H^∞–FUNCTIONAL CALCULUS FOR GENERATORS OF SEMIGROUPS THAT ADMIT LOWER BOUNDS

BENHARD H. HAAK AND PEER CHR. KUNSTMANN

Abstract. We study C_0-semigroups on UMD Banach spaces under the assumption that a single semigroup operator admits a lower bound. We establish boundedness of H^∞-functional calculi for the negative generator of such semigroups. Our approach is based on a dilation argument: combining a recent construction due to Madani with transference results for groups on UMD spaces, we embed the semigroup into a C_0-group on a larger space and transfer functional calculus estimates back to the original generator. As a byproduct, we obtain quantitative exponential lower bounds for the semigroup. We also show that equivalences due to Batty and Geyer, valid in Hilbert spaces, fail in the general Banach space setting.

1. Introduction

The bounded H^∞-functional calculus for negative generators of C_0-semigroups on Banach spaces has become a central tool in modern analysis, with applications ranging from evolution equations to harmonic analysis. For a strongly continuous group $(S(t)) = (e^{-itA})$ on a Hilbert space, Boyadzhiev and de Laubenfels [2] showed that A has a bounded H^∞-functional calculus on a horizontal strip whose height is determined by the growth bounds of the group. This result has been extended to bounded groups on UMD Banach spaces by Hieber and Prüss [7] using transference principles. They showed that, for generators A of bounded C_0-groups, the operator iA admits a bounded H^∞-calculus on a suitable double-sector symmetric about the real axis. This was later extended by Haase [6] to strongly continuous groups on UMD spaces with arbitrary growth bounds, at the expense of obtaining a bounded H^∞ functional calculus on "Venturi regions" which are unions of double sectors and strips.

A natural question is how to obtain such results for semigroups that are not groups. A classical approach consists in dilating a semigroup into a group. For instance, the Sz.-Nagy dilation theorem shows that a contraction semigroup on a Hilbert space admits a unitary dilation, i.e., it can be realized as the compression of a unitary group acting on a larger Hilbert space. Fröhlich and Weis [4] constructed a dilation of C_0-semigroups on Banach spaces with finite cotype by means of square function estimates, thereby providing another route to functional calculus results. Recently, Madani [12] showed that a single lower bound for a bounded operator on a reflexive Banach space suffices to construct an invertible dilation on a reflexive space. The purpose of this paper is to combine Madani's dilation technique with the findings of Haase in order to derive boundedness of the H^∞-functional calculus for negative generators of C_0-semigroups from a simple lower estimate at a single time. More precisely, we show that if a semigroup $(T(t))$ in a UMD space X satisfies

$$\forall x \in X: \; \|T(t_0)x\| \geq c\,\|x\|$$

2020 *Mathematics Subject Classification.* Primary: 47A60, 47D06; Secondary: 46B20, 47A10, 47B40, 47A20.

Key words and phrases. H^∞-functional calculus, C_0-semigroups, dilations, Banach spaces (especially UMD spaces), operator theory.

for some $t_0, c > 0$, then its negative generator A admits a bounded H^∞-calculus on regions that are unions of sectors and half-planes. The argument proceeds by constructing a dilation of the semigroup to a C_0-group on a suitable UMD space Y, using Madani's approach. The functional calculus for the generator of the dilation group in Y that we have by Haase's result is then (partly) transferred back to the negative generator A of the initial semigroup.

We also record a quantitative lower estimate for semigroups that follows from the dilation construction and may be of independent interest.

Notation and Main result. We write $\mathcal{L}(X)$ for the space of bounded operators on a Banach space X. For a fixed operator $T \in \mathcal{L}(X)$ we write $\{T\}'$ for its commutant, i.e. the algebra of all $U \in \mathcal{L}(X)$ such that $UT = TU$. For any nontrivial complex domain $\mathscr{O}$ we denote by $H^\infty(\mathscr{O})$ the space of bounded holomorphic functions on $\mathscr{O}$. For $\sigma \in (0, \pi)$ we denote by Σ_σ the open sector

$$\Sigma_\sigma = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \sigma\}.$$

We call an operator A in a Banach space X sectorial, if A is densely defined with dense range and there exists $\omega \in (0, \pi)$ such that the spectrum $\sigma(A)$ of A is contained in $\overline{\Sigma_\omega}$ and we have, for any $\sigma \in (\omega, \pi)$,

$$\sup\{\|\lambda R(\lambda, A)\|_{\mathcal{L}(X)} : \lambda \in \mathbb{C} \setminus \overline{\Sigma_\sigma}\} < \infty.$$

In this case, we say that A has a bounded $H^\infty(\Sigma_\sigma)$-calculus for a fixed $\sigma > \omega$ if there exists $C > 0$ such that we have

$$\|f(A)\|_{\mathcal{L}(X)} \le C \, \|f\|_{H^\infty(\Sigma_\sigma)}.$$

for all $f \in H^\infty(\Sigma_\sigma)$ with

$$\sup_{z \in \Sigma_\sigma} \frac{|z|^\varepsilon}{1 + |z|^{2\varepsilon}} |f(z)| < \infty$$

for some $\varepsilon > 0$, where the operator $f(A)$ for such f is defined by the absolutely convergent integral

$$f(A) = \frac{1}{2\pi i} \int_{\partial \Sigma_\eta} f(\lambda) R(\lambda, A) \, d\lambda$$

with $\eta \in (\omega, \sigma)$. Here, $\partial \Sigma_\eta$ is parametrized such that Σ_ω lies to the left. If A has a bounded $H^\infty(\Sigma_\sigma)$-calculus in this sense then the map $f \mapsto f(A)$ has indeed a unique extension to an algebra homomorphism $H^\infty(\Sigma_\sigma) \to \mathcal{L}(X)$. For this and further details we refer to [5].

We recall that a Banach space X has the UMD property if the Hilbert transform extends to a bounded operator on $L^p(\mathbb{R}; X)$ for some (equivalently, all) $1 < p < \infty$. We refer to [9] for detailed discussion and references.

