EXACT OBSERVABILITY OF A 1D WAVE EQUATION ON A NON-CYLINDRICAL DOMAIN

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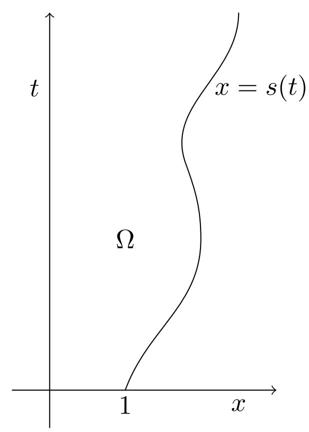
ABSTRACT. We discuss admissibility and exact observability estimates of boundary observation and interior point observation of a one-dimensional wave equation on a time dependent domain for sufficiently regular boundary functions. We also discuss moving observers inside the non-cylindrical domain and simultaneous observability results.

1. INTRODUCTION AND MAIN RESULTS

In this article we are concerned with exact observability of the 1D wave equation on a domain with time-dependent boundary. To be precise, let $s : \mathbb{R}_+ \to (0, \infty)$ and let

$$\Omega = \{(x, t) \in \mathbb{R}^2 : \quad t \geq 0 \text{ and } 0 \leq x \leq s(t)\},$$

Where $s(0) = 1$ and $\|s'(t)\|_{L_\infty(\mathbb{R})} < 1$. The last condition ensures amongst other things that the characteristic emerging from the origin hits the boundary in finite time. Let $f \in L_2([0, 1])$ and $g \in H_0^1([0, 1])$ be initial values. We consider a wave equation on Ω with Dirichlet boundary conditions



$$\text{(W.Eq)} \qquad \begin{cases} u_{tt} - u_{xx} = 0 & (x, t) \in \Omega \\ u(0, t) = u(s(t), t) = 0 & t \geq 0 \\ u(x, 0) = g(x) & x \in [0, 1] \\ u_t(x, 0) = f(x) & x \in [0, 1] \end{cases}$$

1.1. **Existence of solutions.** There are several natural approaches to (W.Eq). One may for example transform the domain Ω to a cylindrical domain. Instead, seeking a natural and more simple approach, we try to develop the solution u into a series of the form

$$u(x, t) := \sum_{n \in \mathbb{Z}} A_n \Big(e^{2\pi in\, \varphi(t+x))} - e^{2\pi in\, \varphi(t-x)} \Big) \tag{1.1}$$

where the coefficients A_n are given by the initial data (g, f). This approach has almost a century of history, dating back to Nicolai [32] in the case of a linear moving boundary $s(t) = 1 + \varepsilon t$ and Moore [30] for general boundary curves (however only asymptotic developments for φ are given). We refer to Donodov [15, 14] for a large number of references. In order to satisfy the Dirichlet boundary condition, we need a solution φ to the functional equation

$$\varphi(t + s(t)) - \varphi(t - s(t)) = 1. \tag{1.2}$$

Because of the importance of this functional equation we fix the notation $\alpha(t) := t + s(t)$ and $\beta(t) := t - s(t)$ and mention that both are strictly increasing bijections from $\mathbb{R}_+$ to $[\pm s(0), \infty)$, respectively. We will also consider $\gamma = \alpha \circ \beta^{-1} : [-s(0), \infty) \to [s(0), \infty)$. Most solutions to (1.2) are useless for our purposes*. On the other hand side, under reasonable assumptions on the boundary function, differentiable solutions to (1.2) are unique, at least up to an additive constant.

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*It is indeed easy to construct solutions depending on an arbitrary function by using the axiom of choice

This is of course what we look for. In some easy cases a differentiable solution φ can be found by calculus, see the following table for some examples. We refer to a detailed discussion on the general situation in the appendix A.

Name	Boundary function		Solution to (1.2)
linear moving boundary	$s(t) = 1 + \varepsilon t$	$\varepsilon \in (0,1)$	$\varphi(t) = \ln(\frac{1+\varepsilon}{1-\varepsilon})^{-1} \ln(1+\varepsilon t)$
parabolic boundary	$s(t) = \sqrt{1+\varepsilon t}$	$\varepsilon \in (0,2)$	$\varphi(t) = \frac{1}{2\varepsilon}\sqrt{\varepsilon^2 + 4\varepsilon t + 4}$
hyperbolic boundary	$s(t) = \frac{1}{\varepsilon}(-1 + \sqrt{1+(1+\varepsilon t)^2})$	$\varepsilon > 0$	$\varphi(t) = \frac{\varepsilon t}{1+\varepsilon t}$
shrinking domain	$s(t) = \frac{1}{1+\varepsilon t}$	$\varepsilon \in (0,1)$	$\varphi(t) = \frac{\varepsilon}{4}(t + \frac{1}{\varepsilon})^2$.

For simplicity of notation, we shall always assume $s(0) = 1$; in case of hyperbolic boundaries some straight-forward modifications have to be made. The common denominator of these examples is the following: $\varphi \in \mathrm{C}^2([-1,\infty))$ and $\varphi'(t) > 0$ for all $t \geq -1$. We call s an **admissible boundary function** if (1.2) admits such a solution φ.

Proposition 1.1. *Let s be an admissible boundary function and assume the initial data $f, g \in \mathscr{D}((0,1))$. Then (g,f) determine uniquely a sequence $(A_n)_{n\in\mathbb{Z}} \in \ell_2$ such that for $t \geq 0$ and $0 \leq x \leq s(t)$, the function (1.1) is the solution of the moving boundary wave equation* (W.Eq)*.*

We start the proof with the following trivial observation.

Lemma 1.2. *For fixed $t_0 \geq 0$, the family $\{e^{2\pi in\,\varphi(x)} : n \in \mathbb{Z}\}$, is a complete orthonormal system in $H := L_2([t_0 - s(t_0), t_0 + s(t_0)], \varphi'(x)\,\mathrm{d}x)$.*

For t_0=0, we obtain as a particular case that the family (b_n) with $b_n(x) = e^{2\pi in\,\varphi(x)}$ is an orthonormal basis in $H := L_2([-1,1], \varphi'(x)\,\mathrm{d}x)$. Since there is $C > 0$ such that $\frac{1}{C} \leq \varphi'(x) \leq C$ on $[0,1]$, we have $L_2([-1,1], \varphi'(x)\,\mathrm{d}x) = L_2([-1,1], \mathrm{d}x)$ as sets with equivalent respective norms†.

Proof of Proposition 1.1 . We let $F(x) = -\int_x^1 f(s)\,\mathrm{d}s$ and

$$h(x) := \begin{cases} \frac{1}{2}g(x) + \frac{1}{2\varphi'(0)}F(x) & \text{for} \quad 0 \leq x \leq 1 \\ -\frac{1}{2}g(-x) + \frac{1}{2\varphi'(0)}F(-x) & \text{for} \quad -1 \leq x < 0 \end{cases}$$

By assumption, $h \in H$ that we develop into the orthonormal basis: $h = \sum_{\mathbb{Z}} \langle h, b_n \rangle\, b_n$. We shall always note

$$A_n = \langle h, b_n \rangle = \int_{-1}^{1} h(x) e^{2\pi in\,\varphi(x)} \varphi'(x)\,\mathrm{d}x \tag{1.3}$$

Since $g(0)$=$g(1)$=0, we have $h(1)$=$h(-1)$=0. Hence the sequences (A_n) and $(n\,A_n)$ are square-summable. Taking sum and difference, we find $F(x) = \varphi'(0)(h(x) + h(-x))$ and $g(x) = h(x) - h(-x)$, so

$$F(x) = \varphi'(0) \sum_{n\in\mathbb{Z}} A_n \Big(e^{2\pi in\varphi(x)} + e^{2\pi in\varphi(-x)} \Big), \qquad x \in [0,1]$$

and

$$g(x) = \sum_{n\in\mathbb{Z}} A_n \Big(e^{2\pi in\varphi(x)} - e^{2\pi in\varphi(-x)} \Big), \qquad x \in [0,1].$$

Since we suppose $f, g \in \mathscr{D}((0,1))$, h satisfies the periodicity condition $h^{(\alpha)}(-1)$=$h^{(\alpha)}(1)$ for all derivative orders $\alpha \geq 0$. As a consequence, the series of F, g and h above may be differentiated term by term. We let

$$u(x,t) := \sum_{n\in\mathbb{Z}} A_n \Big(e^{2\pi in\,\varphi(t+x))} - e^{2\pi in\,\varphi(t-x))} \Big)$$

†In particular, (b_n) is a Riesz basis in $L_2([-1,1])$.

