UNIFORMLY γ–RADONIFYING FAMILIES OF OPERATORS AND THE STOCHASTIC WEISS CONJECTURE

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ABSTRACT. We introduce the notion of uniform γ–radonification of a family of operators, which unifies the notions of R–boundedness of a family of operators and γ–radonification of an individual operator. We study the properties of uniformly γ–radonifying families of operators in detail and apply our results to the stochastic abstract Cauchy problem

$$dU(t) = AU(t)\,dt + B\,dW(t), \quad U(0) = 0.$$

Here, A is the generator of a strongly continuous semigroup of operators on a Banach space E, B is a bounded linear operator from a separable Hilbert space H into E, and W_H is an H–cylindrical Brownian motion. When A and B are simultaneously diagonalisable, we prove that an invariant measure exists if and only if the family

$$\{\sqrt{\lambda}R(\lambda, A)B : \ \lambda \in S_\vartheta\}$$

is uniformly γ–radonifying for some/all $0 < \vartheta < \frac{\pi}{2}$, where S_ϑ is the open sector of angle ϑ in the complex plane. This result can be viewed as a partial solution of a stochastic version of the Weiss conjecture in linear systems theory.

1. Introduction

In recent years it has become apparent that many classical results in operator theory and harmonic analysis can be generalised from their traditional Hilbert space setting to Banach spaces, provided the notion of uniform boundedness is replaced with *R–boundedness*. This notion appeared implicitly in the work of Bourgain [4] and was formalised by Berkson and Gillespie [2] and Clément, de Pagter, Sukochev, and Witvliet [5]. It has accomplished remarkable progress in the theory of parabolic evolution equations. A highlight is the recent solution of the L^p-maximal regularity problem by Weis [45], who proved an extension of the Mihlin multiplier theorem for operator-valued multipliers taking R–bounded values in a UMD space E and used it to deduce that the generator A of a bounded analytic semigroup on a UMD space E has L^p-maximal regularity if and only if $\lambda \mapsto \lambda(\lambda - A)^{-1}$ is R–bounded on $\mathbb{C}_+$. Soon, an alternative approach to the L^p-maximal regularity problem via H^∞–calculus appeared. In the Hilbert space setting this calculus was introduced by McIntosh [31], who characterised it by means of square function estimates. This characterisation extends to Banach spaces, provided the square functions are replaced with γ–radonifying norms [6, 24, 25, 29].

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These developments have been documented in detail in the memoir by Denk, Hieber, and Prüss [7] and the lecture notes by Kunstmann and Weis [26], where extensive references can be found.

In a parallel development, γ–radonifying norms have been used recently to extend the theory of stochastic integration to the Banach space setting, first for operator-valued functions taking values in arbitrary Banach spaces [36, 37] and subsequently for operator-valued processes taking values in UMD spaces [34]. In both papers, the role of the Itô isometry is taken over by an isometry in terms of γ–radonifying norms. Applications to stochastic evolution equations in Banach spaces have been worked out, for linear equations [8, 15, 38] and nonlinear equations [35, 44].

Further applications of R–boundedness and γ–radonifying norms have been given in various areas on analysis, such as harmonic analysis [1, 2, 12, 17, 18, 22], Banach space theory [10, 24, 41, 43], interpolation theory [23], control theory [13, 14, 28, 29], and noncommutative analysis [21, 40]; this list of references is far from complete.

In this paper we unify the notions of R–boundedness (or rather, its Gaussian analogue γ-boundedness) and γ–radonification by introducing the concept of uniformly γ–radonifying families of operators. As we shall demonstrate in Sections 2 and 3 this is a happy marriage: uniformly γ–radonifying families enjoy many of the good properties both of R–bounded families and of γ–radonifying operators.

In Section 4 we apply our abstract results to study some properties of operator-valued Laplace transforms. It turns out that the Laplace transforms of γ–radonifying operators $\Phi : L^2(\mathbb{R}_+; H) \to E$ are uniformly γ–radonifying both in half-planes and in sectors properly contained in $\mathbb{C}_+$.

Natural examples of uniformly γ–radonifying families of operators arise in the theory of stochastic evolution equations. These will be presented in the final Section 5 of the paper, where we apply our results on Laplace transforms to stochastic evolution equations. We prove that a necessary condition for the existence of invariant measures for the linear stochastic Cauchy problem

$$dU(t) = AU(t)\,dt + B\,dW_H(t), \quad U(0) = 0,$$

where W_H is an H–cylindrical Brownian motion and $B : H \to E$ is a bounded operator, is that the family

$$\{\sqrt{\lambda}R(\lambda, A)B : \lambda \in S_\vartheta\}$$

should be uniformly γ–radonifying for all $0 < \vartheta < \frac{\pi}{2}$, where S_ϑ is the open sector of angle ϑ in the complex plane. For simultaneously diagonalisable operators A and B we show that this condition is also sufficient. This result is a partial solution of a stochastic version of the Weiss conjecture in linear systems theory (see [47] and the subsequent work [19, 28, 46, 48, 49]) in which L^2-admissibility of the control operator is replaced with the existence of an invariant measure.

2. Uniformly γ–radonifying families

In this section we introduce the notion of a uniformly γ–radonifying family of operator and study its properties. This notion unifies the concepts of R–boundedness (or rather, γ–boundedness) and γ–radonification, which we shall discuss first.

Let E and F be Banach spaces. A subset $\mathscr{S}$ of $\mathscr{B}(E,F)$ is called *R–bounded* if there exists a constant $C \geqslant 0$ such that for all $n \geqslant 1$, all $x_1, \dots, x_n \in E$, and all $S_1, \dots, S_n \in \mathscr{S}$ we have

$$\mathbb{E}\Big\|\sum_{k=1}^n r_k S_k x_k\Big\|^2 \leqslant C^2 \mathbb{E}\Big\|\sum_{k=1}^n r_k x_k\Big\|^2.$$

Here, $(r_k)_{k\geqslant 1}$ is a *Rademacher sequence*, i.e. a sequence of independent $\{-1, +1\}$-valued random variables on some probability space $(\Omega, \mathbb{P})$ with the property that

$\mathbb{P}(r_k = \pm 1) = \frac{1}{2}$. The least admissible constant C is called the *R–bound* of $\mathscr{S}$, notation $R(\mathscr{S})$. A similar definition may be given in terms of Gaussian sums, which leads to the concept of a γ*–bounded* family with γ*–bound* $\gamma(\mathscr{S})$. By a standard randomisation argument, every R–bounded family $\mathscr{S}$ is γ–bounded with $\gamma(\mathscr{S}) \leqslant R(\mathscr{S})$. If E and F are Hilbert spaces, the notions of γ–boundedness and R–boundedness coincide with that of uniform boundedness and we have $\gamma(\mathscr{S}) = R(\mathscr{S}) = \sup_{S\in\mathscr{S}} \|S\|$.