Theorem 1.1. *Let X be a* UMD *space and let $(T(t))_{t \ge 0}$ be a C_0-semigroup on X with generator $-A$ and with growth bound $\|T(t)\| \le M \, e^{\omega t}$, $t \ge 0$, where $M \ge 1$ and $\omega \in \mathbb{R}$. Assume that the semigroup satisfies the lower bound*

$$\|T(t_0)x\|_X \ge c \, \|x\|, \quad (x \in X) \tag{1.1}$$

for some $t_0, c > 0$. Then, for any $\sigma > \frac{\pi}{2}$ and any $\theta > \omega$, $A + \theta$ admits a bounded $H^\infty(\Sigma_\sigma)$ calculus.

In the final part of this section, we consider a C_0-semigroup $(T(t))_{t \ge 0}$ in an arbitrary Banach space X and assume that $t_0, c > 0$ are such that the lower bound (1.1) holds. As a consequence of Madani's construction that we review in Section 2, we can then show the following.

Proposition 1.2. *Let X be a Banach space and $(T(t))_{t\geq 0}$ a C_0-semigroup on X. Assume $t_0, c > 0$ are such that the lower bound (1.1) holds. Then we have the following:*

a) For all $\alpha > 1$ there exists $m > 0$ such that

$$\|T(t)x\| \geq m\, e^{\nu t}\|x\|, \quad x \in X, t > 0.$$

where $\nu = \frac{1}{t_0}\ln(\frac{c}{\alpha})$.

b) There exists a larger Banach space $Y \supseteq X$, containing X as a closed subspace, and a C_0-group $(S(t))_{t\in\mathbb{R}}$ on Y such that $T(t) = S(t)$ on Y for all $t \geq 0$. Moreover, one can achieve that Y is reflexive if X is reflexive and that Y is UMD *if X is* UMD*.*

Part a) seems not to have been clearly stated in the literature. Considering, in this situation,

$$\gamma(t) := \big(\inf\{\|T(t)x\| : x \in X, \|x\| = 1\}\big)^{-1}, \quad t > 0,$$

one has $\gamma(t+s) \leq \gamma(t)\gamma(s)$ for $s, t > 0$ by the semigroup property which implies exponential boundedness of $\gamma(t)$ for $t \to \infty$. However, boundedness of $\gamma(t)$ as $t \to 0+$ is less clear, as there exist semigroups $(V(t))_{t>0}$ which are strongly continuous on $(0, \infty)$ and for which $\|V(t)\|$ grows arbitrarily fast as $t \to 0+$, see e.g. Example 3.1 below.

If X is a Hilbert space then one can say more than is stated above. In [1, Theorem 7.3] the following is shown.

Theorem 1.3 (Batty-Geyer)**.** *Let $T(t)$ be a strongly continuous semigroup on a Hilbert space X with generator $-A$. Then the following are equivalent.*

a) T satisfies lower bounds, i.e. $\|T(t)x\| \geq c(t)\|x\|$ for some strictly positive function c.

b) There exists a left inverse semigroup S of T on X.

c) There exists a C_0-group extension $(S(t))_{t\in\mathbb{R}}$ of $(T(t))_{t\geq 0}$ on a larger Hilbert space $Y \supseteq X$ that contains X as a closed subspace.

In Proposition 1.2 we have seen that a single lower bound implies a) and c) on arbitrary Banach spaces. This raises the question whether the implication a) $\Rightarrow$ b) remains valid on general Banach spaces, or at least within well-behaved classes such as UMD spaces. This is however not true, as we will show in Example 3.2 below.

The paper is organized as follows: In Section 2 we review Madani's construction of a dilation. In Section 3 we prove Proposition 1.2 and give examples, and in Section 4 we prove Theorem 1.1 and give a more precise result as a corollary to the proof.

2. Madani's construction of the dilation

We review Madani's construction of the dilation from [12]. Since this lies at the heart of all that is to follow we give full details.

2.1. Preliminaries.

We start with with some basic facts that are easily verified.

Let X be a Banach space and $p \in (1, \infty)$. For any operator $U \in \mathcal{L}(X)$ we denote by $\widehat{U} \in \mathcal{L}(\ell_p(\mathbb{N}; X)$ the operator given by $\widehat{U}(x_n)_n := (Ux_n)_n$. We note that $\|\widehat{U}\| = \|U\|$.

If $(T(t))_{t\geq 0}$ is a C_0-semigroup in X then $(\widehat{T}(t))_{t\geq 0}$ is a C_0-semigroup in $\ell_p(\mathbb{N}; X)$. Strong continuity follows from strong continuity on finite sequences and

$$\sup_{t\in[0,1]} \|\widehat{T}(t)\| = \sup_{t\in[0,1]} \|T(t)\| < \infty.$$

Now suppose that F is a closed subspace of $\ell_p(\mathbb{N};X)$. We denote by Y_F the quotient space $\ell_p(\mathbb{N};X)/F$ and write $[x] = x + F$ for the co-class of $x \in \ell_p(\mathbb{N};X)$.

Lemma 2.1. *If F is invariant under $\widehat{U}$, where $U \in \mathcal{L}(X)$, then $\Phi_F(U)[x] = [\widehat{U}x]$ defines an operator $\Phi_F(U) \in \mathcal{L}(Y_F)$ and we have $\|\Phi_F(U)\| \leq \|U\|$.*

If $(T(t))_{t\geq 0}$ is a C_0-semigroup in X and F is invariant under all operators $\widehat{T}(t)$, $t \geq 0$, then $\big(\Phi_F(T(t))\big)_{t\geq 0}$ is a C_0-semigroup in Y_F.

Proof. The assertion on $\Phi_F(U)$ is standard. The semigroup property of $\big(\Phi_F(T(t))\big)$ is inherited from $(\widehat{T}(t))$ and strong continuity follows from

$$\|\Phi_F(T(t))[x] - [x]\|_{Y_F} = \|[\widehat{T}(t)x - x]\|_{Y_F} \leq \|\widehat{T}(t)x - x\|_{\ell_p(\mathbb{N};X)} \to 0 \ (t \to 0+)$$

for all $x \in \ell_p(\mathbb{N};X)$. □

We shall also need the right shift R on $\ell_p(\mathbb{N};X)$ defined by $R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$. We note that R is an isometry onto its range and that R commutes with every operator $\widehat{U}$, $U \in \mathcal{L}(X)$.

