OBSERVATION OF VOLTERRA SYSTEMS WITH SCALAR KERNELS

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Abstract. Volterra observations systems with scalar kernels are studied. New sufficient conditions for admissibility of observation operators are developed. The obtained results are applied to time-fractional diffusion equations of distributed order.

1. Introduction

Consider the following scalar abstract Volterra system

$$x(t) = x_0 + \int_0^t a(t-s)\, Ax(s)\, ds, \qquad t \geq 0. \tag{1}$$

Here, the operator A is supposed to be a closed operator with dense domain on a Banach space X, $x_0 \in X$, the kernel function $a \in L^1_{\text{loc}}$ is supposed to be of subexponential growth so that its Laplace transform $\widehat{a}(\lambda)$ exists for all λ with positive real part, and it is assumed that (1) is *parabolic* in the sense of Prüss [26], that is

- (P1) $\widehat{a}(\lambda) \neq 0$ and $\frac{1}{\widehat{a}(\lambda)} \in \varrho(A)$ for all λ with positive real part,
- (P2) there exists a constant $M \geq 0$ such that $\|(1 - \widehat{a}(\lambda)A)^{-1}\| \leq M$ for all λ with positive real part.

In addition, we always assume that the kernel function a is 1-*regular*, that is, there is a constant $K > 0$ such that

$$|\lambda \widehat{a}'(\lambda)| \leq K |\widehat{a}(\lambda)| \tag{2}$$

for all λ with positive real part.

In Prüss [26, Theorem I.3.1] it is shown that under these assumptions, equation (1) admit a unique *solution family*, i.e. a family of bounded linear operators $(S(t))_{t\geq 0}$ on X, such that

- (a) $S(0) = I$ and $S(\cdot)$ is strongly continuous on $\mathbb{R}_+$.
- (b) $S(t)$ commutes with A, which means $S(t)(D(A)) \subset D(A)$ for all $t \geq 0$, and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$.
- (c) For all $x \in D(A)$ and all $t \geq 0$ the resolvent equations hold:

$$S(t)x = x + \int_0^t a(t-s) AS(s)x\, ds. \tag{3}$$

Moreover, $S \in C((0,\infty), \mathcal{B}(X))$ and $\|S(t)\| \leq K$.

For some results we need in addition that $-A$ a sectorial operator of type $\omega \in (0,\pi)$ or that the kernel a is *sectorial of angle* $\theta \in (0,\pi)$. Recall that $-A$ is called *sectorial operator of type* $\omega \in (0,\pi)$, if the operator A is a closed operator with dense domain on X having its spectrum contained in some open sectorial region of the complex plane, symmetric to the real axis and open to the left:

$$\sigma(A) \subseteq -\Sigma_\omega \qquad \text{where} \qquad \Sigma_\omega = \{z \in \mathbb{C} : \ |\arg(z)| < \omega\}$$

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for some $\omega \in (0, \pi)$. Moreover, the resolvent of A is supposed to satisfy a growth condition of the type $\|\lambda R(\lambda, A)\| \le M$ uniformly on each sector $\Sigma_{\pi-\omega-\varepsilon}$. Typical examples of such operators are generators of bounded strongly continuous semi-groups, where $\omega \le {}^{\pi}\!/_{2}$. We mention that 'sectoriality' may have different meanings for different authors in the literature.
The kernel a is called *sectorial of angle* $\theta \in (0, \pi)$ if

$$\widehat{a}(\lambda) \in \Sigma_\theta \qquad \text{for all } \lambda \text{ with positive real part.}$$

In particular, when $-A$ and a are both sectorial in the respective sense with angles that sum up to a constant strictly inferior to π, the Volterra equation is parabolic.

The purpose of this article is to present conditions for the admissibility of observation operators to parabolic Volterra equations, that is, we consider the 'observed' system

$$\text{(V)} \qquad \begin{cases} x(t) = x_0 + \displaystyle\int_0^t a(t-s)\, Ax(s)\, ds, & t \ge 0, \\ y(t) = Cx(t). \end{cases}$$

The operator C in the second line is supposed to be an operator from X into another Banach space Y that acts as a bounded operator from $X_1 \to Y$ where $X_1 = \mathcal{D}(A)$ is endowed by the graph norm of A. In order to guarantee that the output function lies locally in L_2 we are interested in the following property.