Throughout this paper, unless otherwise stated H is a separable infinite-dimensional Hilbert space and E is a Banach space. Let $(\gamma_k)_{k\geqslant 1}$ be a *Gaussian sequence*, i.e., a sequence of independent real-valued standard Gaussian random variables on some probability space $(\Omega, \mathbb{P})$. A linear operator $T : H \to E$ is called γ*–radonifying* if for some orthonormal basis $(h_k)_{k\geqslant 1}$ of H the sum

$$\sum_{k\geqslant 1} \gamma_k T h_k$$

converges in $L^2(\Omega; E)$. If this is the case, the sum $\sum_{k\geqslant 1} \gamma_k T h_k$ converges in E almost surely and in $L^p(\Omega; E)$ for all $1 \leqslant p < \infty$, for every orthonormal basis $(h_k)_{k\geqslant 1}$ of H. The linear space of all γ–radonifying operators from H to E is denoted by $\gamma(H, E)$. Endowed with the norm

$$\|T\|^2_{\gamma(H,E)} := \mathbb{E}\Big\|\sum_{k\geqslant 1} \gamma_k T h_k\Big\|^2,$$

which is independent of the basis $(h_k)_{k\geqslant 1}$, the space $\gamma(H, E)$ is a Banach space. Furthermore, it is a two-sided operator ideal in $\mathscr{B}(H, E)$, the space of all bounded linear operators from H to E. For proofs and more information we refer to the review paper [33].

Definition 2.1. A subset $\mathscr{T}$ of $\mathscr{B}(H, E)$ is *uniformly* γ*–radonifying* if for all orthonormal bases $(h_k)_{k\geqslant 1}$ of H and sequences $(T_k)_{k\geqslant 1}$ in $\mathscr{T}$ the Gaussian sum $\sum_{k\geqslant 1} \gamma_k T_k h_k$ converges in $L^2(\Omega; E)$.

It is important to note that this definition refers to *all* orthonormal bases of H. Evidently, this definition trivializes for finite-dimensional Hilbert spaces; it is mainly for this reason that we restrict our attention to infinite-dimensional H.

By considering the constant sequence $T_k = T$ we see that every operator T in a uniformly γ–radonifying subset $\mathscr{T}$ of $\mathscr{B}(H, E)$ is γ–radonifying, i.e., $\mathscr{T} \subseteq \gamma(H, E)$.

We begin our investigations with proving some simple permanence properties of uniformly γ–radonifying families of operators, resembling those of R–bounded and γ–bounded families of operators. In what follows, $\mathscr{T}$ denotes a subset of $\mathscr{B}(H, E)$.

Proposition 2.2 (Strong closure)**.** *If $\mathscr{T}$ is uniformly γ–radonifying, then the closure $\overline{\mathscr{T}}$ in the strong operator topology of $\mathscr{B}(H, E)$ is uniformly γ–radonifying.*

Proof. Let $(\overline{T}_k)_{k\geqslant 1}$ be a sequence in $\overline{\mathscr{T}}$. Given an $\varepsilon > 0$ and an orthonormal basis $(h_k)_{k\geqslant 1}$ of H, choose a sequence $(T_k)_{k\geqslant 1}$ in $\mathscr{T}$ such that $\|\overline{T}_k h_k - T_k h_k\| < 2^{-k}\varepsilon$ for all $k \geqslant 1$. Then, for all $1 \leqslant M \leqslant N$,

$$\begin{aligned}
\Big(\mathbb{E}\Big\|\sum_{k=M}^{N} \gamma_k \overline{T}_k h_k\Big\|^2\Big)^{\frac{1}{2}} &\leqslant \Big(\mathbb{E}\Big\|\sum_{k=M}^{N} \gamma_k T_k h_k\Big\|^2\Big)^{\frac{1}{2}} + \Big(\mathbb{E}\Big\|\sum_{k=M}^{N} \gamma_k (\overline{T}_k h_k - T_k h_k)\Big\|^2\Big)^{\frac{1}{2}} \\
&\leqslant \Big(\mathbb{E}\Big\|\sum_{k=M}^{N} \gamma_k T_k h_k\Big\|^2\Big)^{\frac{1}{2}} + \sum_{k=M}^{N} \|\overline{T}_k h_k - T_k h_k\| \\
&\leqslant \Big(\mathbb{E}\Big\|\sum_{k=M}^{N} \gamma_k T_k h_k\Big\|^2\Big)^{\frac{1}{2}} + \varepsilon.
\end{aligned}$$

The result follows by letting $M, N \to \infty$. □

Lemma 2.3. *If $\mathscr{T}$ is uniformly γ–radonifying, then for all orthonormal bases $(h_k)_{k\geqslant 1}$ of H we have*

$$\lim_{n\to\infty} \Big(\sup_T \ \mathbb{E}\Big\| \sum_{k=n}^{\infty} \gamma_k T_k h_k \Big\|^2 \Big) = 0,$$

where the supremum is taken over all sequences $T = (T_k)_{k\geqslant 1}$ in $\mathscr{T}$.

Proof. If the lemma was false, we could find an orthonormal basis $(h_k)_{k\geqslant 1}$ of H, a number $\delta > 0$, an increasing sequence of indices $1 \leqslant n_1 < N_1 < n_2 < N_2 < \dots$, and for each $j = 1, 2, \dots$ a finite set of operators $T_{n_j}, \dots, T_{N_j} \in \mathscr{T}$ such that

$$\mathbb{E}\Big\| \sum_{k=n_j}^{N_j} \gamma_k T_k h_k \Big\|^2 \geqslant \delta^2, \quad j = 1, 2, \dots$$

Putting $T_k := 0$ if $N_j < k < n_{j+1}$ for some $j \geqslant 0$ (with the convention that $N_0 = 0$) we obtain a sequence $(T_k)_{k\geqslant 1}$ for which the sum $\sum_{k\geqslant 1} \gamma_k T_k h_k$ fails to converge in $L^2(\Omega; E)$, and we have arrived at a contradiction. □

Proposition 2.4 (Absolute convex hull)**.** *If $\mathscr{T}$ is uniformly γ–radonifying, then the absolute convex hull of $\mathscr{T}$ is uniformly γ–radonifying.*

Proof. Considering real and complex parts separately and possibly replacing some of the γ_k by $-\gamma_k$, it suffices to prove the statement in the lemma for the convex hull of $\mathscr{T}$. Furthermore, by the contraction principle for Banach space-valued Gaussian sums, $\mathscr{T} \cup \{0\}$ is uniformly γ–radonifying and therefore we may assume that $0 \in \mathscr{T}$.