Since $\varphi \in \mathrm{C}^2([-1,\infty))$, u is twice differentiable and with respect to x and t. Moreover, partial derivatives can be calculated term by term. As an immediate consequence, $u_{xx} - u_{tt} = 0$ in the interior domain Ω°. Moreover, u satisfies the Dirichlet condition since for $x = 0$

$$u(0,t) = \sum_{n\in\mathbb{Z}} A_n \left(e^{2\pi i n\, \varphi(t))} - e^{2\pi i n\, \varphi(t))} \right) = 0$$

whereas for $x = s(t)$, thanks to the functional equation (1.2),

$$\begin{aligned} u(s(t),t) &= \sum_{n\in\mathbb{Z}} A_n \left(e^{2\pi i n\, \varphi(t+s(t))} - e^{2\pi i n\, \varphi(t-s(t))} \right) \\ &= \sum_{n\in\mathbb{Z}} A_n e^{2\pi i n\, \varphi(t+s(t))} \left(1 - e^{-2\pi i n}\right) = 0. \end{aligned}$$

Finally, $u(x,0) = g(t)$ and $u_t(x,0) = f(t)$ by direct calculation. □

The series representation of the solution is the key to obtain explicit and precise constants for admissibility and exact observability in different situations, since they can be played back to classical Fourier analysis.
Let us fix some often appearing constants:

$$\begin{aligned} m(t) &= \min\{\varphi'(x) :\ x \in [t - s(t), t + s(t)]\} \qquad \text{and} \\ M(t) &= \max\{\varphi'(x) :\ x \in [t - s(t), t + s(t)]\}. \end{aligned} \tag{1.4}$$

Since on $[0,1]$, $m(0) \le \varphi'(x) \le M(0)$, we may use the unweighted Poincaré inequality on $[0,1]$ to show that

$$\left\|(g,f)\right\|^2_{H_0^1([0,1];\frac{\mathrm{d}x}{\varphi'(x)})\times L_2([0,1];\frac{\mathrm{d}x}{\varphi'(x)})} := \left\|\nabla g\right\|^2_{L_2([0,1];\frac{\mathrm{d}x}{\varphi'(x)})} + \left\|f\right\|^2_{L_2([0,1];\frac{\mathrm{d}x}{\varphi'(x)})}. \tag{1.5}$$

is an equivalent to $\|g\|^2_{L_2([0,1];\frac{\mathrm{d}x}{\varphi'(x)})} + \|g'\|^2_{L_2([0,1];\frac{\mathrm{d}x}{\varphi'(x)})} + \|f\|^2_{L_2([0,1];\frac{\mathrm{d}x}{\varphi'(x)})}$. The notation

$$\|(g,f)\|^2_{H_0^1\times L_2} := \|g'\|^2_{L_2(0,1)} + \|f\|^2_{L_2(0,1)}$$

(without specifying intervals or weights) always refers to the unweighted norms on $[0, s(0)] = [0,1]$.

Proposition 1.3. *We have the following estimate*

$$8\pi^2 m(0) \sum_{n\in\mathbb{Z}} n^2 |A_n|^2 \quad \le \quad \|(g,f)\|^2_{H_0^1\times L_2} \quad \le \quad 8\pi^2 M(0) \sum_{n\in\mathbb{Z}} n^2|A_n|^2,$$

where the constants are given by (1.4).

Proof. Recall that $g(x) = h(x) - h(-x)$ and $F(x) = h(x) + h(-x)$ on $[0,1]$. Therefore

$$\begin{aligned} \left\|(g,f)\right\|^2_{H_0^1\times L_2} &= \left\|g'\right\|^2_{L_2([0,1])} + \left\|F'\right\|^2_{L_2([0,1])} \\ &= \left\|h'(\cdot) + h'(-(\cdot))\right\|^2_{L_2([0,1])} + \left\|h'(\cdot) - h'(-(\cdot))\right\|^2_{L_2([0,1])} \\ &= 2\left\|h'\right\|^2_{L_2([0,1])} + 2\left\|h'(-\cdot)\right\|^2_{L_2([0,1])} = 2\left\|h'\right\|^2_{L_2([-1,1])} \end{aligned}$$

by parallelogram identity. Estimating the maximum of φ' and $\frac{1}{\varphi'}$ on $[-1,1]$ allows to relate $\left\|h'\right\|^2_{L_2([-1,1],\varphi'(x)\,\mathrm{d}x)}$ and $\left\|h'\right\|^2_{L_2([-1,1])}$, and the result follows by Parseval's identity. □

Observe that for the concrete examples we discuss later, the minimum respectively maximum is easy to calculate; we obtain therefore explicit constants in Proposition 1.3.

1.2. **Energy estimates.** Define the energy of the problem (W.Eq) as

$$E_u(t) = \tfrac{1}{2}\int_0^{s(t)} |u_x(x,t)|^2 + |u_t(x,t)|^2 \,\mathrm{d}x.$$

for all $t \geq 0$. When $t = 0$, we see that $E_u(0) = \frac{1}{2}\|(g,f)\|^2_{H_0^1\times L_2(0,1)}$. In the case of a 1D-wave equation with time-invariant boundary (i.e. $s \equiv 1$) the energy is constant. In time-dependent domains it decays when $s'(t) > 0$ and increases when $s'(t) < 0$.

Lemma 1.4. *The function $t \mapsto E_u(t)$ is decreasing for $t \geq 0$ if $s'(t) > 0$ and increasing when $s'(t) < 0$. More precisely,*

$$\tfrac{d}{dt}E_u(t) = \tfrac{s'(t)}{2}(s'(t)^2 - 1)\,|u_x(s(t),t)|^2. \tag{1.6}$$

Proof. Differentiating the constant zero function $u(s(t),t)$ with respect to t yields $u_t(s(t),t) = -s'(t)\,u_x(s(t),t)$. We use this twice in the following calculation.

$$\begin{aligned}
\tfrac{d}{dt}E_u(t) &= \tfrac{1}{2}s'(t)(u_t^2+u_x^2)\big|_{x=s(t)} + \tfrac{1}{2}\int_0^{s(t)} \tfrac{\partial}{\partial t}(u_t^2+u_x^2)\,\mathrm{d}x \\
&= \tfrac{s'(t)}{2}(1+s'(t)^2)\,(u_x^2)\big|_{x=s(t)} + \int_0^{s(t)} (u_t u_{tt} + u_x u_{tx})\,\mathrm{d}x \\
&= \tfrac{s'(t)}{2}(1+s'(t)^2)\,(u_x^2)\big|_{x=s(t)} + \int_0^{s(t)} (u_t u_{xx} + u_x u_{tx})\,\mathrm{d}x \\
\text{(integration by parts)}\quad &= \tfrac{s'(t)}{2}(1+s'(t)^2)\,(u_x^2)\big|_{x=s(t)} + \Big[u_t u_x\Big]_{x=0}^{x=s(t)} \\
&= \tfrac{s'(t)}{2}(1+s'(t)^2)\,(u_x^2)\big|_{x=s(t)} + u_t u_x\big|_{x=s(t)} \\
&= \tfrac{s'(t)}{2}(s'(t)^2-1)\,|u_x(s(t),t)|^2.
\end{aligned}$$

Recall that $\|s'\|_\infty < 1$ to conclude that $\mathrm{sign}(\frac{d}{dt}E_u(t)) = -\mathrm{sign}(s'(t))$. $\square$

Proposition 1.5. *For (W.Eq) the following energy estimate holds*

$$\tfrac{m(t)}{2M(0)}\,\big\|(g,f)\big\|^2_{H_0^1\times L_2} \quad\leq\quad E_u(t) \quad\leq\quad \tfrac{M(t)}{2m(0)}\,\big\|(g,f)\big\|^2_{H_0^1\times L_2} \tag{1.7}$$

where the constants are given by (1.4).

Proof. Taking term by term derivatives in (1.1) gives

$$\begin{aligned}
u_x(x,t) &= 2\pi i \sum_{n\in\mathbb{Z}} nA_n\big(\varphi'(t+x)e^{2\pi in\,\varphi(t+x)} + \varphi'(t-x)e^{2\pi in\,\varphi(t-x)}\big) \\
u_t(x,t) &= 2\pi i \sum_{n\in\mathbb{Z}} nA_n\big(\varphi'(t+x)e^{2\pi in\,\varphi(t+x)} - \varphi'(t-x)e^{2\pi in\,\varphi(t-x)}\big)
\end{aligned}$$

Therefore, using parallelogram identity as in the proof of Proposition 1.3,

$$\begin{aligned}
2E_u(t) &= \int_0^{s(t)} \big|u_x(x,t)\big|^2 + \big|u_t(x,t)\big|^2\,\mathrm{d}x \\
&= 8\pi^2\Big(\int_0^{s(t)}\Big|\sum_{n\in\mathbb{Z}} nA_n\varphi'(t+x)e^{2\pi in\,\varphi(t+x)}\Big|^2\,\mathrm{d}x \;+\; \int_0^{s(t)}\Big|\sum_{n\in\mathbb{Z}} nA_n\varphi'(t-x)e^{2\pi in\,\varphi(t-x)}\Big|^2\,\mathrm{d}x\Big) \\
&= 8\pi^2\int_{t-s(t)}^{t+s(t)}\Big|\sum_{n\in\mathbb{Z}} nA_n\big(\varphi'(y)e^{2\pi in\,\varphi(y)}\big)\Big|^2\,\mathrm{d}y.
\end{aligned}$$

This yields the double inequality

$$4\pi^2 m(t)\,a(t) \leq\; E_u(t) \;\leq\; 4\pi^2 M(t)\,a(t)$$

where

$$a(t) = \int_{t-s(t)}^{t+s(t)}\Big|\sum_{n\in\mathbb{Z}} nA_n e^{2\pi in\,\varphi(y)}\Big|^2\varphi'(y)\,\mathrm{d}y.$$

By Lemma 1.2 and Proposition 1.3 we conclude. □

2. Point Observations

2.1. Boundary Observation. Recall the notation $\alpha(t) = t+s(t)$, $\beta(t) = t-s(t)$ and $\gamma = \alpha\circ\beta^{-1}$.

