ADMISSIBILITY OF UNBOUNDED OPERATORS AND WELLPOSEDNESS OF LINEAR SYSTEMS IN BANACH SPACES

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ABSTRACT. We study linear systems, described by operators A, B, C for which the state space X is a Banach space. We suppose that $-A$ generates a bounded analytic semigroup and give conditions for admissibility of B and C corresponding to those in G. WEISS' conjecture. The crucial assumptions on A are boundedness of an H^∞–calculus or suitable square function estimates, allowing to use techniques recently developed by N. KALTON and L. WEIS. For observation spaces Y or control spaces U that are not Hilbert spaces we are led to a notion of admissibility extending previous considerations by C. LE MERDY. We also obtain a characterisation of wellposedness for the full system. We give several examples for admissible operators including point observation and point control. At the end we study a heat equation in $X = L^p(\Omega)$, $1 < p < \infty$, with boundary observation and control and prove its wellposedness for several function spaces Y and U on the boundary $\partial\Omega$.

1. INTRODUCTION

In this paper we study linear control systems of the form

$$\left\{\begin{array}{lcll} x'(t) + Ax(t) & = & Bu(t), & t \in [0,\tau) \\ x(0) & = & x_0, & \\ y(t) & = & Cx(t), & t \in [0,\tau) \end{array}\right. \tag{1}$$

where $0 < \tau \le \infty$, $-A$ is the generator of a C_0–semigroup $T(\cdot)$ in a Banach space X, and y and u take values in Banach spaces Y and U, respectively. If $B : U \to X$ and $C : X \to Y$ are bounded and the spaces X, Y and U are of finite dimensions then (1) is the setup of classical linear systems theory. There is a large literature (cf., e.g., the bibliography in [6]) on the case where X is an infinite–dimensional Hilbert space. This allows applications to partial differential equations, but in order to model, e.g., observation on the boundary or control from the boundary one has to deal with "unbounded" operators C and B. Writing $B(Z,W)$ for the space of all bounded linear operators from a Banach space Z to a Banach space W and following the literature, cf., e.g., [13], one only requires $C \in B(X_1, Y)$ and $B \in B(U, X_{-1})$, where X_1 denotes the domain $\mathcal{D}(A)$ of A equipped with the graph norm and X_{-1} denotes the completion of X with respect to the norm $\|(\lambda_0 - A)^{-1} \cdot \|_X$ for a fixed λ_0 in the resolvent set $\rho(A)$ of A (all those norms are equivalent).

The usual choice for function spaces in which observations y should lie or controls u are taken from is $L^2([0,\tau), Y)$ and $L^2([0,\tau), U)$, respectively (which is also the natural one if X, Y, U are Hilbert spaces). An observation operator $C \in B(X_1, Y)$ is called *finite-time admissible* for A (cf. [34, 13]) if, for some (and hence for all) $\tau \in (0,\infty)$, there exists $M = M_\tau > 0$ such that $\|CT(\cdot)x\|_{L^2([0,\tau),Y)} \le M\|x\|_X$ for all $x \in X_1$, which implies that $CT(\cdot)$ extends to a bounded linear operator from X to $L^2([0,\tau), Y)$. This notion is invariant under scalings $e^{-\alpha \cdot}T(\cdot)$ of the semigroup $T(\cdot)$, and if $T(\cdot)$ is exponentially stable, it is equivalent to *infinite-time admissibility* (henceforth called *admissibility* for short), that is to existence of $M > 0$ satisfying

$$\Big(\int_0^\infty \|CT(t)x\|_Y^2 \, dt\Big)^{1/2} \le M\|x\|_X, \quad x \in X_1.$$

The notion of *finite-time admissibility* for a control operator $B \in B(U, X_{-1})$ is dual, i.e., for some (and hence all) $\tau \in (0,\infty)$ there exists $M = M_\tau > 0$ such that

$$\Big\|\int_0^\tau T_{-1}(\tau - t)Bu(t)\, dt\Big\|_X \le M\|u\|_{L^2([0,\tau),U)}, \quad u \in L^2([0,\tau), U),$$

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where the integral is taken in X_{-1} but its value is required to lie in X. Here $T_{-1}(\cdot)$ denotes the unique extension of the semigroup $T(\cdot)$ to the extrapolation space $X_{-1} \supset X$. Then $T_{-1}(\cdot)$ is a C_0-semigroup in X_{-1} and its generator $-A_{-1}$ is an extension of $-A$ (cf. [8]). Again, the notion is invariant under scalings, and for $T(\cdot)$ exponentially stable it is equivalent to *infinite-time admissibility* (henceforth called *admissibility* for short), i.e., to existence of a constant $M > 0$ satisfying

$$\left\| \int_0^\infty T_{-1}(t)Bu(t)\,dt \right\|_X \le M\|u\|_{L^2(\mathbb{R}_+,U)}, \quad u \in L^2(\mathbb{R}_+,U).$$

In this paper we concentrate on the parabolic case, and we assume from now on that the semigroup $T(\cdot)$ is bounded and analytic. Then A is a sectorial operator in X of type $< \pi/2$ (and A_{-1} is sectorial in X_{-1} of the same type). G. Weiss [35] observed that admissibility of an observation operator $C \in B(X_1, Y)$ implies boundedness of the set

$$W_C := \{\lambda^{1/2} C(\lambda+A)^{-1} :\ \lambda > 0\} \subset B(X,Y), \tag{2}$$

and conjectured that the converse holds in Hilbert spaces. This has become known as the Weiss conjecture (actually, this is the form it takes for bounded analytic semigroups). The nice feature is, if this is true, then it is possible to check for admissibility by looking at resolvents of the operator A. In system theoretic terms this means that a property in the *state space* may be checked by conditions in the *frequency domain*, i.e., by conditions on *Laplace transform images*.
The conjecture has been disproved in general (see [15, 14]). Similarly, admissibility of a control operator $B \in B(U, X_{-1})$ always implies boundedness of the set

$$W_B := \{\lambda^{1/2}(\lambda+A_{-1})^{-1}B :\ \lambda > 0\} \subseteq B(U,X), \tag{3}$$

but the converse is not even true in Hilbert spaces.
There were, however, positive results on the Weiss conjecture for bounded analytic semigroups, if $T(\cdot)$ is a diagonal semigroup, or more general, a normal semigroup ([35]). It was Le Merdy who realised a connection to the H^∞–calculus (for this notion we refer to Section 2). His result ([24]) combined with [5] yields the following.

Theorem 1.1 (Le Merdy)**.** *Let X, Y, and U be Hilbert spaces. Let A have dense range and an $H^\infty(\Sigma_\omega)$–calculus for some $0 < \omega < \pi/2$. Then $C \in B(X_1, Y)$ and $B \in B(U, X_{-1})$ are admissible if and only if $W_C \subset B(X,Y)$ from (2) and $W_B \subset B(U,X)$ from (3) are bounded.*

Remark 1.2.

(a) The assumption on A is equivalent to the fact that A has dense range, $-A$ generates a bounded analytic semigroup and A has an $H^\infty(\Sigma_\nu)$–calculus for some $0 < \nu < \pi$ (cf. [26, 8.Thm]).

(b) Since A is sectorial the set $\{\lambda^{1/2} A^{1/2}(\lambda+A)^{-1} :\ \lambda > 0\} \subset B(X)$ is always bounded (the proof is similar to what is done in Remark 4.3 below). Hence, if the Weiss conjecture on observation operators holds for an operator A, then $A^{1/2}$ must be admissible, i.e.,

$$\left(\int_0^\infty \left\|(tA)^{1/2} T(t)x\right\|_X^2 \tfrac{dt}{t} \right)^{1/2} \le M\|x\|_X, \quad x \in X_1, \tag{4}$$

holds for some $M > 0$. This is called a *quadratic estimate* for A. Le Merdy [24, Thm 4.1] used (4) to show that boundedness of $W_C \subset B(X,Y)$ implies admissibility of $C \in B(X_1,Y)$. The arguments apply in general Banach spaces X and Y.

(c) By works of McIntosh and others [1, 5, 26, 27] it is known that, if X is a Hilbert space, then A has an $H^\infty(\Sigma_\omega)$–calculus for some $\omega < \pi/2$ if and only if (4) and

$$\Big(\int_0^\infty \left\|(tA')^{1/2} T(t)'x'\right\|_{X'}^2 \tfrac{dt}{t} \Big)^{1/2} \le M\|x'\|_{X'} \tag{5}$$

hold for some $M > 0$ (cf. [26]). Le Merdy showed ([24, Thm. 5.2]) that, in general, (4) does not imply (5).

(d) As a by-product of our arguments, we will obtain below (Theorem 4.6) that (5) and boundedness of $W_B \subset B(U,X)$ imply admissibility of $B \in B(U,X_{-1})$ in general Banach spaces X and U. This may be obtained from Le Merdy's theorem by dualisation if X and U are reflexive, but not in the general case we consider.

(e) When leaving the Hilbert space setting, the simultaneous validity of quadratic estimates for both operators A and A' seem, however, to be extremely rare in applications. Consider, e.g., the operator $A = -\Delta$ on $X = L^p(\mathbb{R}^n)$, $1 < p < \infty$. Then A satisfies quadratic estimates only if $p \leq 2$ whereas quadratic estimates for A' require $p' \leq 2$, i.e., $p \geq 2$ (cf. [5]).

In this paper we investigate the situation for Banach spaces X, Y, and U. To give a flavour of our main results we state the following for the case that only the state space X is a Banach space but U and Y are Hilbert spaces (which includes the case that U and Y are finite-dimensional). Here we assume that X has Pisier's property (α) (cf. Definition 3.16), an assumption that holds in particular for Lebesgue-spaces L^p, Sobolev-spaces W^m_p, Bessel-potential spaces H^s_p, or Besov-spaces $B^s_{p,q}$, provided $p, q \in [1, \infty)$.

Theorem 1.3. *Let X have property (α). Suppose that A has dense range and an $H^\infty(\Sigma_\omega)$–calculus for some $0 < \omega < \pi/2$. Let Y and U be Hilbert spaces and $C \in B(X_1, Y)$, $B \in B(U, X_{-1})$. Then C and B are admissible if and only if the sets $W_C \subset B(X, Y)$ from* (2) *and $W_B \subset B(U, X)$ from* (3) *are l–bounded.*

Here, l–boundedness is a notion which is equivalent to boundedness in Hilbert spaces, but stronger than boundedness in general Banach spaces (we refer to Section 3 for definition and properties).

Remark 1.4.

(a) In contrast to the validity of both (4) and (5), the assumption on an H^∞–calculus for A is a reasonable one outside Hilbert spaces. It is satisfied for "many" differential operators in divergence form, such as elliptic differential operators with Hölder continuous coefficients and common boundary conditions, Schrödinger operators with singular potentials and many Stokes operators (cf. [18, 21] and references therein).

(b) In the proof we make use of function spaces $l(\mathbb{R}_+, Z)$ and their completions $l(L^2(\mathbb{R}_+), Z)$ where Z is a Banach space. These spaces were introduced by Kalton and Weis ([20], [19]) who showed that, in a Banach space X with finite cotype, boundedness of the H^∞–calculus for a sectorial operator A is equivalent to the validity of square function estimates for A and A' based on norms in $l(\mathbb{R}_+, \frac{dt}{t}, X)$ and $l(\mathbb{R}_+, \frac{dt}{t}, X')$, respectively. If X is a Hilbert space then $l(I, X) = L^2(I, X)$ and we are back in the situation of Remark 1.2 (c). For general Banach spaces the l-space may be viewed as a version of $L^2(I, X)$, but a version which carries much more of the Hilbert space structure of $L^2(I)$ than the Bochner space (see Section 3 for more details).

(c) For a space $L^p(\Omega)$, $1 < p < \infty$, the space $l(L^2(\mathbb{R}_+), X)$ equals $L^p(\Omega, L^2(\mathbb{R}_+))$, which shows that the result by Kalton and Weis is the general Banach space analog of a result in [5]: in a reflexive L^p-space, A has an $H^\infty(\Sigma_\omega)$-calculus for some $0 < \omega < \pi/2$, if and only if

$$\left\| \Big(\int_0^\infty \big|(tA)^{1/2} T(t) f\big|^2 \, \frac{dt}{t} \Big)^{1/2} \right\|_p \leq M \|f\|_p, \quad f \in L^p \tag{6}$$

$$\left\| \Big(\int_0^\infty \big|(tA')^{1/2} T(t)' g\big|^2 \, \frac{dt}{t} \Big)^{1/2} \right\|_{p'} \leq M \|g\|_{p'}, \quad g \in L^{p'}. \tag{7}$$

This kind of square function estimates is familiar from harmonic analysis.

(d) The result of [5] mentioned in (*c*) motivated Le Merdy [25] to show the following in spaces $X = L^p(\Omega)$ and $Y = L^q(\Sigma)$ where $p, q \in (1, \infty)$. : if $C \in B(X_1, Y)$ and (6) holds for A, then W_C is R–bounded (which is equivalent to l–boundedness here, cf. Section 3) if and only if

$$\left\| \left(\int_0^\infty \big|CT(t)f\big|^2 \, dt \right)^{1/2} \right\|_q \leq M \|f\|_p, \tag{8}$$

for some constant $M > 0$. In [25], an operator $C \in B(X_1, Y)$ satisfying (8) is called *R–admissible.*

In view of this remark, we replace, for general Banach spaces X, Y and U, the function spaces $L^2(\mathbb{R}_+, Y)$ and $L^2(\mathbb{R}_+, U)$ by the spaces $l(\mathbb{R}_+, Y)$ and $l(\mathbb{R}_+, U)$. We are thus led to the concept of l–admissibility which coincides with admissibility in case Y and U are Hilbert spaces, and with R–admissibility in case $X = L^p(\Omega)$, $Y = L^q(\Sigma)$, $U = L^r(\Sigma')$ where $p, q, r \in (1, \infty)$. A rough version of our main results then reads as follows.

Theorem 1.5. *Let X, Y and U have property (α). Suppose that A has dense range and an $H^\infty(\Sigma_\omega)$–calculus for some $0 < \omega < \pi/2$. Then $C \in B(X_1, Y)$, $B \in B(U, X_{-1})$ are l–admissible if and only if the sets $W_C \subset B(X, Y)$ from (2) and $W_B \subset B(U, X)$ from (3) are l–bounded.*

We also study *wellposedness* of the system (1). The idea of this concept is to have, for each $\tau > 0$, continuous dependency of the output $(x(\tau), y|_{[0,\tau)})$ of the system on the input $(x_0, u|_{[0,\tau)})$ (cf. [37]). Since the output of (1) is given by

$$x(\tau) = T(\tau)x_0 + \int_0^\tau T_{-1}(s)Bu(\tau - s)\,ds$$
$$y(\cdot) = CT(\cdot)x_0 + CT_{-1}(\cdot)B * u$$

one has to study continuity of four different maps: continuity of $x_0 \mapsto T(\tau)x_0$ is clear since $T(\cdot)$ is a C_0–semigroup, continuity of $x_0 \mapsto CT(\cdot)x_0$ and $u \mapsto \int_0^\tau T_{-1}(s)Bu(\tau - s)\,ds$ just means admissibility of C and B, respectively, and all that is left to study is continuity of the *input–output–map* $\mathbb{F} : u \mapsto CT_{-1}(\cdot)B * u$. Again, the choice of function spaces for u and y plays a decisive role. Besides spaces $L^2(I, U)$, $L^2(I, Y)$ for Hilbert spaces U, Y and spaces $l(I, U)$, $l(I, Y)$ for Banach spaces U, Y we shall also consider continuity of the input–output–map $L^p(I, U) \to L^p(I, Y)$ where $1 < p < \infty$ and U, Y are Banach spaces. Clearly, the input–output–map is a (singular) convolution operator and we obtain our characterisations of wellposedness of the input–output–map (Theorem 4.10) by application of Fourier multiplier results from [20] and [33]. Finally, we combine our results on l–admissibility of unbounded observation and control operators and on wellposedness of the input–output–map and obtain a characterisation of wellposedness for the full system (1) (see Theorem 4.11 and Corollary 4.12).