Fix an orthonormal basis $(h_k)_{k\geqslant 1}$ of H and an $\varepsilon > 0$, and choose $n_0 \geqslant 1$ so large that

$$\sup_T \ \mathbb{E}\Big\| \sum_{k=n_0}^{\infty} \gamma_k T_k h_k \Big\|^2 < \varepsilon^2.$$

Let $(S_k)_{k\geqslant 1}$ be a sequence in $\mathrm{conv}(\mathscr{T})$ and fix indices M, N satisfying $n_0 \leqslant M \leqslant N$. Noting that

$$\mathrm{conv}(\mathscr{T}) \times \dots \times \mathrm{conv}(\mathscr{T}) = \mathrm{conv}(\mathscr{T} \times \dots \times \mathscr{T})$$

we can find $\lambda_1, \dots, \lambda_N \in [0, 1]$ with $\sum_{j=1}^N \lambda_j = 1$ such that $S_k = \sum_{j=1}^N \lambda_j T_{jk}$ with $T_{jk} \in \mathscr{T}$ for all $k = M, \dots, N$. Then,

$$\Big(\mathbb{E}\Big\| \sum_{k=M}^{N} \gamma_k S_k h_k \Big\|^2\Big)^{\frac12} \leqslant \sum_{j=1}^N \lambda_j \Big(\mathbb{E}\Big\| \sum_{k=M}^{N} \gamma_k T_{jk} h_k \Big\|^2\Big)^{\frac12} < \sum_{j=1}^N \lambda_j \varepsilon = \varepsilon.$$

□

Combining Propositions 2.2 and 2.4 we obtain that the strongly closed absolutely convex hull of every uniformly γ–radonifying set is uniformly γ–radonifying. As in the case of R–boundedness, cf. [7, 26, 45], this may be used to show that uniform γ–radonification is preserved by taking integral means. In this way a number of well-known R–boundedness results can be carried over to uniformly γ–radonifying families. To give a few examples we formulate analogues of [26, Corollary 2.14] and [45, Propositions 2.6 and 2.8].

Proposition 2.5. *Let (S, μ) be a σ-finite measure space and let $\mathscr{T}$ be a uniformly γ–radonifying subset of $\mathscr{B}(H, E)$. If $f : S \to \mathscr{B}(H, E)$ is strongly μ-measurable (in* *the sense that $s \mapsto f(s)h$ is strongly μ-measurable for all $h \in H$) with $f(s) \in \mathscr{T}$ for μ-almost all $s \in S$, then for all $\phi \in L^1(S,\mu)$ the operator*

$$T_\phi h := \int_S \phi(s) f(s)h \, d\mu(s), \quad h \in H,$$

belongs to $\mathscr{B}(H,E)$ and the family $\{T_\phi : \ \|\phi\|_1 \leqslant 1\}$ is uniformly γ–radonifying.

Proposition 2.6. *Let $G \subseteq \mathbb{C}$ be an open domain and let $f : G \to \mathscr{B}(H,E)$ be an analytic function with $f(z) \in \gamma(H,E)$ for all $z \in G$. Then for every compact subset $K \subseteq G$ the family $\{f(z) : \ z \in K\}$ is uniformly γ–radonifying.*

Proposition 2.7. *Let $G \subseteq \mathbb{C}$ be a simply connected Jordan domain such that $\mathbb{C} \setminus G$ has nonempty interior. Let $f : \overline{G} \to \mathscr{B}(H,E)$ be uniformly bounded and strongly continuous, analytic on G, and assume that $\{f(z) : \ z \in \partial G\}$ is uniformly γ–radonifying. Then $\{f(z) : \ z \in \overline{G}\}$ is uniformly γ–radonifying.*

In the situation of Proposition 2.7 it follows that $f(z) \in \gamma(H,E)$ for all $z \in \overline{G}$. It will follow from Theorem 3.3 below that $f : \overline{G} \to \gamma(H,E)$ is continuous.

The next lemma shows that uniform γ–radonification is preserved under left and right multiplication.

Proposition 2.8 (Ideal property)**.** *Let $\tilde{H}$ be a separable infinite-dimensional Hilbert space and $\tilde{E}$ a Banach space. If $\mathscr{T}$ is a uniformly γ–radonifying subset of $\mathscr{B}(H,E)$ and $R : \tilde{H} \to H$ and $S : E \to \tilde{E}$ are bounded operators, then $S\mathscr{T}R$ is a uniformly γ–radonifying subset of $\mathscr{B}(\tilde{H},\tilde{E})$.*

Proof. The left ideal property is trivial. To prove the right ideal property we first consider the case of complex scalars. For $\tilde{H} = H$ the right ideal property then follows from the well-known fact that the convex hull of the unitary operators on H space are uniformly dense in the closed unit ball of $\mathscr{B}(H)$ (this is the so-called Russo-Dye theorem [42]; at the cost of picking up a constant 2 we could alternatively use the elementary fact that every operator in $\mathscr{B}(H)$ of norm less than $\frac{1}{2}$ is a convex combination of at most four unitaries).

In the case of different Hilbert spaces H and $\tilde{H}$ write $TR = TRU^* \circ U$, where U is an isometry from $\tilde{H}$ onto H and note that $\mathscr{T}RU^*$ is uniformly γ–radonifying by the preceding observation.

In the case of real scalars, let $(\tilde{h}_k)_{k\geqslant 1}$ be an orthonormal basis of $\tilde{H}$ and write $\sum_{k\geqslant 1} \gamma_k T_k S\tilde{h}_k = \sum_{k\geqslant 1} \gamma_k T_k^{\mathbb{C}} S^{\mathbb{C}} \tilde{h}_k^{\mathbb{C}}$, where $T_k^{\mathbb{C}}$ and $S^{\mathbb{C}}$ are the complexifications of T_k and S, and $\tilde{h}_k^{\mathbb{C}} = \tilde{h}_k + i0$. Since $(\tilde{h}_k^{\mathbb{C}})_{k\geqslant 1}$ is an orthonormal basis for $\tilde{H}^{\mathbb{C}}$, the right-hand side converges in $L^2(\Omega; E^{\mathbb{C}})$. □

We continue with a preliminary boundedness result for uniformly γ–radonifying families. It will be strengthened in Theorem 2.10 below.

Proposition 2.9. *Let $\mathscr{T}$ be a uniformly γ–radonifying subset of $\mathscr{B}(H,E)$. Then $\mathscr{T}$ is a bounded subset of $\gamma(H,E)$.*

Proof. The fact that $\mathscr{T}$ is contained in $\gamma(H,E)$ has already been noted. It suffices to prove that $\sup_{k\geqslant 1} \|T_k\|_{\gamma(H,E)} < \infty$ for every sequence $(T_k)_{k\geqslant 1}$ in $\mathscr{T}$.

Fix an orthonormal basis $(h_k)_{k\geqslant 1}$ of H. By Proposition 2.8 and a closed graph argument, there exists a constant $C \geqslant 0$ such that for all $S \in \mathscr{B}(H)$ we have

$$\mathbb{E}\Big\| \sum_{k\geqslant 1} \gamma_k T_k S h_k \Big\|^2 \leqslant C^2 \|S\|^2.$$

In particular, for all $x^* \in E^*$ this implies

$$\begin{aligned}\sum_{k\geqslant 1} \big|(Sh_k, T_k^* x^*)_H\big|^2 &= \mathbb{E}\Big|\sum_{k\geqslant 1} \gamma_k \langle T_k S h_k, x^*\rangle\Big|^2 \\ &= \mathbb{E}\Big|\Big\langle \sum_{k\geqslant 1} \gamma_k T_k S h_k, x^*\Big\rangle\Big|^2 \leqslant C^2\|S\|^2\|x^*\|^2.\end{aligned}$$

Taking $S := h_n \otimes T_n^* x^*$ with $n \geqslant 1$ fixed it follows that

$$\|T_n^* x^*\|_H^4 \leqslant C^2 \|T_n^* x^*\|^2 \|x^*\|^2.$$

Therefore, $\sup_{n\geqslant 1} \|T_n\| \leqslant C$. Next, by Lemma 2.3 we can find $N \geqslant 1$ so large that

$$\sup_{k\geqslant 1} \mathbb{E}\Big\|\sum_{n=N+1}^{\infty} \gamma_n T_k h_n\Big\|^2 \leqslant 1.$$