Theorem 2.1. *For any admissible boundary curve $s(t)$ and solution u to the moving boundary wave equation* (W.Eq) *given by (1.1) the following double inequality holds:*

$$2\frac{m(\beta^{-1}(0))}{M(0)}\left\|(g,f)\right\|^2_{H^1_0\times L_2} \quad\leq\quad \int_0^{\gamma(0)}\left|u_x(0,t)\right|^2\,\mathrm{d}t \quad\leq\quad 2\frac{M(\beta^{-1}(0))}{m(0)}\left\|(g,f)\right\|^2_{H^1_0\times L_2} \tag{2.1}$$

In particular, with the observations $C\psi = \psi_x(0)$ the problem (W.Eq) *is exactly observable in time τ if and only if $\tau \geq \gamma(0)$.*

Proof. Differentiating u term by term, and evaluating at $x = 0$ we have for all $\tau > 0$

$$\|u_x(0,t)\|_{L_2(0,\tau,\frac{1}{\varphi'(t)})} = \int_0^\tau\left|4\pi i\sum_{n\in\mathbb{Z}} n\,A_n\varphi'(t)e^{2\pi in\,\varphi(t)}\right|^2\frac{\mathrm{d}t}{\varphi'(t)}.$$

Consider $\beta(t) = t-s(t)$ with domain $t\in[0,+\infty)$. Clearly, $\beta(t)$ is strictly increasing and since $\beta(0) = -1 < 0$, there exist a unique t_0 such that $\beta(t_0) = 0$. Let $\tau_0 := t_0+s(t_0) = \gamma(0)$. Then, by Lemma 1.2,

$$\|u_x(0,t)\|^2_{L_2(0,\tau_0,\frac{1}{\varphi'(t)})} = 16\pi^2\sum_{n\in\mathbb{Z}} n^2|A_n|^2$$

Clearly,

$$\tfrac{1}{M(t_0)}\|u_x(0,t)\|^2_{L_2(0,\tau_0)} \leq \|u_x(0,t)\|^2_{L_2(0,\tau_0,\frac{1}{\varphi'(t)})} \leq \tfrac{1}{m(t_0)}\|u_x(0,t)\|^2_{L_2(0,\tau_0)}.$$

Combining this with Proposition 1.3, we find our double inequality. From this is obvious that observation times $\tau\geq\tau_0$ suffice. On the other hand, if $\tau<\tau_0$, $\|u_x(0,t)\|^2_{L_2(0,\tau,\frac{1}{\varphi'(t)})}$ and $\sum n^2|A_n|^2$ cannot be comparable, which is easy to see by a change of variables bringing it back the the standard trigonometric orthonormal basis of $L_2(0,1)$. This shows, again by Proposition 1.3, that exact observation is impossible. □

Theorem 2.2. *For the solution u given by (1.1) to the moving boundary wave equation* (W.Eq) *the following double inequality holds:*

$$C_1\left\|(g,f)\right\|^2_{H^1_0\times L_2} \quad\leq\quad \int_0^{\gamma^{-1}(0)}\left|u_x(s(t),t)\right|^2\mathrm{d}t \quad\leq\quad C_2\left\|(g,f)\right\|^2_{H^1_0\times L_2} \tag{2.2}$$

where $C_1 = \frac{m(0)}{2M(0)(1+\|s'\|_\infty)}(1+\frac{m(t_0)}{M(t_0)})^2$ and $C_2 = \frac{M(0)}{2m(0)(1-\|s'\|_\infty)}(1+\frac{M(t_0)}{m(t_0)})^2$.
In particular, with the observations $M(t)\psi = \psi_x(s(t))$ the problem (W.Eq) *is exactly observable in time τ if and only if $\tau\geq\gamma^{-1}(0)$.*

Proof. Next we consider observation on the right boundary $x = s(t)$. As in the proof of Theorem 2.1, let t_0 be such that $\beta(t_0) = t_0-s(t_0) = 0$ and define $\tau_0 := \gamma^{-1}(0)$. Taking the derivative of $u(x,t)$ with respect to x term by term, substituting $x = s(t)$ and exploiting (1.2) yields

$$\begin{aligned} u_x(s(t),t) &= 2\pi i\sum_{n\in\mathbb{Z}} n\,A_n\Big(e^{2\pi in\,\varphi(t+s(t))}\varphi'(t+s(t))) + e^{2\pi in\,\varphi(t-s(t))}\varphi'(t-s(t))\Big) \\ &= 2\pi i\sum_{n\in\mathbb{Z}}\varphi'(t-s(t))e^{2\pi in\,\varphi(t-s(t))}n\,A_n\Big(1+\frac{\varphi'(t+s(t))}{\varphi'(t-s(t))}\Big)\end{aligned} \tag{2.3}$$

Then

$$(1+\tfrac{m(t_0)}{M(t_0)}) \leq \Big(1+\frac{\varphi'(t+s(t))}{\varphi'(t-s(t))}\Big) \leq (1+\tfrac{M(t_0)}{m(t_0)}) \tag{2.4}$$

Let $\omega(t) = \frac{1-s'(t)}{\varphi'(t-s(t))}$. Then

$$\left\|u_x(s(t),t)\right\|^2_{L_2(0,\tau_0,\omega(t)\,\mathrm{d}t)} \sim 4\pi^2\int_0^{\tau_0}\left|\sum_{n\in\mathbb{Z}} e^{2\pi in\,\varphi(t-s(t))}n\,A_n\right|^2\varphi'(t-s(t))(1-s'(t))\,\mathrm{d}t$$

where the equivalence comes from (2.4). We make the change of variables $\xi = \varphi(t-s(t))$ and observe that (1.2) gives an upper bound of the integral to be $\varphi(\beta(\tau_0))) = 1 + \varphi(\beta(0))$. So

$$\left\|u_x(s(t),t)\right\|^2_{L_2(0,\tau_0,\omega(t)\,\mathrm{d}t)} \sim 4\pi^2 \int_{\varphi(\beta(0))}^{\varphi(\beta(0))+1} \Big|\sum_{n\in\mathbb{Z}} e^{2\pi i n\xi} n A_n\Big|^2 \,\mathrm{d}\xi = 4\pi^2 \sum_{n\in\mathbb{Z}} n^2|A_n|^2$$

We summarise:

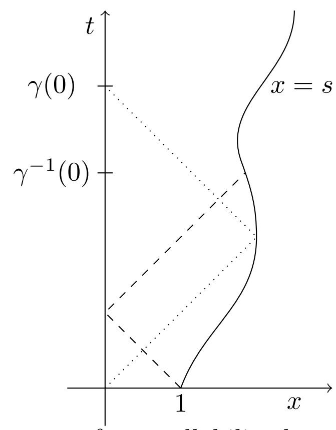
$$4\pi^2(1+\tfrac{m(t_0)}{M(t_0)})^2 \sum_{n\in\mathbb{Z}} n^2|A_n|^2 \;\le\; \left\|u_x(s(t),t)\right\|^2_{L_2(0,\tau_0,\omega(t)\,\mathrm{d}t)} \;\le\; 4\pi^2(1+\tfrac{M(t_0)}{m(t_0)})^2 \sum_{n\in\mathbb{Z}} n^2|A_n|^2$$

We conclude the proof observing that $\frac{1-\|s'\|_\infty}{M(0)} \le \omega(t) \le \frac{1+\|s'\|_\infty}{m(0)}$ which allows to remove the weight function:

$$\tfrac{4\pi^2 m(0)}{1+\|s'\|_\infty}(1+\tfrac{m(t_0)}{M(t_0)})^2 \sum_{n\in\mathbb{Z}} n^2|A_n|^2 \;\le\; \left\|u_x(s(t),t)\right\|^2_{L_2(0,\tau_0)} \;\le\; \tfrac{4\pi^2 M(0)}{1-\|s'\|_\infty}(1+\tfrac{M(t_0)}{m(t_0)})^2 \sum_{n\in\mathbb{Z}} n^2|A_n|^2$$

We conclude using Proposition 1.3. □

Let us finish this paragraph with a little observation. The optimal times for boundary observations given in Theorems 2.1 and 2.2 are precisely the times where a characteristic emerging from the left (resp. right) boundary point $x = 0$, resp. $x = 1$ hit again the boundary curve, see the picture on the right.
A second remark is that since $u(s(t),t) = 0$, taking derivative with respect to t gives $s'(t)u_x(s(t),t) = -u_t(s(t),t)$. We may hence replace u_x by u_t in the inequality (2.2), at the only price to modify the constants by a factor $\|s'\|_\infty$.



Somehow a similar result to Theorem 2.2 in a dual setting in terms of controllability have been shown in [13] for the special case of a linear moving wall $s(t) = 1 + \varepsilon t$ by a transformation to a cylindrical domain proposed by Miranda [29]. The minimal control time estimate was however far from optimal. Their result (again only for the linear moving wall case) was subsequently improved in [34] who found the same minimal control time as ourselves by a different method[‡].

2.2. **Internal Point observation.** Next, we turn our attention to observation on an internal point. In the situation where $s(t) = 1$ and hence $\varphi(x) = x$, the solution u to (W.Eq) is given by a sine-series (due to Dirichlet boundary conditions),

$$u(x,t) = \sum_{n\in\mathbb{Z}} a_n e^{i\pi n t} \,\sin\big(n\pi x\big).$$

Consequently, internal point observation at $x{=}a$ is not possible when $a \in \mathbb{Q}$ since then infinitely many terms in the sum vanish, independently of the leading coefficient. One way to counter this problem is to obtain observability results for the average of $|u|^2$ in a small neighbourhood of a fixed internal point a, see [17]. It is also well known that another way to counter this problem is to consider a moving interior point, see for example [8, 22, 21]. We follow in this article the idea that fixed domain with moving observers should somehow behave similar to moving domains with fixed observers. The following result confirms this intuition: for any fixed point $a \in (0,1)$, consider a Neumann observer defined by $Cu = u_x(a,t)$ to the solution u of the moving boundary wave equation (W.Eq).