The paper is organised as follows: in Section 2 we recall basic facts on sectorial operators and the H^∞-calculus. In Section 3 we give an introduction to l-spaces and briefly survey the properties we shall need in the sequel and the role they play in characterisations of the H^∞–calculus. The quoted results are due N. Kalton and L. Weis ([20]) with exception of Theorem 3.18 which is new. We also review some notions from Banach space geometry. Section 4 contains our main results on [l–]admissibility (Theorems 4.2 and 4.5) as well as on [l–]wellposedness of the input-output map (Theorem 4.10) and of the full system (Theorem 4.11, Corollary 4.12). These results are proved in Section 5 as well as Theorems 1.3 and 1.5. The final Section 6 contains examples of unbounded [l–]admissible observation and control operators, which are obtained via our main results. We concentrate on the case $A = -\Delta$ in L^p, $1 < p < \infty$, and give several examples, in particular point observation and control as well as observation on and control from the boundary.

At the end we set up a well–posed linear system for a controlled heat equation with state space $L^p(\Omega)$ where $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ is a bounded and smooth domain. This is the L^p–version of a system, which – for $p = 2$ – was studied in [2].

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2. Preliminaries on H^∞–calculus and semigroups

We start with a brief overview of the H^∞–calculus for sectorial operators. Details heron may be found in e.g. in [26, 27, 5, 20]. Given some $\theta \in (0, \pi]$ let $S(\theta)$ be the open sector of all $z \in \mathbb{C}\backslash\{0\}$ such that $|\arg(z)| < \theta$. Let Γ_θ be the counterclockwise orientated boundary of $S(\theta)$. The set of all bounded and holomorphic functions f on $S(\theta)$ is denoted by $H^\infty(S(\theta))$. These functions form a Banach algebra for the norm $\|f\|_\infty := \sup\{|f(z)| : \ z \in S(\theta)\}$. Let $H_0^\infty(S(\theta))$ be the sub-algebra of all functions f, for which a number $s > 0$ exists such that

$$|f(z)| = O(|z|^s) \text{ at zero} \quad \text{and} \quad |f(z)| = O(|z|^{-s}) \text{ at infinity.}$$

A closed linear operator is said to be *sectorial of type* ω, $\omega \in (0, \pi)$, if its spectrum is contained in the closure of $S(\omega)$, and if for any $\theta \in (\omega, \pi)$, there is a constant C_θ such that

$$\|zR(z, A)\| \le C_\theta, \qquad z \notin \overline{S(\theta)}.$$

It is clear that if $-A$ generates a bounded strongly continuous semigroup, then A is sectorial of type $\pi/2$. Furthermore, $-A$ generates a bounded analytic semigroup if and only if A is densely defined and sectorial of type $\omega < \pi/2$.

Now assume that A is a densely defined sectorial operator of type ω. Let $\theta \in (\omega, \pi)$ and $f \in H_0^\infty(S(\theta))$. Then setting

$$f(A) := \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A)\, dz,$$

where $\Gamma = \Gamma_\nu$ for some $\nu \in (\omega, \theta)$ defines a bounded linear operator on X. By Cauchy's theorem this definition does not depend on the choice of $\nu \in (\omega, \theta)$. Notice that the imposition of a dense domain of A is not necessary for the above definition, but since we intend to treat only generators of semigroups we will not go into details here.

The mapping $f \mapsto f(A)$ is an algebra homomorphism from $H_0^\infty(S(\theta))$ to $B(X)$. Moreover, the definition of $f(A)$ satisfies the following important consistency property with the ad-hoc functional calculus for rational functions: let p, q are polynomials with all zeros of q in the resolvent set of A such that $f := p/q \in H_0^\infty(S(\theta))$, then $f(A) = p(A)(q(A))^{-1}$.

Definition 2.1. Let A be a densely defined sectorial operator of type $\omega \in (0, \pi)$ on a Banach space X and let $\theta \in (\omega, \pi)$. Then A is said to admit a *bounded $H^\infty(S(\theta))$–calculus* if there is a constant $M \geq 0$ such that

$$\|f(A)\| \leq M\, \|f\|_\theta, \qquad f \in H_0^\infty(S(\theta)).$$

If a densely defined sectorial operator A has dense range, then by [5, Theorem 3.8] it is also injective. In this case, there is a natural extension of the above definition of $f(A)$ to arbitrary functions $f \in H^\infty(S(\theta))$ [26, 27]. However, in general $f(A)$ then is a closed but possibly unbounded operator on X. An application of the closed graph theorem reveals that A admits a bounded $H^\infty(S(\theta))$–calculus in the above sense if and only if $f(A)$ is bounded for any $f \in H^\infty(S(\theta))$.

In [26] it is shown, that in Hilbert spaces X the property of a densely defined sectorial operator A with dense range to possess a bounded $H^\infty(S(\theta))$–calculus or not, is strongly connected to quadratic estimates, which we will recall in the sequel:

If A is sectorial of type ω and if F is a non-zero function belonging to $H_0^\infty(S(\theta))$ for some $\theta \in (\omega, \pi)$, we set

$$\|x\|_F := \left(\int_0^\infty \left\| F(tA)x \right\|_X^2 \frac{dt}{t} \right)^{\frac{1}{2}}, \qquad x \in X.$$

Note that $\|x\|_F$ may be equal to $+\infty$. These square functions were introduced by McIntosh in [26], see also [27]. The next theorem is originally stated for Hilbert spaces X only, but the proof extends in verbatim to the Banach space case.

Theorem 2.2 ([27, Thm. 5])**.** *Let A be a densely defined sectorial operator of type ω on a Banach space X, and assume that A has dense range. Let $F, G \in H_0^\infty(S(\theta)) \setminus \{0\}$, where $\theta > \omega$. Then there exist two positive constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1 \|x\|_G \leq \|x\|_F \leq c_2 \|x\|_G, \qquad x \in X.$$

This leads to the following

Definition 2.3. Let A be as in Theorem 2.2, and let $F \in H_0^\infty(S(\theta)) \setminus \{0\}$, where $\theta > \omega$. We say that A has a quadratic estimate if there is a constant $c > 0$ such that

$$\|x\|_F \leq c\|x\|_X, \qquad x \in X.$$

By the above theorem, this definition does not dependent on F.

In [26, Section 8] it is shown that on Hilbert spaces a densely defined sectorial operator A of type ω with dense range has a bounded H^∞–calculus if and only if A and its dual A' admit quadratic estimates. This is not the case on non-Hilbertian spaces, as was shown in [5, Section 6]. However, when considering Banach spaces of finite cotype the H^∞–calculus still admits a similar characterisation if we change the notion of quadratic estimates. This is the topic of the next section.

3. Square function estimates and generalised square functions

Consider the Hilbert space $X = L^2(\Omega)$. For classical quadratic estimates the norm of $t \mapsto \varphi(tA)x$ is taken in $L^2(\mathbb{R}_+, dt/t, L^2(\Omega))$. When generalising this to spaces $X = L^p(\Omega)$, one may consider the function norm in $L^2(\mathbb{R}_+, dt/t, L^p(\Omega))$. But it turns out that, for characterising boundedness of the H^∞–calculus, the function norm in $L^p(\Omega, L^2(\mathbb{R}_+, dt/t))$ is the right one (cf. [5]). Notice that by Fubini's theorem both norms coincide when $p = 2$.

In the sequel we will, following N. Kalton and L. Weis [20], survey a notion of square function estimates based on a generalisation of the norm of $L^p(\Omega, L^2(\mathbb{R}_+, dt/t))$ to arbitrary Banach spaces X.

Let $f \in L^p(\Omega, L^2(I))$ for some measure spaces Ω, I. Then, for an orthonormal system (e_n) of $L^2(I)$ and the bilinear product $\langle f, g\rangle = \int fg$ one has

$$\int_I |f(\omega,t)|^2 dt = \sum_{n\in\mathbb{N}} |\langle f(\omega), \bar{e}_n\rangle|^2.$$

If $g_1, \ldots, g_N$ are independent $N(0,1)$–distributed Gaussian random variables, then for any series of complex numbers $\alpha_1, \ldots, \alpha_N$ one has

$$\sum_{n=1}^N |\alpha_n|^2 = \mathbb{E}\Big|\sum_{n=1}^N g_n \alpha_n\Big|^2.$$

Both equations together lead to

$$\begin{aligned}\|f\|_{L^q(L^2)} &= \left(\int_\Omega \left(\int_I \big|f(t)(\omega)\big|^2 dt\right)^{q/2} d\omega\right)^{1/q}\\ &= \left(\int_\Omega \left(\lim_{N\to\infty} \mathbb{E}\Big|\sum_{n=1}^N g_n \underbrace{\langle f(\cdot)(\omega), \bar{e}_n\rangle}_{=:u_f(\bar{e}_n)(\omega)}\Big|^2\right)^{q/2} d\omega\right)^{1/q}\end{aligned}$$

Now, a Khintchine type result that may be found, e.g., in [7, Chap. 12] yields

$$= C_q\left(\int_\Omega \lim_{N\to\infty} \mathbb{E}\Big|\sum_{n=1}^N g_n u_f(\bar{e}_n)(\omega)\Big|^q d\omega\right)^{1/q}.$$

By monotone convergence, this equals

$$= C_q \lim_{N\to\infty}\left(\mathbb{E}\Big\|\sum_{n=1}^N g_n u_f(\bar{e}_n)\Big\|^q_{L^q(\Omega)}\right)^{1/q},$$

and Kahane's inequality finally gives

$$\sim \lim_{N\to\infty}\left(\mathbb{E}\Big\|\sum_{n=1}^N g_n u_f(\bar{e}_n)\Big\|^2_{L^q(\Omega)}\right)^{1/2}.$$

The above calculation yields an equivalent formula which does no longer depend on the function space structure of $L^q(\Omega)$ and allows the following generalisation:

Definition 3.1 ([20, Def. 4.1])**.** Let H be a Hilbert space and X a Banach space. We denote by $l_+(H, X)$ the space of all linear operators $u : H \to X$ such that

$$\|u\|_l = \sup\left\{\left(\mathbb{E}\Big\|\sum g_n u(e_n)\Big\|^2\right)^{1/2} : (e_n) \textit{ is a finite orthonormal system in } H\right\} < \infty.$$

By $l(H, X)$ we denote the closure of the finite dimensional operators in $l_+(H, X)$.

We note, that $l(H, X)$ is a Banach space, that is contained in $B(H, X)$ ([20]). For the most important case $H = L^2(I, \mu)$ there is a class of generating functions which may be defined as follows (see [20, Definition 4.5]): Let $\mathcal{P}_2(I, \mu, X)$ be the class of all Bochner-measurable functions from I to X for which

$x'(f) \in L^2(I,X)$ whenever $x' \in X'$. For $f \in \mathcal{P}_2(I,\mu,X)$ we define the operator $u_f \in B(L^2(I,\mu),X)$ such that for $x' \in X'$ and $h \in L^2(I)$

$$\langle u_f h, x'\rangle_{X,X'} = \int_I \langle f(t), x'\rangle_{X,X'} h(t)\, d\mu(t).$$

For the details of this definition (e.g., the question why $u_f \in B(H,X'')$ can be regarded as an element of $B(H,X)$) we refer to [20, Definition 4.5] or [9, Section 5.5]. Then if $u_f \in l_+(L^2(I),X)$ define

$$\|f\|_{l(I,X)} := \|u_f\|_{l(L^2(I),X)}.$$

The space of all $f \in \mathcal{P}_2(I,\mu,X)$ for which $u_f \in l_+(L^2(I),X)$ or $u_f \in l(L^2(I),X)$ is called $l_+(I,X)$ or $l(I,X)$ respectively. Contrary to the case $X = L^q(I)$, for arbitrary Banach spaces X, $l(I,X)$ does not have to be a Banach space, see [20, Remark 4.7]. However, $l(I,X)$ is dense in $l(L^2(I),X)$ [ibidem]. The following shows that $L^2(I)\otimes X$ is a subset of $l(I,X)$:

Lemma 3.2. *If φ is an element of $L^2(I)$ and $x \in X$, then $f(t) := \varphi(t)x$ defines an element of $l(I,X)$ with $\|f\|_l = \|\varphi\|_{L^2}\|x\|_X$.*

Proof. Let (e_n) a fixed orthonormal system in $L^2(I)$. Then by definition

$$\left(\mathbb{E}\left\|\sum_{k=1}^n g_k (e_k|\varphi)x\right\|_X^2\right)^{\frac12} = \left(\mathbb{E}\left|\sum_{k=1}^n g_k (e_k|\varphi)\right|^2\right)^{\frac12}\|x\|_X = \left(\sum_{k=1}^n \big|(e_k|\varphi)\big|^2\right)^{\frac12}\|x\|_X \le \|\varphi\|_{L^2}\|x\|_X.$$

Now, letting $n \to \infty$ we obtain the desired result by Parseval's equality. □

Next we discuss how to extend a bounded operator $S : H_1 \to H_2$ to an operator $S^{\otimes} : l(H_1,X) \to l(H_2,X)$. This is simply done by $S^{\otimes}(u) := u \circ S'$. By [20, Proposition 4.4] we have that $\|S^{\otimes}\| \le \|S\|$. This result has some remarkable applications (see [20, Example 4.9]). Notice that for non-Hilbertian spaces X, (b) is in contrast to the behaviour of the Bochner-space $L^2(I,X)$:

Remark 3.3.

(a) Since for intervals $I \subseteq J$ the zero extension of functions of $L^2(I)$ to J and the restriction of functions of $L^2(J)$ to I are both bounded operators, the restriction $l(J,X) \to l(I,X)$ and the extension $l(I,X) \to l(J,X)$ are bounded (of norm at most one).

(b) Let f be in $l(\mathbb{R}^n,X) \cap L^1(\mathbb{R}^n,X)$. Then we may extend the Fourier transform $\mathcal{F}$ and obtain $\mathcal{F}^{\otimes} f \in l(\mathbb{R}^n,X)$ and $\|\mathcal{F}^{\otimes} f\|_{l(\mathbb{R}^n,X)} = \|f\|_{l(\mathbb{R}^n,X)}$.