But then,

$$\sup_{k\geqslant 1} \|T_k\|_{\gamma(H,E)} \leqslant 1 + CN.$$

□

The next result explains our terminology 'uniformly γ–radonifying':

Theorem 2.10. *Let $\mathscr{T}$ be a uniformly γ–radonifying subset of $\mathscr{B}(H,E)$. Then there exists a constant $C \geqslant 0$ such that for all orthonormal bases $h = (h_k)_{k\geqslant 1}$ of H and all sequences $(T_k)_{k\geqslant 1}$ in $\mathscr{T}$ we have*

$$\mathbb{E}\Big\|\sum_{k\geqslant 1} \gamma_k T_k h_k\Big\|^2 \leqslant C^2.$$

Proof. Let

$$W := \overline{\operatorname{abs\,conv}}(\mathscr{T}) \times \overline{\operatorname{abs\,conv}}(\mathscr{T}) \times \dots$$

where $\operatorname{abs\,conv}(\mathscr{T})$ denotes the absolute convex hull of $\mathscr{T}$ and the closure is taken in the norm of $\gamma(H,E)$. Note that W is absolutely convex. Let $l(\mathscr{T})$ denote the vector space of all sequences $T = (T_k)_{k\geqslant 1}$ in $\gamma(H,E)$ such that $cT \in W$ for some $c > 0$. In view of Propositions 2.2, 2.4, and Proposition 2.9 we may endow $l(\mathscr{T})$ with the norm

$$\|T\|_{l(\mathscr{T})} := \inf\Big\{\frac{1}{c} : \ c > 0, \ cT \in W\Big\} + \sup_{k\geqslant 1} \|T_k\|_{\gamma(H,E)}.$$

It is routine to check that the normed space $l(\mathscr{T})$ is a Banach space.

We fix an orthonormal basis $(\overline{h}_k)_{k\geqslant 1}$ in H and consider the bilinear operator $\beta : \mathscr{B}(H) \times l(\mathscr{T}) \to L^2(\Omega; E)$ defined by

$$\beta(S,T) := \sum_{k\geqslant 1} \gamma_k T_k S \overline{h}_k.$$

Note that this sum converges in $L^2(\Omega; E)$ thanks to Propositions 2.2, 2.4, and 2.8. By the closed graph theorem there is a constant $c \geqslant 0$ such that for all $S \in \mathscr{B}(H)$ and $T \in l(\mathscr{T})$,

$$\mathbb{E}\Big\|\sum_{k\geqslant 1} \gamma_k T_k S \overline{h}_k\Big\|^2 \leqslant c^2 \|S\|^2 \|T\|_{l(\mathscr{T})}^2.$$

For sequences $(T_k)_{k\geqslant 1}$ in $\mathscr{T}$ we have $\inf\big\{\frac{1}{c} : \ c > 0, \ cT \in W\big\} \leqslant 1$ and consequently $\|T\|_{l(\mathscr{T})} \leqslant 1 + \sup_{k\geqslant 1} \|T_k\|_{\gamma(H,E)} \leqslant 1 + M$, where $M := \sup_{T\in\mathscr{T}} \|T\|_{\gamma(H,E)}$. Hence,

$$\mathbb{E}\Big\|\sum_{k\geqslant 1} \gamma_k T_k S \overline{h}_k\Big\|^2 \leqslant c^2 (1+M)^2 \|S\|^2.$$

Finally, if $(h_k)_{k\geqslant 1}$ is an arbitrary orthonormal basis of H we let U be the unitary operator defined by $U\overline{h}_k = h_k$ and obtain, for all sequences $(T_k)_{k\geqslant 1}$ in $\mathscr{T}$,

$$\mathbb{E}\Big\|\sum_{k\geqslant 1}\gamma_k T_k h_k\Big\|^2 = \mathbb{E}\Big\|\sum_{k\geqslant 1}\gamma_k T_k U\overline{h}_k\Big\|^2 \leqslant c^2(1+M)^2\|U\|^2 = c^2(1+M)^2.$$

□

For a uniformly γ–radonifying family $\mathscr{T}$ in $\gamma(H,E)$ we define

$$\|\mathscr{T}\|^2_{\text{unif-}\gamma} := \sup_h \sup_T \ \mathbb{E}\Big\|\sum_{k\geqslant 1}\gamma_k T_k h_k\Big\|^2,$$

where the fist supremum is taken over all orthonormal bases of H and the second over all sequences in $\mathscr{T}$. Inspection of the proofs of Propositions 2.2, 2.4, and 2.8 shows that we have

$$\|\overline{\mathscr{T}}\|_{\text{unif-}\gamma} = \|\mathscr{T}\|_{\text{unif-}\gamma},$$

where $\overline{\mathscr{T}}$ is the strong closure of $\mathscr{T}$,

$$\begin{aligned}\|\text{abs conv}(\mathscr{T})\|_{\text{unif-}\gamma} &= \|\mathscr{T}\|_{\text{unif-}\gamma} \quad \text{(real scalars)},\\ \|\text{abs conv}(\mathscr{T})\|_{\text{unif-}\gamma} &\leqslant 2\|\mathscr{T}\|_{\text{unif-}\gamma} \quad \text{(complex scalars)},\end{aligned} \tag{2.1}$$

and

$$\|R\mathscr{T}S\|_{\text{unif-}\gamma} \leqslant \|R\|\,\|\mathscr{T}\|_{\text{unif-}\gamma}\,\|S\|. \tag{2.2}$$

Using (2.1) we obtain analogous bounds for the sets discussed in the Propositions 2.5, 2.6, and 2.7.

We proceed with some applications of Theorem 2.10. The first two results clarify the relation between uniform γ–radonification and γ–boundedness.

Corollary 2.11. *If $\mathscr{T}$ is uniformly γ–radonifying, then:*

(a) *$\mathscr{T}$ is R–bounded with $R(\mathscr{T}) \leqslant \sqrt{\tfrac12\pi}\|\mathscr{T}\|_{\text{unif-}\gamma}$.*
(b) *$\mathscr{T}$ is γ–bounded with $\gamma(\mathscr{T}) \leqslant \|\mathscr{T}\|_{\text{unif-}\gamma}$;*

Proof. We shall prove part (a). Since every R–bounded set is γ–bounded with the same boundedness constant, the γ-boundedness assertion in (b) follows directly from (a), but this argument produces an additional constant $\sqrt{\frac12\pi}$. The sharper constant 1 is obtained by noting that for the proof of (b), Rademacher variables can be replaced by Gaussians and the first inequality in (2.3) can be omitted and we may replace the role of Rademachers by Gaussians in the last step of the argument.