[‡]Caution: when writing out the parametrisation of the boundary integral in [34, formula (2.2)], the authors forget a factor $(1+\varepsilon)^{1/2}$. This wrong factor then appears in many subsequent estimates in their paper.

Theorem 2.3. *Let s be an monotonic admissible boundary curve and φ be a C^2-solution to (1.2). Assume additionally that φ' is strictly decreasing if $s(\cdot)$ is increasing or that φ' is strictly increasing if $s(\cdot)$ is decreasing, respectively.*
Then solution u to the wave equation (W.Eq) *satisfies the following double inequality:*

$$C_1(a)\,\big\|(g,f)\big\|^2_{H^1_0\times L_2} \quad\le\quad \int_0^{a+\gamma(-a)}\big|u_x(a,t)\big|^2\,\mathrm{d}t \quad\le\quad C_2(a)\,\big\|(g,f)\big\|^2_{H^1_0\times L_2},$$

where the constants C_1 and C_2 depend only on $s(\cdot)$ and a. We provide them explicitly in the proof.

Proof. Let $t_1=\beta^{-1}(-a)$ and $\tau_a=a+\gamma(-a)$. Term by term differentiation of (1.1) with respect to x gives

$$u_x(a,t)=2\pi i\sum_{n\in\mathbb{Z}} n\,A_n\Big(e^{2\pi in\,\varphi(t+a)}\varphi'(t+a)+e^{2\pi in\,\varphi(t-a)}\varphi'(t-a)\Big)$$

First we suppose that φ' is strictly decreasing. We first calculate a weighted L_2-norm with $\omega_a(t)=\frac{1}{\varphi'(t-a)}$:

$$A-B\quad\le\quad\|u_x(a,t)\|_{L_2(0,\tau_a,\omega_a(t)\,\mathrm{d}t)}\quad\le\quad A+B$$

with

$$A:=2\pi\Big\|\sum_{n\in\mathbb{Z}} n\,A_n e^{2\pi in\,\varphi(t-a)}\varphi'(t-a)\Big\|_{L_2(0,\tau_a,\omega_a(t)\,\mathrm{d}t)}$$
$$B:=2\pi\Big\|\sum_{n\in\mathbb{Z}} n\,A_n e^{2\pi in\,\varphi(t+a)}\varphi'(t+a)\Big\|_{L_2(0,\tau_a,\omega_a(t)\,\mathrm{d}t)}.$$

To estimate A, the change of variables $s=t-a$ together with Lemma 1.2 therefore gives

$$A^2=4\pi^2\sum_{n\in\mathbb{Z}} n^2|A_n|^2.$$

For B, we have

$$B^2=4\pi^2\int_0^{\tau_a}\Big|\sum_{n\in\mathbb{Z}} n\,A_n(e^{2\pi in\,\varphi(t+a)}\varphi'(t+a))\Big|^2\omega_a(t)\,\mathrm{d}t$$

Since φ' is strictly decreasing, $0<\frac{\varphi'(t+a)}{\varphi'(t-a)}<1$ for all $t\in[0,\tau_a]$ and so $q_a:=\max_{[0,\tau_a]}\frac{\varphi'(t+a)}{\varphi'(t-a)}<1$. We then have

$$\begin{aligned}B^2&\le 4\pi^2 q_a\int_0^{\tau_a}\Big|\sum_{n\in\mathbb{Z}} n\,A_n e^{2\pi in\,\varphi(t+a)}\varphi'(t+a))\Big|^2\tfrac{1}{\varphi'(t+a)}\,\mathrm{d}t\\&=4\pi^2 q_a\int_a^{a+\tau_a}\Big|\sum_{n\in\mathbb{Z}} n\,A_n e^{2\pi in\,\varphi(s)}\Big|^2\varphi'(s)\,\mathrm{d}s\end{aligned}$$

Recall that $a+\tau_a=2a+\gamma(-a)$. Since $s'\ge 0$, we have $\gamma'\ge 1$ and so $2a+\gamma(-a)\le\gamma(a)$. By Lemma 1.2 we infer

$$B^2\le 4\pi^2 q_a\int_a^{\gamma(a)}\Big|\sum_{n\in\mathbb{Z}} n\,A_n e^{2\pi in\,\varphi(s)}\Big|^2\varphi'(s)\,\mathrm{d}s=4\pi^2 q_a\sum_{n\in\mathbb{Z}} n^2|A_n|^2.$$

Putting both on A and B estimates together, and using Proposition 1.3, we get the lower estimate

$$\begin{aligned}\|u_x(a,t)\|^2_{L_2(0,\tau_a)}&\ge m(t_1)\,\big\|u_x(a,t)\big\|^2_{L_2(0,\tau_a,\omega_a(t)\,\mathrm{d}t)}\\&\ge 4\pi^2 m(t_1)(1-\sqrt{q_a})^2\sum_{n\in\mathbb{Z}} n^2|A_n|^2\\&\ge C_1(a)\,\big\|(g,f)\big\|^2_{H^1_0\times L_2}\end{aligned}$$

with $C_1(a)=\frac{m(t_1)}{2M(0)}(1-\sqrt{q_a})^2$. The upper estimate is similar; we find $C_2(a)=\frac{M(t_1)}{2m(0)}(1+\sqrt{q_a})^2$. In the case where φ' is strictly increasing we use $\widetilde{\omega_a}(t)=\frac{1}{\varphi'(t+a)}$ as a weight function and change the rôles of A and B. The result follows the same lines then. ∎

We observe that the same proof also gives the double inequality

$$C_1(a)\,\big\|(g,f)\big\|^2_{H_0^1\times L_2}\quad\le\quad\int_0^{a+\gamma(-a)}\big|u_t(a,t)\big|^2\,\mathrm{d}t\quad\le\quad C_2(a)\,\big\|(g,f)\big\|^2_{H_0^1\times L_2}.$$

Discussion. One may formulate (W.Eq) as an abstract non-autonomous Cauchy problem, for example as follows: let $H_t = L_2([0,s(t)])$ and define

$$\mathscr{D}(A(t)) = H_0^1([0,s(t)]\cap H^2([0,s(t)])\qquad\text{and}\qquad A(t)f=f''$$

Then $A(t)$ is the generator of an analytic semigroup on H_t. For $t\ge 0$, we let $\mathscr{H}_t = H_0^1([0,s(t)])\times L_2([0,s(t)])$ and

$$\mathscr{D}(\mathfrak{A}(t)) = \mathscr{D}(A(t))\times H_0^1([0,s(t)])\qquad\text{and}\qquad \mathfrak{A}(t)=\begin{pmatrix}0 & I\\ A(t) & 0\end{pmatrix}.$$

With this notation (W.Eq) rewrites as

$$\begin{cases}x'(t)=\mathfrak{A}(t)x(t)\\ x(0)=x_0=(g,f)\in\mathscr{H}_0.\end{cases}\tag{2.5}$$

The observation of $t\mapsto u_x(a,t)$ discussed in the theorem is then realised with observation operators $C(t):\mathscr{D}(\mathfrak{A}(t))\to\mathbb{C}$ defined by $C(t)(v,w)^t = v_x(a)$. Theorem 2.3 states in particular exact observability on $[0,\tau]$ if and only if $\tau\ge a+\gamma(-a)$. It is remarkable that this holds true, although, for a dense subset of values of t_0 (precisely if $a/s(t_0)\in\mathbb{Q}$) the "frozen" evolution equations

$$x'(t)+\mathfrak{A}(t_0)x(t)=0\qquad y(t)=C(t)x(t)$$

are *not* exactly observable by the sine-series argument given above for the case $s(t)=1$. This could now lead to the intuition that the non-observability on for all $t>0$ such that $a/s(t)\in\mathbb{Q}$ is an "almost everywhere phenomenon", and may be ignored. This idea is partially contradicted by the following result, where the observation position depends on time and may be such that the ratio $a(t)/s(t)\in\mathbb{Q}$ for all $t>0$.

Theorem 2.4. *Let $s(t)=1+\varepsilon t$ and $a(t)=as(t)$ for some $a\in(0,1)$. Then the solution u to the wave equation* (W.Eq) *satisfies the following admissibility and observation inequality:*

$$C_1(a,\varepsilon)\,\big\|(g,f)\big\|^2_{H_0^1\times L_2}\quad\le\quad\int_0^{\frac{2}{1-\varepsilon}}\big|u_t(a(t),t)\big|^2\,\mathrm{d}t\quad\le\quad C_2(a,\varepsilon)\,\big\|(g,f)\big\|^2_{H_0^1\times L_2}$$

The constants C_1 and C_2 depend only on a and ε. We provide them explicitly in the proof.