(c) If $f \in l(\mathbb{R}_+,X)$ for some Banach space X, then the Laplace transform

$$\hat f(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt$$

exists at least on the open right half plane $\mathbb{C}_0$. This is due to the fact that setting $e_\lambda := \exp(\lambda\cdot)$ we have $\widehat{f}(\lambda) = u_f(e_{-\lambda})$ and thus $\|\widehat{f}(\lambda)\|_X \le \|e_{-\lambda}\|_{L^2}\,\|f\|_l = (2Re(\lambda))^{-1/2}\,\|f\|_l$.

Moreover, there is a sort of analogue of Hölder's inequality for l–norms:

Proposition 3.4 ([20, Cor. 5.5])**.** *Let (I,μ) be a σ–finite measure space. Let $f \in l(I,X)$ and $g \in l(I,X')$. Then*

$$\int_I \big|\langle f(t), g(t)\rangle\big|\, d\mu(t) \le \|f\|_{l(I,X)}\|g\|_{l(I,X')}.$$

The analysis of bounded families of operators on $l(H,X)$ (see Proposition 3.9 below) motivates the following definition:

Definition 3.5. Let X, Y be Banach spaces and $\mathcal{T}$ be a set of operators in $B(X,Y)$.

(a) The set $\mathcal{T}$ is called *l–bounded* if there is a constant C such that for all $m \in \mathbb{N}$, $T_1,\dots,T_m \in \mathcal{T}$ and $x_1,\dots,x_m \in X$

$$\left(\mathbb{E}\left\|\sum_{n=1}^m g_n T_n x_n\right\|_Y^2\right)^{1/2} \le C\left(\mathbb{E}\left\|\sum_{n=1}^m g_n x_n\right\|_X^2\right)^{1/2} \tag{9}$$

where (g_n) is a sequence of independent, $N(0,1)$–distributed Gaussian variables.

(b) The set $\mathcal{T}$ is called *R–bounded* if there is a constant C such that above inequality (9) holds when substituting the Gaussian variables by a sequence (r_n) of independent $\{\pm 1\}$–distributed Bernoulli variables. Notice that the Rademacher functions $r_n(t) := \text{sign } \sin(2^n \pi t)$ form such a sequence on $[0,1]$.

(c) The infimum of all constants C, for which the above inequality holds, is called the *l–bound* (respectively the *R–bound*) of the set $\mathcal{T}$.

Remark 3.6. Clearly R–boundedness implies uniform boundedness. The converse holds provided that X has cotype 2 and Y has type 2 (cf. [12, Ex. 6.13] for a counterexample). In particular, if $X = Y$ this holds true only for spaces that are isomorphic to Hilbert spaces. Notice that R–bounded sets are always l–bounded (cf. [32, Sect. 3]) whereas the converse holds only if both X and Y have finite cotype (see [7, Proposition 12.11 and Theorem 12.27]).
Recall that a Banach space X is said to be of *type* $p \in [1,2]$ if

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^m g_n x_n\Big\|_X^2\Big)^{1/2} \le C\Big(\sum_{n=1}^m \|x_n\|_X^p\Big)^{1/p},$$

whereas X is said to be of cotype $q \in [2,\infty]$ if

$$\Big(\sum_{n=1}^m \|x_n\|_X^q\Big)^{1/2} \le C\Big(\mathbb{E}\Big\|\sum_{n=1}^m g_n x_n\Big\|_X^2\Big)^{1/2}$$

with obvious modification for $q = \infty$. Here, of course, C is required to be independent of the finite set $(x_n)_{n=1}^m$ in X. Observe that any Banach space X has cotype ∞ and type 1; X is said to have *nontrivial type* if X has type p for some $p > 1$. In this case, X also has finite cotype ([29, Theorem 4.6.20]).

Lemma 3.7 ([3, Lemma 3.3])**.** *Let X and Y be Banach spaces and $\mathcal{T} \subset B(X,Y)$. If $\mathcal{T}$ is l–bounded (R–bounded) with bound M, then its closed absolute-convex hull with respect to the strong operator topology is l–bounded (R–bounded) too with a bound of at most $2M$.*

The following lemma is an important application of the foregoing result:

Lemma 3.8 ([20, Lemma 5.8])**.** *Let (I,μ) be a σ-finite measure space and let $N(t)$ be a strongly measurable mapping from I to $B(X,Y)$. Suppose that $\{N(t) : t \in I\}$ is an l–bounded set with bound C. For scalar-valued functions $h \in L^1(I)$ and $x \in X$ we define*

$$N_h(x) := \int_I h(t)N(t)x\, d\mu(t)$$

Then the set $\{T_h : \|h\|_{L^1(I)} \le 1\}$ is l–bounded in $B(X,Y)$ with bound less or equal to $2C$.

To see the link between l–bounded sets and l–norms, we cite

Proposition 3.9 ([20, Proposition 4.11])**.** *Let I be an interval, $g \in L^1_{loc}(I)$, $g > 0$ a.e. and $\mu = g(t)\,dt$. Let $N : I \to B(X,Y)$ be a strongly continuous map. Then the set $\mathcal{T} = \{N(t) : t \in I\}$ is l–bounded with bound C if and only if for all $f \in l(I,X)$ one has*

$$\|N(\cdot)f(\cdot)\|_{l(I,d\mu,Y)} \le C\|f\|_{l(I,d\mu,X)}.$$

We recall a dualisation result

Lemma 3.10 ([17, Lem. 3.1])**.** *Let X,Y be Banach spaces of nontrivial type. Then $\mathcal{T} \subseteq B(X,Y)$ is l–bounded (or equivalently R–bounded) if and only if $\mathcal{T}' \subseteq B(Y',X')$ is.*

Now we shall come back to the announced link between square function estimates and boundedness of the H^∞–calculus.

Definition 3.11. A sectorial operator A on a Banach space X is called *l–sectorial* of type ω_l if the set $\{\lambda(\lambda+A)^{-1} : \lambda \in S(\pi-\theta)\}$ is l–bounded for every $\theta \in (\omega_l, \pi)$.

Definition 3.12. Let A be an densely defined l–sectorial operator of type ω on X and for some $\theta > \omega$ let φ be a function in $H_0^\infty(S(\theta))\backslash\{0\}$. We say that A satisfies a *square-function estimate* for φ if there exists a constant $M > 0$ such that

$$\forall x \in X : \quad \|\varphi(\cdot A)x\|_{l(\mathbb{R}_+,\frac{dt}{t},X)} \le M\|x\|_X.$$

By [20, Prop. 7.7] (the analogue of Theorem 2.2 cited above), the property of A to satisfy square function estimates (or not) does not depend on the particular choice of the function $\varphi \in H_0^\infty(S(\theta))\backslash\{0\}$.

Theorem 3.13 ([20, Thm. 7.2])**.** *Let A be a densely defined l–sectorial operator of type ω with dense range. If A and its dual A' satisfy square-function estimates, then A has a bounded H^∞–calculus on X. If X has finite cotype, then the converse holds, too.*

For the last result in this section we recall some geometric properties of Banach spaces related to *randomised sums*, i.e., to sums $\sum_{k\le n}\chi_k x_k$ where the x_k are elements of a given Banach space and the χ_k are real-valued and symmetric random variables. The following inequality may be found, e.g., in [7, 12.2].

Proposition 3.14 (Contraction principle)**.** *Let $1 \le p < \infty$ and consider the randomised sum $\sum_{k\le n}\chi_k x_k$ on a Banach space X. Then for any choice of complex numbers a_k with $|a_k| \le 1$,*

$$\Big(\mathbb{E}\Big\|\sum_{k\le n} a_k\chi_k x_k\Big\|^p\Big)^{1/p} \le 2\Big(\mathbb{E}\Big\|\sum_{k\le n}\chi_k x_k\Big\|^p\Big)^{1/p}.$$

We mention an important consequence of the contraction principle. A proof may be found e.g. [21, Cor. 2.17].

Lemma 3.15 ([20, Lemma 5.9])**.** *Let (I,μ) be a σ-finite measure space and let $N(t)$ be strongly integrable. Suppose that there exists a constant C such that for all $x \in X$*

$$\int_I \|N(t)x\|\,d\mu(t) \le C\|x\|.$$

Then, using the notations of 3.8, the set $\{N_h \mid h \in L^\infty(I), \|h\| \le 1\}$ is l–bounded with a bound of at most $2C$.

If we take not one, but two independent sequences of independent symmetric, real valued random variables (χ_i), (χ'_j) then their products $\chi_i(\cdot)\chi'_j(\cdot)$ will be prevail their symmetry but lose their independence. This can easily be seen since for independent $\{\pm 1\}$–distributed Bernoulli variables r_1, r_2, r'_1, r'_2 the case

$$r_1 r'_1 = 1, \quad r_1 r'_2 = 1, \quad r_2 r'_1 = 1, \quad r_2 r'_2 = -1$$

is impossible. For the general case of symmetric real valued random variables notice that their signs are Rademacher random variables. Since products of independent random variables are not independent we do not have a contraction principle for such product-randomised sums in general spaces X.

Definition 3.16 (Pisier [30])**.** We say that X has property (α), if for two independent sequences (r_n), (r'_n) of Rademacher sequences there is a constant C so that for all $x_{ij} \in X$ and $|a_{ij}| \le 1$ the estimate

$$\Big(\mathbb{E}\,\mathbb{E}'\Big\|\sum_{i,j=1}^n r_i r'_j a_{ij} x_{ij}\Big\|^2\Big)^{1/2} \le C\Big(\mathbb{E}\,\mathbb{E}'\Big\|\sum_{i,j=1}^n r_i r'_j x_{ij}\Big\|^2\Big)^{1/2} \tag{10}$$

holds.

Remark 3.17. If a Banach space X has property (α) then by [30, Remark 2.2] X does not contain l_∞^n's uniformly, which by [7, Theorem 14.1] implies that X has finite cotype. Examples of spaces with property (α) are, e.g., L^p–spaces for $p \in [1,\infty)$.

Now we present a powerful extension theorem whose implications are crucial for the characterisations in the next section. It will be proved in Section 5.

Theorem 3.18. *Let H_1 and H_2 be two separable Hilbert spaces. Let Y be a Banach space with property (α). Let $\mathcal{A}$ be a bounded set in $B(H_1,H_2)$. Then the set $\mathcal{A}^\otimes := \{A^\otimes : A \in \mathcal{A}\}$ is l–bounded in $B\big(l(H_1,Y), l(H_2,Y)\big)$.*

The following corollary was found in the case of reflexive L^p-spaces X and Y by Le Merdy [25, Proposition 3.3]. Here in fact we apply the above Theorem for $H_1 = L^2(I)$ and $H_2 = \mathbb{C}$.

Corollary 3.19. *Let X and Y be Banach spaces and Y have property (α). Let $\varphi \in B(X, l(I,Y))$. Then the set*

$$\Big\{\int_I a(t)\varphi(t)dt : \quad a \in L^2(I), \|a\|_2 \le 1\Big\}$$

is l–bounded in $B(X,Y)$.

Applying this to the special functions $h_\lambda(t) := \lambda^{1/2} e^{-\lambda t}$, $Re(\lambda) > 0$ yields the following result in the case that Y has property (α). The assertion even holds without any restriction on Y, as L. Weis pointed out to the authors:

Proposition 3.20. *Let X, Y be Banach spaces and $N : \mathbb{R}_+ \to B(X,Y)$ be strongly measurable. If $M > 0$ and $N(\cdot)x \in l(\mathbb{R}_+, Y)$ with an estimate $\|N(\cdot)x\|_l \le M\|x\|$ for all $x \in X$, then for every $\theta < \pi/2$ the set $\{\lambda^{1/2}\widehat{N}(\lambda) : \lambda \in S(\theta)\}$ is l–bounded where $\widehat{N}(\cdot)$ denotes the Laplace transform of N.*

Now we give a useful corollary of Theorem 3.18 for $H_1 = \mathbb{C}$ and $H_2 = L^2(I)$:

Corollary 3.21. *Let X be a Banach space and U be a Banach space with property (α). If $\varphi : I \to B(U,X)$ is strongly measurable such that $\| \int_I \varphi(t)u(t)\,dt\|_X \le K\|u\|_l$ for all $u \in l(I,U)$, then the set*

$$\Big\{\int_I a(t)\varphi(t)\,dt : \ a \in L^2(I), \|a\|_2 \le 1\Big\}$$

is l–bounded in $B(U,X)$.

It seems not to be clear if an analogue of Proposition 3.20 holds in the situation of 3.21, too.

4. Main results

Let U, X and Y be Banach spaces. We now consider linear control systems of the forms

$$\begin{cases} x'(t) + Ax(t) &= 0, \quad t > 0,\\ x(0) &= x_0,\\ y(t) &= Cx(t), \quad t > 0,\end{cases} \qquad \text{and} \qquad \begin{cases} x'(t) + Ax(t) &= Bu(t), \quad t > 0,\\ x(0) &= x_0.\end{cases} \tag{11}$$

on $[0,\infty)$. Here, $C \in B(X_1, Y)$ and $B \in B(U, X_{-1})$ are unbounded with respect to X.

Admissibility of observation and control operators

Definition 4.1. Let $-A$ be generator of a bounded strongly continuous semigroup $T(\cdot)$ on X and $C \in B(X_1, Y)$. Then C is called an *l–admissible observation operator* for A, if there exists a $M > 0$ such that for all $x \in X$ one has

$$\|CT(\cdot)x\|_{l(\mathbb{R}_+,Y)} \le M\|x\|_X.$$

We shall establish a characterisation of l–admissibility of observation operators for an l–bounded analytic semigroup, that is an analytic semigroup with $\{T(z) : \ z \in S(\theta)\}$ being l–bounded for some positive angle θ. Indeed, if $-A$ denotes the generator of such a semigroup, an equivalent formulation is that A is densely defined and l–sectorial of type $\omega_l < \pi/2$ (this can be shown similar to the proof [33, Thm. 2.10]). The following theorem extends a result obtained by Le Merdy [25] for L^p–spaces.

Theorem 4.2. *Let X and Y be Banach spaces and let A be densely defined l–sectorial operator of type $\omega_l < \pi/2$ that has dense range. Consider an observation operator $C \in B(X_1, Y)$ and let $W_C := \{\lambda^{1/2}C(\lambda + A)^{-1} : \lambda > 0\}$.*

(a) *If Y has property (α), then l–admissibility of C implies l–boundedness of W_C.*
(b) *If A satisfies square function estimates in the sense of Definition 3.12, then l–boundedness of W_C implies l–admissibility of C.*

Remark 4.3. Notice that the assumption of square function estimates is necessary: Let Γ be the positively orientated boundary of some sector $S(\theta)$ with $\theta \in (\omega_l, \pi)$. Since for positive numbers α, β with $\alpha + \beta = 1$ we have

$$\lambda^\alpha A^\beta(\lambda + A)^{-1}x = \tfrac{1}{2\pi i}\int_\Gamma \frac{\lambda^\alpha z^\beta}{\lambda - z} R(z,A)x\,dz = \tfrac{1}{2\pi i}\int_\Gamma \frac{\lambda^\alpha z^{\beta-1}}{\lambda - z}\, zR(z,A)x\,dz,$$

the uniform boundedness of the functions $h_\lambda(z) := \frac{\lambda^\alpha z^{\beta-1}}{\lambda-z}$ in $L^1(\Gamma)$ implies together with Lemma 3.8 l–boundedness of $\{\lambda^\alpha A^\beta(\lambda+A)^{-1} : \lambda > 0\}$. In particular, the choice $\alpha=\beta=1/2$ shows l–boundedness of the set W_C in case $C = A^{1/2}$. Therefore, if for any Banach space Y and any operator $C \in B(X_1, Y)$, l–boundedness of the set W_C implies l–admissibility of C, then $A^{1/2}$ must be l–admissible for A proving that A satisfies square function estimates. This shows that the assumption of square function estimates in the foregoing theorem cannot be omitted.