Definition 1.1. A bounded linear operator $C : X_1 \to Y$ is called *finite-time admissible* for the Volterra equation (1) if there are constants $\eta, K > 0$ such that

$$\left(\int_0^t \|CS(r)x\|^2\, dr \right)^{1/2} \le Ke^{\eta t}\|x\|$$

for all $t \ge 0$ and all $x \in \mathcal{D}(A)$.

The notion of admissible observation operators is well studied in the literature for Cauchy systems, that is, $a \equiv 1$, see for example [17], [27], and [28]. Admissible observation operators for Volterra systems are studied in [12], [18], [19] and [22].
The Laplace transform of S, denoted by H, is given by

$$H(\lambda)x = \frac{1}{\lambda}(I - \hat{a}(\lambda)A)^{-1}x, \qquad \operatorname{Re}\lambda > 0.$$

The following necessary condition for admissibility was shown in [19].

Proposition 1.2. *If C is a finite-time admissible observation operator for the Volterra equation (1), then there is a constant $M > 0$ such that*

$$\text{(4)} \qquad \|\sqrt{\operatorname{Re}\lambda}\, CH(\lambda)\| \le M, \qquad \operatorname{Re}\lambda > 0.$$

In [19] it is shown that (4) is also sufficient for admissibility if X is a Hilbert space, Y is finite-dimensional and A generates a contraction semigroup. However, in general this condition is not sufficient (see e.g. [17]).
We show that the slightly stronger growth condition on the resolvent

$$\sup_{r>0} \left\| \, (1 + \log^+ r)^\alpha r^{1/2} CH(r) \, \right\| < \infty,$$

is sufficient for admissibility if $\alpha > 1/2$ (see Theorem 3.6). This result generalises the sufficient condition of Zwart [29] for Cauchy systems to general Volterra systems (1).
Our second main result, Theorem 3.1 provides a subordination argument to obtain admissibility for the observed Volterra equation from the admissibility of the observation operator for the underlying Cauchy problem. In the particular case of

diagonal semigroups and one-dimensional output spaces Y this improves a direct Carleson measure criterion from Haak, Jacob, Partington and Pott [12].
We proceed as follows. In Section 2 we obtain an integral representation for the solution family $(S(t))_{t\geq 0}$ and several regularity results of the corresponding kernel. Section 3 is devoted to sufficient condition for admissibility of observation operators. A subordination result as well as a general sufficient condition is obtain. Several examples are included as well.
To enhance readability of the calculations, for rest of this article, K denotes some positive constant that may change from one line to the other unless explicitly quantified.

2. Regularity transfer

The main result of this section is formulated in the following proposition. Let $s(t,\mu)$ denote the solution of the scalar equation

$$s(t,\mu)+\mu\int_0^t a(t-r)s(r,\mu)\,dr = 1 \qquad t>0, \mu\in\mathbb{C}.$$

Proposition 2.1. *In addition to the general assumptions, we suppose that A generates a bounded C_0-semigroup $(T(t))_{t\geq 0}$ and that the kernel a is sectorial of angle $\theta < {}^{\pi}\!/_{2}$. Then there exists a family of functions $(v_t)_{t>0}$ such that*

$$\mathcal{L}(v_t)(\mu) = s(t,\mu) \quad \textit{and} \quad S(t) = \int_0^\infty v_t(s)T(s)\,ds$$

satisfying

(a) $\sup_{t>0}\|v_t\|_{L^1(\mathbb{R}_+)} < \infty$
(b) $\|v_t\|_{L^2(\mathbb{R}_+)} \leq K(t^{-\theta/\pi} + t^{+\theta/\pi})$ *where K depends only on θ and the constant in (2).*
(c) $\|v_t\|_{W^{1,1}(\mathbb{R}_+)} \leq K(1 + t^{-\frac{2\theta}{\pi}} + t^{+\frac{2\theta}{\pi}})$.

Moreover, for $\gamma\in[0,1]$, $|\mu^\gamma s(t,\mu)| \leq Kt^{-\frac{2\gamma\theta}{\pi}}$.

For the proof of this proposition the following two lemmas are needed.