Fix $T_1,\dots,T_n \in \mathscr{T}$ and vectors $g_1,\dots,g_n \in H$. Let $(h_k)_{k\geqslant 1}$ be an orthonormal basis of H and define $S \in \mathscr{B}(H)$ by $Sh_k = g_k$ for $k = 1,\dots,n$ and $Sh_k = 0$ for $k \geqslant n+1$. If $(r_k)_{k\geqslant 1}$ is a Rademacher sequence, then

$$\begin{aligned}\|Sh\| &= \Big\|\sum_{k=1}^n (h,h_k)_H g_k\Big\|_H \leqslant \sum_{k=1}^n \big|(h,h_k)_H\big|\,\|g_k\|_H\\ &\leqslant \|h\|_H\Big(\sum_{k=1}^n \|g_k\|_H^2\Big)^{\frac12} = \|h\|_H\Big(\mathbb{E}\Big\|\sum_{k=1}^n r_k g_k\Big\|^2\Big)^{\frac12}.\end{aligned}$$

Hence, estimating Rademachers with Gaussians and using (2.2),

$$\begin{aligned}\mathbb{E}\Big\|\sum_{k=1}^n r_k T_k g_k\Big\|^2 &\leqslant \tfrac12\pi\,\mathbb{E}\Big\|\sum_{k=1}^n \gamma_k T_k S h_k\Big\|^2\\ &\leqslant \tfrac12\pi\|\mathscr{T}\|^2_{\text{unif-}\gamma}\|S\|^2 \leqslant \tfrac12\pi\|\mathscr{T}\|^2_{\text{unif-}\gamma}\mathbb{E}\Big\|\sum_{k=1}^n r_k g_k\Big\|^2.\end{aligned} \tag{2.3}$$

□

The next example shows that even for Hilbert spaces E, a γ–bounded family of operators in $\gamma(H,E)$ need not be uniformly γ–radonifying.

Example 2.12. Let $(h_k)_{k\geqslant 1}$ be an orthonormal basis for an infinite-dimensional Hilbert space H and let P_n be the orthogonal projection onto the span of the vector h_n. The family $\{P_n : n \geqslant 1\}$ is uniformly bounded, hence γ–bounded, in $\mathscr{B}(H)$ and fails to be uniformly γ–radonifying, as is immediate by considering the sum $\sum_{k\geqslant 1} \gamma_k P_k h_k$.

The next corollary identifies γ–bounded sets as the class of 'multipliers' for uniformly γ–radonifying sets:

Corollary 2.13. *For a subset $\mathscr{S}$ of $\mathscr{B}(E,F)$ the following assertions are equivalent:*

(a) *$\mathscr{S}$ is γ–bounded;*
(b) *$\mathscr{S}T$ is a uniformly γ–radonifying subset of $\mathscr{B}(H,F)$ for every $T \in \gamma(H,E)$;*
(c) *$\mathscr{S}\mathscr{T}$ is a uniformly γ–radonifying subset of $\mathscr{B}(H,F)$ for every uniformly γ–radonifying subset $\mathscr{T}$ of $\mathscr{B}(H,E)$.*

In the situation of (c) *we have* $\|\mathscr{S}\mathscr{T}\|_{\text{unif-}\gamma} \leqslant \gamma(\mathscr{S})\|\mathscr{T}\|_{\text{unif-}\gamma}$.

Proof. The implication (a)⇒(c) and the estimate are immediate consequences of the definitions, and the implication (c)⇒(b) is trivial. To prove (b)⇒(a) we fix an orthonormal basis $(h_k)_{k\geqslant 1}$ in H and denote by $\mathscr{S}^\infty = \mathscr{S} \times \mathscr{S} \times \dots$ the set of all sequences in $\mathscr{S}$. By Theorem 2.10, for each $T \in \gamma(H,E)$ we have

$$\sup_{S\in\mathscr{S}^\infty} \mathbb{E}\Big\| \sum_{k\geqslant 1} \gamma_k S_k T h_k \Big\|^2 < \infty,$$

This induces a well-defined linear operator

$$U : \gamma(H,E) \to l^\infty(\mathscr{S}^\infty; L^2(\Omega; E)),$$

which is bounded by the closed graph theorem. This means that for some constant $C \geqslant 0$ we have

$$\sup_{S\in\mathscr{S}^\infty} \mathbb{E}\Big\| \sum_{k\geqslant 1} \gamma_k S_k T h_k \Big\|^2 \leqslant C^2 \|T\|^2_{\gamma(H,E)}.$$

Now fix arbitrary $S_1, \dots, S_n \in \mathscr{S}$ and $x_1, \dots, x_n \in E$, and define $T \in \gamma(H,E)$ by $Th_k = x_k$ for $k = 1, \dots, n$ and $Th_k = 0$ for $k \geqslant n+1$. Choosing $S_k \in \mathscr{S}$ for $k \geqslant n+1$ arbitrary, we obtain

$$\mathbb{E}\Big\| \sum_{k=1}^n \gamma_k S_k x_k \Big\|^2 \leqslant \mathbb{E}\Big\| \sum_{k\geqslant 1} \gamma_k S_k T h_k \Big\|^2 \leqslant C^2 \|T\|^2_{\gamma(H,E)} = C^2 \mathbb{E}\Big\| \sum_{k=1}^n \gamma_k x_k \Big\|^2.$$

□

As an immediate consequence of the previous two results we note:

Corollary 2.14. *If $\mathscr{T}$ is uniformly γ–radonifying in $\mathscr{B}(\tilde{H}, H)$ and $\mathscr{S}$ is uniformly γ–radonifying in $\mathscr{B}(H,E)$, then $\mathscr{S}\mathscr{T}$ is uniformly γ–radonifying in $\mathscr{B}(\tilde{H}, E)$ and*

$$\|\mathscr{S}\mathscr{T}\|_{\text{unif-}\gamma} \leqslant \|\mathscr{S}\|_{\text{unif-}\gamma} \|\mathscr{T}\|_{\text{unif-}\gamma}.$$

If E does not contain a closed subspace isomorphic to c_0, then by a result of Hoffmann-Jørgensen and Kwapień [16, 27], a Gaussian sum converges in $L^2(\Omega; E)$ if and only if its partial sums are bounded in $L^2(\Omega; E)$. In combination with Theorem 2.10 we obtain the following equivalent condition for uniform γ–radonification:

Corollary 2.15. *Let E be a Banach space not containing a copy of c_0. Then a subset $\mathscr{T}$ of $\gamma(H,E)$ is uniformly γ–radonifying if and only if there exists a constant $C \geqslant 0$ such that for all integers $n \geqslant 1$, all orthonormal $h_1,\dots,h_n \in H$, and all $T_1,\dots,T_n \in \mathscr{T}$,*

$$\mathbb{E}\Big\|\sum_{k=1}^{n}\gamma_k T_k h_k\Big\|^2 \leqslant C^2.$$

In this situation, $\|\mathscr{T}\|_{\text{unif-}\gamma} \leqslant C$.

Here is a simple application.