2.2. **Madani's dilation.** We assume that $T \in \mathcal{L}(X)$ satisfies the lower bound

$$\|Tx\| \geq c\,\|x\|, \qquad x \in X, \tag{2.1}$$

where $c > 0$. Madani constructed a dilation of T to an isomorphism S on a Banach space Y that contains (an isomorphic copy of) X as a closed subspace ([12]). Moreover, Y is reflexive if X is reflexive, which was the main motivation for the construction.

We denote by $j : X \to \ell_p(\mathbb{N};X)$ the injection $j(x) = (x, 0, 0, \ldots)$. The idea is to construct a suitable $G \in \mathcal{L}(\ell^p(\mathbb{N};X))$ with closed range and commuting with $\widehat{T}$ such that, for $F = \operatorname{ran}(G)$ and the quotient space $Y := Y_F = \ell_p(\mathbb{N};X)/\operatorname{ran}(G)$ with natural quotient map q and $S := \Phi_F(T)$, we have

$$\begin{array}{ccc} \ell_p(\mathbb{N};X) & \xrightarrow{\ \widehat{T}\ } & \ell_p(\mathbb{N};X) \\ \downarrow q & & \downarrow q \\ Y & \xrightarrow{\ S\ } & Y \\ \uparrow \iota & & \uparrow \iota \\ X & \xrightarrow{\ T\ } & X \end{array} \qquad (j : X \to \ell_p(\mathbb{N};X)) \tag{2.2}$$

with $\iota := q \circ j : X \to Y$ being an isomorphism onto its range and S being an isomorphism of Y. This was achieved by Madani [12] elaborating on some ideas from [1].

Theorem 2.2 (Madani [12])**.** *Fix $\alpha > 1$ and $p \in (1,\infty)$ such that $\alpha \geq 2^{1-1/p}$. Let $T \in \mathcal{L}(X)$ satisfy (2.1) where $c > 0$. Let $G := I - \frac{\alpha}{c}\widehat{T}R$.*

(i) Then $F := \operatorname{ran}(G)$ is closed and for $Y := Y_F$ with quotient map q the map $\iota := q \circ j : X \to Y$ is an isomorphism onto its range and we have

$$\|\iota x\|_Y \leq \|x\|_X \leq \alpha\|\iota x\|_Y, \quad x \in X.$$

(*ii*) *The operator* $S := \Phi_F(T) : Y \to Y$ *is an isomorphism with* $\|S\| \le \|T\|$ *and* $\|S^{-1}\| \le \frac{\alpha}{c}$ *and the following diagram commutes.*

$$\begin{array}{ccc} Y & \xrightarrow{\ S\ } & Y \\ {\scriptstyle\iota}\uparrow & & \uparrow{\scriptstyle\iota} \\ X & \xrightarrow{\ T\ } & X \end{array}$$

(*iii*) *If* X *is reflexive then* Y *is reflexive.*

Remark 2.3. (a) For fixed $\alpha > 1$ we can choose $p \in (1, \infty)$ such that $2^{1-1/p} \le \alpha$. Hence $\|S^{-1}\|$ can be made arbitrarily close to $\frac{1}{c}$.

(b) The construction respects other geometric properties of the Banach space X. Below we shall use that Y is UMD if X is UMD, see, e.g. [9, Proposition 4.2.17]. But also a lot of other properties persist such as, e.g., uniform convexity or B-convexity or Pisier's property (α). We do not elaborate on this further.

Proof of Theorem 2.2. **Step 1: The construction of F.** Let $G = I - \frac{\alpha}{c}\widehat{T}R$. By the assumption on α and p, we have

$$(|a| + |b|)^p \le |\alpha a|^p + |\alpha b|^p \qquad a, b \in \mathbb{C}. \tag{2.3}$$

By the triangle inequality and (2.1) we infer

$$\|Gx\|_{\ell_p(\mathbb{N};X)} \ge (\alpha - 1)\|x\|, \qquad x \in \ell_p(\mathbb{N}; X),$$

and so $F = \mathrm{ran}\,(G)$ is closed by $\alpha > 1$.

Step 2: Lower bounds for the injection ι. Recall we defined $Y = \ell_p(\mathbb{N}; X)/F$ with quotient map q as well as $\iota = q \circ j$. This implies immediately $\|\iota\| \le 1$. On the other hand, (2.3) ensures

$$\|y\|_X^p \le (\|y - z\|_X + \|z\|_X)^p \le \|\alpha(y - z)\|_X^p + \|\alpha z\|_X^p$$

Applying this inequality as well as (2.1) iteratively to a sequence $(z_k) \in \ell_p(\mathbb{N}; X)$, we obtain

$$\begin{aligned} \|x\|_X^p &\le \|\alpha(x - z_1)\|_X^p + \|\alpha z_1\|_X^p \\ &\le \|\alpha(x - z_1)\|_X^p + \|\tfrac{\alpha}{c}Tz_1\|_X^p \\ &\le \|\alpha(x - z_1)\|_X^p + \|\alpha(\tfrac{\alpha}{c}Tz_1 - z_2)\|_X^p + \|\alpha z_2\|_X^p \\ &\le \|\alpha(x - z_1)\|_X^p + \|\alpha(\tfrac{\alpha}{c}Tz_1 - z_2)\|_X^p + \|\tfrac{\alpha}{c}Tz_2\|_X^p \\ &\le \|\alpha(x - z_1)\|_X^p + \|\alpha(\tfrac{\alpha}{c}Tz_1 - z_2)\|_X^p + \|\alpha(\tfrac{\alpha}{c}Tz_2 - z_3)\|_X^p + \|\alpha z_3\| \end{aligned}$$

etc. We infer that

$$\|x\|_X^p \le \alpha^p\|(x - z_1)\|_X^p + \alpha^p \sum_{k=1}^{\infty} \|\tfrac{\alpha}{c}Tz_k - z_{k+1}\|_X^p = \alpha^p \big\|j(x) - (I - \tfrac{\alpha}{c}\widehat{T}R)z\big\|_{\ell_p(\mathbb{N};X)}.$$

This implies $\|x\|_X \le \alpha\|\iota(x)\|_Y$, so that $\iota : X \to Y$ is injective and has closed range.