Lemma 2.2. *In addition to the general assumption on the kernel a, we suppose that a is sectorial of angle $\theta\leq\pi$. Let $\rho_0 := 2\theta/\pi$. Then there exists a constant $c>0$ such that*

$$|\widehat{a}(\lambda)| \geq \begin{cases} c|\lambda|^{-\rho_0} & |\lambda|\geq 1 \\ c|\lambda|^{\rho_0} & |\lambda|\leq 1 \end{cases}$$

for all $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda > 0$.

Proof. We borrow the argument from the proof of [25, Proposition 1]: we start with the analytic completion of the Poisson formula for the harmonic function $H(\lambda) = \arg\widehat{a}(\lambda)$, that is,

$$\log\widehat{a}(\lambda) = \kappa_0 + \frac{i}{\pi}\int_{-\infty}^{\infty}\left[\frac{1-i\rho\lambda}{\lambda - i\rho}\right] h(i\rho)\frac{d\rho}{1+\rho^2},$$

where $\kappa_0\in\mathbb{R}$ is a constant. An easy calculation shows

$$|\operatorname{Re}\log\widehat{a}(\lambda)| \leq \kappa_0 + \rho_0|\log\lambda|$$

for real $\lambda>0$, and thus

$$|\widehat{a}(\lambda)| = e^{\log(|\widehat{a}(\lambda)|)} = e^{\operatorname{Re}\log\widehat{a}(\lambda)} \geq \begin{cases} c\lambda^{-\rho_0} & \lambda\geq 1 \\ c\lambda^{\rho_0} & 0\leq\lambda\leq 1 \end{cases},$$

where $c := e^{-\kappa_0} > 0$. This estimate, together with [26, Lemma 8.1] stating the existence of a constant $c > 0$ such that $c^{-1} \leq \left|\widehat{a}(|\lambda|)/\widehat{a}(\lambda)\right| \leq c$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$ completes the proof. □

Lemma 2.3. *Let $\theta \in (0, \pi)$. Then there exists $c_\theta > 0$ such that*

$$1 + |\lambda| \leq C_\theta\, |1 + \lambda| \tag{5}$$

for all $\lambda \in \Sigma_{\pi-\theta}$.

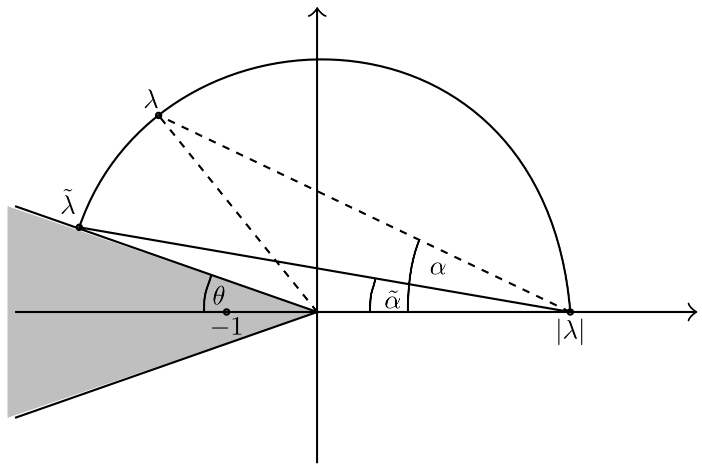

FIGURE 1. Illustration of (5)

Proof. Clearly, $\alpha > \widetilde{\alpha}$, see Figure 1. Since $\widetilde{\alpha} = \frac{\theta}{2}$, the assertion follows then from the fact that $\frac{|1+\lambda|}{1+|\lambda|} = \frac{\sin(\alpha)}{\sin(\theta-\alpha)} \geq \sin(\alpha) \geq \sin(\theta/2)$. □

Proof of Proposition 2.1. (a) is [26, Proposition I.3.5]. This latter result is also the principal inspiration of the next part:

(b) Let $\sigma(\lambda,\mu) = \big(\mathcal{L}s(\cdot,\mu)\big)(\lambda)$, i.e. $\sigma(\lambda,\mu) = \frac{1}{\lambda(1+\mu\widehat{a}(\lambda))}$. Fix $t > 0$ and $\varepsilon > 0$. Then

$$s(t,\mu) = \tfrac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{\lambda t}\sigma(\lambda,\mu)\,d\lambda.$$