Corollary 2.16 (Fatou lemma)**.** *Let E be a Banach space not containing a copy of c_0. Let $(\mathscr{T}_n)_{n\geqslant 1}$ be an increasing sequence of uniformly γ–radonifying sets in $\gamma(H,E)$ satisfying $\sup_{n\geqslant 1}\|\mathscr{T}_n\|_{\text{unif-}\gamma} < \infty$. Then $\mathscr{T} := \bigcup_{n\geqslant 1}\mathscr{T}_n$ is uniformly γ–radonifying and*

$$\|\mathscr{T}\|_{\text{unif-}\gamma} \leqslant \sup_{n\geqslant 1}\|\mathscr{T}_n\|_{\text{unif-}\gamma}.$$

Proof. Let $(T_k)_{k\geqslant 1}$ be a sequence in $\mathscr{T}$. For each $m \geqslant 1$ choose $N_m \geqslant 1$ such that $T_1,\dots,T_m \in \mathscr{T}_{N_m}$. For all orthonormal $h_1,\dots,h_m \in H$ we have

$$\mathbb{E}\Big\|\sum_{k=1}^{m}\gamma_k T_k h_k\Big\|^2 \leqslant \|\mathscr{T}_{N_m}\|_{\text{unif-}\gamma} \leqslant C,$$

where $C := \sup_{n\geqslant 1}\|\mathscr{T}_n\|_{\text{unif-}\gamma}$. □

The condition $c_0 \not\subseteq E$ cannot be omitted:

Example 2.17. Let $(e_k)_{k\geqslant 1}$ and $(u_k)_{k\geqslant 1}$ denote the standard unit bases of ℓ^2 and c_0, respectively. It is a classical example of Linde and Pietsch [30] that the operator $T \in \mathscr{B}(\ell^2, c_0)$ defined by $Te_k = (\ln(k+1))^{-\frac{1}{2}}u_k$ fails to be γ–radonifying but satisfies

$$\sup_{n\geqslant 1}\,\mathbb{E}\Big\|\sum_{k=1}^{n}\gamma_k Te_k\Big\|^2 < \infty.$$

Let P_k be the rank one projection $e_k\otimes e_k$ in ℓ^2. Then the sets $\mathscr{T}_n := \{TP_1,\dots,TP_n\}$ satisfy the assumptions of Corollary 2.16, but their union fails to be uniformly γ–radonifying.

3. Uniformly γ–radonifying families and compactness in $\gamma(H,E)$

For an operator $T \in \gamma(H,E)$ we define μ_T as the distribution of the random variable $\sum_{k\geqslant 1}\gamma_k Th_k$, where $(h_k)_{k\geqslant 1}$ is an arbitrary orthonormal basis of H. The measure μ_T is a centred Gaussian Radon measure on E which does not depend on the choice of the basis $(h_k)_{k\geqslant 1}$ and whose covariance operator equals TT^*. For more information on Gaussian measures we refer to [3, Chapter 3], whose terminology we follow.

The first result of this section gives a necessary and sufficient condition for relative compactness in the space $\gamma(H,E)$. In a rephrasing in terms of sequential convergence in $\gamma(H,E)$, this result is due to Neidhardt [39]. For reasons of self-containedness we shall give a different proof based on a characterisation of compactness in Lebesgue-Bochner spaces.

Theorem 3.1. *For a subset $\mathscr{T}$ of $\gamma(H,E)$ the following assertions are equivalent:*

(a) *The set $\mathscr{T}$ is relatively compact in $\gamma(H,E)$;*
(b) *The set $\{\mu_T : T \in \mathscr{T}\}$ is uniformly tight, and for all $x^* \in E^*$ the set $\{T^*x^* : T \in \mathscr{T}\}$ is relatively compact in H.*

Proof. Let $(h_k)_{k\geqslant 1}$ be a fixed orthonormal basis of H and define, for $T \in \mathscr{T}$, the random variable $X_T \in L^2(\Omega; E)$ as $X_T := \sum_{k\geqslant 1} \gamma_k T h_k$. Since $T \mapsto X_T$ defines an isometry from $\gamma(H, E)$ onto a closed subspace of $L^2(\Omega; E)$, $\mathscr{T}$ is relatively (weakly) compact in $\gamma(H, E)$ if and only if $\{X_T : T \in \mathscr{T}\}$ is relatively (weakly) compact in $L^2(\Omega; E)$. With this in mind, the proof of the theorem will be based on the following compactness result of Diaz and Mayoral [9]: for $1 \leqslant p < \infty$, a subset A of $L^p(\Omega; E)$ is relatively weakly compact if and only if the following three conditions are satisfied:

(i) The set A is uniformly p-integrable;
(ii) The set of distributions $\{\mu_f : f \in A\}$ is uniformly tight;
(iii) The set $\{\langle f, x^*\rangle : f \in A\}$ is relatively weakly compact in $L^p(\Omega)$ for all $x^* \in E^*$.

An elementary proof of this result, valid for arbitrary Banach function spaces over $(\Omega, \mathbb{P})$ with order continuous norm, may be found in [32].

(a)⇒(b): The uniform tightness of the set $\{\mu_T : T \in \mathscr{T}\}$ follows from the Diaz-Mayoral result. For every $x^* \in E^*$ the set $\{T^*x^* : T \in \mathscr{T}\}$ is relatively compact in H, since it is the image of the relatively compact set $\mathscr{T}$ under the continuous mapping from $\gamma(H, E)$ into H given by $T \mapsto T^*x^*$.

(b)⇒(a): The relative compactness of $\{T^*x^* : T \in \mathscr{T}\}$ in H implies the relative compactness in $L^2(\Omega)$ of the random variables $\{\langle X_T, x^*\rangle : T \in \mathscr{T}\}$. To see this, just note that

$$\|\langle X_{T_1}, x^*\rangle - \langle X_{T_2}, x^*\rangle\|^2_{L^2(\Omega)} = \|\langle X_{T_1 - T_2}, x^*\rangle\|^2_{L^2(\Omega)} = \|T_1^* x^* - T_2^* x^*\|^2_H.$$

By [3, Theorem 3.8.11], uniformly tight families of centred Gaussian E-valued random variables are uniformly square integrable and therefore (b) implies (a) by another application of the compactness result of Diaz and Mayoral. □

Corollary 3.2. *Let $\mathscr{T}$ be a subset in $\gamma(H, E)$ which is* dominated *by some fixed element $S \in \gamma(H, E)$, in the sense that for all $x^* \in E^*$,*

$$0 \leqslant \|T^*x^*\|_H \leqslant \|S^*x^*\|_H.$$

Then the following assertions are equivalent:

(a) *$\mathscr{T}$ is relatively compact in $\gamma(H, E)$;*
(b) *For all $x^* \in E^*$ the set $\{T^*x^* : T \in \mathscr{T}\}$ is relatively compact in H.*

Proof. A standard domination result for Gaussian measures (see [33, Theorem 8.8]) implies that if $\mathscr{T}$ is dominated, then the family $\{\mu_T : T \in \mathscr{T}\}$ is uniformly tight. The result now follows from Theorem 3.1. □

The second main result of this section is the following characterisation of relative compactness in $\gamma(H, E)$ of uniformly γ–radonifying families.

Theorem 3.3. *Let $\mathscr{T}$ be uniformly γ–radonifying subset of $\gamma(H, E)$. The following assertions hold:*

(a) *$\mathscr{T}$ is relatively compact in $\gamma(H, E)$ if and only if $\mathscr{T}h$ is relatively compact in E for all $h \in H$;*
(b) *$\mathscr{T}$ is relatively weakly compact in $\gamma(H, E)$ if and only if $\mathscr{T}h$ is relatively weakly compact in E for all $h \in H$.*

Proof. The relative (weak) compactness of $\mathscr{T}$ in $\gamma(H, E)$ clearly implies the relative (weak) compactness of $\mathscr{T}h$ in E for all $h \in H$, so we only need to prove the converse statements. Throughout the proof we fix an orthonormal basis $(h_k)_{k\geqslant 1}$ of H.