Step 3: Construction and invertibility of S. Since $\widehat{T}$ and G commute, $F = \mathrm{ran}\,(G)$ is invariant under $\widehat{T}$, so that

$$S = \Phi_F(T) : Y \to Y, \qquad S([z]) = [\widehat{T}z].$$

is well-defined, cf. Lemma 2.1. For every $z \in \ell_p(\mathbb{N}; X)$ we have

$$Gz = z - \tfrac{\alpha}{c}\widehat{T}Rz \in \mathrm{ran}\,(G) = F.$$

Hence in the quotient space Y,

$$[z] = [\tfrac{\alpha}{c}\widehat{T}Rz]. \tag{2.4}$$

Now let $L[y] := \frac{\alpha}{c}[Rz]$ for any $z \in [y]$. This is well-defined, since $RG = GR$ implies that $F = \operatorname{ran}(G)$ is invariant under R. Now we calculate

$$SL[z] = \tfrac{\alpha}{c}\, S[Rz] = \tfrac{\alpha}{c}\, [\widehat{T}Rz] = \tfrac{\alpha}{c}\, [\widehat{T}Rz] \overset{(2.4)}{=\!=} [z]$$

so that L is a right inverse of S. In the same way,

$$LS[z] = L[\widehat{T}z] = \tfrac{\alpha}{c}\, [\widehat{T}Rz] \overset{(2.4)}{=\!=} [z].$$

We observe that $\|S\| \le \|T\|$ and that $\|S^{-1}\| = \|L\| \le \frac{\alpha}{c}$. □

Remark 2.4. *Theorem 2.2 extends to settings where $T_1 \in \mathcal{L}(X,Y)$ admits lower bounds: let X, Y be Banach spaces, and $\|T_1x\|_Y \ge c\,\|x\|_X$. Then T_1 admits a dilation to an invertible operator S. The idea is to consider the space $Z := \ell_p(\mathbb{Z}_{\le 0}; X) \oplus \ell_p(\mathbb{Z}_{>0}; Y)$ and the bounded operator T given by $T(z_n)_n = (w_n)_n$ where for all $n \neq 0$, $w_n = z_{n+1}$ and $w_0 = \frac{1}{c}T_1z_1$. Now, $\|Tz\|_Z \ge \|z\|_Z$ and we can apply Theorem 2.2. As in the main result, if X, Y are reflexive, so is Z, and the same is true for the* UMD *property.*

Remark 2.5. *Let $P : \ell_p(\mathbb{N}; X) \to X$ be the projection onto the first component, so that $P \circ j = I_X$. We observe that, in the above construction, no bounded projection $\pi : Y \to X$ with the property $P = \pi \circ q$ can exist (unless $X = \{0\}$, of course). Indeed, observe that $I_X = P \circ G \circ j$. Hence, assuming $P = \pi \circ q$ leads by $q \circ G = 0$ to the contradiction $I_X = \pi \circ q \circ G \circ j = 0$.*

The following corollary sharpens some of Madani's findings slightly.

Corollary 2.6 (Madani [12])**.** *Under the assumptions of Theorem 2.2, the map $\Phi := \Phi_F : \{T\}' \to \mathcal{L}(Y)$, $U \mapsto \Phi(U) := \Phi_F(U)$ is an algebra homomorphism satisfying*

$$\frac{1}{\alpha}\|U\| \le \|\Phi(U)\| \le \|U\|, \qquad U \in \{T\}',$$

for which the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\ \Phi(U)\ } & Y \\ {\scriptstyle\iota}\uparrow & & \uparrow{\scriptstyle\iota} \\ X & \xrightarrow{\ \ U\ \ } & X \end{array}$$

commutes, i.e. $\Phi(U)\iota = \iota U$. More precisely, for all $U \in \{T\}'$ and $x \in X$, we have

$$\frac{1}{\alpha}\|Ux\|_X \le \|\Phi(U)\iota x\|_Y \le \|Ux\|_X. \tag{2.5}$$

Proof. Let $U \in \{T\}'$, so that U and T commute. Then also $\widehat{U}$ and $\widehat{T}$ commute, and hence $\widehat{U}$ and G commute, which implies that $\widehat{U}$ leaves $F = \operatorname{ran}(G)$ invariant. Thus $\Phi(U) := \Phi_F(U)$ is well-defined, cp. Lemma 2.1.

Clearly, $\Phi(I_X) = I_Y$, and by definition Φ is linear. Now let $U, V \in \{T\}'$. Then

$$\Phi(UV)[z] = [\widehat{UV}z], \qquad \Phi(U)\Phi(V)[z] = [\widehat{U}\,\widehat{V}z].$$

Since $\widehat{UV} = \widehat{U}\,\widehat{V}$ on $\ell_p(\mathbb{N}; X)$, the multiplicativity follows. Finally, observe

$$\Phi(U)\iota x = \Phi(U)[j(x)] = [\widehat{U}j(x)] = [j(Ux)] = \iota(Ux)$$

Using the norm equivalence on $\iota(X)$ from Theorem 2.2 and the previous identity we obtain, for all $x \in X$,

$$\frac{1}{\alpha}\|Ux\|_X \leq \|\Phi(U)\iota x\|_Y = \|\iota(Ux)\|_Y \leq \|Ux\|_X.$$

This implies $\frac{1}{\alpha}\|U\|_{\mathcal{L}(X)} \leq \|\Phi(U)\|_{\mathcal{L}(Y)}$ on the one hand, while $\|\Phi(U)[z]\| = \|[\widehat{U}z]\| \leq \|U\|_{\mathcal{L}(X)}\|[z]\|$ implies $\|\Phi(U)\|_{\mathcal{L}(Y)} \leq \|U\|_{\mathcal{L}(X)}$. □