Then, by partial integration

$$\begin{aligned} s(t,\mu) &= \lim_{R\to\infty} \tfrac{1}{2\pi i}\Big[\tfrac{1}{t}e^{\lambda t}\sigma(\lambda,\mu)\Big]_{\lambda=\varepsilon-iR}^{\lambda=\varepsilon+iR} - \tfrac{1}{2\pi i}\int_{\varepsilon-iR}^{\varepsilon+iR} \tfrac{1}{t}e^{\lambda t}\frac{d}{d\lambda}\sigma(\lambda,\mu)\,d\lambda \\ &= -\tfrac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \tfrac{1}{t}e^{\lambda t}\frac{d}{d\lambda}\sigma(\lambda,\mu)\,d\lambda \end{aligned}$$

An elementary calculation gives

$$\frac{d}{d\lambda}\frac{1}{\lambda(1+\mu\widehat{a}(\lambda))} = -\frac{1+\mu\widehat{a}(\lambda)\Big(1+\Big(\frac{\lambda\widehat{a}'(\lambda)}{\widehat{a}(\lambda)}\Big)\Big)}{\lambda^2(1+\mu\widehat{a}(\lambda))^2}$$

By 1-regularity of the kernel, $\left|\frac{\lambda\widehat{a}'(\lambda)}{\widehat{a}(\lambda)}\right| \le C$ and so Lemma 2.3 yields for any $\delta > 0$,

$$\begin{aligned}
&\left(\int_{-\infty}^{\infty}\Big|s(t,\delta+iy)\Big|^2\,dy\right)^{1/2}\\
\le\ & C_\theta(1+C)\frac{e^{\varepsilon t}}{2\pi t}\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^2{+}x^2)(1+|\delta{+}iy||\widehat{a}(\varepsilon{+}ix)|)}\,dx\right)^2dy\right)^{1/2}\\
\le\ & \sqrt{2}C_\theta(1+C)\frac{e^{\varepsilon t}}{2\pi t}\left(\int_{0}^{\infty}\left(\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^2{+}x^2)(1+|y||\widehat{a}(\varepsilon{+}ix)|)}\,dx\right)^2dy\right)^{1/2}\\
\le\ & \sqrt{2}C_\theta(1+C)\frac{e^{\varepsilon t}}{2\pi t}\int_{-\infty}^{\infty}\left(\int_{0}^{\infty}\frac{1}{(\varepsilon^2{+}x^2)^2(1+|y||\widehat{a}(\varepsilon{+}ix)|)^2}\,dy\right)^{1/2}dx\\
=\ & \sqrt{2}C_\theta(1+C)\frac{e^{\varepsilon t}}{2\pi t}\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^2{+}x^2)|\widehat{a}(\varepsilon+ix)|^{1/2}}\left(\int_{0}^{\infty}\frac{1}{(1+u)^2}\,du\right)^{1/2}dx\\
=\ & \sqrt{2}C_\theta(1+C)\frac{e^{\varepsilon t}}{2\pi t}\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^2{+}x^2)|\widehat{a}(\varepsilon+ix)|^{1/2}}\,dx
\end{aligned}$$

Now we split the integral into two parts, by considering the cases $\varepsilon^2{+}x^2 \ge 1$ and $\varepsilon^2{+}x^2 < 1$ to apply Lemma 2.2 which is controlling $|1/\widehat{a}|$. Substituting $x = \varepsilon t$ in both parts easily gives

$$\|s(t,\cdot)\|_{H^2} \le \widetilde{C}_\theta\frac{e^{\varepsilon t}}{t}\big(\varepsilon^{-1-\theta/\pi} + \varepsilon^{-1+\theta/\pi}\big),$$

which yields the assertion by letting $\varepsilon = 1/t$.

(c) We argue in the same spirit as above: by partial integration

$$\frac{d}{d\mu}\Big(\mu s(t,\mu)\Big) \quad=\quad \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\tfrac{1}{t}e^{\lambda t}\frac{d^2}{d\mu\,d\lambda}\Big(\mu\sigma(\lambda,\mu)\Big)\,d\lambda$$

An elementary calculation gives

$$\frac{d^2}{d\lambda\,d\mu}\frac{\mu\widehat{a}(\lambda)}{(\lambda(1+\mu\widehat{a}(\lambda)))^2} = \frac{1+\mu\widehat{a}(\lambda)\Big(1+2\Big(\frac{\lambda\widehat{a}'(\lambda)}{\widehat{a}(\lambda)}\Big)\Big)}{\lambda^2(1+\mu\widehat{a}(\lambda))^3}$$