The proof of (a) is divided into two steps.

Step 1 – Fix $n \geqslant 1$ and let P_n be the orthogonal projection in H onto the span H_n of $h_1, \dots, h_n$. The set $\mathscr{T}_n = \{TP_n : T \in \mathscr{T}\}$ is relatively compact in $\gamma(H,E)$ by Corollary 3.2.

Step 2 – Assume that $\mathscr{T}$ is not relatively compact; we shall prove that $\mathscr{T}$ is not uniformly γ–radonifying.

Since $\mathscr{T}$ is not totally bounded, we can find an $\varepsilon > 0$ such that $\mathscr{T}$ cannot be covered with finitely many 3ε-balls. We shall construct an increasing sequence of positive integers $0 = N_0 < N_1 < \dots$ and a sequence $T_1, T_2, \dots$ of elements of $\mathscr{T}$ such that for all $m \geqslant 1$ we have

$$\mathbb{E}\Big\| \sum_{m=N_{k-1}+1}^{N_k} \gamma_k T_m h_k \Big\|^2 \geqslant 4\varepsilon^2.$$

The 3ε-ball with centre 0 does not cover $\mathscr{T}$, and therefore we may pick $T_1 \in \mathscr{T}$ such that $\|T_1\|_{\gamma(H,E)} \geqslant 3\varepsilon$. Choose the index $N_1 \geqslant 1$ in such a way that

$$\mathbb{E}\Big\| \sum_{k=1}^{N_1} \gamma_k T_1 h_k \Big\|^2 \geqslant \varepsilon^2.$$

We claim that for some $T_2 \in \mathscr{T}$ we have

$$\mathbb{E}\Big\| \sum_{k=N_1+1}^{\infty} \gamma_k T_2 h_k \Big\|^2 \geqslant 4\varepsilon^2.$$

Suppose this claim was false. Denoting by $Q_{N_1} = I - P_{N_1}$ the orthogonal projection onto $H_{N_1}^{\perp}$, this would mean that $\|TQ_{N_1}\|_{\gamma(H,E)} < 2\varepsilon$ for all $T \in \mathscr{T}$. Then for all $T \in \mathscr{T}$ we have

$$T = TP_{N_1} + TQ_{N_1} \in \mathscr{T}_{N_1} + \mathscr{B}(2\varepsilon),$$

where $\mathscr{B}(2\varepsilon)$ is the 2ε-ball in $\gamma(H,E)$ centred at 0. By Step 1 we can cover $\mathscr{T}_{N_1}$ with finitely many ε-balls, and therefore we can cover $\mathscr{T}$ with finitely many 3ε-balls. This contradiction proves the claim. Now choose the index $N_2 \geqslant N_1 + 1$ in such a way that

$$\mathbb{E}\Big\| \sum_{k=N_1+1}^{N_2} \gamma_k T_2 h_k \Big\|^2 \geqslant \varepsilon^2.$$

It is clear that this construction can be continued inductively.

Let $S_k := T_m$ if $N_{m-1} + 1 \leqslant k \leqslant N_m$ for some $m \geqslant 1$. Then $(S_k)_{k\geqslant 1}$ is a sequence in $\mathscr{T}$ for which the sum $\sum_k \gamma_k S_k h_k$ fails to converge.

Next we prove (b). We say that the sequence $(y_n)_{n\geqslant 1}$ is a *convex tail subsequence* of a sequence $(x_n)_{n\geqslant 1}$ in E if each y_n is a convex combination of elements of the tail sequence $(x_k)_{k\geqslant n}$. Note that if $\lim_{n\to\infty} x_n = x$ strongly or weakly, then also $\lim_{n\to\infty} y_n = x$ strongly or weakly. We shall use of the following weak compactness criterium [11, Corollary 2.2]: a subset K of a Banach space X is relatively weakly compact if and only if every sequence in K has a strongly convergent convex tail subsequence.

After these preparations we turn to the proof of (b). Let $(T_k)_{k\geqslant 1}$ be a sequence in $\mathscr{T}$. By a diagonal argument we find a subsequence $(T_{k_j})_{j\geqslant 1}$ such that the weak limit $\lim_{j\to\infty} T_{k_j} h_n$ exists for every $n \geqslant 1$. By a standard corollary to the Hahn-Banach theorem and a diagonal argument we find a convex tail subsequence $(S_j)_{j\geqslant 1}$ of $(T_{k_j})_{j\geqslant 1}$ such that the strong limit $\lim_{j\to\infty} S_j h_n$ exists for every $n \geqslant 1$. By the uniform boundedness of $\mathscr{T}$, the strong limit $Sh := \lim_{j\to\infty} S_j h$ exists for all $h \in H$. Now part (a) implies that $\lim_{j\to\infty} S_{k_j} = S$ in $\gamma(H,E)$. Hence, by the above criterium, $\mathscr{T}$ is relatively weakly compact. □

The following example shows that uniformly γ–radonifying families in $\gamma(H,E)$ need not be relatively compact in $\gamma(H,E)$, even in the case where E is a Hilbert space.

Example 3.4. Let $(e_k)_{k\geqslant 1}$ be the standard unit basis of ℓ^2 and fix an arbitrary nonzero element $h\in\ell^2$. We check that the family

$$\mathscr{T} := \{h\otimes e_j :\ j\geqslant 1\}$$

is a uniformly γ–radonifying subset of $\gamma(\ell^2) := \gamma(\ell^2,\ell^2)$. Taking this for granted for the moment, noting that $\{Th :\ T\in\mathscr{T}\}$ fails to be relatively compact in ℓ^2 it follows from Theorem 3.3 that $\mathscr{T}$ fails to be relatively compact in $\gamma(\ell^2)$.

If $(T_k)_{k\geqslant 1}$ is a sequence in $\mathscr{T}$, say $T_k = h\otimes e_{j_k}$, then for all $1\leqslant M\leqslant N$ we have

$$\mathbb{E}\Big\|\sum_{k=M}^{N}\gamma_k T_k e_k\Big\|^2 = \mathbb{E}\Big\|\sum_{k=M}^{N}\gamma_k (h,e_k)e_{j_k}\Big\|^2 = \sum_{k=M}^{N}\|(h,e_k)e_{j_k}\|^2 = \sum_{k=M}^{N}|(h,e_k)|^2.$$

As $M,N\to\infty$ the right-hand side tends to 0, which proves that $\sum_{k\geqslant 1}\gamma_k T_k e_k$ converges in $L^2(\Omega;\ell^2)$.

Our next aim is to show that for every Banach space E there exists a relatively compact $\mathscr{T}$ set in $\gamma(\ell^2,E)$ which fails to be uniformly γ–radonifying.