3. Results and examples concerning semigroups

Proof of Proposition 1.2. Fixing $\alpha > 1$ we choose $p \in (1,\infty)$ such that $\alpha \geq 2^{1-1/p}$. Applying Theorem 2.2 and Corollary 2.6 to $T = T(t_0)$ we obtain Y, $\iota : X \to Y$ and $\Phi : \{T(t_0)\}' \to \mathcal{L}(Y)$. We define $S(t) := \Phi(T(t))$ for $t \geq 0$, which is a C_0-semigroup in Y by Lemma 2.1, and have that $S(-t_0) : Y \to Y$ is bijective with $\|S(t_0)^{-1}\| \leq \frac{\alpha}{c}$. But a C_0-semigroup extends to a C_0-group if one of the semigroup operators is invertible, see e.g. [8, Theorem 16.3.6]. Hence we obtain a C_0-group $(S(t))_{t\in\mathbb{R}}$ in Y extending the given semigroup.
We now examine the growth bound of $(S(-t))_{t\geq 0}$. We have, for $0 \leq \delta \leq t_0$,

$$\|S(-nt_0 - \delta)\| \leq K e^{\ln(\frac{\alpha}{c})n}$$

where $K = \max\{\|S(-\delta)\| : 0 \leq \delta \leq t_0\}$. A standard calculation thus yields

$$m\, e^{\nu t}\, \|x\| \leq \|S(t)x\| \quad x \in X, t > 0,$$

for $\nu = \frac{1}{t_0}\ln(\frac{c}{\alpha})$ and some $m > 0$. Now by (2.5) we obtain, for any $x \in X$,

$$\|T(t)x\|_X \geq \|\Phi(T(t))\iota x\|_Y = \|S(t)\iota x\| \geq m\, e^{\nu t}\, \|\iota x\| \geq \frac{m}{\alpha}\, e^{\nu t}\, \|x\|. \qquad \square$$

Example 3.1. *The following example shows that strong continuity of a semigroup on $(0,\infty)$ does not imply growth bounds as $t \to 0+$. In particular, a function $\gamma : (0,\infty) \to [0,\infty)$ satisfying $\gamma(t+s) \leq \gamma(t)\gamma(s)$ for all $s,t > 0$ can grow arbitrarily fast for $t \to 0+$.*
Consider $X = L^p(\mathbb{R}_+) \times L^p(\mathbb{R}_+)$ and let

$$A\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \phi(x) & -e^x \\ 0 & \phi(x) \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function of class C^1. Let us assume additionally $\phi(0) = \phi'(0) = 0$ and that $\phi' : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing with $\phi'(x) \to \infty$ for $x \to \infty$ and $\phi'(\log x) \leq x$ for $x > 1$. A direct calculation shows

$$\exp(tA) = e^{-t\,\phi(x)}\begin{pmatrix} 1 & te^x \\ 0 & 1 \end{pmatrix},$$

and it follows that $\|\exp(tA)\| \simeq \sup_{x>0}\, e^{-t\,\phi(x)}\max\{1, te^x\}$. Now $\sup_{x>0} e^{-t\,\phi(x)} = 1$ and for $t \in (0,1)$ we thus have

$$\sup_{x>0}\, e^{-t\,\phi(x)}\max\{1, te^x\} = \max\{1, \sup\{te^{x-t\phi(x)} : x \geq \log(1/t)\}\}$$

and

$$\sup_{x\geq\log(1/t)} t\exp(x - t\,\phi(x)) = t\exp\Big(t \sup_{x\geq\log(1/t)} \big(\frac{x}{t} - \phi(x)\big)\Big)$$

Let $\phi^*(s) := \sup_{x>0}\ (sx - \phi(x))$ denote the convex conjugate, or Young dual function of ϕ and observe that, for $s > 1$ and under our assumptions, the sup *is attained at x_0 with $\phi'(x_0) = s \geq \phi'(\log s)$ which means $x_0 \geq \log s$. Hence*

$$\sup_{x\geq\log(1/t)} \left(\frac{x}{t} - \phi(x)\right) = \phi^*\left(\frac{1}{t}\right), \qquad t \in (0,1).$$

As a consequence, the operator norm of the semigroup can grow arbitrarily fast as $t \to 0+$. For example, if $\phi(x) \sim x \log^{\circ k}(x)$, then $\phi^*(s) \sim \exp^{\circ(k+1)}(s)$ as $s \to \infty$.

Example 3.2. *The following example shows there exist semigroups on non-Hilbertian Banach spaces that do allow a lower estimate, but no left inverse semigroup.*

Recall that every infinite dimensional, non-Hilbert Banach space contains a closed non-complemented subspace, see [11]. So let Z a UMD *Banach space, and $W \subseteq Z$ a non-complemented, closed subspace. We let $1 < p < \infty$,*

$$X = L^p(\mathbb{R}_-; Z) \oplus L^p(\mathbb{R}_+; W),$$

and $(T(t))_{t\geq 0}$ the left shift semigroup on X. It is isometric on X since we shift from W into Z and both carry the same norm. Fix $t_0 > 0$. We let

$$j_0 : \quad \begin{cases} W & \to & X \\ w & \mapsto & t_0^{-1/p}\, \mathbb{1}_{(0,t_0)} \otimes w \end{cases}, \qquad j_1 : \quad \begin{cases} Z & \to & X \\ z & \mapsto & t_0^{-1/p}\, \mathbb{1}_{(-t_0,0)} \otimes z \end{cases},$$

$$M_0 : \quad \begin{cases} X & \to & W \\ f & \mapsto & t_0^{1/p-1} \displaystyle\int_0^{t_0} f(t)\, dt \end{cases}.$$

Observe that $M_0 j_0 = I_W$ and that $j_1 I_W = T(t_0) j_0$. Now assume towards a contradiction that $S(t_0) \in \mathcal{L}(X)$ is a left inverse to $T(t_0)$. Then

$$P := M_0 S(t_0) j_1 \in \mathcal{L}(Z)$$

satisfies

$$P^2 = (M_0 S(t_0) j_1)^2 = M_0 S(t_0)\, T(t_0) j_0\, M_0 S(t_0) j_1 = M_0 j_0 M_0 S(t_0) j_1 = P$$

so that P is a projection satisfying $\operatorname{ran}(P) \subseteq \operatorname{ran} M_0 \subseteq W$. *Finally, for $w \in W$,*

$$Pw = M_0 S(t_0)\, j_1\, w = M_0 S(t_0)\, T(t_0) j_0\, w = M_0 j_0 w = w,$$

which contradicts W being not complemented. This example shows that the obstruction to a generalization of Theorem 1.3 lies in Banach space geometry.