Proof. Recall that the solution u of the equation (W.Eq) can be written in the form (1.1) with $\varphi(t)=C_\varepsilon\,\ln(1{+}\varepsilon t)$, see the table on page 2. Taking the derivative respected to t gives

$$u_t(x,t)=2\pi i\sum_{n\in\mathbb{Z}}nA_n\Big(e^{2\pi in\,\varphi(t+x)}\varphi'(t{+}x)-e^{2\pi in\,\varphi(t-x)}\varphi'(t{-}x)\Big)$$

Substituting $x=a(t)$, we get

$$u_t(a(t),t)=2\pi i\sum_{n\in\mathbb{Z}}nA_n\Big(e^{2\pi in\,\varphi(t+a(1+\varepsilon t))}\varphi'(t+a(1{+}\varepsilon t))-e^{2\pi in\,\varphi(t-a(1+\varepsilon t))}\varphi'(t-a(1{+}\varepsilon t))\Big)$$

By calculation, we have the followings identities

$$\begin{aligned}\varphi(t\pm a(1{+}\varepsilon t)) &= \varphi(t)+\varphi(\pm a)\\ \varphi_t(t\pm a(1{+}\varepsilon t)) &= \tfrac{1}{\varepsilon}\varphi'(t)\varphi'(\pm a)\end{aligned}$$

Plugging them into the preceding equation we get

$$\begin{aligned}u_t(a(t),t) &= \tfrac{2\pi i}{\varepsilon}\sum_{n\in\mathbb{Z}}A_n\Big(e^{2\pi in\,(\varphi(t)+\varphi(a))}\varphi'(t)\varphi'(a)-e^{2\pi in\,(\varphi(t)+\varphi(-a))}\varphi'(t)\varphi'(-a))\Big)\\ &= \tfrac{2\pi i}{\varepsilon}\sum_{n\in\mathbb{Z}}A_ne^{2\pi in\,\varphi(t)}\varphi'(t)\Big(e^{2\pi in\,\varphi(a)}\varphi'(a)-e^{2\pi in\,\varphi(-a)}\varphi'(-a)\Big)\end{aligned}$$

Let $t_0 = \frac{1}{1-\varepsilon}$. Then $[t_0 - s(t_0), t_0 + s(t_0)] = [0, \frac{2}{1-\varepsilon}]$ and so, using Lemma 1.2,

$$\begin{aligned}&\big\|u_t(a(t),t)\big\|^2_{L_2(0,\frac{2}{1-\varepsilon},\frac{1}{\varphi'(t)})}\\ &= \tfrac{4\pi^2}{\varepsilon^2}\int_0^{\frac{2}{1-\varepsilon}}\Big|\sum_{n\in\mathbb{Z}} e^{2\pi in\,\varphi(t)}\varphi'(t)\,nA_n\Big(e^{2\pi in\,\varphi(a)}\varphi'(a)-e^{2\pi in\,\varphi(-a)}\varphi'(-a)\Big)\Big|^2\tfrac{1}{\varphi'(t)}\,\mathrm{d}t\\ &= \tfrac{4\pi^2}{\varepsilon^2}\sum_{n\in\mathbb{Z}} n^2|A_n|^2\big|e^{2\pi in\,\varphi(a)}\varphi'(a)-e^{2\pi in\,\varphi(-a)}\varphi'(-a))\big|^2\end{aligned}$$

Now we need to estimate the multiplicative term

$$\begin{aligned}M_n^2 &= \big|e^{2\pi in\,\varphi(a)}\varphi'(a)-e^{2\pi in\,\varphi(-a)}\varphi'(-a))\big|^2\\ &= \varphi'(a)^2+\varphi'(-a)^2-2\varphi'(a)\varphi'(-a)\cos\Big(2\pi n(\varphi(a)-\varphi(-a))\Big).\end{aligned}$$

Clearly, $(\varphi'(a)-\varphi'(-a))^2 \le M_n^2 \le (\varphi'(a)+\varphi'(-a))^2$; by direct calculation,

$$(\varphi'(a)-\varphi'(-a))^2 \;=\; C_\varepsilon^2\frac{4\varepsilon^4a^2}{(1-\varepsilon^2a^2)^2}\quad\text{and}\quad(\varphi'(a)+\varphi'(-a))^2 \;=\; C_\varepsilon^2\frac{4\varepsilon^2}{(1-\varepsilon^2a^2)^2}$$

Therefore, by Proposition 1.3,

$$\tfrac{16\pi^2\varepsilon^2a^2}{(1-\varepsilon^2a^2)^2\eta_\varepsilon^2}\sum_{n\in\mathbb{Z}}n^2|A_n|^2 \quad\le\quad \|u_t(a(t),t)\|^2_{L_2(0,\frac{2}{1-\varepsilon},\frac{1}{\varphi'(t)})}\quad\le\quad \tfrac{16\pi^2}{(1-\varepsilon^2a^2)^2\eta_\varepsilon^2}\sum_{n\in\mathbb{Z}}n^2|A_n|^2$$

Now we apply Proposition 1.3 to conclude. We find

$$C_1(a,\varepsilon)=\tfrac{1-\varepsilon}{1+\varepsilon}\tfrac{2\varepsilon^2a^2}{(1-\varepsilon^2a^2)^2\eta_\varepsilon^2}\quad\text{and}\quad C_2(a,\varepsilon)=\tfrac{1+\varepsilon}{1-\varepsilon}\tfrac{2}{(1-\varepsilon^2a^2)^2\eta_\varepsilon^2}.$$ □

2.3. **Simultaneous exact observability.** A last result in this section concerns simultaneous exact observability : consider a system of two coupled 1D wave equations, one of which has a fixed boundary, and the second has the moving domain $0 \le x \le s(t)$ as above. Assume that we can observe only the combined force exerted by the strings at the common endpoint $\varphi(t) = u_x^{(1)}(0,t) + u_x^{(2)}(0,t)$, for $t \in [0,T]$. The question is whether we can still exactly observe all initial data. Our system is defined as

$$(W_2)\qquad\left\{\begin{array}{ll} u_{tt}-u_{xx}=0 & (x,t)\in\Omega\\ v_{tt}-v_{xx}=0 & -1\le x\le 0\\ u(0,t)=u(s(t),t)=v(-1,t)=v(0,t)=0 & t\ge 0\\ u(x,0)=g(x), u_t(x,0)=f(x) & x\in[0,1]\\ v(x,0)=\widetilde{g}(x), v_t(x,0)=\widetilde{f}(x) & x\in[-1,0]\end{array}\right.$$

Theorem 2.5. *Let $s(\cdot)$ be an admissible boundary curve and assume additionally that either*

$$\liminf_{t\to\infty}\gamma'(t)>1\qquad or\qquad \gamma'(t)=1+ax^{-\delta}+o(t^{-\delta}),\quad 0<\delta<1, a>0.$$

Moreover assume that φ' is bounded on $\mathbb{R}_+$. Let (u,v) be the solution to (W_2). Then, for all $\lambda > 0$ there exists $\tau_0 > 2$ such that for all $\tau \ge \tau_0$

$$\lambda\Big(\big\|(g,f)\big\|^2_{H_1^0\times L_1}+\big\|(\widetilde{g},\widetilde{f})\big\|^2_{H_1^0\times L_2}\Big)\quad\le\quad\int_0^\tau\big|u_x(0,t)+v_x(0,t)\big|^2dt \tag{2.6}$$

Our assumptions include the cases of linear moving boundaries, parabolic boundaries and hyperbolic boundaries. However, for the shrinking domain they are not satisfied.

Proof. By the triangle inequality we have

$$\Big(\int_0^\tau\big|u_x(0,t)+v_x(0,t)\big|^2\,\mathrm{d}t\Big)^{1/2}\ge A(\tau)-B(\tau)$$

where

$$A(\tau)=\Big(\int_0^\tau\big|v_x(0,t)\big|^2\,\mathrm{d}t\Big)^{1/2}\quad\text{and}\quad B(\tau)=\Big(\int_0^\tau\big|u_x(0,t)\big|^2\,\mathrm{d}t\Big)^{1/2}$$

It is well known that the solution v of the wave equation with the fixed boundary can be expressed as a pure sine series

$$v(x,t)=\sum_{n\in\mathbb{Z}} a_n e^{\pi i n\, t}\ \sin\bigl(n\pi x\bigr), \tag{2.7}$$

where $(na_n)_{n\in\mathbb{Z}}\in\ell_2$ and hence $(a_n)_{n\in\mathbb{Z}}\in\ell_2$. Consequently, for all $t\geq 0$, the energy of v is constant: indeed, by direct computation,

$$E_v(t)=\tfrac{1}{2}\int_0^1 \bigl|\tfrac{\partial v(x,t)}{\partial t}\bigr|^2+\bigl|\tfrac{\partial v(x,t)}{\partial x}\bigr|^2\,\mathrm{d}x=\tfrac{\pi^2}{2}\sum_{n\in\mathbb{Z}} n^2 a_n^2$$

We also have

$$\int_0^2 \bigl|v_x(0,t)\bigr|^2\,\mathrm{d}t=\ \int_0^2\Bigl|\sum_{n\in\mathbb{Z}}\pi n a_n e^{i\pi n t}\Bigr|^2\,\mathrm{d}t=\ 4E_v(0).$$

Hence, using periodicity of v, we obtain (recall $\tau\geq 2$)

$$A(\tau)^2=\int_0^\tau \bigl|v_x(0,t)\bigr|^2\,\mathrm{d}t\ \geq\ 4\lfloor\tfrac{\tau}{2}\rfloor\ E_v(0)$$