We now turn to control operators:

Definition 4.4. Consider the system (11) for a generator $-A$ of a C_0-semigroup $T(\cdot)$ and a control operator $B \in B(U, X_{-1})$. Then B is called an *l–admissible control operator* for A, if the integral in the following formula exists as Pettis integral in X_{-1}, takes values in X and allows the estimate

$$\left\| \int_0^\infty T_{-1}(t)Bu(t)\,dt \right\| \le K\|u\|_{l(\mathbb{R}_+,U)}.$$

Notice that in reflexive Banach spaces U and X of finite cotype, B is an l–admissible control operator for A if and only if B' is an l–admissible observation operator for A'. In the general case, the following holds true:

Theorem 4.5. *Let X and U be Banach spaces and let A be a densely defined l–sectorial operator of type $\omega_l < \pi/2$ with dense range. Consider a control operator $B \in B(U, X_{-1})$ and let $W_B := \{\lambda^{1/2}(\lambda+A_{-1})^{-1}B : \lambda > 0\}$.*

(a) *If U has property (α), then l–admissibility of B implies l–boundedness of W_B in $B(U, X)$.*
(b) *If A' satisfies a square function estimate in the sense of Definition 3.12 on X', then l–boundedness of W_B in $B(U, X)$ yields l–admissibility of B.*

A review of the proof of the foregoing theorem, which will be given in Section 5 shows that the following result for the case of classical $L^2(\mathbb{R}_+, X)$–norms can be derived by slight modifications. Notice that this characterisation is new for non-reflexive spaces.

Theorem 4.6. *Let X and U be Banach spaces and let $T(\cdot)$ be a bounded and analytic semigroup $T(\cdot)$ whose generator $-A$ has dense range. Assume A' to satisfy a quadratic estimate as in* (5)*. Then $B \in B(U, X_{-1})$ is an admissible control operator for A if and only if the operators $\lambda^{1/2}(\lambda+A)^{-1}B$ are uniformly bounded in $B(U, X)$ for $\lambda > 0$.*

Similar to the classical Bochner-norm theory, l–admissibility may also be considered on finite time intervals:

Definition 4.7. Let $\tau > 0$. Then an observation operator $C \in B(X_1, Y)$ is said to be *l–admissible on* $[0, \tau]$, if the estimate

$$\|CT(\cdot)x\|_{l([0,\tau],Y)} \le M\|x\|_X$$

holds for every $x \in X_1$. Analogously a control operator $B \in B(U, X_{-1})$ is called *l–admissible on* $[0, \tau]$ provided that the following integral exists as a Pettis integral in X_{-1}, takes values in X and allows the estimate

$$\left\| \int_0^\tau T_{-1}(\tau - s)Bu(s)\,ds \right\|_X \le K\|u\|_{l([0,\tau],U)}.$$

Lemma 4.8. *Let $T(\cdot)$ be an l–bounded analytic semigroup with generator $-A$ and let $B : U \to X_{-1}$ and $C : X_1 \to Y$ be bounded.*

(a) *B is finite-time l–admissible for A if and only if for any $\alpha > 0$, B is l–admissible for the scaled semigroup $e^{-\alpha\cdot}T(\cdot)$.*
(b) *If C is finite-time l–admissible for A if and only if for any $\alpha > 0$, C is l–admissible for the scaled semigroup $e^{-\alpha\cdot}T(\cdot)$.*

In particular, finite-time l–admissibility does not depend on the length τ of the given time interval.

Proof. The proof is a standard argument exploiting the semigroup property and a geometric series. □

Remark 4.9. If $C \in B(X,Y)$ and $T(\cdot)$ is a bounded and analytic semigroup, then C is always finite-time l–admissible. Analogously the corresponding assertion for $B \in B(U,X)$ is true. This is an immediate consequence of the estimate

$$\|f\|_{l([0,\tau],X)} \leq \int_0^\tau t^{1/2}\|f'(t)\|_X\,dt + \tau^{1/2}\|f(\tau)\|_X,$$

for $f \in C([0,\tau]) \cap C^1((0,\tau])$ (see [20, Example 4.6]) and the fact that for bounded and analytic semigroups we have $\|\frac{d}{dt}T_t x\| \leq \frac{M}{t}\|x\|$.

Wellposedness of the full system

Now we consider wellposedness of the full linear system

$$\begin{cases} x'(t) + Ax(t) &= Bu(t), \quad t > 0, \\ x(0) &= x_0, \\ y(t) &= Cx(t), \quad t > 0. \end{cases} \tag{12}$$

For the concept of wellposedness we refer to the remarks in the introduction and to [37] where the case $y \in L^2([0,\tau),Y)$, $u \in L^2([0,\tau),U)$ for Hilbert spaces Y and U, and the case $y \in L^p([0,\tau),Y)$, $u \in L^p([0,\tau),U)$, $p \in [1,\infty]$, for Banach spaces X, Y, U is studied.

In view of the results presented so far in this section we shall also consider the case of l–spaces instead of L^2 or L^p. The observation $y(\cdot)$ in (12) is given by $y(\cdot) = CT(\cdot)x_0 + CT(\cdot)B * u$. Hence it rests to study continuous dependence of $y(\cdot) = CT(\cdot)B * u$ on $u(\cdot)$. Again, this is best done on the interval $[0,\infty)$. Below we give conditions for $\|y\|_{L^p(\mathbb{R}_+,Y)} \leq M\|u\|_{L^p(\mathbb{R}_+,U)}$, $1 < p < \infty$, and for $\|y\|_{l(L^2(\mathbb{R}_+),Y)} \leq M\|u\|_{l(L^2(\mathbb{R}_+),U)}$.

Let $-A$ be the generator of a bounded analytic semigroup on X. Then the *input-output map* $\mathbb{F}_\infty$ is given by $\mathbb{F}_\infty(u) := CT_{-1}(\cdot)B * u$. As it is done in [23, Sect. 4] for bounded operators $C \in B(X,Y)$ and $B \in B(U,X)$, the map $\mathbb{F}_\infty$ can be regarded as a Fourier multiplier. Note that our assumption on A implies that the imaginary axis with possible exception of zero is contained in the resolvent set of A. In order to consider the symbol of $\mathbb{F}_\infty$ we resort to a construction due to G. Weiss ([34, 36]): the *Yoshida–extension* C_Λ of $C : X_1 \to Y$ is given by

$$x \in \mathcal{D}(C_\Lambda) \text{ and } C_\Lambda x = y \quad \Longleftrightarrow y = \lim_{\lambda\to+\infty} C\lambda(\lambda + A)^{-1}x.$$

We recall that the transfer function, i.e., the Laplace transform image of $CT_{-1}(\cdot)B$, is called *regular* if resolvents of A_{-1} map the range $B(U)$ of the operator B into $\mathcal{D}(C_\Lambda)$ (there are several equivalent formulations of regularity, cf. [36, 37, 38]). We also call the system (12) *regular* if it has a regular transfer function.

In the sequel we shall only consider regular systems, i.e., those for which the Laplace transform of $CT_{-1}(\cdot)B$ can be written as $\lambda \mapsto C_\Lambda(\lambda + A_{-1})^{-1}B$. We call the input-output map *L^p–wellposed*, if $\mathbb{F}_\infty$ is bounded from $L^p(\mathbb{R}_+,U)$ to $L^p(\mathbb{R}_+,Y)$. In [4, Proposition 1] it is shown that a necessary condition for L^p–boundedness of $\mathbb{F}_\infty$ (for a regular system) is R–boundedness of the symbol on its Lebesgue–points, i.e., R–boundedness of the set

$$\{C_\Lambda(i\xi + A_{-1})^{-1}B : \xi \neq 0\} \subset B(U,Y). \tag{13}$$

To achieve a sufficient condition we need the notion of a UMD space. A Banach space X is said to have the UMD property if the Hilbert–transform is a bounded operator on $L^p(\mathbb{R},X)$ for some (and thus all) $p \in (1,\infty)$. UMD refers to the fact that X is a UMD space if and only if all martingale difference sequences induce an unconditional decomposition of the space $L^p(\Omega,X)$. Typical examples of UMD spaces are $L^p(\Omega)$–spaces, Sobolev spaces $W_p^s(\Omega)$ or Besov spaces $B_{p,q}^s(\Omega)$ for $p,q \in (1,\infty)$ and their closed subspaces. For equivalent definitions, properties and further references on UMD–spaces we refer, e.g., to [21]. We mention here that UMD–spaces have nontrivial type and thus finite cotype (cf. [21, Ch. 3]), whence by Remark 3.6 the notions of R–boundedness and l–boundedness coincide in UMD–spaces.

If the system (12) is regular and U and Y are UMD–spaces then, by the Mikhlin-Weis theorem [33, Theorem 3.4], R–boundedness of the set of operators in (13) together with R–boundedness of the following set

$$\{\xi \frac{d}{d\xi} C_\Lambda(i\xi + A_{-1})^{-1}B : \xi \neq 0\} \subset B(U,Y) \tag{14}$$

is sufficient to ensure L^p–boundedness of $\mathbb{F}_\infty$ for $1 < p < \infty$. Note that

$$\xi \frac{d}{d\xi} C_\Lambda(i\xi + A_{-1})^{-1}B = -i\, \xi^{1/2} CR(i\xi, -A) \cdot \xi^{1/2} R(i\xi, -A_{-1})B, \quad \xi \neq 0,$$

since $C \in B(X_1, Y)$ and $B \in B(U, X_{-1})$. If U and Y have property (α) and B and C are l–admissible then Theorems 4.2 and 4.5 show that the set in (14) is l–bounded, hence R–bounded (see Remark 3.6). For UMD–spaces U, Y with property (α), L^p–wellposedness of $\mathbb{F}_\infty$ for a regular system in this situation is thus characterised by l–boundedness of (13).

Besides L^p–wellposedness it seems natural to take into account also a notion of *l–wellposedness* of the input-output map. For this notion we require the convolution operator $\mathbb{F}_\infty$ to be bounded from $l(\mathbb{R}_+, U)$ to $l(\mathbb{R}_+, Y)$. Again considering $\mathbb{F}_\infty$ for a regular system as a Fourier multiplier we obtain by Proposition 3.9 equivalence of l–wellposedness and l–boundedness of the set in (13). Notice that the condition that is only *necessary* in the L^p–case is *necessary and sufficient* in the l–case without any geometric assumptions on the spaces U, X, or Y.
We sum up the above arguments in the following theorem.

Theorem 4.10. *Let X, U and Y be Banach spaces. Let $-A$ be the generator of an l–bounded analytic semigroup on X. Assume that the observation and control operators B, C are l–admissible and that the system is regular. Then the following are equivalent:*

(a) *The input-output map $\mathbb{F}_\infty$ is l–wellposed.*
(b) *The set $\{C_\Lambda(i\xi + A_{-1})^{-1}B : \xi \neq 0\}$ is l–bounded in $B(U, Y)$.*

Moreover, (a) and (b) are implied by each of the following conditions:

(c) *The input-output map $\mathbb{F}_\infty$ is L^p–wellposed for some $p \in (1, \infty)$.*
(d) *The input-output map $\mathbb{F}_\infty$ is L^p–wellposed for all $p \in (1, \infty)$.*

If U and Y are UMD spaces which have property (α) then (a) implies (d), whence all four assertions are equivalent.

This finally yields a characterisation for *l–wellposedness of the full system* (12), by which we mean that B and C are l–admissible and that $\mathbb{F}_\infty$ is l–wellposed.

Theorem 4.11. *Let X, U, Y be a Banach spaces. Assume that $-A$ is the generator of an l–bounded analytic semigroup, that A has dense range. Let $C \in B(X_1, Y)$ and $B \in B(U, X_{-1})$, assume that the system* (12) *is regular and consider the following sets:*

$$\big\{\sqrt{\lambda}C(\lambda + A)^{-1} : \lambda > 0\big\} \subset B(X, Y), \tag{15}$$

$$\big\{\sqrt{\lambda}(\lambda + A_{-1})^{-1}B : \lambda > 0\big\} \subset B(U, X), \tag{16}$$

$$\big\{C_\Lambda(\lambda + A_{-1})^{-1}B : \lambda \in i\mathbb{R} \setminus \{0\}\big\} \subset B(U, Y). \tag{17}$$

(a) *If* (12) *is l–wellposed and U and Y have property (α), then the sets in* (15), (16) *and* (17) *are l–bounded.*
(b) *If A, A' have square function estimates and the sets in* (15), (16) *and* (17) *are l–bounded, then* (12) *is l–wellposed.*

Corollary 4.12. *Let X be a Banach space of finite cotype, and U and Y have property (α). Assume that $-A$ generates an l–bounded analytic semigroup, that A has dense range and a bounded $H^\infty(S(\theta))$–calculus for some $\theta < \pi/2$. Let $C \in B(X_1, Y)$ and $B \in B(U, X_{-1})$, assume that the system* (12) *is regular. Then the system* (12) *is l–wellposed if and only if the sets in* (15), (16) *and* (17) *are l–bounded.*
If U and Y are Hilbert spaces, then (12) *is L^2–wellposed if and only if the sets in* (15) *and* (16) *are l–bounded, and the set in* (17) *is bounded.*

Remark 4.13. In general, the determination of C_Λ or $\mathcal{D}(C_\Lambda)$ is not easy (see [34, 36]). In applications, however, one may use the following argument, which often gives sufficient information. Suppose that Z is a Banach space satisfying $X_1 \subset Z \subset X$ with continuous injections such that the part A_Z of A in Z is sectorial and densely defined. Suppose that $\widetilde{C} \in B(Z, Y)$ is an extension of $C \in B(X_1, Y)$. Then $Z \subset \mathcal{D}(C_\Lambda)$ and C_Λ is an extension of $\widetilde{C}$. Indeed, for $z \in Z$ we have by the assumptions on A_Z

$$\lambda(\lambda + A)^{-1}z = \lambda(\lambda + A_Z)^{-1}z \to z \qquad \text{in } \|\cdot\|_Z, \quad (\lambda \to +\infty)$$

which implies $C\lambda(\lambda+A)^{-1}z \to \widetilde{C}z(\lambda\to+\infty)$ in $\|\cdot\|_Y$ by continuity of $\widetilde{C}: Z \to Y$.
We thus find the following sufficient condition for regularity: Suppose that Z is a Banach space as described above such that C has a continuous extension $\widetilde{C}: Z \to Y$. Let $W := ((1+A_{-1})Z, \|(1+A_{-1})^{-1}\cdot\|_Z)$. Then W is Banach space, $X \subset W \subset X_{-1}$ with continuous injections, and the part $(A_{-1})_W$ of A_{-1} in W is densely defined and sectorial in W. If $B(U) \subset W$ then the system (12) is regular.