4. Proof of the main result

As mentioned in the introduction, the argument relies on Madani's dilation that we reviewed in Section 2 and the functional calculus for groups obtained by Haase on strips using transference methods. For a nontrivial complex domain $\mathscr{O}$ we denote by $H_1^\infty(\mathscr{O})$ the space of functions $f \in H^\infty(\mathscr{O})$ for which $z \mapsto zf'(z)$ is bounded on $\mathscr{O}$, and we define $\mathcal{E}(\mathscr{O}) := \{f \in H_1^\infty(\mathscr{O}) : \sup_{z\in\mathscr{O}}(1+|z|)^2|f(z)| < \infty\}$. For $\alpha \in \mathbb{R}$ and $\beta > 0$ we define open half-planes $\mathbb{H}_\alpha$ and and open strips St_β by

$$\mathbb{H}_\alpha = \{z \in \mathbb{C} :\ \mathrm{Re}(z) > \alpha\}, \qquad \mathrm{St}_\beta = \{z \in \mathbb{C} : |\mathrm{Re}(z)| < \beta\}.$$

The result we shall use states that, for a group $(U(t))_{t\in\mathbb{R}}$ in a UMD space with $\|U(t)\| \le Me^{\omega|t|}$, $t \in \mathbb{R}$, the generator has a bounded $H_1^\infty(\mathrm{St}_\eta)$-calculus for any $\eta > \omega$. The precise construction of the functional calculus is reviewed in the proof.

Proof of Theorem 1.1 . Replacing A by $A + \omega + \delta$ with $\delta > 0$, if necessary, we may suppose $\omega < 0$ for the growth bound, i.e. exponential stability of the semigroup, and have to show that A has a bounded $H^\infty(\Sigma_\sigma)$-calculus for all $\sigma > \frac{\pi}{2}$.

Part b) of Proposition 1.2 and subsequent remarks ensure that the space Y we obtained by application of Theorem 2.2 and Corollary 2.6 to $T := T(t_0)$ is again a UMD space and that the C_0-semigroup $\big(\Phi(T(t))\big)_{t\ge 0}$ extends to a C_0-group $(S(t))_{t\in\mathbb{R}}$ in Y. These operators satisfy

$$S(t)\iota = \iota T(t) \qquad (t \ge 0). \tag{4.1}$$

Let $-A$ denote the generator of $(T(t))$ and $-A_Y$ the generator of $(S(t))$ (and not $i\,A_Y$ which is customary if one only deals with groups). From Proposition 1.2 we know that

$$m\, e^{\nu t}\,\|y\| \le \|S(t)y\| \le M\, e^{\omega t}\,\|y\| \qquad (y \in Y, t \ge 0)$$

for some $\nu < \omega < 0$, and some $m, M > 0$. In particular, the group type of S (see [5, p.302]) is $\le -\nu$ and the spectrum of A_Y lies in the strip

$$\mathrm{St}_{-\nu} = \{z : \quad |\mathrm{Re}(z)| \le -\nu\}.$$

Fix $\eta > -\nu$. Let us write $\mathcal{E}(\mathrm{St}_\eta) := \{f \in H_1^\infty(\mathrm{St}_\eta) : \ |f(z)| = O((1+|z|)^{-2})\}$. Then by Haase's generalisation [6, Theorem 3.6] of the Hieber–Prüss theorem [7], A_Y admits a bounded $H_1^\infty(\mathrm{St}_\eta)$-functional calculus, that is, there exists some $C > 0$ such that, for $f \in \mathcal{E}(\mathrm{St}_\eta)$ and $\sigma \in (-\nu, \eta)$, the absolutely convergent integral

$$f(A_Y) = \frac{1}{2\pi i}\int_{\partial\,\mathrm{St}_\sigma} f(z)R(z, A_Y)\,dz \tag{4.2}$$

satisfies $\|f(A_Y)\| \le C\|f\|_{H_1^\infty(\mathrm{St}_\eta)}$, see [6, Cor. 3.4]. Now consider $f \in \mathcal{E}(\mathbb{H}_{-\eta})$. By holomorphy, the right hand boundary of the curve integral in (4.2) can be shifted to the right, i.e. from $\mathrm{Re}(z) = \sigma$ to $\mathrm{Re}(z) = r$, for any $r > \eta$. Letting $r \to +\infty$, uniform boundedness of the resolvents and the decay of f make this curve integral vanish, so that

$$f(A_Y) = \frac{-1}{2\pi i}\int_{-\sigma+i\mathbb{R}} f(z)R(z, A_Y)\,dz \tag{4.3}$$

with absolutely converging integrals, which gives us a functional calculus on $\mathcal{E}(\mathbb{H}_{-\eta})$ for A_Y with the estimate $\|f(A_Y)\| \le C\|f\|_{H_1^\infty(\mathbb{H}_{-\eta})}$. The absolutely converging integral

$$f(A) := \frac{1}{2\pi i}\int_{\mathrm{Re}(z)=-\eta} f(z)\,R(z, A)\,dz \tag{4.4}$$

defines a bounded operator $f(A) \in \mathcal{L}(X)$ for all $f \in \mathcal{E}(\mathbb{H}_{-\eta})$. By the properties obtained in Corollary 2.6, we may apply Laplace transform to (4.1) and obtain

$$R(\lambda, A_Y) = \Phi(R(\lambda, A)) \qquad (\mathrm{Re}(\lambda) < -\omega).$$

Furthermore this entails $f(A_Y) = \Phi(f(A))$. Using Corollary 2.6 again we conclude

$$\|f(A)\|_{\mathcal{L}(x)} \le \alpha\|\Phi(f(A))\|_{\mathcal{L}(Y)} = \alpha\|f(A_Y)\|_{\mathcal{L}(Y)} \le \alpha C\|f\|_{H_1^\infty(\mathcal{E}(\mathbb{H}_{-\eta}))}.$$

We also have

$$\Phi((fg)(A)) = (fg)(A_Y) = f(A_Y)g(A_Y) = \Phi(f(A))\Phi(g(A)) = \Phi(f(A)g(A)),$$

so that the assignement $f \mapsto f(A)$ is multiplicative. Let

$$\varrho(z) = \frac{z}{(1+\eta'+z)^2} \qquad \varrho_n(z) = \varrho(nz) - \varrho(\frac{z}{n}).$$

We set $f_n(z) := \varrho_n(z)^2 f(z)$. Then $f_n \in \mathcal{E}(\mathbb{H}_{-\eta})$, $f_n \to f$ pointwise, boundedly, and

$$\|f_n(A)\| \le C\|f_n\|_{H_1^\infty(\mathbb{H}_{-\eta})} \le C'\|f\|_{H_1^\infty(\mathbb{H}_{-\eta})}.$$

By the convergence lemma, see e.g. [5, Chapter 5], $f(A)$ is bounded on X and $f_n(A) \to f(A)$ strongly, i.e. A has a $H_1^\infty(\mathbb{H}_{-\eta})$ functional calculus. Shifting both the operator and the function class, we obtain that $A+\eta$ has a bounded $H_1^\infty(\mathbb{H}_0)$ functional calculus and it is well known, see e.g. [6, Lemma 4.5], that $A+\eta$ then has a bounded $H^\infty(\Sigma_\sigma)$ functional calculus for all $\sigma > \frac{\pi}{2}$.