Next we turn to an estimate for $B(\tau)$. Recall that

$$u_x(0,t)=4\pi i\sum_{n\in\mathbb{Z}} nA_n\varphi'(t)e^{2\pi in\,\varphi(t)}$$

Let $t_0=0$ and $t_n=\gamma^{(n)}(t_0)$. By construction of t_n and (1.2),

$$\varphi(t_{n+1})-\varphi(t_n)=\varphi(\gamma(t_n))-\varphi(t_n)=1.$$

Hence, by Lemma 1.2, $e^{2\pi in\,\varphi(x)}$ is an orthonormal system on $L_2([t_n,t_{n+1}],\varphi'(t)\,\mathrm{d}t)$.
An inspection of the proof of Theorems A.1 and A.2 shows that if $\liminf_{t\to\infty}\gamma'>1$, $t_n\to+\infty$ exponentially, whereas the asymptotics $\gamma'(t)=1+at^{-\delta}+o(t^{-\delta})$ ensures $t_n\sim cn^{1/\delta}$. Let $N(\tau)$ be the unique integer satisfying $t_n\leq\tau<t_{n+1}$. Let $C=\sup\{\varphi'(t):\ t\geq 0\}$. Then

$$\begin{aligned}
B(\tau)=\int_0^\tau \bigl|u_x(0,t)\bigr|^2\,\mathrm{d}t &\leq \int_0^\tau \bigl|u_x(0,t)\bigr|^2\tfrac{1}{\varphi'(t)}\,\mathrm{d}t\\
&\leq C\sum_{j=0}^{N(\tau)}\int_{t_j}^{t_{j+1}}\bigl|u_x(0,t)\bigr|^2\tfrac{1}{\varphi'(t)}\,\mathrm{d}t\\
&\leq 16\pi^2 C(N(\tau)+1)\sum_{n\in\mathbb{Z}} n^2|A_n|^2\\
&\leq \tfrac{2C}{m(0)}(N(\tau)+1)\Bigl(\|g^{(1)}(x)\|^2_{H_1^0(0,1)}+\|f^{(1)}(x)\|^2_{L_2(0,1)}\Bigr).
\end{aligned}$$

We obtained so far that

$$\begin{aligned}
&\int_0^\tau \bigl|u_x(0,t)+v_x(0,t)\bigr|^2\,\mathrm{d}t\ \geq\ A(\tau)^2-B(\tau)^2\\
&\geq 4\lfloor\tfrac{\tau}{2}\rfloor\ E_v(0)-\tfrac{2C}{m(0)}(N(\tau)+1)\Bigl(\|g^{(1)}(x)\|^2_{H_1^0(0,1)}+\|f^{(1)}(x)\|^2_{L_2(0,1)}\Bigr)
\end{aligned}$$

The first term grows linearly in τ. The second term is $o(\tau)$ since in case of exponential growth of the sequence t_n, $N(\tau)$ behaves logarithmically and in case that $t_n\sim cn^{1/\delta}$, $N(\tau)\sim\tau^\delta$ with $\delta<1$. Hence, the difference tends to infinity with $\tau\to+\infty$, which means that for all $\lambda>0$ there exists $\tau_0>0$ such that for $\tau\geq\tau_0$,

$$\begin{aligned}
\int_0^\tau \bigl|u_x(0,t)+v_x(0,t)\bigr|^2\,\mathrm{d}t\ &\geq 2\lambda\bigl(E(u)(0)+E_v(0)\bigr)\\
&=\lambda\Bigl(\bigl\|(g,f)\bigr\|^2_{H_0^1\times L_2}+\bigl\|(\widetilde{g},\widetilde{f})\bigr\|^2_{H_0^1\times L_2}\Bigr).
\end{aligned}$$

□

3. Additional results

Variants of the construction. We mention that for our usual choice of φ, a series of the type

$$u(x,t) := \sum_{n=0}^{\infty} C_n \int_{t-x}^{t+x} \exp(2\pi i n \varphi(y))\, \mathrm{d}y$$

will solve the wave equation on the moving boundary domain with boundary condition $u(0,t) = u_t(s(t),t) = 0$.

3.1. **Duality.** Without detailed proofs we state dual results to our results formulated as null-controllability in the sense of 'transposition'.

Dirichlet control on boundary. Let s be an admissible boundary curve, v the solution to the wave equation on Ω. Let $(Gv)(t) = (v(0,t), v(s(t),t))$ be the trace of v on the two boundary points. Then for either choice, $\zeta(t) = (y(t),0)$ or $\zeta(t) = (0,y(t))$ the boundary controlled wave equation

$$\begin{cases} v_{tt} - v_{xx} &= 0 & (x,t) \in \Omega \\ (Gv)(t) &= \zeta(t) & t \geq 0 \\ v(x,0) &= g \in L_2([0,1]) & x \in [0,1] \\ v_t(x,0) &= f \in H^{-1}([0,1]) & x \in [0,1] \end{cases} \tag{3.1}$$

is null-controllable in times $\tau = \gamma(0)$ in case $\zeta(t) = (y(t),0)$ and in time $\tau = \gamma^{-1}(0)$ in case $\zeta(t) = (0,y(t))$. The null control can be achieved by the control function $y(t) = -u_x(0,t)$, or $y(t) = -u_x(s(t),t)$, respectively where $u(\cdot)$ is the solution to (W.Eq).

Simultaneous Null Control. Next we focus on the dual statement to Theorem 2.3 in terms of null-controllability. Instead of one wave equation on Ω, we consider two wave equations with mixed boundary conditions, one on the cylindrical domain $[0,a] \times \mathbb{R}_+$ and one on the non-cylindrical domain $\{(x,t) : \ a \leq x \leq s(t)\}$. Both equations are coupled via the control function ζ in the following way:

$$\begin{cases} v_{tt} - v_{xx} = 0 & 0 \leq x \leq a \\ w_{tt} - w_{xx} = 0 & a \leq x \leq s(t) \\ v(0,t) = w(s(t),t) = 0 & t \geq 0 \\ v(a-,t) = w(a+,t) & t \geq 0 \\ v_x(a-,t) - w_x(a+,t) = \zeta(t) & t \geq 0 \\ v(x,0) = g(x), \quad v_t(x,0) = f(x) & x \in [0,a] \\ w(x,0) = g(x), \quad w_t(x,0) = f(x) & x \in [a,1] \end{cases} \tag{3.2}$$

Then Theorem 2.3 implies that (3.2) is null-controllable in time $\tau \geq a + \gamma(-a)$. The control can be achieved by letting $\zeta(t) = u_x(a,t)$ where $u(\cdot)$ is the solution to (W.Eq).

3.2. **Boundary stabilization.** Finally we consider a linear boundary stabilisation of the wave equation (W.Eq) by a feedback of the Neumann observation on the moving boundary. Since the boundary depends on time, it seems reasonable to consider time-dependent boundary feedbacks as well. We are thus lead to study for a positive function λ

$$\begin{cases} u_{tt} - u_{xx} = 0 & (x,t) \in \Omega \\ u(0,t) = 0 & t \geq 0 \\ u_t(s(t),t) = -\lambda(t) u_x(s(t),t) & t \geq 0 \\ u(x,0) = g(x) & x \in [0,1] \\ u_t(x,0) = f(x) & x \in [0,1] \end{cases} \tag{3.3}$$

The solution of a wave equation of the general form $u(x,t) = a(t+x) + b(t-x)$. The Dirichlet boundary condition on x=0 forces $a = -b$. Next, we find $u(x,0) = a(x) - a(-x) = g$ and $u_t(x,0) = a'(x) - a'(-x) = f$. Hence, $g' + f = 2a'$ fixes a' (in an L_2 sense) on $[0,1]$ whereas

$g' - f = 2a'(-x)$ fixes a' on $[-1, 0)$. Most interestingly is the impact of the boundary condition $u_t + \lambda u_x\big|_{x=s(t)} = 0$: we get

$$\begin{aligned} 0 &= u_t(s(t),t) + \lambda(t) u_x(s(t),t)) \\ &= (a'(t+s(t)) - a'(t-s(t))) + \lambda(t)\,(a'(t+s(t)) + a'(t-s(t))) \\ &= (1+\lambda(t))a'(t+s(t)) - (1-\lambda(t))a'(t-s(t)) \end{aligned} \tag{3.4}$$

so that

$$\frac{a'(t+s(t))}{a'(t-s(t))} = \frac{1-\lambda(t)}{1+\lambda(t)} \qquad \text{or} \qquad \frac{a'\circ\gamma}{a'} = \frac{1-\lambda}{1+\lambda}\circ\beta^{-1}, \tag{3.5}$$

where we re-used our definition $\alpha(t) = t+s(t)$, $\beta(t) = t-s(t)$, and $\gamma = \alpha\circ\beta^{-1}$. Since the initial data fixes a' on $[-1,1] = [-1, \gamma(-1)]$, this fixes the function a' for all $t > -1$ by iteration of γ. The problem (3.3) has therefore a uniquely determined solution u_λ. Let us turn to the calculation of the actual energy of the solution:

$$\begin{aligned} E_\lambda(t) &= \tfrac{1}{2}\int_0^{s(t)} |u_t(x,t)|^2 + |u_x(x,t)|^2 \,\mathrm{d}x \\ &= \tfrac{1}{2}\int_0^{s(t)} |a'(t+x) - a'(t-x)|^2 + |a'(t+x) + a'(t-x)|^2 \,\mathrm{d}x \\ &= \int_0^{s(t)} |a'(t+x)|^2 + |a'(t-x)|^2 \,\mathrm{d}x = \int_{t-s(t)}^{t+s(t)} |a'(y)|^2 \,\mathrm{d}y \\ &= \int_{\beta(t)}^{\alpha(t)} |a'(y)|^2 \,\mathrm{d}y. \end{aligned}$$