By 1-regularity of the kernel, $\left|\frac{\lambda\widehat{a}'(\lambda)}{\widehat{a}(\lambda)}\right| \le C$ and so Lemma 2.3 yields for any $\delta > 0$,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left|\frac{d}{d\mu}\Big(\mu s(t,\delta+iy)\Big)\right|dy\\
\le\ & C_\theta(1+2C)\frac{e^{\varepsilon t}}{2\pi t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^2{+}x^2)(1+|\delta{+}iy||\widehat{a}(\varepsilon{+}ix)|)^2}\,dx\,dy\\
\le\ & C_\theta(1+2C)\frac{e^{\varepsilon t}}{2\pi t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^2{+}x^2)(1+|y||\widehat{a}(\varepsilon{+}ix)|)^2}\,dx\,dy\\
=\ & 2C_\theta(1+2C)\frac{e^{\varepsilon t}}{2\pi t}\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^2{+}x^2)}\frac{1}{|\widehat{a}(\varepsilon{+}ix)|}\int_{0}^{\infty}\frac{1}{(1+u)^2}\,du\,dx\\
=\ & C_\theta(1+2C)\frac{e^{\varepsilon t}}{2\pi t}\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^2{+}x^2)}\frac{1}{|\widehat{a}(\varepsilon{+}ix)|}\,dx\\
\le\ & K\big(t^{-\frac{2\theta}{\pi}} + t^{+\frac{2\theta}{\pi}}\big)
\end{aligned}$$

by choosing $\varepsilon = {}^1\!/\!_t$. This shows that $f_t(\mu) = \frac{d}{d\mu}\Big(\mu s(t,\mu)\Big) \in H^1(\mathbb{C}_+)$. Note that $\mathbb{C}_+ := \Sigma_{\frac{\pi}{2}}$ is the open right half plane. We may apply Hardy's inequality (see e.g. [8, p.198], [14, Theorem 4.2]),

$$\int_0^\infty \frac{|\check{f}_t(r)|}{r}\,dr \le \tfrac{1}{2}\int_{-\infty}^\infty |f_t(i\omega)|\,d\omega$$

so that $\frac{\check{f}_t(r)}{r} \in L^1(\mathbb{R}_+)$ is Laplace transformable for every $t > 0$. Since

$$\mathcal{L}\Big(\frac{\check{f}_t(r)}{r}\Big)(\sigma) = \int_\sigma^\infty f_t(\mu)\,d\mu = \sigma s(t,\sigma),$$

we find that $\mu \mapsto \mu s(t,\mu) \in H^\infty(\mathbb{C}_+)$ with a norm controlled by a multiple of $(t^{\frac{+2\theta}{\pi}} + t^{-\frac{2\theta}{\pi}})$. This implies that $v_t' \in L^1(\mathbb{R}_+)$. Together with (a) the claim follows.

Finally, the same technique gives an estimate for the growth of $s(t,\mu)$:

$$\begin{aligned}
\mu^\gamma s(t,\mu) &\le K\frac{|\mu|^\gamma e^{\varepsilon t}}{t}\int_{-\infty}^\infty \frac{1}{(\varepsilon^2+r^2)(1+|\mu||\widehat{a}(\varepsilon+ir)|)}\,dr\\
&\le K\frac{e^{\varepsilon t}}{t}\int_{-\infty}^\infty \frac{1}{(\varepsilon^2+r^2)|\widehat{a}(\varepsilon+ir)|^\gamma}\,\frac{|\mu|^\gamma|\widehat{a}(\varepsilon+ir)|^\gamma}{(1+|\mu||\widehat{a}(\varepsilon+ir)|)}\,dr\\
&\le K\frac{e^{\varepsilon t}}{t}\int_{-\infty}^\infty \frac{1}{(\varepsilon^2+r^2)|\widehat{a}(\varepsilon+ir)|^\gamma}\,dr\\
&\overset{\varepsilon=\frac{1}{t}}{\le} K(t^{-\frac{2\gamma\theta}{\pi}} + t^{+\frac{2\gamma\theta}{\pi}}).
\end{aligned}$$

□

3. Sufficient conditions for finite-time admissibility

In this section we present the two main results of this paper.

