[A1] *$f(z, \cdot)$ is Σ-measurable for every $z \in O$,*

[A2] *$f(\cdot, t)$ is holomorphic for every $t \in \Omega$,*

[A3] *$\int |f(\cdot, t)| \mu(\mathrm{d}t)$ is locally bounded.*

Then $z \mapsto \int_\Omega f(z,t)\, \mu(\mathrm{d}t)$ is holomorphic and may be differentiated under the integral. More precisely, for each $n \in \mathbb{N}_0$:

[C1] *$D_z^n f$ is $\mathrm{Borel}(O) \otimes \Sigma$-measurable and, for every $\emptyset \neq A \subseteq O$, the function $t \mapsto \sup_{z \in A} |D_z^n f(z,t)|$ is Σ-measurable,*

[C2] *If $K \subseteq O$ is compact, then $\sup_{z \in K} \int_\Omega |D_z^n f(z,t)|\, \mu(\mathrm{d}t) < \infty$.*

[C3] *$z \mapsto \int_\Omega f(z,t)\, \mu(\mathrm{d}t)$ is holomorphic on O with*

$$D_z^n \int_\Omega f(z,t)\, \mu(\mathrm{d}t) = \int_\Omega D_z^n f(z,t)\, \mu(\mathrm{d}t).$$

Assertion 3 in Theorem 5.3 is covered (literally) by part b) of Theorem 4.3. The joint measurability assertion in C1 follows directly from a) and Lemma 5.1; and the remaining part of C1 follows since the supremum is effectively a supremum over a countable subset of A. (Mattner employs the same argument). Finally, C2 is a straightforward consequence of a) and the following general result.

Lemma 5.4. *Let $O \subseteq \mathbb{C}^d$ open, E a Banach lattice and $F : \Omega \to E$ holomorphic. Then for each compact $K \subseteq O$ the set $F(K)$ is order bounded in E, i.e., there is $0 \leq u \in E$ such that $|F(z)| \leq u$ for all $z \in K$.*

Proof. We only treat the case $d = 1$, the general case being analogous (but more technical to write down). By compactness of K it suffices to prove that each point $a \in O$ has a neighborhood U_a such that $F(U_a)$ is order-bounded. Without loss of generality, $a = 0$. Write F as a convergent power series

$$F(z) = \sum_{n=0}^{\infty} u_n z^n \qquad (|z| < r)$$

for some $r > 0$ and $u_n \in E$. Making r smaller if necessary, we have

$$\sum_{n=0}^{\infty} \|u_n\| r^n < \infty.$$

The series $u := \sum_{n=0}^{\infty} |u_n| r^n$ is (norm-absolutely) convergent and we obtain for $|z| < r$ that $|F(z)| \le u$, as claimed. □

Alternatively, one may prove Lemma 5.4 by means of the Cauchy formula, and this is exactly what Mattner does in [18] for $E = \mathrm{L}_1(\Omega)$.

Remark 5.5. By adapting the proof, Lemma 5.4 can be generalized from Banach lattices to ordered Banach spaces with generating positive cone, but we refrain from doing so here. The result may be known, but we do not know of a direct reference, hence we have included the simple proof. Basically, the argument is present in the proof of [20, Lemma on p.72], already mentioned in Remark 4.5, 2).

In the remainder of this section, we show how Theorem 4.3 can be derived from Theorem 5.3, as long as one supposes in addition that φ is integration against a finite measure. Since the latter is no restriction, by Remark 3.3, this yields an alternative proof of Theorem 4.3 for the case p=1. (With a little more effort, one would also obtain a proof for the case $p \ge 1$.) Although Mattner only treated the single-variable case, we expect his approach to work also for functions of several variables.

Sketch of an alternate proof of Theorem 4.3 building on [18]. As already said, our point of departure is as follows: $p = 1$, f is already known to be joint measurable (Lemma 5.1) and μ is σ-finite (by the same argument as (1) in our original proof). Moreover, b) holds, i.e., the function $z \mapsto \int_\Omega F(z)h \,\mathrm{d}\mu$ is holomorphic for all $h \in \mathrm{L}_\infty(\Omega)$. Finally, φ is integration with respect to a complex measure ν.

From b) and the boundedness of F it follows from Theorem 2.1 that F is holomorphic. Then from b) and Lemma 2.4 it follows that $D^\alpha F$ is represented by $D_z^\alpha f(z, \cdot)$, and hence a) is proved.

Next, by Fubini-Tonelli, f is $|\nu| \times \mu$-integrable and we can interchange the order of integration. This yields the first part of c) (measurability and integrability of $t \mapsto \varphi(f_t)$ as well as the norm estimate) and d).

The remaining part of c) is now proved exactly as in the original proof. □

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