Since A is sectorial of angle $\frac{\pi}{2}$ and $0 \in \varrho(A)$, [10, Prop. 6.10] shows that A has a bounded $H^\infty(\Sigma_\sigma)$ functional calculus for all $\sigma > \frac{\pi}{2}$. □

An inspection of the proof shows that a little more can be said. For $a, r \in \mathbb{R}$ and $\frac{\pi}{2} < \sigma < \pi$ we consider the following sets

$$\mathbb{K}_{\sigma,a,r} := (a+\Sigma_\sigma) \cup \{z : \operatorname{Re}(z) > r\} = (a+\Sigma_\sigma) \cup \mathbb{H}_r. \tag{4.5}$$

For such sets the following folklore lemma becomes relevant.

Lemma 4.1. *Let $\eta > 0$. Then for any $\varepsilon > 0$, $a \in \mathbb{R}$, and $\sigma > \frac{\pi}{2}$ there exists $C_{\varepsilon,a,\sigma} > 0$ such that*

$$\|f\|_{H_1^\infty(\mathbb{H}_{-\eta})} \le C_{\varepsilon,\sigma} \, \|f\|_{H^\infty(\mathbb{K}_{\sigma,a,-\eta-\varepsilon})}$$

Proof. We use Cauchy's formula and combine the estimates from two repesentations. We limit our attention to the case $a > -\eta$, since otherwise $\mathbb{K}_{\sigma,a,-\eta} = a + \Sigma_\sigma$.

For $z \in \mathbb{H}_{-\eta}$ we distinguish 2 cases: we fix $\sigma' \in (\frac{\pi}{2}, \sigma)$ and define the bounded zone K by $\operatorname{Re}(z) > -\eta$ and

$$|z-a| \le \max\big\{2a, \frac{\eta+a}{|\cos(\sigma')|}\big\}.$$

For $z \in K$ we use

$$z\, f'(z) = \frac{1}{2\pi i} \int_{|\zeta - z| = \varepsilon/2} \frac{z\, f(\zeta)}{(\zeta - z)^2}\, d\zeta,$$

leading to

$$|z\, f'(z)| \le \; \frac{1}{2\pi}\, \pi\varepsilon \frac{4|z|}{\varepsilon^2}\, \|f\|_{H^\infty(\mathbb{K}_{\sigma,a,-\eta-\varepsilon})} \le \; \frac{4M}{\varepsilon}\, \|f\|_{H^\infty(\mathbb{K}_{\sigma,a,-\eta-\varepsilon})}.$$

For z in the complement $\mathbb{H}_{-\eta} \setminus K$ we use the fact that $|\arg(z)| \le \sigma' < \sigma$, which allows a suitable choice of $\delta \in (0, \sin(\sigma - \sigma'))$ to write

$$z\, f'(z) = \frac{1}{2\pi i} \int_{|\zeta - z| = \delta|z-a|} \frac{z f(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

Using $|z-a| \ge 2a$ we have

$$\frac{1}{2}|z| \le |z-a| \le \frac{3}{2}|z|,$$

which leads to

$$|z f'(z)| \le \frac{2\pi\delta|z-a||z|}{2\pi\delta^2|z-a|^2}\, \|f\|_{H^\infty(\mathbb{K}_{\sigma,a,-\eta-\varepsilon})} \le \frac{6}{\delta}\, \|f\|_{H^\infty(\mathbb{K}_{\sigma,a,-\eta-\varepsilon})}.$$

□

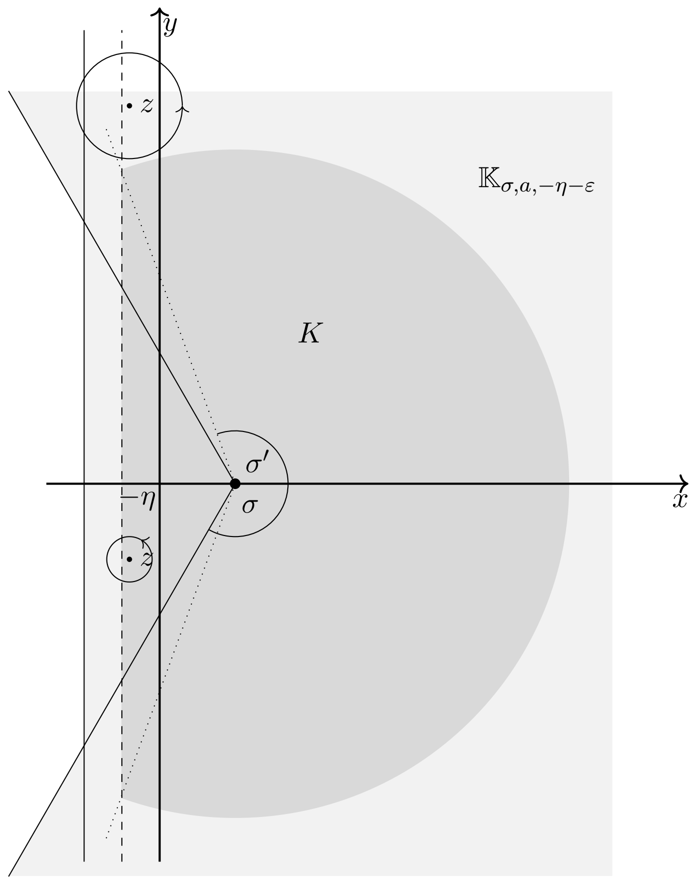

FIGURE 1. The region $\mathbb{K}_{\sigma,a,-\eta-\varepsilon}$, two cases for a point z in the half-plane $\mathbb{H}_{-\eta}$ and respective integration paths.