Theorem 3.1. *Let A generate an exponentially stable strongly continuous semigroup $(T(t))_{t\ge0}$ and let $C : X_1 \to Y$ be bounded. Further we assume that the kernel $a \in L^1_{loc}(\mathbb{R}_+)$ is of sub-exponential growth, 1-regular and sectorial of angle $\theta < \pi/2$. Then finite-time admissibility of C for the semigroup $(T(t))_{t\ge0}$ implies that of C for the solution family $(S(t))_{t\ge0}$.*

Proof. We first note that the assumptions of the theorem apply that equation (1) is parabolic. By Proposition 2.1 there exists a family of functions v_t such that $\|v_t\|_{L^2(\mathbb{R}_+)} \le K(t^{-\theta/\pi} + t^{+\theta/\pi})$ for some constant $K > 0$ independent of $t > 00$ and

$$S(t) = \int_0^\infty v_t(r)\,T(r)\,dr, \qquad t > 0.$$

For $x \in \mathcal{D}(A)$ we have thus

$$CS(t)x = \int_0^\infty v_t(r)\,CT(r)x\,dr.$$

Note that finite-time admissibility of C for $(T(t))_{t\ge0}$ implies the existence of a constant $M > 0$ such that

$$\|CT(\cdot)x\|_{L^2(\mathbb{R}_+)} \le M\|x\|, \qquad x \in \mathcal{D}(A),$$

thanks to the exponential stability of $(T(t))_{t\ge0}$. Thus the result follows from Cauchy-Schwarz inequality. □

By replacing the Cauchy-Schwarz inequality by Hölder's inequality, similar arguments can be used to obtain sufficient conditions for L^p-admissibility.

Corollary 3.2. *Assume in addition to the hypotheses of the theorem that one of the following conditions is satisfied:*

(a) *Y is finite-dimensional, X is a Hilbert space and A generates a contraction semigroup;*
(b) *X is a Hilbert space and A generates a normal, analytic semigroup;*
(c) *A generates an analytic semigroup and $(-A)^{1/2}$ is an finite-time admissible observation operator for $(T(t))_{t\geq 0}$.*

If there exists a constant $M > 0$ such that

$$\|C(\lambda - A)^{-1}\| \leq \frac{M}{\sqrt{\operatorname{Re}\lambda}}, \qquad \operatorname{Re}\lambda > 0, \tag{6}$$

then C is a finite-time admissible observation operator for $(S(t))_{t\geq 0}$.

Proof. Under the assumption of the corollary, the inequality (6) implies that C is a finite-time admissible observation operator for $(S(t))_{t\geq 0}$, see [16], [13], [23]. Thus the result follows from Theorem 3.1. □

The following corollary is an immediate consequence of the Carleson measure criterion of Ho and Russell [15].

Corollary 3.3. *Assume in addition to the hypotheses of the theorem that A admits a Riesz basis of eigenfunctions (e_n) on a Hilbert space X with corresponding eigenvalues λ_n. If $Y = \mathbb{C}$ and if*

$$\mu = \sum_n |Ce_n|^2 \delta_{-\lambda_n}$$

is a Carleson measure on $\mathbb{C}_+$, then C is finite-time admissible for the solution family $(S(t))_{t\geq 0}$.

A nice sufficient condition for admissibility for Cauchy problems is given by Zwart [29]. For the convenience of the reader we reproduce it here:

Theorem 3.4 (Zwart)**.** *Let A be the infinitesimal generator of an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space H and let $C : X_1 \to Y$ be bounded, where Y is another Hilbert space. If for some $\alpha > 1/2$,*

$$\sup_{\operatorname{Re}\lambda > 0} \left\| \, (1 + \log^+ \operatorname{Re}\lambda)^\alpha (\operatorname{Re}(\lambda))^{1/2} CR(\lambda, A) \right\| < \infty, \tag{7}$$

then C is a finite-time admissible observation operator.

Notice that the condition (7) can be reformulated by saying that in the sense of Evans, Opic and Pick (see [10, 9, 11])

$$\forall x \in X : \qquad \|CR(\cdot, A)x\|_{1/2,\infty.\mathbb{A}} < \infty$$

where $\mathbb{A} = (0, \alpha)$, see also Cobos, Frenandez-Cabrera and Triebel [7] for logarithmic type interpolation functors.
Combining Theorem 3.1 and Theorem 3.4 we receive the following corollary.