Remark 4.14. If, in the proofs, we make use of Proposition 3.20 instead of Corollary 3.19, some assumptions on Y in the foregoing results may be weakened: Theorem 4.2 (a) is valid without any assumption on Y; in the last assertion of Theorem 4.10, it is sufficient that Y is a UMD space (implying finite cotype); Theorem 4.11 (a) is valid without any assumption on Y.

5. Proofs of the main theorems

Proof of Theorem 4.2. The necessity follows immediately from $\lambda^{1/2}C(\lambda+A)^{-1}x = \int_0^\infty \lambda^{1/2}e^{-\lambda t}CT(t)x$, by Proposition 3.20. We now prove the sufficiency part. It is well known from [5, Thm 3.8] that for sectorial operators dense range implies injectivity. We may thus write

$$CT(t) = CA^{-1/2}(tA)^{1/2}T(t)t^{-1/2} = CA^{-1/2}\varphi_0(tA)t^{-1/2}$$

where $\varphi_0(z) := z^{1/2}e^{-z}$. We decompose $\varphi_0(z) = \varphi(z)\psi(z)$ where

$$\varphi(z) := z^\alpha(1+z)^{-1}, \qquad \psi(z) := z^{1/2-\alpha}(1+z)e^{-z}$$

for some $\alpha \in (0, 1/2)$. Let Γ be the positively orientated boundary of $S(\theta)$ where $\theta \in (\omega_l, \pi/2)$. Then, for $x \in \mathcal{R}(A^{1/2}(I+A)^{-1})$

$$CA^{-1/2}\varphi_0(tA)x = CA^{-1/2}\tfrac{1}{2\pi i}\int_\Gamma \varphi(tz)R(z,A)x\,dz.$$

By [17, Lemma 4.2], this equals

$$\begin{aligned} &= CA^{-1/2}\tfrac{1}{2\pi i}\int_\Gamma \varphi(tz)z^{-1/2}A^{1/2}R(z,A)x\,dz \\ &= \tfrac{1}{2\pi i}\int_\Gamma \varphi(tz)z^{1/2}CR(z,A)x\,\tfrac{dz}{z} =: K(t)x. \end{aligned}$$

In the last equality we made use of $x \in \mathcal{R}(A^{1/2}(I+A)^{-1})$. Now

$$\big\|K(t)x\big\| \le \tfrac{1}{2\pi}\int_\Gamma |\varphi(tz)|\,\big\|z^{1/2}CR(z,A)x\big\|\tfrac{|dz|}{|z|}.$$

For $z \in \Gamma$,

$$z^{1/2}CR(z,A) = |z|^{1/2}C(|z|+A)^{-1}\,\big[2\cosh(\pm\theta/2)zR(z,A) - I\big].$$

Hence l–sectoriality of A and l–boundedness of W_C yield that $\{z^{1/2}CR(z,A):\ z\in\Gamma\}$ is l–bounded. Scaling invariance of Γ and the measure dz/z implies that $h_t(z) := \varphi(tz)$ is uniformly bounded in $L^1(\Gamma, |dz|/|z|)$, whence the set $\{K(t):\ t>0\} \subset B(X,Y)$ is l–bounded by Lemma 3.8.
Now we conclude that

$$\begin{aligned} \big\|CT(t)x\big\|_{l(\mathbb{R}_+,Y)} &= \big\|CA^{-1/2}\varphi(tA)\psi(tA)t^{-1/2}x\big\|_{l(\mathbb{R}_+,Y)} \\ &= \big\|K(t)\psi(tA)x\big\|_{l(\mathbb{R}_+,\frac{dt}{t},Y)} \\ &\le c_1\big\|\psi(tA)x\big\|_{l(\mathbb{R}_+,\frac{dt}{t},X)} \le c_2\big\|x\big\|, \end{aligned}$$

where we used Proposition 3.9 and the assumed square function estimate for A in the last two steps. □

Notice that some slight modifications in the above proof yield a short proof of [24, Thm. 4.1].

Proof of Theorem 1.3. By [17, Lem. 3.1] property (α) implies property (Δ). Therefore, by [17, Thm. 5.3] A is R–sectorial of angle $\omega_R = \theta =< \pi/2$. By Remark 3.6 A is l–sectorial of the same angle, so $-A$ generates an l–bounded analytic semigroup. Property (α) implies finite cotype of X and the boundedness of the H^∞–calculus of A ensures square function estimates for A and A' by Theorem 3.13. Therefore, Theorem 4.2 applies. □

In order to prove Theorem 4.5 we first state the following representation lemma:

Lemma 5.1. *Let A be a densely defined sectorial operator of type $\omega < \pi/2$ with dense range on a Banach space X. Let $B \in B(U, X_{-1})$ a control operator and assume the set $W_B := \{\lambda^{1/2}(\lambda + A_{-1})^{-1}B : \lambda > 0\}$ to be uniformly bounded in $B(U, X)$. Then for all $\alpha \in (0, 1/2)$ and $u \in U$ the following representation is valid in X_{-1}:*

$$T_{-1}(t)Bu = \tfrac{1}{2\pi i}\int_\Gamma \lambda^{-\alpha}e^{-\lambda t}(A_{-1})^\alpha R(\lambda, A_{-1})Bu\, d\lambda. \tag{18}$$

Here Γ is the positively orientated boundary of some sector $S(\theta)$ with $\theta \in (\omega, \pi/2)$.

Proof. Let

$$\varphi_n(z) := n(n+z)^{-1} - \tfrac{1}{n}(\tfrac{1}{n}+z)^{-1} = z(n+z)^{-1}(\tfrac{1}{n}+z)^{-1}(n - \tfrac{1}{n}).$$

notice that $\varphi_n \in H_0^\infty(S(\theta))$ for all $\theta \in (0, \pi)$. Moreover, $\varphi_n(A_{-1})x \in \mathcal{D}(A_{-1}) \cap \mathcal{R}(A_{-1})$ for all $x \in X_{-1}$ and $\varphi_n(A_{-1})$ is an approximate identity, that is $\varphi_n(A_{-1})x \to x$ for $n \to \infty$ and $x \in X_{-1}$. We have $T_{-1}(t)Bu = \lim_{n\to\infty} T_{-1}(t)\varphi_n(A_{-1})Bu$ and write, for any $n \in \mathbb{N}$,

$$\begin{aligned} T_{-1}(t)\varphi_n(A_{-1})Bu &= (A_{-1})^\alpha T_{-1}(t)(A_{-1})^{-\alpha}\varphi_n(A_{-1})Bu \\ &= (A_{-1})^\alpha \tfrac{1}{2\pi i}\int_\Gamma z^{-\alpha}e^{-tz}\varphi_n(z)R(z, A_{-1})Bu\, dz. \end{aligned}$$

Here we use that $z^{-\alpha}\varphi_n(z) \in H_0^\infty$ for all $\alpha \in (0,1)$. Next we want to show, that $(A_{-1})^\alpha$ and the integral commute. By closedness of $(A_{-1})^\alpha$ and Hille's theorem we have to verify that the following Bochner integral exists

$$\int_\Gamma z^{-\alpha}e^{-tz}\varphi_n(z)(A_{-1})^\alpha R(z, A_{-1})Bu\, dz \tag{19}$$

in X_{-1}. Moreover, we shall show that the integrand may be estimated in norm and uniformly in n by an integrable function, which enables use of dominated convergence later on. We have $\|(A_{-1})^\alpha R(z, A_{-1})\| \le M|z|^{\alpha-1}$ and shall exploit the additional assumption on W_B. First we double the resolvent:

$$R(z, A_{-1})Bu = \int_{\gamma_z} R(\lambda, A_{-1})^2 Bu\, d\lambda, \quad \text{where } \gamma_z = \{z + se^{i\arg z} : s > 0\}.$$

Notice that the integral converges absolutely. Now, by $\big\|(A_{-1})^\alpha R(\lambda, A_{-1})^2 B\big\| \le M|\lambda|^{\alpha-1}|\lambda|^{-1/2}$ and the assumption $\alpha < 1/2$ the Bochner integral

$$\int_{\gamma_z} (A_{-1})^\alpha R(\lambda, A_{-1})^2 Bu\, d\lambda$$

exists in X_{-1} and satisfies an estimate against $c|z|^{\alpha-1/2}$. Therefore, also the integral in (19) exists as a Bochner integral. Indeed, by $\|\varphi_n\|_{H^\infty(S(\theta))} \le M < \infty$ the integrand satisfies an estimate against $c'|z|^{-1/2}\exp(-tRe(z))$ in norm, uniformly in $n \in \mathbb{N}$.
This shows the formula

$$T_{-1}(t)\varphi_n(A_{-1})Bu = \int_\Gamma z^{-\alpha}e^{-tz}\varphi_n(z)(A_{-1})^\alpha R(z, A_{-1})Bu\, dz,$$

and the desired result follows by dominated convergence theorem. □

Proof of Theorem 4.5. A is l–sectorial of type $\omega_l < \pi/2$. Chose some $\sigma \in (\omega_l, \pi/2)$ and let $\Gamma := \partial S(\sigma)$ the positively orientated integration path as above. By Lemma 5.1 we have

$$T_{-1}(t)Bu(t) = \tfrac{1}{2\pi i}\int_\Gamma z^{-\alpha}e^{-\lambda t}(A_{-1})^\alpha(\lambda + A_{-1})^{-1}Bu(t)\, d\lambda$$

for all $\alpha \in (0, 1/2)$. Let $x' \in \mathcal{D}(A')$. Then

$$\begin{aligned} &\big|\langle \int_0^\infty T_{-1}(t)Bu(t)\, dt, x'\rangle\big| \\ = \;& \tfrac{1}{2\pi}\big|\langle \int_0^\infty T_{-1}(t/2)\int_\Gamma \lambda^{-\alpha}e^{-\lambda/2\, t}(A_{-1})^\alpha(\lambda + A_{-1})^{-1}Bu(t)\, d\lambda\, dt, x'\rangle\big| \end{aligned}$$

Setting $h_t(\lambda) := t^{1/2-\alpha} e^{-\lambda/2 t} \lambda^{-\alpha-1/2}$ and $R(\lambda) := \lambda^{1/2}(\lambda+A_{-1})^{-1}B$ we obtain

$$\leq \ \tfrac{1}{2\pi} \int_0^\infty \big| \big\langle \int_\Gamma h_t(\lambda)\ R(\lambda)\, d\lambda\ u(t), t^{\alpha-1/2} \big((A_{-1})^\alpha T_{-1}(t/2)\big)' x' \big\rangle \big|\, dt$$

Applying Proposition 3.4 this may be estimated by

$$\leq \ \tfrac{1}{2\pi} \big\| t \mapsto \int_\Gamma h_t(\lambda)\ R(\lambda)\, d\lambda\ u(t) \big\|_{l(\mathbb{R}_+,X)} \big\| t \mapsto t^{\alpha-1/2} \big((A_{-1})^\alpha T_{-1}(t/2)\big)' x' \big\|_{l(\mathbb{R}_+,X')}.$$

The assumed square function estimate for A', applied to the function $\phi_\alpha(z) = z^\alpha e^{-z/2}$ yields

$$\leq \ M \|x'\| \big\| t \mapsto \big(\int_\Gamma h_t(\lambda) R(\lambda)\, d\lambda \big)\ u(t) \big\|_{l(\mathbb{R}_+,X)}. \tag{20}$$

By an argument similar to that in the proof of 4.2 above, the functions $\lambda \mapsto \lambda h_t(\lambda)$ are uniformly bounded in $L^1(\Gamma, |d\lambda|/|\lambda|)$ for $t > 0$. Thus by Lemma 3.8 the set $\{\int_\Gamma h_t(\lambda)R(\lambda)\, d\lambda : t > 0\} \subset B(X)$ is l–bounded. We use Proposition 3.9, and since $\mathcal{D}(A')$ is norming for X, we obtain l–admissibility of B.

Now let U have finite cotype and B be l–admissible. We have

$$\lambda^{1/2}(\lambda+A_{-1})^{-1}B = \int_0^\infty \lambda^{1/2} e^{-\lambda t} \cdot T_{-1}(t)B\, dt,$$

and uniform boundedness of the functions $t \mapsto h_\lambda(t) := \lambda^{1/2} e^{-\lambda t}$, $\lambda > 0$, in $L^2(\mathbb{R}_+)$ yields l–boundedness of $W_B \subseteq B(U, X)$ by Corollary 3.21. □

Proof of Theorem 1.5. The proof is very similar to that of Theorem 1.3 above and makes use of Theorem 4.5. □

Now we turn to the proof of Theorem 3.18. We first show the following lemma:

Lemma 5.2. *Let Y be a Banach space with property (α). Then there exists a constant $c > 0$ such that for all $J, K, N \in \mathbb{N}$, and independent sequences of independent Gaussian random variables (g_n) and (g'_k) and complex numbers (α_{njk}) the following estimate holds true:*

$$\Big(\mathbb{E}\, \mathbb{E}' \Big\| \sum_{k=1}^K \sum_{n=1}^N g'_k g_n \sum_{j=1}^J \alpha_{njk} y_{nj} \Big\|_Y^2 \Big)^{1/2} \leq\ c \max_{n=1,\dots,N} \big\| (\alpha_{njk})_{jk} \big\|_{l_2^K \to l_2^J} \Big(\mathbb{E}\, \mathbb{E}' \Big\| \sum_{k=1}^K \sum_{n=1}^N g'_k g_n y_{nk} \Big\|_Y^2 \Big)^{1/2}.$$

Proof. The proof is based on two observations: First, by Remark 3.17 property (α) implies finite cotype, and thus by [7, 12.11, 12.27] Gaussian and Bernoulli sums have equivalent p-th moments. Therefore secondly property (α) implies that there exists some $C_Y > 0$ depending only upon the property (α)–constant such that

$$\frac{1}{C_Y} \mathbb{E} \Big\| \sum_{(k,j)} g_{(k,j)} y_{kj} \Big\|^2 \leq \mathbb{E}\, \mathbb{E}' \Big\| \sum_{k,j}^{K,J} g_k g'_j y_{kj} \Big\|^2 \leq C_Y \mathbb{E} \Big\| \sum_{(k,j)} g_{(k,j)} y_{kj} \Big\|^2$$

holds (see, e.g., [21, II, Lemma 4.11]). We now prove the lemma: consider the matrices $M_n := (\alpha_{njk})_{j,k}$. We extend them to a block-diagonal matrix of $J{\times}K$–matrices by setting $\beta_{(j,m),(n,k)} := \delta_{nm}\alpha_{njk}$, $m = 1, \dots, N$. Then

$$\begin{aligned} &\Big(\mathbb{E}\, \mathbb{E}' \Big\| \sum_{k=1}^K \sum_{n=1}^N g'_k g_n \sum_{j=1}^J \alpha_{njk} y_{nj} \Big\|_Y^2 \Big)^{1/2} \\ \leq &C_Y \Big(\mathbb{E} \Big\| \sum_{(k,n)} g_{(k,n)} \sum_{(j,m)} \beta_{(j,m),(n,k)} y_{(j,m)} \Big\|_Y^2 \Big)^{1/2}. \end{aligned}$$