The following is a corollary to the proof of Theorem 1.1.

Corollary 4.2. *Under the assumptions of Theorem 1.1 the operator A has a bounded $H^\infty(\mathbb{K}_{\sigma,a,-\theta})$-calculus for any $\theta > \omega$, $\sigma > \frac{\pi}{2}$, and $a \in \mathbb{R}$.*

Here, for $f \in H^\infty(\mathbb{K}_{\sigma,a,-\theta})$ satisfying in addition $|f(z)| = O((1+|z|)^{-\delta})$ for some $\delta > 0$, the operator A is defined by the absolutely convergent integral

$$f(A) = \frac{1}{2\pi i}\int_{\partial\mathbb{K}_{\sigma',a,-\theta'}} f(\lambda)R(\lambda, A)\, d\lambda,$$

where $\sigma' \in (\frac{\pi}{2}, \sigma)$ and $\theta' \in (\omega, \theta)$. This defines a functional calculus $f \mapsto f(A)$, and boundedness of an $H^\infty(\mathbb{K}_{\sigma,a,-\theta})$-calculus means that there exists $C > 0$ with $\|f(A)\|_{\mathcal{L}(X)} \leq C\|f\|_{H^\infty(\mathbb{K}_{\sigma,a,-\theta})}$ for all such f. Again, we then have a unique extension to a bounded algebra homomorphism $H^\infty(\mathbb{K}_{\sigma,a,-\theta}) \to \mathcal{L}(X)$ via the convergence lemma.

Proof. Again, we resort to the case $\omega < 0$ and have to show the assertion for $\theta = 0$. Since we saw in the proof that A has a bounded $H_1^\infty(\mathbb{H}_{-\eta})$ functional calculus for all $\eta > -\nu$, Lemma 4.1

implies that it also has a bounded $H^\infty(\mathbb{K}_{\sigma,a,-\eta})$ functional calculus for all $\eta > -\nu$, all $\sigma > \frac{\pi}{2}$ and all $a \in \mathbb{R}$. The freedom of $a \in \mathbb{R}$ allows shifting, so that $A + \eta$ has a bounded $H^\infty(\mathbb{K}_{\sigma,a,0})$ functional calculus for all $a \in \mathbb{R}$ and all $\sigma > \frac{\pi}{2}$. Now we can mimic the main idea of the proof of [10, Prop. 6.10]. Let Γ be the boundary curve of $\mathbb{K}_{\sigma',a,-\theta'}$ where $\sigma' \in (\frac{\pi}{2}, \sigma)$ and $\theta' \in (\omega, 0)$. Then for bounded holomorphic functions $f \in \mathbb{K}_{\sigma,a,0}$ with $|f(z)| = O((1+|z|)^{-\delta}$ for $|z| \to \infty$ and some $\delta > 0$, we have three convergent integrals,

$$\begin{aligned}\tfrac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda,A)\,d\lambda &= \tfrac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda,A_\eta)\,d\lambda + \tfrac{\eta}{2\pi i}\int_\Gamma f(\lambda)R(\lambda,A+\eta)R(\lambda,A))\,d\lambda \\ &= f(A_\eta) + \tfrac{\eta}{2\pi i}\int_\Gamma f(\lambda)R(\lambda,A+\eta)R(\lambda,A))\,d\lambda.\end{aligned}$$

but the right hand side allows an upper estimate against $\|f\|_\infty$. The standard approximation argument then gives a bounded $H^\infty(\mathbb{K}_{\sigma,a,0})$-functional calculus for A. ☐

References

[1] Charles K.J. Batty, Felix Geyer, Lower bounds for unbounded operators and semigroups, Journal of Operator Theory 78, No.2 (2017), 473–500

[2] Khristo Boyadzhiev, Ralph Delaubenfels, Spectral Theorem for Unbounded Strongly Continuous Groups on a Hilbert Space, Proceedings of the American Mathematical Society, Vol. 120, No. 1 (1994), 127–136.

[3] Klaus.-Jochen Engel and Rainer Nagel, One-parameter semigroups for linear evolution equations. Berlin: Springer 2000.

[4] Andreas M. Fröhlich, Lutz Weis, H^∞-calculus and dilatations. Bulletin de la Société Mathématique de France Vol.: 134, no 4 (2006), 487–508.

[5] Markus Haase, The Functional Calculus for Sectorial Operators, Operator Theory: Advances and Applications, Birkhäuser Basel (2006).

[6] Markus Haase, A transference principle for general groups and functional calculus on UMD spaces. Math. Ann. 345 (2009), 245–265.

[7] Matthias Hieber and Jan Prüss, Functional calculi for linear operators in vector-valued Lp-spaces via the transference principle. Adv. Differential Equations, 3(6):847–872, 1998.

[8] Einar Hille and Ralph S. Phillips, Functional Analysis and Semigroups (Colloquium Publications AMS), 1957.

[9] Tuomas Hytönen, Jan van Neerven, Mark Veraar and Lutz Weis, Analysis in Banach Spaces: Volume I: Martingales and Littlewood-Paley Theory. (Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge; Vol. 63). Springer (2016).

[10] Nigel Kalton, Peer Kunstmann, and Lutz Weis, Perturbation and interpolation theorems for the H^∞-calculus with applications to differential operators. Math. Ann. 336, No. 4, 747-801 (2006), erratum ibid. 357, No. 2, 801-804 (2013).

[11] Joram Lindenstrauss, Lior Tzafriri, On the complemented subspaces problem. Israel Journal of Mathematics, 9, 263-269 (1971).

[12] Belabbas Madani, On the invertibility of bounded operators with a lower bound on reflexive Banach spaces, and applications to C_0–semigroups. Archiv der Mathematik, Volume 126 (2026), pages 305–313.

Université de Bordeaux, Institut de Mathématiques de Bordeaux, 351 cours de la Libération, F – 33405 Talence, France
Email address: bernhard.haak@math.u-bordeaux.fr

Karlsruhe Institute of Technology (KIT), Institute for Analysis, Englerstr. 2, D – 76128 Karlsruhe, Germany
Email address: peer.kunstmann@kit.edu