Corollary 3.5. *Let in addition to the assumptions of Theorem 3.4, a be of subexponential growth, 1-regular and sectorial of type $< \pi/2$. Then C is finite-time admissible for the solution family $\big(S(t)\big)_{t\geq 0}$.*

In some situations, the condition of sectoriality of angle $< \pi/2$ in the above corollary may be inconvenient. Under much weaker assumptions one can also obtain admissibility by the following direct argument.

Theorem 3.6. *Assume that A is a closed operator with dense domain on X, the kernel function $a \in L^1_{loc}$ is of sub-exponential growth, 1-regular, and (1) is parabolic. Let $C : X_1 \to Y$ be bounded and assume that for some $\alpha > 1/2$,*

$$\sup_{r>0} \big\| \ (1+\log^+ r)^\alpha r^{1/2} CH(r) \, \big\| < \infty. \tag{8}$$

Then C is finite-time admissible for $(S(t))_{t\ge 0}$.

Note that the exponent $\alpha > 1/2$ is optimal in the sense that for $\alpha < 1/2$ it is even wrong in the case $a \equiv 1$, see [20]. About the case $\alpha = 1/2$ nothing is known at the moment.

Proof. Let $\lambda \in \mathbb{C}_+$ and let φ such that $\lambda = |\lambda| e^{2i\varphi}$. Then, by resolvent identity,

$$\begin{aligned}
& (1+(\log^+(\operatorname{Re}\lambda))^\alpha \lambda^{1/2} CH(\lambda) \\
= \;& (1+(\log^+(\operatorname{Re}\lambda))^\alpha \lambda^{-1/2} C \frac{1}{\widehat{a}(\lambda)} R(\frac{1}{\widehat{a}(\lambda)}, A) \\
= \;& (1+\log^+|\lambda|)^\alpha |\lambda|^{1/2} CH(|\lambda|)\, e^{-i\varphi} \frac{\widehat{a}(|\lambda|)}{\widehat{a}(\lambda)} \Big[I + (\frac{1}{\widehat{a}(|\lambda|)} - \frac{1}{\widehat{a}(\lambda)}) R(\frac{1}{\widehat{a}(\lambda)}, A) \Big] \\
= \;& (1+\log^+|\lambda|)^\alpha |\lambda|^{1/2} CH(|\lambda|)\, e^{-i\varphi} \Big[I + (\frac{\widehat{a}(\lambda)}{\widehat{a}(|\lambda|)} - 1)(I - \widehat{a}(\lambda)A)^{-1} \Big].
\end{aligned}$$

By [26, Lemma 8.1], $c^{-1} \le \big|\widehat{a}(|\lambda|)/\widehat{a}(\lambda)\big| \le c$ for some $c > 0$. This together with the parabolicity of (1) yields uniform boundedness of expression in brackets and so the assumed estimate (8) gives

$$\|\lambda \mapsto CH(r+\lambda)\|_{H^\infty(\mathbb{C}_+)} \le K(1+\log^+ r)^{-\alpha} r^{-1/2}. \tag{9}$$

Since $(S(t))_{t\ge 0}$ is bounded,

$$\|\lambda \mapsto H(r+\lambda)x\|_{H^2(\mathbb{C}_+)} = \|e^{-rt} S(t)x\|_{H^2(\mathbb{C}_+)} \le K r^{-1/2} \, \|x\| \qquad \forall r > 0$$

and together with (9), we infer

$$\|\lambda \mapsto CH(r+\lambda)^2 x\|_{H^2(\mathbb{C}_+)} \le \frac{K}{(1+\log^+ r)^\alpha r} \, \|x\| \qquad \forall r > 0. \tag{10}$$

Moreover, the estimate

$$\big\|\lambda \mapsto \frac{1}{r+\lambda} CH(r+\lambda)x\big\|_{H^2(\mathbb{C}_+)} \le \big\|\lambda \mapsto CH(r+\lambda)x\big\|_{H^\infty(\mathbb{C}_+)} \big\|\lambda \mapsto \frac{1}{r+\lambda}\big\|_{H^2(\mathbb{C}_+)}$$

implies

$$\big\|\lambda \mapsto \frac{1}{r+\lambda} CH(r+\lambda)x\big\|_{H^2(\mathbb{C}_+)} \le \frac{K}{(1+\log^+ r)^\alpha r} \|x\| \qquad \forall r > 0. \tag{11}$$