Now, by [7, Lemma 12.17] we have an estimate against

$$\leq C_Y\ \big\| \big(\beta_{(j,m),(n,k)}\big) \big\|_{l_2^{J\times N} \to l_2^{N\times K}} \Big(\mathbb{E} \Big\| \sum_{(j,m)} g_{(j,m)} y_{(j,m)} \Big\|_Y^2 \Big)^{1/2}.$$

The norm of the matrix $(\beta_{(j,m),(n,k)})$ may be simplified due to its block-diagonal structure and becomes

$$=C_Y \max_{n=1,\dots,N}\{\|(\alpha_{njk})\|_{l_2^K\to l_2^J}\}\Big(\mathbb{E}\Big\|\sum_{(j,m)} g_{(j,m)}y_{(j,m)}\Big\|_Y^2\Big)^{1/2}$$
$$\le C_Y^2 \max_{n=1,\dots,N}\{\|(\alpha_{njk})\|_{l_2^K\to l_2^J}\}\Big(\mathbb{E}\,\mathbb{E}'\Big\|\sum_{j,n=1}^{J,N} g_j g_n' y_{jn}\Big\|_Y^2\Big)^{1/2}. \qquad \square$$

Proof of Theorem 3.18. The l–boundedness of $\mathcal{A}^{\otimes}$ is equivalent to the uniform boundedness (in $N\in\mathbb{N}$) of the diagonal operators

$$(A_\nu^{\otimes}) : l\big(l_2^N(\mathbb{Z}), l(H_1,Y)\big) \to l\big(l_2^N(\mathbb{Z}), l(H_2,Y)\big).$$

for some $A_1,\dots,A_N\in\mathcal{A}$. Let $N\in\mathbb{N}$ and fix an operator $v: l_2^N(\mathbb{Z})\to l(H_1,Y)$. Let (e_n) be the canonical basis of l_2^N and (f_k) some countable orthonormal system in H_2. Then

$$\begin{aligned}
\|(A_\nu^{\otimes})v\|_{l(l_2^N(\mathbb{Z}),l(H_2,Y))} &= \Big(\mathbb{E}\Big\|\sum_{n=1}^N g_n(A_\nu^{\otimes})v(e_n)\Big\|_{l(H_2,Y)}^2\Big)^{1/2}\\
&= \Big(\mathbb{E}\Big\|\sum_{n=1}^N g_n A_n^{\otimes}v(e_n)\Big\|_{l(H_2,Y)}^2\Big)^{1/2}\\
&= \Big(\mathbb{E}\Big\|\sum_{n=1}^N g_n v(e_n)A_n'\Big\|_{l(H_2,Y)}^2\Big)^{1/2} \qquad (21)\\
&= \Big(\mathbb{E}\lim_{K\to\infty}\mathbb{E}'\Big\|\sum_{k=1}^K g_k'\Big(\sum_{n=1}^N g_n v(e_n)A_n'\Big)(f_k)\Big\|_Y^2\Big)^{1/2},\\
&= \lim_{K\to\infty}\Big(\mathbb{E}\,\mathbb{E}'\Big\|\sum_{n,k}^{N,K} g_k' g_n v(e_n)(A_n' f_k)\Big\|_Y^2\Big)^{1/2}.
\end{aligned}$$

The last equality holds by the dominated convergence theorem. For K fixed, the subspace *span* $\{A_n'(f_k) : n=1,\dots,N,\ k=1,\dots,K\}$ is finite-dimensional. Call its dimension J and chose some orthonormal basis (h_j) of this subspace. We then may write $A_n'(f_k) =: \sum_j \alpha_{njk}h_j$. Therefore

$$\Big(\mathbb{E}\,\mathbb{E}'\Big\|\sum_{n,k}^{N,K} g_k' g_n v(e_n)(A_n' f_k)\Big\|_Y^2\Big)^{1/2}$$
$$= \Big(\mathbb{E}\,\mathbb{E}'\Big\|\sum_{n,k}^{N,K} g_k' g_n \sum_j \alpha_{njk} v(e_n)(h_j)\Big\|_Y^2\Big)^{1/2}.$$

By Lemma 5.2 we have the estimate

$$\le C' \max_{n=1,\dots,N}\|(\alpha_{njk})_{jk}\|_{l_2^K\to l_2^J}\Big(\mathbb{E}\Big\|\sum_{n,k} g_k' g_n v(e_n)(h_k)\Big\|_Y^2\Big)^{1/2}.$$

Next we show that the norms $\|(\alpha_{njk})_{jk}\|_{l_2^K\to l_2^J}$ are uniformly bounded. To this end, write

$$\|(\alpha_{njk})\|_{l_2^K\to l_2^J} = \sup_{\|(\lambda_k)\|_{l_2^K}\le 1}\Big\|(\sum_k \alpha_{njk}\lambda_k)\Big\|_{l_2^J} = \sup_{(\lambda_k)}\Big\|\sum_{j,k}\alpha_{njk}\lambda_k h_j\Big\|_{H_1}$$
$$= \sup_{(\lambda_k)}\Big\|A_n'(\sum_{k=1}^K \lambda_k f_k)\Big\|_{H_1} \le \|A_n'\|_{H_2\to H_1} \le M.$$

Notice that the above estimates are independent of he choice of N,K,J. To obtain the desired result, we now apply (21) backward. $\square$

Remark 5.3. In general, property (α) cannot be omitted in the foregoing theorem. However, if $H_1 = \mathbb{C}$ we may apply [16, Lemma 3.1] in the above proof after equation (21) which shows that in this special case the result remains true even if Y has finite cotype.

Proof of Corollary 3.19. Setting $H_1 = L^2(I)$ and $H_2 = \mathbb{C}$ we obtain from Theorem 3.18 for

$$a^{\otimes} : \begin{cases} l(I,Y) & \to & Y \\ f & \mapsto & \int_I a(t)f(t)dt \end{cases}$$

that $\{a^{\otimes} :\ a \in \mathcal{U}\}$ is l–bounded in $B(l(I,Y),Y)$ whence $\{a^{\otimes} \circ \varphi :\ a \in \mathcal{U}\}$ is l–bounded in $B(X,Y)$. By the above remark, Corollary 3.19 holds even under the weaker assumption of finite cotype for Y. □

Proof of Corollary 3.21. Let (a_k) be a dense sequence in the unit ball $\mathcal{U}$ of $L^2(I)$. Let $n \in \mathbb{N}$ and $u_1, \dots, u_n \in U$. Then we have by assumption

$$\begin{aligned}\Big(\mathbb{E}\Big\|\sum_{k=1}^n g_k \int_I a_k(t)\varphi(t)u_k\,dt\Big\|_X^2\Big)^{1/2} &= \Big(\mathbb{E}\Big\|\int_I \varphi(t)\Big(\sum_{k=1}^n g_k(a_k(t)\otimes u_k)\Big)dt\Big\|_X^2\Big)^{1/2} \\ &\le K\Big(\mathbb{E}\Big\|\sum_{k=1}^n g_k(a_k\otimes u_k)\Big\|_{l(\mathbb{R}_+,U)}^2\Big)^{1/2} \\ &= K\Big(\mathbb{E}\Big\|\sum_{k=1}^n g_k u_{a_k\otimes u_k}\Big\|_{l(L^2(\mathbb{R}_+),U)}^2\Big)^{1/2}.\end{aligned}$$

The operators $T_k : \lambda \mapsto \lambda a_k$ are bounded from $\mathbb{C}$ to $L^2(I)$, thus by Theorem 3.18 the set $\{T_k^{\otimes} : k \in \mathbb{N}\}$ is l–bounded from $l(\mathbb{C}, U)$ to $l(L^2(I), U)$. If $h \in L^2(I)$ and $u_k \in U \simeq l(\mathbb{C}, U)$ we have $(T_k^{\otimes} u_k)(h) = (a_k|h)u_k$ and therefore we have $T_k^{\otimes} = u_{a_k\otimes u_k}$. This yields

$$\Big(\mathbb{E}\Big\|\sum_{k=1}^n g_k \int_I a_k(t)\varphi(t)u_k\,dt\Big\|_X^2\Big)^{1/2} \le K'\Big(\mathbb{E}\Big\|\sum_{k=1}^n g_k u_k\Big\|_U^2\Big)^{1/2}.$$

Now Lemma 3.7 shows the assertion. □

6. Examples

In this section we present several l–admissible observation and control operators in L^p–spaces, and we prove l–wellposedness of a controlled heat equation. First we recall some notation (cf., e.g., [7], [21], [11]).

Definition 6.1. Let $r_j(\cdot)$ be the sequence of Rademacher functions. For a Banach space X, define

$$\mathrm{Rad}(X) := \big\{(x_j)_{j\in\mathbb{N}} : \sum_{j=1}^{\infty} r_j(\cdot)x_j \text{ converges in } L^2([0,1],X)\big\}.$$

When equipped with one of the following (by Kahane's inequality equivalent) norms $\|(x_j)\|_{\mathrm{Rad}_p(X)}$, $1 \le p < \infty$, given by

$$\big\|(x_j)_j\big\|_{\mathrm{Rad}_p(X)} := \big\|\sum_j r_j(\cdot)x_j\big\|_{L^p([0,1],X)},$$

$\mathrm{Rad}(X)$ becomes a Banach space.

Remark 6.2. A set $\{T_j :\ j \in \mathbb{N}\} \subseteq B(X,Y)$ is R–bounded if and only if the corresponding diagonal operator $\mathbb{T} := (T_j)$ is bounded from $\mathrm{Rad}_p(X)$ to $\mathrm{Rad}_p(Y)$. Moreover the norm of $\mathbb{T}$ equals the R–bound of the set $\{T_j :\ j \in \mathbb{N}\}$. By Fubini's theorem the mapping

$$I_X : L^p(\Omega, \mathrm{Rad}_p(X)) \to \mathrm{Rad}_p(L^p(\Omega, X))$$

given by $I_X f := (f_j)_{j\in\mathbb{N}}$ for $f(\cdot) = \big(f_j(\cdot)\big)_{j\in\mathbb{N}} \in L^p(\Omega, \mathrm{Rad}_p(X))$ defines an isometry. Moreover, whenever $f \in L^1(\mathbb{R}^n, \mathrm{Rad}_p(X))$,

$$\mathcal{F}_n f = \big(\mathcal{F}_n f_j\big)_{j\in\mathbb{N}} \quad \text{and} \quad \mathcal{F}_n^{-1} f = \big(\mathcal{F}_n^{-1} f_j\big)_{i\in\mathbb{N}},$$

where $\mathcal{F}_n$ denotes Fourier transform on $\mathbb{R}^n$.

We turn to the examples. As state space we take $X = L^p(\mathbb{R}^n)$ where $1 < p < \infty$. As state equation we consider the homogeneous heat equation $\frac{d}{dt}v(t) = \Delta_n v(t)$ for functions $v : [0,\infty) \to L^p(\mathbb{R}^n)$ where Δ_n is the realisation of the Laplace operator Δ in $X = L^p(\mathbb{R}^n)$. For simplicity, we shall give examples of l–admissible operators for $A := \epsilon - \Delta_n$ where $\epsilon > 0$. Those operators are finite time l–admissible for $A_0 := -\Delta_n$. The arguments below may be adapted to give examples of l–admissible operators for A_0 but that would involve *homogeneous* Besov spaces which we want to avoid. In the following we let $1 \le k \le n$, and in case $k < n$, we write (x, y) for the variable in $\mathbb{R}^n$ where $x \in \mathbb{R}^{n-k}$ and $y \in \mathbb{R}^k$. We also understand $L^p(\mathbb{R}^0) = L^p(\{0\}) = \mathbb{C}$.

Observation operators

We consider unbounded observation operators C acting from $X = L^p(\mathbb{R}^n)$ to $Y = L^p(\mathbb{R}^{n-k})$ which are of the form $(C_\psi f)(x) = \langle \psi, f(x,\cdot)\rangle_k$ where $\psi \in \mathcal{S}'(\mathbb{R}^k)$. l–Admissibility of such operators is characterised by

Proposition 6.3. *Let $1 < p < \infty$, $n \ge k \ge 1$, $X = L^p(\mathbb{R}^n)$, $Y = L^p(\mathbb{R}^{n-k})$, $A = \epsilon - \Delta_n$, $\epsilon > 0$. Then $\psi \in \mathcal{S}'(\mathbb{R}^k) \cap H^{-2}_{p'}(\mathbb{R}^k)$ induces an l–admissible observation operator C_ψ from X to Y if and only if the set*

$$\left\{ \mu\big\langle (\mu^2 + \epsilon - \Delta_k)^{-1}\psi, \cdot \big\rangle : \mu > 0 \right\} \subset B(L^p(\mathbb{R}^k), \mathbb{C}) \tag{22}$$

is R–bounded.

Observe that both X and Y are spaces of finite cotype whence R–boundedness coincides with l–boundedness. Also l–admissibility coincides with the notion of R–admissibility from [25]. The restriction to distributions in $H^{-2}_{p'}(\mathbb{R}^k)$ is necessary to ensure that $C \in B(X_1, Y)$.

Proof. We fix ψ and write $C = C_\psi$ for short. Since A has a bounded H^∞-calculus we know by Theorem 4.2 that C is l–admissible if and only if the set

$$\left\{ \sqrt{\lambda} C(\lambda + A)^{-1} : \lambda > 0 \right\} \subset B(L^p(\mathbb{R}^n), L^p(\mathbb{R}^{n-k}))$$

is l–bounded (which is equivalent to R–bounded here). This proves the claim for $k = n$, so let $k < n$. We write (x, y) for the variable in $\mathbb{R}^n$ where $x \in \mathbb{R}^{n-k}$ and $y \in \mathbb{R}^k$. In a first step we write $T_{m_\lambda} := \sqrt{\lambda}\, C(\lambda + A)^{-1}$ as a Fourier multiplier operator $L^p(\mathbb{R}^n) = L^p(\mathbb{R}^{n-k}, L^p(\mathbb{R}^k)) \to L^p(\mathbb{R}^{n-k})$ with an operator valued symbol $m_\lambda : \mathbb{R}^{n-k} \to B(L^p(\mathbb{R}^k), \mathbb{C}), \xi \mapsto m_\lambda(\xi)$. To this end we write $\mathcal{F}_{n-k}$ for the Fourier transform on $\mathbb{R}^{n-k}$ and denote by $\hat{f}(\xi, y)$ the partial Fourier transform with respect to $x \to \xi$. Writing $A_k := \epsilon - \Delta_k$, we have

$$\sqrt{\lambda} C(\lambda + A)^{-1} f = \mathcal{F}^{-1}_{n-k}\big(\xi \mapsto \sqrt{\lambda}\, \langle \psi, (\lambda + |\xi|^2 - A_k)^{-1} \hat{f}(\xi, \cdot)\rangle\ \big),$$

i.e., $T_{m_\lambda} f = \mathcal{F}^{-1}_{n-k}(\xi \mapsto m_\lambda(\xi)\mathcal{F}_{n-k} f(\xi))$ where $m_\lambda(\xi) = \sqrt{\lambda}\, \langle \psi, (\lambda + |\xi|^2 - A_k)^{-1} \cdot\rangle : L^p(\mathbb{R}^k) \to \mathbb{C}$ for $\xi \in \mathbb{R}^{n-k}$.