Using the boundary conditions of (3.3), we get from the first equality together with (3.5)

$$\begin{aligned} E'_\lambda(t) &= \tfrac{s'(t)}{2}\left(|u_t(s(t),t)|^2 + |u_x(s(t),t)|^2\right) + \big[u_t(x,t)u_x(x,t)\big]_{x=0}^{x=s(t)} \\ &= \left(\tfrac{s'(t)}{2}(1+\lambda(t)^2) - \lambda(t)\right)|u_x(s(t),t)|^2 \\ &= \tfrac{2s'(t)(1+\lambda(t)^2 - 4\lambda(t)}{(1+\lambda(t))^2}|a'(t-s(t))|^2. \end{aligned} \tag{3.6}$$

It is obvious that the energy decays if $s' < 0$, for whatever choice of $\lambda > 0$. In the case that $s' > 0$, a simple calculation shows that the energy decays strictly for $\lambda \in (a_s, b_s)$ where

$$a_s = \tfrac{1}{\|s'\|_\infty}\left(1 - \sqrt{1 - \|s'\|_\infty^2}\right) \qquad \text{and} \qquad b_s = \tfrac{1}{\|s'\|_\infty}\left(1 + \sqrt{1 - \|s'\|_\infty^2}\right)$$

Observe that $1 \in (a_s, b_s)$.

(a) If we calculate, for fixed t, the optimal value for a time-varying coefficient $\lambda(t)$ in (3.6) we find the maybe surprising result $\lambda(t) = 1$ for all $t > 0$. Indeed, in this case $a'(t+s(t)) = 0$ for all $t > 0$ and, whence $a(t)$ is constant for $t > 1$. We observe therefore extinction in finite time: precisely $u(x,t) = 0$ for $\min(t+x, t-x) > 1$. Inspecting the illustration on page 6, this corresponds to the time $t = \gamma^{-1}(0)$, i.e. the time the characteristic emerging from x=1 needs to come back to the moving boundary after reflection on the axis x=0. This phenomenon is well known in the case of the time-independent case $s(t)$=1, see e.g. [23, Theorem 0.5]

We now discuss what happens for fixed $\lambda \in (a_s, b_s)$, $\lambda \neq 1$ and increasing boundary curves. First, (3.5) implies that $a'\circ\gamma = q\cdot a'$ where $q = \frac{1-\lambda}{1+\lambda}$ satisfies $|q| < 1$. We let $t_0 = 0$ and $t_{n+1} = \gamma(t_n)$, $n \geq 0$. Then (t_n) is an increasing sequence and $E(t_n) \geq E(t) \geq E(t_{n+1})$ for $t \in [t_n, t_{n+1}]$ by monotony. Writing $x_0 = -1$ and $x_{n+1} = \gamma x_n$,

$$\begin{aligned} E(t_n) &= \int_{\beta(t_n)}^{\alpha(t_n)} |a'(y)|^2 \,\mathrm{d}y = \int_{x_n}^{x_{n+1}} |a'(y)|^2 \,\mathrm{d}y \\ &= \int_{\gamma(x_{n-1})}^{\gamma(x_n)} |a'(y)|^2 \,\mathrm{d}y = \int_{x_{n-1}}^{x_n} |a'(\gamma(x))|^2 \gamma'(x) \,\mathrm{d}x \end{aligned}$$

$$= q^2 \int_{x_{n-1})}^{\gamma(x_n)} |a'(x)|^2 \gamma'(x)\ \mathrm{d}x$$

so that

$$q^2(\min \gamma')\ E_\lambda(t_{n-1}) \quad \leq \quad E_\lambda(t_n) \quad \leq \quad q^2(\max \gamma')\ E_\lambda(t_{n-1}) \tag{3.7}$$

where the minimum and maximum is calculated on $[x_{n-1}, x_n]$.

(b) As a by-product of the proof of Theorem A.2, we know that whenever $\gamma'(x) = 1+a(1-\delta)x^{-\delta} + o(x^{-\delta})$ (with $a > 0, \delta > 0, \delta \neq 1$), then $t_n \sim n^{1/\delta}$. Therefore, approximately $n = t^\delta$ iterations are necessary to reach t from $t_0 = 0$ so that

$$E(t) \lesssim \ln(t^\delta) q^{2t^\delta} = \delta \ln(t) \exp(2t^\delta \ln |q|).$$

Since $\ln |q| < 0$ this implies an exponential type of decay of the energy, but in a manner that cannot be observed in the case when $s(t)$=1, since then the solution u_λ is given by a semigroup.

(c) Let $s(t) = 2 - \frac{1}{1+t}$. In this case, for any $\lambda > 0$, the energy *eventually* decays, as can be seen by looking at (3.6). Moreover, we can explicitly calculate

$$\gamma(t) = t + 4 - \tfrac{4}{3+t+\sqrt{t^2+6t+5}} \qquad \gamma'(t) \sim 1 + 2t^{-2} + O(t^{-3}) \quad \text{at infinity}.$$

Hence γ is not of the form discussed in (b). Since we have no monotony of E_λ on $(0, \infty)$, but only on some interval $[a, \infty)$, (3.7) takes the form

$$q^2\gamma'(x_n))\ E_\lambda(t_{n-1}) \quad \leq \quad E_\lambda(t_n) \quad \leq \quad q^2\gamma'(x_{n-1}))\ E_\lambda(t_{n-1}),$$

but only for $n \geq n_0$. Observe however that the orbits t_n and x_n grow asympotically linear in n. Therefore, when iterating the double energy inequality above, the infinite product $\prod_{n \geq n_0} \gamma'(x_n)$ converges to some strictly positive qualtity. Using $q^2 < 1$, we conclude that the system energy has exponential decay $E(t) \sim_\lambda \exp(t \ln(q^2)) E(0)$, for all $\lambda > 0$.

Appendix A. Differentiable solutions for general boundary functions

In this section we discuss the solvability of (1.2) by a differentiable function φ. Our hypotheses are that the boundary function s be of class C^1 at least and that $\lim_{t\to\infty} s'(t) = \jmath$ exists. This last condition is of course only of interest if we seek for solutions φ satisfying (1.2) for $t \in \mathbb{R}_+$, since it can easily be arranged if we consider only $t \in [0, \tau]$.

Let $s(\cdot)$ be of class C^1 and $\|s'\|_\infty < 1$. Let $\alpha(t) = t + s(t)$ and $\beta(t) = t - s(t)$. Both functions, α and β are strictly increasing and continuous. Moreover, $\alpha(t) = \alpha(0)+t\alpha'(\xi_t) > \alpha(0)+t(1-\|s'\|_\infty)$ yields $\lim_{t\to+\infty} \alpha(t) = +\infty$. Hence α is a bijection from $[0, \infty)$ to $[1, \infty)$; similarly β is a bijection from $[0, \infty)$ to $[-1, \infty)$. We then consider the bijection

$$\gamma := \alpha \circ \beta^{-1} : [-1, \infty) \to [+1, \infty).$$

Observe that

$$\gamma'(t) = \frac{\alpha' \circ \beta^{-1}}{\beta' \circ \beta^{-1}} = \frac{1 + s'(\beta^{-1}(t))}{1 - s'(\beta^{-1}(t))}, \tag{A.1}$$

so that γ is strictly increasing by $\|s'\|_\infty < 1$. The sign of $s'(\beta^{-1}(t))$ determines whether γ is strictly contractive or strictly expansive. We also note for further reference that if $s \in \mathrm{C}^2$,

$$\gamma''(t) = \frac{2s''(\beta^{-1}(t))}{(1 - s'(\beta^{-1}(t)))^3}.$$

The functional equation (1.2) can now be rephrased as

$$\varphi \circ \gamma = \varphi + 1. \tag{A}$$

This equation is known as 'Abel's equation' and intensively studied, see for example [24, 25] and references therein.
We will consider only the case where $\lim s'(t) = \jmath$ exists. Since $s(t) > 0$ for all t, $\lim s'(t) = \jmath < 0$ is impossible. We may therefore either have $\jmath = 0$ or $\jmath \in (0,1)$. We first discuss the situation of a non-zero limit, which means that $\gamma'(t) \to \ell = \frac{1+\jmath}{1-\jmath} > 1$.

Theorem A.1. *Let $\ell > 1$ and assume that $\gamma'(x) = \ell + \mathcal{O}(x^{-\delta})$ for $\delta > 0$. Then Abel's equation (A) admits a strictly increasing solution $\varphi \in \mathrm{C}^1([-1,\infty))$. If additionally $\gamma \in \mathrm{C}^2[0,\infty)$, $\gamma'' = \mathcal{O}(x^{-1-\delta})$ and γ' is decreasing, then φ is of class $\mathrm{C}^2([-1,\infty))$.*

We mention as a simple example that for linear moving wall as well as the hyperbolic boundary the hypothesis of the preceding theorem are satisfied.