Since $\frac{d}{d\lambda} H(\lambda) = \Big(\frac{\lambda \widehat{a}'(\lambda)}{\widehat{a}(\lambda)}\Big) H(\lambda)^2 - \frac{1}{\lambda}\big(1 + \frac{\lambda \widehat{a}'(\lambda)}{\widehat{a}(\lambda)}\big) H(\lambda)$ we infer from (10) and (11) that

$$\Big\|\mu \mapsto \frac{d}{d\mu} CH(r+\mu)x\Big\|_{H^2(\mathbb{C}_+)} \le \frac{K}{(1+\log^+ r)^\alpha r} \|x\| \qquad \forall r > 0.$$

Finally, (inverse) Laplace transform yields

$$\Big\|t \mapsto rte^{-rt} CS(t)x\Big\|_{L^2(\mathbb{R}_+)} \le \frac{K}{(1+\log^+ r)^\alpha} \|x\| \qquad \forall r > 0$$

and that is the estimate we need in the following dyadic decomposition argument: notice that $xe^{-x} \ge 2e^{-2}$ for $x \in [1,2]$. Fix some $t_0 > 0$. Then,

$$\int_0^{t_0} \big\|CS(t)x\big\|^2 \, dt \quad = \quad \sum_{n=1}^\infty \int_{t_0 2^{-n}}^{t_0 2^{-n+1}} \big\|CS(t)x\big\|^2 \, dt$$

$$\leq \frac{e^4}{4} \sum_{n=1}^{\infty} \int_{t_0 2^{-n}}^{t_0 2^{-n+1}} \left\| t 2^n t_0^{-1} e^{t 2^n t_0^{-1}} CS(t)x \right\|^2 dt$$
$$\leq K \sum_{n=1}^{\infty} \frac{1}{(1+\log^+(2^n t_0^{-1}))^{2\alpha}} \|x\|^2 \leq K\|x\|^2.$$

□

4. Example

In this section we apply the obtained results to time-fractional diffusion equations of distributed order.

Let A generate an exponentially stable strongly continuous semigroup $(T(t))_{t\geq 0}$. For $\omega > 0$ and $0 < \alpha < \beta \leq 1$ we study a time-fractional diffusion equation of distributed order of the form

$$\begin{aligned} \omega D_t^\alpha x(t) + D_t^\beta x(t) &= Ax(t), \quad t \geq 0, \\ x(0) &= x_0, \end{aligned} \tag{12}$$

where $D_t^\alpha x = \left(-\frac{\partial}{\partial t}\right)^\alpha x$ denotes the Caputo derivative of x, given by the Phillips functional calculus of the right shift semigroup, that is,

$$D_t^\gamma x(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} x'(s)\, ds.$$

for $\gamma \in (0,1)$. Time-fractional diffusion equations of distributed order have attracted attention as a possible tool for the description of anomalous diffusion and relaxation phenomena in many areas such as turbulence, disordered medium, intermittent chaotic systems, mathematical finance and stochastic mechanics. For further information on time-fractional diffusion equations of distributed order we refer the reader to [1, 2, 3, 4, 5, 6, 21, 24].

Using the Laplace transform equation (12) is equivalent to

$$x(t) = x_0 + \int_0^t a(t-s)\, Ax(s)\, ds,$$

where

$$a(t) = a(t) = t^{\beta-1} E_{\beta-\alpha,\beta}(-\omega t^{\beta-\alpha})$$

Here $E_{\gamma,\delta}$, where $\gamma, \delta > 0$, denotes the Mittag-Leffler function

$$E_{\gamma,\delta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k + \delta)}$$

The Laplace transformation of the kernel a is given by

$$\hat{a}(\lambda) = \frac{\lambda^{-\alpha}}{\omega + \lambda^{\beta-\alpha}}.$$

Thus the kernel a satisfies the assumption of Theorem 3.1.

We note that this example does e.g. not satisfy the assumption of [12, Theorem 3.10] due to the 'mixed' growth conditions near infinity and the origin, such that, even when A is the Dirichlet Laplacian on a bounded domain, the latter result cannot be used to guarantee admissibility whereas a 'standard' Carleson measure criterion and the subordination result of Corollary 3.3 still applies.

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