We make the following observation: Let $(\lambda_j)_{j\in\mathbb{N}}$ be some sequence in $(0,\infty)$ and assume the set $\{T_{m_\lambda} : \lambda > 0\} \subset B(X, Y)$ to be R–bounded. Consider the following diagram:

$$\begin{array}{ccc}
\mathrm{Rad}_p(L^p(\mathbb{R}^{n-k}), L^p(\mathbb{R}^k)) & \xrightarrow{(T_{m_{\lambda_j}})} & \mathrm{Rad}_p(L^p(\mathbb{R}^{n-k}, \mathbb{C})) \\
{\scriptstyle I_{L^p(\mathbb{R}^k)}}\updownarrow & & \updownarrow{\scriptstyle I_{\mathbb{C}}} \\
L^p(\mathbb{R}^{n-k}, \mathrm{Rad}_p(L^p(\mathbb{R}^k))) & \xrightarrow{(\widetilde{T}_{m_{\lambda_j}})} & L^p(\mathbb{R}^{n-k}, \mathrm{Rad}_p(\mathbb{C}))
\end{array}$$

Notice that by Remark 6.2 for functions $f = (f_j)$ of the Schwartz class $\mathcal{S}(\mathbb{R}^{n-k}, \mathrm{Rad}_p(L^p(\mathbb{R}^k)))$, one has

$$\widetilde{T}_{m_{\lambda_j}} f = \mathcal{F}^{-1}_{n-k}(\xi \mapsto m_{\lambda_j}(\xi)\hat{f}(\xi, \cdot)) = \big(\mathcal{F}^{-1}_{n-k}\big)\big[\xi \mapsto \big(m_{\lambda_j}(\xi)\big)\big(\hat{f}(\xi, \cdot)\big)\big],$$

indicating that $\widetilde{T}_{m_{\lambda_j}}$ is in fact a bounded Fourier multiplier. Thus by the necessity criterion [4, Prop. 1] the set $\big\{(m_{\lambda_j}(\xi))_j : \xi \in \mathbb{R}^{n-k}\big\}$ is R–bounded in $B(\mathrm{Rad}_p(L^p(\mathbb{R}^k)), \mathrm{Rad}_p(\mathbb{C}))$.

Now we show R–boundedness of the set $\{m_\lambda(\xi) : \lambda > 0, \xi \in \mathbb{R}^{n-k}\}$ in $B(L^p(\mathbb{R}^n), \mathbb{C})$: Let $(\xi_j)_j \subseteq \mathbb{R}^{n-k}$, $(\lambda_j)_j \subseteq \mathbb{R}_+$ and $(f_j)_j \subseteq L^p(\mathbb{R}^k)$ be some finite sequences, which for convenience of notation we extend by zero elements to infinite sequences. We set $f_{jl} := f_j \delta_{jl}$, where δ_{jl} denotes the Kronecker symbol. Then

$$\begin{aligned}
\mathbb{E}\Big\|\sum_j \epsilon_j m_{\lambda_j}(\xi_j) f_j\Big\|_{\mathbb{C}}^2 &= \mathbb{E}\,\mathbb{E}'\Big\|\sum_{j,l} \epsilon_j \epsilon_l' m_{\lambda_j}(\xi_j) f_{jl}\Big\|_{\mathbb{C}}^2 \quad \text{by [7, 11.2]} \\
&= \mathbb{E}\Big\|\sum_l \epsilon_l \big(m_{\lambda_j}(\xi_l)\big)\big(f_{jl}\big)_j\Big\|_{\mathrm{Rad}_p(\mathbb{C})}^2 \\
&\le M\,\mathbb{E}\Big\|\sum_l \epsilon_l \big(f_{jl}\big)_j\Big\|_{\mathrm{Rad}_p(L^p(\mathbb{R}^k))}^2 \\
&= M\,\mathbb{E}\,\mathbb{E}'\Big\|\sum_{j,l} \epsilon_j \epsilon_l' f_{jl}\Big\|_{L^p(\mathbb{R}^k)}^2 \\
&= M\,\mathbb{E}\Big\|\sum_j \epsilon_j f_j\Big\|_{L^p(\mathbb{R}^k)}^2 \quad \text{again by [7, 11.2].}
\end{aligned}$$

Thus necessity of (22) is proved. On the other hand, by [11, Thm. 3.2] the set $\{T_{m_\lambda} : \lambda > 0\}$ is R–bounded provided the symbols $m_\lambda(\xi)$ satisfy R–versions of the conditions in Mikhlin's multiplier theorem. To be precise, we shall show that for any multi-index $\alpha \in \mathbb{N}_0^{n-k}$ the set

$$\{\xi^\alpha m_\lambda^{(\alpha)}(\xi) : \xi \neq 0, \lambda > 0\}$$

is R–bounded (in fact it would suffice to show this for $\alpha \le (1,\dots,1)$, but the proof below shows it for all $\alpha \in \mathbb{N}_0^{n-k}$ without extra effort). By a simple induction one can show that this is equivalent to R–boundedness of all sets

$$\{\widehat{D}^\alpha m_\lambda(\xi) : \xi \neq 0, \lambda > 0\}$$

where $\widehat{D}^\alpha := \widehat{D}_1^{\alpha_1} \cdots \widehat{D}_{n-k}^{\alpha_{n-k}}$ and $\widehat{D}_j := \xi_j \frac{\partial}{\partial \xi_j}$.
Another induction shows that, for any multi-index α, there exists a $\varphi_{\alpha,\nu} : \mathbb{R}^{n-k} \to \mathbb{C}$ of class C^∞ which is homogeneous of degree 2ν (i.e., $\varphi_{\alpha,\nu}(\rho\xi) = \rho^{2\nu}\varphi_{\alpha,\nu}(\xi)$ for $\xi \in \mathbb{R}^{n-k}$, $\rho \in \mathbb{R}$), such that

$$\widehat{D}^\alpha m_\lambda(\xi) = \sqrt{\lambda}\sum_{\nu=0}^{|\alpha|} \varphi_{\alpha,\nu}(\xi)\langle \psi, (\lambda + |\xi|^2 + A_k)^{-(\nu+1)} \cdot \rangle,$$

where $A_k := \epsilon - \Delta_k$. We write $\lambda = \sigma^2$ and $\mu^2 = \sigma^2 + |\xi|^2$ with $\sigma, \mu > 0$. By homogeneity and the domination $\sigma < \mu$ we only have to show the R–boundedness of the sets

$$\big\{\mu^{2\nu+1}\big\langle (\mu^2 + A_k)^{-(\nu+1)}\psi, \cdot \big\rangle : \mu > 0\big\} \subset B(L^p(\mathbb{R}^k), \mathbb{C})$$

for $\nu \in \mathbb{N}_0$. Since $\{\mu^2(\mu^2 + A_k)^{-1} : \mu > 0\}$ is an R–bounded subset of $B(L^p(\mathbb{R}^k))$ it is sufficient to have R–boundedness for $\nu = 1$, i.e. R–boundedness of the set

$$\big\{\mu\big\langle (\mu^2 + \epsilon - \Delta_k)^{-1}\psi, \cdot \big\rangle : \mu > 0\big\}$$

as asserted. □

Remark 6.4. In Remark 3.6 we mentioned that bounded sets are R–bounded in $B(X, Y)$ if X has cotype 2 and Y has type 2. This applies, e.g., to sets in $B(L^p(\mathbb{R}^k), \mathbb{C})$ if $p \le 2$.

Application 6.5. In virtue of the above remark we consider the case $p \le 2$ and $\epsilon > 0$. Then for $\psi \in H_{p'}^{-2}(\mathbb{R}^k)$ condition (22) is equivalent to boundedness of $\{\lambda^{1/2}\Delta_k(\lambda - \Delta_k)^{-1}\psi\}$ in $H_{p'}^{-2}(\mathbb{R}^k)$. By [31, 1.14.2] this is equivalent to $\psi \in (H_{p'}^{-2}(\mathbb{R}^k), L^{p'}(\mathbb{R}^k))_{1/2,\infty} = B_{p',\infty}^{-1}(\mathbb{R}^k) = (B_{p,1}^1(\mathbb{R}^k))'$.

Example 6.6. We are in particular interested in the case $\psi = \delta_0 \in \mathcal{S}'(\mathbb{R}^k)$. Then C_ψ may be interpreted as observation on an $(n-k)$–dimensional linear subspace of $\mathbb{R}^n$. By Proposition 6.3 and the following Proposition 6.7 we obtain that $\psi = \delta_0 \in \mathcal{S}'(\mathbb{R}^k)$ induces an l–admissible observation operator if $p \in (k, \infty)$. Taking $k = 1$ we obtain that observation on an $(n-1)$–dimensional linear subspace always

defines an l–admissible observation (for $A = \epsilon - \Delta$) on $X{=}L^p(\mathbb{R}^n)$, $1 < p < \infty$. But one has to take p large when modelling observation on a lower dimensional subspace. In particular, one has to take $p > n$ when modelling point observation.
In [28, Ex. 3.2, p. 50] it is shown that $\delta_0 \in B^s_{p',q}(\mathbb{R}^k)$ for $s \le -k/p$ and $q = \infty$ and that this result is optimal in the sense that it becomes false for strictly bigger s or for $s = -k/p$ and finite q. Thus $\delta_0 \in B^{-1}_{p',\infty}(\mathbb{R}^k)$ if and only if $k \le p$. Together with the consideration in Application 6.5 this shows that the bound on p we obtain in Proposition 6.7 cannot be improved. It also shows that we may still take $p = 2$ for $k = 2$.

Proposition 6.7. *Let $k \in \mathbb{N}$ and $\epsilon > 0$. Then, for $p \in (k, \infty)$, the set*

$$\{\mu\langle(\mu^2 + \epsilon - \Delta_k)^{-1}\delta_0, \cdot\rangle : \mu > 0\}$$

is an R-bounded subset of $B(L^p(\mathbb{R}^k), \mathbb{C})$.

Proof. First observe that by Lemma 3.10, R-boundedness of the set in question is equivalent to the existence of a constant $C > 0$ such that

$$\Big\|\Big(\sum_j |\alpha_j \mu_j (\mu_j^2 + \epsilon - \Delta_k)^{-1}\delta_0|^2\Big)^{1/2}\Big\|_{p'} \le C\big(\sum_j |\alpha_j|^2\big)^{1/2}$$

for any choice of $\mu_j > 0$ and scalars α_j. This estimate surely holds if

$$C := \Big\|\sup_{\mu>0} |\mu(\mu^2 + \epsilon - \Delta_k)^{-1}\delta_0|\Big\|_{p'} < \infty. \tag{23}$$

We use the representation of resolvents in terms of the heat semigroup $(T_k(t))$ on $\mathbb{R}^k$. This yields, for $x \in \mathbb{R}^k$,

$$\mu((\mu^2 + \epsilon - \Delta_k)^{-1}\delta_0)(x) = \int_0^\infty \mu e^{-(\mu^2+\epsilon)t}(T_k(t)\delta_0)(x)\,dt. \tag{24}$$

Now we use the Gaussian kernel

$$(T_k(t)\delta_0)(x) = ct^{-k/2}e^{-b\frac{|x|^2}{t}}$$

and sup over $\mu > 0$ in the integrand in (24). We obtain

$$\sup_{\mu>0} |\mu((\mu^2 + \epsilon - \Delta_k)^{-1}\delta_0)(x)| \le c' \int_0^\infty t^{-(k+1)/2}e^{-\epsilon t}e^{-b\frac{|x|^2}{t}}\,dt.$$

Taking the norm $\|\cdot\|_{p'}$ and substituting $x = y\sqrt{t}$ we have for C from (23)

$$C \le c'\big\|y \mapsto e^{-b|y|^2}\big\|_{p'} \int_0^\infty t^{-(k+1)/2}e^{-\epsilon t}t^{k/(2p')}\,dt$$

The integral on the right hand side is finite for $-k(1 - 1/p') > -1$, i.e., for $p > k$. □

Remark 6.8. For $p > 2$, R–boundedness of general subsets of $B(L^p(\Omega), \mathbb{C})$ or, in the dual situation, of $B(\mathbb{C}, L^{p'}(\Omega))$ may be proved using the following abstract characterisation of R–boundedness based on the results [10, Thms VI.4.2',VI.4.5']. This was pointed out to the authors by L. Weis.

Theorem 6.9. *Let (Ω, Σ, μ) be a σ–finite measure space, $p \in (2, \infty)$ and (f_j) a sequence in $L^{p'}(\Omega)$. Let the operators $T_j : L^p(\Omega) \to \mathbb{C}$ be given by $T_j(g) := \int_\Omega g f_j\,d\mu$ and let operators $S_j : \mathbb{C} \to L^{p'}(\Omega)$ be given by $S_j(\lambda) := \lambda f_j$. Then the following characterisations hold true:*

(a) *The set $\{T_j : j \in \mathbb{N}\}$ is R–bounded if and only if there exists some positive function $w \in L^{\frac{p}{p-2}}(\Omega)$ with $\|w\|_{\frac{p}{p-2}} \le 1$ such that the functions f_j are uniformly bounded in $L^2(\Omega, w^{-1}d\mu)$.*

(b) *The set $\{S_j : j \in \mathbb{N}\}$ is R–bounded if and only if there exists a positive function $w \in L^{\frac{p}{p-2}}(\Omega)$ with $\|w^{-1}\|_{\frac{p}{p-2}} \le 1$ such that the functions f_j are uniformly bounded in $L^2(\Omega, w\,d\mu)$.*

Control operators

Observe that, in the situation discussed above, the dual operator C'_ψ of C_ψ is given by $g \mapsto g \otimes \psi$. Indeed, for $g \in L^{p'}(\mathbb{R}^{n-k})$, $f \in L^p(\mathbb{R}^n)$ we have

$$\langle g, C_\psi f\rangle = \int_{\mathbb{R}^{n-k}} g(x)\langle \psi, f(x,\cdot)\rangle\, dx = \langle \psi, \int g(x) f(x,\cdot)\, dx\rangle = \langle g\otimes\psi, f\rangle.$$

Hence we obtain the following characterisation by dualising Proposition 6.3 (recall also Lemma 3.10).

Proposition 6.10. *Let $1 < p < \infty$, $n \geq k \geq 1$, $X = L^p(\mathbb{R}^n)$, $U = L^p(\mathbb{R}^{n-k})$, $A = \epsilon - \Delta_n$, $\epsilon > 0$. Then for $\phi \in \mathcal{S}'(\mathbb{R}^k) \cap H_p^{-2}(\mathbb{R}^k)$ the operator $B_\phi : g \mapsto g \otimes \phi$ is an l–admissible control operator if and only if the set*

$$\left\{\, \mu(\mu^2 + \epsilon - \Delta_k)^{-1}\phi :\ \mu > 0\right\} \subset B(\mathbb{C}, L^p(\mathbb{R}^k)) \tag{25}$$

is R–bounded.