Proof of Theorem A.1. Put $\psi = \ell^{\varphi}$. Then ψ satisfies the Schröder equation $\psi \circ \gamma = \ell\psi$. Since $\gamma(-1) = +1$ and γ has no fixed points (otherwise $s(t) = 0$), $\gamma(x) > x$ for all $x \geq -1$. Observe that by assumption, there exists some $\xi > 0$ such that $\gamma'(x) \geq \frac{1+\ell}{2} > 1$ for all $x \geq \xi$. Let $a_0 = -1$ and $a_n = \gamma^{(n)}(a_0)$. If (a_n) were bounded, we could extract a subsequence that converges to a fixed point of γ. So $a_n \to \infty$. Let k be such that $a_k > \xi$. Hence

$$a_{n+k+1} - \xi \ \geq \ \gamma(a_{n+k}) - \gamma(\xi) \ > \ \tfrac{1+\ell}{2}(a_{n+k} - \xi)$$

shows that $a_n \to +\infty$ exponentially. By monotonicity of γ we infer the same for $\gamma^{(n)}(x) \geq a_n$ for all $x \geq -1$. This, together with $\gamma'(x) = \ell + \mathcal{O}(x^{-\delta})$ shows that

$$P(x) = \prod_{n=0}^{\infty} \frac{\gamma'(\gamma^{(n)}(x))}{\ell}$$

converges absolutely and uniformly on $[-1,\infty)$. P vanishes nowhere and satisfies $P \circ \gamma = \frac{\ell}{\gamma'} P$. We define

$$\psi(x) := \int_1^x P(t)\,dt + C$$

where the constant C is to be determined. By construction, ψ is strictly increasing and satisfies

$$\psi \circ \gamma(x) = \int_{\gamma(-1)}^{\gamma(x)} P(t)\,dt + C = \ell \int_{-1}^{x} P(t)\,dt + C = \ell \int_{-1}^{1} P(t)\,dt + \ell\psi + C(1-\ell)$$

So that, letting $C = \frac{\ell}{\ell-1}\int_{-1}^{1} P(t)\,dt > 0$ ensures $\psi \circ \gamma = \ell\psi$ as required. Then $\varphi := \frac{\ln \psi}{\ln(\ell)}$ is of class C^1, strictly increasing.
If additionally γ' decreases towards ℓ at infinity, a new lecture of the above growth rate of (x_n) shows that $\limsup \frac{\ell^n}{x_n} \leq 1$ for any $x_0 \geq -1$. Therefore, the (termwise differentiated product P) yields a series

$$\sum_n \gamma''(x_n)\Big(\prod_{j=0}^{n-1} \gamma'(x_j)\Big)\Big(\prod_{k\neq n} \frac{\gamma'(x_n)}{\ell}\Big)$$

that converges normally on $[-1,\infty)$. We infer that P is of class C^1, hence ψ and φ of class C^2. $\square$

In the situation that $\lim s'(t) = \jmath = 0$ and hence $\lim \gamma'(t) = 1$ things are more delicate. If γ is such that $\gamma'(x) = 1 + o(x^{-\delta})$ at infinity, for all x, y,

$$\lim_{n\to\infty} \frac{\gamma^{(n+1)}(x) - \gamma^{(n)}(x)}{\gamma^{(n+1)}(y) - \gamma^{(n)}(y)} = 1.$$

We leave the proof as exercise, as it is a modification of [24, Lemma 7.3]. Consequently, whenever

$$\varphi(x) := \lim_{n\to\infty} \frac{\gamma^{(n)}(x) - \gamma^{(n)}(x_0)}{\gamma^{(n+1)}(x_0) - \gamma^{(n)}(x_0)}$$

exists, φ is a solution to Abel's equation (A). This is the P. Lévy's algorithm, see e.g. [24, Chapter VII]. In order to ensure existence of a solution we will in general have to get a finer control of the asymptotics. The next result in this direction is based on ideas of Szekeres [35, Theorem 1c], see also [24, Theorem 7.2]). The principal idea is similar to Theorem A.1, but we have to transform differently and to be more careful how to construct an infinite product.

Theorem A.2. *If $\gamma'(x) = 1 + a(1-\delta)x^{-\delta} + o(x^{-\delta})$ at infinity, where $a > 0$ and $\delta > 0$, $\delta \neq 1$, then Abel's equation (A) has a strictly positive and strictly increasing* C^1*-solution φ.*

We mention as an example that the parabolic and shrinking domains mentioned in the introduction satisfy the hypothesis of the theorem.

Proof. First observe that $\frac{\gamma(x)}{x} = 1 + ax^{-\delta} + o(x^{-\delta})$, by integrating γ' on $[0, x]$ or $[x, \infty)$ according to $\delta < 1$ or $\delta > 1$. First we transform our problem into a multiplicative version. To this end, let $g : [-1, \infty) \to (0, \infty)$ be a C^1-function. Then, whenever φ solves Abel's equation (A), $\psi(x) = g(x)\varphi'(x)$ satisfies

$$(\psi \circ \gamma)(x) = g(\gamma(x))\varphi'(\gamma(x)) = g(\gamma(x))\frac{\varphi'(x)}{\gamma'(x)} = \frac{g(\gamma(x))}{g(x)\gamma'(x)}\psi(x) =: m(x)\psi(x)$$

Let $x_n = \gamma^{(n)}(x)$. If (x_n) were bounded, it would converge to a fixed point of γ — but there is none. So $x_n \to +\infty$. Assume that we chose the function g such that

$$\sum_n \left| \frac{g(x_n)\gamma'(x_n)}{g(x_{n+1})} - 1 \right| \tag{A.2}$$

converges uniformly on compact intervals. Then the infinite product

$$P(x) = \prod_{n=0}^{\infty} \tfrac{1}{m(\gamma^{(n)}(x))} = \prod_{n=0}^{\infty} \frac{g(x_n)\gamma'(x_n)}{g(x_{n+1})}, \tag{A.3}$$

defines a continuous function P that solves $\psi \circ \gamma = m \cdot \psi$. From P we then easily regain φ. We chose $g(x) = \gamma(x)^{1-\delta}$. Then $P(x) > 0$ for all x. Moreover we have the following asymptotics for $x \to \infty$:

$$\begin{aligned} 1 - \gamma'(x)\left(\frac{x}{\gamma(x)}\right)^{1-\delta} &= 1 - \tfrac{1}{(1+ax^{-\delta}+r_1(x))^{1-\delta}}\left(1 + a(1-\delta)x^{-\delta} + \widetilde{r_1}(x)\right) \\ &= 1 - \left(1 - a(1-\delta)x^{-\delta} + r_2(x)\right)\left(1 + a(1-\delta)x^{-\delta} + \widetilde{r_2}(x)\right) \\ &= a^2(1-\delta)^2x^{-2\delta} + r(x). \end{aligned}$$

where $r_1, r_2, \widetilde{r_1}\widetilde{r_2} = o(x^{-\delta})$ and $r = o(x^{-2\delta})$ for $x \to \infty$. Next, we need a growth rate for the orbits $x_n = \gamma^{(n)}(x_0)$: Observe that $a = \lim_{n\to\infty} \frac{\gamma(x_n)-x_n}{x_n^{1-\delta}} = \lim_{n\to\infty} \frac{x_{n+1}-x_n}{x_n^{1-\delta}}$. Rewriting the right hand side we obtain

$$a = \lim_{n\to\infty} (x_n^\delta - x_{n+1}^\delta)\left(\tfrac{x_{n+1}}{x_n}\right)^{-\delta} \frac{\frac{x_{n+1}}{x_n} - 1}{\left(\frac{x_{n+1}}{x_n}\right)^{-\delta} - 1}.$$

Using $\frac{x_{n+1}}{x_n} = \frac{\gamma(x_n)}{x_n} \to 1$ as $n \to \infty$ the last fraction has limit $-1/\delta$ and we obtain

$$\delta a = \lim_{n\to\infty} (x_{n+1}^\delta - x_n^\delta).$$

Taking Cesaro sums,

$$\delta a = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1}(x_{j+1}^\delta - x_j^\delta) = \lim_{n\to\infty} \frac{1}{n}x_n^\delta.$$

We infer finally $x_n \sim c\, n^{1/\delta}$ when $n \to \infty$. Putting both parts together,

$$\left| \frac{g(x_n)\gamma'(x_n)}{g(x_{n+1})} - 1 \right| = a^2(1-\delta)^2x_n^{-2\delta} + r(x_n) = a^2(1-\delta)^2n^{-2} + r(x_n)$$

where $r(x_n) = o(n^{-2})$. Therefore (A.2) converges absolutely and uniformly on compact intervals so that (A.3) converges to a strictly positive function P. For $C > 0$ to be determined in a moment, we let

$$\varphi(x) := C\int_1^x \frac{P(t)}{\gamma(t)^{1-\delta}}\,dt.$$

P and γ being strictly positive, φ is positive, strictly increasing and of class C^1. Moreover,

$$\begin{aligned}\varphi(\gamma(x)) &= C\int_{\gamma(-1)}^{\gamma(x)} \frac{P(t)}{\gamma(t)^{1-\delta}}\,dt \;=\; C\int_{-1}^{x} \frac{P(\gamma(s))}{\gamma(\gamma(s))^{1-\delta}}\gamma'(s)\,ds \\ &= C\int_{-1}^{x} \frac{P(s)m(s)}{\gamma(\gamma(s))^{1-\delta}}\gamma'(s)\,ds \;=\; C\int_{-1}^{x} \frac{P(t)}{\gamma(t)^{1-\delta}}\,dt \\ &= \varphi(x) + C\int_{-1}^{1} \frac{P(t)}{\gamma(t)^{1-\delta}}\,dt,\end{aligned}$$

so that adjusting C (the integral being strictly positive) we obtain a solution of Abel's equation (A). □

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