Example 6.11. Again we are interested in the case $\phi = \delta_0 \in \mathcal{S}'(\mathbb{R}^k)$. Here $k = n$ corresponds to point control and $k = 1$ corresponds to control from a hyperplane. By dualising the assertion of Proposition 6.7 we obtain that $\phi = \delta_0 \in \mathcal{S}'(\mathbb{R}^k)$ induces an l–admissible control operator if $p' \in (k, \infty)$, i.e., if $p \in (1, \frac{k}{k-1})$. Hence control from a hyperplane is always l–admissible, but control from a point needs $p \in (1, \frac{n}{n-1})$. Again, we may take $p = 2$ for $k = 2$ by Peetre's result.

Remark 6.12. Combining the results of Examples 6.6 and 6.11, we see that as far as admissibility is concerned, we may combine observation on hyperplanes with control from hyperplanes in any dimension n for any $p \in (1, \infty)$. In particular, we may combine point observation and point control in dimension $n = 1$ for any $p \in (1, \infty)$. But the combination of point observation and point control in dimension $n = 2$ requires $p = 2$, and it is not possible in dimensions $n \geq 3$. However, we may, e.g., combine point observation with control from a hyperplane by taking $p > n$, etc.

A controlled heat equation

Next we illustrate our results with a controlled heat equation. The example is very much inspired by [2] where the same problem is studied in the state space $X = L^2(\Omega)$, i.e., in a Hilbert space context. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega \in C^\infty$ and $1 < p < \infty$. Denote the the outer normal unit vector on $\partial\Omega$ by $\nu : \partial\Omega \to \mathbb{R}^n$. We consider the following problem

$$\begin{cases} x'(t) - \Delta x(t) &= 0, \quad (t > 0) \\ \frac{\partial x(t)}{\partial \nu}\big|_{\partial\Omega} &= u(t), \quad (t > 0) \\ x(0) &= x_0 \\ y(t) &= x(t)\big|_{\partial\Omega}, \quad (t > 0), \end{cases} \tag{26}$$

where $x(\cdot)$ takes values in $X := L^p(\Omega)$, and $u(\cdot)$ and $y(\cdot)$ take values in function spaces on the boundary. For the modelling we follow closely [2]. We let $A := -\Delta$ with homogeneous Neumann boundary conditions, i.e., we have $D(A) = \{x \in W_p^2(\Omega) : \frac{\partial x}{\partial \nu}\big|_{\partial\Omega} = 0\}$ due to the smoothness of $\partial\Omega$. Similar to [2] we are only interested in l–admissibility for finite time intervals. This of course is equivalent to discussing l–admissibility on $\mathbb{R}_+$ for $1 + A$. The operator $1 + A$ has a bounded H^∞-calculus on each sector $S(\omega)$, $\omega > 0$, and $-A$ generates a bounded analytic semigroup which is even R-analytic of angle $\pi/2$ (cf., e.g., [21]). Moreover, $D(A^{1/2}) = W_p^1(\Omega)$ in this case, and this space equals the Bessel potential space $H_p^1(\Omega)$ (cf. [31]).

Example 6.13. Denoting the Dirichlet trace operator $\gamma_0 : x \mapsto x\big|_{\partial\Omega}$ by C we are looking for a space Y on $\partial\Omega$ such that $C : D(A) \to Y$ is l–admissible on finite time intervals for the operator A. We will do so by finding spaces Y such that $C : D(A) \to Y$ is l–admissible for $1 + A$. Since $1 + A$ has a bounded H^∞-calculus, we know by Theorem 4.2 that l–admissibility of $C : D(A) \to Y$ is equivalent to

$$\{\lambda^{1/2} C(\lambda + 1 + A)^{-1} : \lambda > 0\} \quad \text{is } R\text{-bounded in } B(X, Y).$$

This certainly holds if C is bounded in $\|\cdot\|_{Z\to Y}$ where $D(A) \subset Z \subset X$ is a Banach space such that

$$\{\lambda^{1/2}(\lambda + 1 + A)^{-1} : \lambda > 0\} \quad \text{is } R\text{-bounded in } B(X, Z). \tag{27}$$

Since

$$\{\lambda^{1/2}(1 + A)^{1/2}(\lambda + 1 + A)^{-1} : \lambda > 0\} \quad \text{is } R\text{-bounded in } B(X),$$

(cf. [22, Lem. 10]), condition (27) holds for $Z = D((1+A)^{1/2}) = D(A^{1/2}) = H_p^1(\Omega)$. Since $\gamma_0 : H_p^1(\Omega) \to B_{p,p}^{1-1/p}(\partial\Omega)$ is continuous (cf. [31]), we obtain that C is l–admissible for $1+A$ if we choose $Y = B_{p,p}^{1-1/p}(\Omega)$.
Another possible choice of Z is provided by [12, Lem. 6.10]: condition (27) also holds for the real interpolation space $(X, D(A))_{1/2,2} = B_{p,2}^1(\Omega)$ (we refer to [31] for its definition). Since $\gamma_0 : B_{p,2}^1(\Omega) \to B_{p,2}^{1-1/p}(\partial\Omega)$ is continuous (cf. [31]), we obtain that C is l–admissible for $1+A$ if we choose $Y = B_{p,2}^{1-1/p}(\Omega)$. Since Besov spaces $B_{p,q}^\alpha$ grow with $q \in [1,\infty]$, the smallest space Y we thus obtained is $B_{p,\min(p,2)}^{1-1/p}(\partial\Omega)$. Observe that, for $p = 2$, we have

$$B_{p,p}^1(\Omega) = B_{p,2}^1(\Omega) = H_2^1(\Omega) \quad \text{and} \quad Y = B_{p,p}^{1-1/p}(\partial\Omega) = B_{2,2}^{1/2}(\partial\Omega) = H_2^{1/2}(\partial\Omega)$$

which is just the space obtained in [2]. Notice, however, that our proof, being based on Theorem 4.2, is essentially different from the proof given there.
Concerning the control operator B we proceed again as in [2]. Multiplying the heat equation (26) with a fixed function $v \in C^\infty(\overline{\Omega})$ and integrating by parts we obtain

$$\langle x'(t), v\rangle_\Omega + \langle \nabla x(t), \nabla v\rangle_\Omega = \int_{\partial\Omega} u(t) v \, d\sigma,$$

where $\langle\cdot,\cdot\rangle_\Omega$ denotes the usual duality pairing on $L^p(\Omega) \times L^{p'}(\Omega)$ and σ denotes the surface measure on $\Gamma := \partial\Omega$. Denoting extensions of the usual $L^2(\Gamma)$-duality by $\langle\cdot,\cdot\rangle_\Gamma$ we thus have

$$\int_{\partial\Omega} u(t) v \, d\sigma = \langle u(t), \gamma_0 v\rangle_\Gamma,$$

which means that $B = \gamma_0' = C'$ if we identify X_{-1} with the dual space of $(D(A'))$. Notice that $A' = -\Delta$ with Neumann boundary conditions in $X' = L^{p'}(\Omega)$. Since the spaces involved are reflexive, we obtain that $B : U \to X_{-1}$ is a finite time l–admissible control operator for the operator A if and only if $C : D(A') \to U'$ is a finite time l–admissible observation operator for the operator A'. By our previous results the largest space we obtain for finite time l–admissibility of $B : U \to X_{-1}$ is thus $U = (B_{p',\min(2,p')}^{1-1/p'}(\partial\Omega))' = B_{p,\max(2,p)}^{-1/p}(\partial\Omega)$. Again, for $p = 2$, we reobtain the space $U = B_{2,2}^{-1/2}(\partial\Omega) = H_2^{-1/2}(\partial\Omega)$ as it was obtained in [2] by a different proof.
We remark that the result (for observation operators) from [25], which is restricted to the case $Y = L^q(\Omega)$, may not be applied to this example in order to derive the results above.

Notice that if $\Psi : B_{p,\min(2,p)}^{1-1/p}(\partial\Omega) \to \mathbb{C}^k$ and $\Phi : \mathbb{C}^m \to B_{p,\max(2,p)}^{-1/p}(\partial\Omega)$ are bounded operators then, by what was shown above, $\Psi C : X_1 \to \mathbb{C}^k$ and $B\Phi : \mathbb{C}^m \to X_{-1}$ are finite time l–admissible for A. Since $\mathbb{C}^k$, $\mathbb{C}^m$ are Hilbert spaces this means that these operators are finite time admissible in the usual L^2-sense.

Example 6.14. We consider in particular the case of averaged observation and piecewise constant control (in [2] called type 2 output and and type 2 input, respectively). To this end we fix smoothly bounded open subsets $\Gamma_1, \ldots, \Gamma_k, Q_1, \ldots, Q_m \subset \partial\Omega$ where $k, m \in \mathbb{N}$ and define

$$\Psi y := (|\Gamma_j|^{-1} \int_{\Gamma_j} y \, d\sigma)_{j=1}^k, \qquad \Phi(u_j)_{j=1}^m := \sum_{j=1}^m u_j \mathbb{1}_{Q_j}.$$

We have to check that $\Psi : B_{p,\min(2,p)}^{1-1/p}(\partial\Omega) \to \mathbb{C}^k$ and $\Phi : \mathbb{C}^m \to B_{p,\max(2,p)}^{-1/p}(\partial\Omega)$ are bounded. For Φ this boils down to the question when $\mathbb{1}_Q \in B_{p,q}^\alpha(\partial\Omega)$ for a smoothly bounded open subset $Q \subset \partial\Omega$. By [31, 2.8.7] the operator $v \mapsto \mathbb{1}_Q v$ is bounded on $B_{p,q}^\alpha$ for $\alpha \in [0, 1/p)$. Applying this to a function v equal to one on Q we obtain $\mathbb{1}_Q \in B_{p,q}^\alpha(\partial\Omega)$ for $\alpha < 1/p$, hence in particular $\mathbb{1}_Q \in B_{p,q}^{-1/p}(\partial\Omega)$ and boundedness of Φ follows. For the boundedness of Ψ we have to show that $\mathbb{1}_{\Gamma_j} \in (B_{p,q}^{1-1/p}(\partial\Omega))' = B_{p',q'}^{-1/p'}(\partial\Omega)$, which is clear from the preceding argument.

We turn to wellposedness and regularity of the full system (26). We formulate the result as a theorem making use of the admissibility results presented above. All that is left is the study of the input–output function.

Theorem 6.15. *Let $1 < p < \infty$ and $X = L^p(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is a smoothly bounded open set. Let A, B, C be as above, and let Φ, Ψ be as in Example 6.14.*

(a) *For $q \in [\min(p,2), \max(p,2)]$ and $Y = B^{1-1/p}_{p,q}(\partial\Omega)$, $U = B^{-1/p}_{p,q}(\partial\Omega)$, the system (A,B,C) is regular and l-wellposed on finite time intervals.*

(b) *For $Y = \mathbb{C}^k$ and $U = \mathbb{C}^m$ the system $(A, B\Phi, \Psi C)$ is regular and wellposed on finite time intervals.*

Proof. (a). By Theorem 4.11 (or rather Corollary 4.12) we have to check that $(\lambda + A_{-1})^{-1}$ maps $B(U)$ into $\mathcal{D}(C_\Lambda)$ and that $\{C_\Lambda(\lambda + A_{-1})^{-1}B : \lambda \in \mathbb{C}_+\}$ is an R–bounded subset of $B(U,Y)$. We shall use interpolation. To this end we recall that R–boundedness can be interpolated by the real and the complex method if the spaces involved have nontrivial type [18, Prop. 3.7]. The set $\{(\lambda + A)^{-1} : \lambda \in \mathbb{C}_+\} \subset B(L^p(\Omega), D(A))$ is R–bounded since A is R–sectorial in $L^p(\Omega)$. By self–duality we also obtain that $\{(\lambda + A_{-1})^{-1} : \lambda \in \mathbb{C}_+\} \subset B(D(A')', L^p(\Omega))$ is R–bounded. Here we recall that X_{-1} is canonically isomorphic to $D(A')'$.
Case $q = p$: By complex interpolation $[\cdot,\cdot]_{1/2}$ we obtain that $\{(\lambda + A_{-1})^{-1} : \lambda \in \mathbb{C}_+\}$ is R–bounded $(H^1_{p'}(\Omega))' \to H^1_p(\Omega)$. The part of A in $Z_p := H^1_p(\Omega) = D(A^{1/2})$ is sectorial and densely defined, and $W_p := ((1+A_{-1})H^1_p(\Omega), \|(1+A_{-1})\cdot\|_{H^1_p(\Omega)})$ equals $(H^1_{p'}(\Omega))'$ with equivalent norms. The operators $C = \gamma_0 : H^1_p(\Omega) \to B^{1-1/p}_{p,p}(\partial\Omega)$ and $B = \gamma_0' : B^{-1/p}_{p,p}(\partial\Omega) \to \big(H^1_{p'}(\Omega)\big)'$ are bounded. For $Y = B^{1-1/p}_{p,p}(\partial\Omega)$ and $U = B^{-1/p}_{p,p}(\partial\Omega)$, we hence obtain regularity of the system by Remark 4.13 and l–wellposedness by Corollary 4.12.
Case $q = 2$: By real interpolation $(\cdot,\cdot)_{1/2,2}$ we obtain that $\{(\lambda + A_{-1})^{-1} : \lambda \in \mathbb{C}_+\}$ is R–bounded. $(B^1_{p',2}(\Omega))' \to B^1_{p,2}(\Omega)$. The part of A in $Z_2 := B^1_{p,2}(\Omega) = (X, D(A))_{1/2,2}$ is sectorial and densely defined, and $W_2 := ((1 + A_{-1})B^1_{p,2}(\Omega), \|(1 + A_{-1}) \cdot \|_{B^1_{p,2}(\Omega)})$ equals $(B^1_{p',2}(\Omega))'$ with equivalent norms. The operators $C = \gamma_0 : B^1_{p,2}(\Omega) \to B^{1-1/p}_{p,2}(\partial\Omega)$ and $B = \gamma_0' : B^{-1/p}_{p,2}(\partial\Omega) \to \big(B^1_{p',2}(\Omega)\big)'$ are bounded. For $Y = B^{1-1/p}_{p,2}(\partial\Omega)$ and $U = B^{-1/p}_{p,2}(\partial\Omega)$, we hence obtain regularity of the system by Remark 4.13 and l–wellposedness by Corollary 4.12.
For q between 2 and p we obtain the spaces $Y = B^{1-1/p}_{p,q}(\partial\Omega)$ and $U = B^{-1/p}_{p,q}(\partial\Omega)$ by complex interpolation between the cases $q = 2$ and $q = p$. Observe that we may apply Remark 4.13 to the space $Z_q := [Z_p, Z_2]_\theta$ where $q^{-1} = (1-\theta)/p + \theta/2$.
(b). Use (a) and Example 6.14 and the last assertion of Corollary 4.12. □

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