Example 3.5. Let $(h_k)_{k\geqslant 1}$ denote the standard unit basis of ℓ^2. Define $S\in\mathscr{B}(\ell^2)$ to be the right shift, i.e. $Sh_k = h_{k+1}$ for all $k\geqslant 1$. For $T\in\gamma(\ell^2,E)$ let $\mathscr{S}_T := \{TS^n :\ n\in\mathbb{N}\}$. This set is bounded in $\gamma(\ell^2,E)$ and for all $n\in\mathbb{N}$ we have

$$\|S^{n*}T^*x^*\|_{\ell^2}^2 \leqslant \|T^*x^*\|_{\ell^2}^2.$$

Also, for all $x^*\in E^*$ we have $\lim_{n\to\infty} S^{n*}T^*x^* = 0$ strongly, and therefore $\mathscr{S}_T$ is relatively compact by Corollary 3.2.

In what follows we take for E the scalar field $\mathbb{K}$. Define $M_1 = 1$ and, inductively, $M_{n+1} := M_n + n$ for $n\geqslant 1$. Consider the operators $T_n : \ell^2\to\mathbb{K}$ defined by $T_n h_{M_{n+1}} = 1$ and $T_n h_k = 0$ for $k\neq M_{n+1}$. Trivially, this operator is γ–radonifying with $\|T_n\|_{\gamma(\ell^2,\mathbb{K})} = 1$. Let $m_k^n := n-j$ for $k = M_n + j$, $j = 1,\dots,n$. Then,

$$\Big(\mathbb{E}\Big\|\sum_{k\geqslant 1}\gamma_k T_n S^{m_k^n} h_k\Big\|^2\Big)^{\frac12} = \Big(\mathbb{E}\Big\|\sum_{k=M_n+1}^{M_{n+1}}\gamma_k T_n h_{M_{n+1}}\Big\|^2\Big)^{\frac12} = (M_{n+1}-M_n)^{\frac12} = n^{\frac12}$$

and similarly,

$$\Big(\mathbb{E}\Big\|\sum_{k\geqslant 1}\gamma_k T_{n'} S^{m_k^n} h_k\Big\|^2\Big)^{\frac12} = 0, \qquad n\neq n'.$$

Define $T:\ell^2\to\mathbb{K}$ by $T := \sum_{n\geqslant 1}\frac{1}{n^2}T_{2^n}$. Note that $T\in\gamma(\ell^2,\mathbb{K})$. By the contraction principle, for all $n\geqslant 1$ we have

$$\begin{aligned}\Big(\mathbb{E}\Big\|\sum_{k=1}^{M_{2^n+1}}\gamma_k T S^{m_k^{2^n}} h_k\Big\|^2\Big)^{\frac12} &\geqslant \Big(\mathbb{E}\Big\|\sum_{k=M_{2^n}+1}^{M_{2^n+1}}\gamma_k T S^{m_k^{2^n}} h_k\Big\|^2\Big)^{\frac12} \\ &= \frac{1}{n^2}\Big(\mathbb{E}\Big\|\sum_{k=M_{2^n}+1}^{M_{2^n+1}}\gamma_k T_{2^n} S^{m_k^{2^n}} h_k\Big\|^2\Big)^{\frac12} = \frac{1}{n^2}\cdot 2^{\frac{n}{2}}.\end{aligned}$$

Since n is arbitrary, this implies that the family $\mathscr{S}_T = \{TS^m :\ m\in\mathbb{N}\}$ fails to be uniformly γ–radonifying. Note that this family is bounded, hence γ-bounded, in $\mathscr{L}(\ell^2,\mathbb{K})$.

4. Laplace transforms

Let I be a countable index set. A sequence $(h_i)_{i\in I}$ in a Hilbert space H is said to be a *Hilbert sequence* if there exists a constant $C>0$ such that for all scalar sequences $\alpha\in\ell^2(I)$,

$$\Big\|\sum_{i\in I}\alpha_i h_i\Big\|_H^2 \leqslant C^2\sum_{i\in I}|\alpha_i|^2.$$

The infimum of all admissible constants C will be called the *Hilbert constant* of the sequence $(h_i)_{i\in I}$, cf. [51, Section 1.8]. The usefulness of this notion is explained by the following result [15, Proposition 2.1]:

Proposition 4.1. *Let $T\in\gamma(H,E)$ be given. If $(h_i)_{i\in I}$ is a Hilbert sequence in H, then the Gaussian sum $\sum_{i\in I}\gamma_i Th_i$ converges in $L^2(\Omega;E)$ and we have*

$$\mathbb{E}\Big\|\sum_{i\in I}\gamma_i Th_i\Big\|^2 \leqslant C^2\|T\|^2_{\gamma(H,E)},$$

where C is the Hilbert constant of $(h_i)_{i\in I}$.

Example 4.2. Let $(\lambda_n)_{n\geqslant 1}$ be a sequence in $\mathbb{C}_+$ which is *properly spaced* in the sense that

$$\inf_{m\neq n}\Big|\frac{\lambda_m-\lambda_n}{\mathrm{Re}\,(\lambda_n)}\Big|>0.$$

Then the functions

$$f_n(t):=\sqrt{\mathrm{Re}\,(\lambda_n)}e^{-\lambda_n t},\quad n\geqslant 1,$$

define a Riesz sequence on the closure of their span in $L^2(\mathbb{R}_+)$, i.e., there are constants $0<c\leqslant C<\infty$ such that

$$c^2\sum_{n\geqslant 1}|\alpha_n|^2\leqslant\Big\|\sum_{n\geqslant 1}\alpha_n f_n\Big\|^2\leqslant C^2\sum_{n\geqslant 1}|\alpha_n|^2$$

for all sequences $(\alpha_n)_{n\geqslant 1}\in\ell^2$; see [20, Theorem 1, (3)$\Leftrightarrow$(5)]. In particular, $(f_n)_{n\geqslant 1}$ is a Hilbert sequence in $L^2(\mathbb{R}_+)$. From this one easily deduces that for any $b>0$ and $\rho\in[0,1)$ the functions

$$f_n(t)=e^{-bt+2\pi i(n+\rho)t},\quad n\in\mathbb{Z},$$

define a Hilbert sequence in $L^2(\mathbb{R}_+)$. This has been shown by direct computation in [15, Example 2.5], where the bound $1/\sqrt{1-e^{-2b}}$ was obtained for its Hilbert constant. Note that this bound is independent of ρ.

The next proposition is well-known and shows that a sequence is a Hilbert sequence if it is not 'too far' from being orthogonal. For the reader's convenience we include an elementary proof.

Proposition 4.3. *Let $(h_n)_{n\in\mathbb{Z}}$ be a sequence in H. If there exists a function $\phi:\mathbb{N}\to\mathbb{R}_+$ such that for all $n\geqslant m\in\mathbb{Z}$ we have $\big|(h_n,h_m)_H\big|\leqslant\phi(n-m)$ and $\sum_{j\in\mathbb{N}}\phi(j)<\infty$, then $(h_n)_{n\in\mathbb{Z}}$ is Hilbert sequence.*

Proof. Let $(\alpha_n)_{n\in\mathbb{Z}}$ be scalars. Then

$$\begin{aligned}\Big\|\sum_{n=-N}^{N}\alpha_n h_n\Big\|^2 &= \sum_{n=-N}^{N}|\alpha_n|^2\|h_n\|^2+2\mathrm{Re}\sum_{-N\leqslant n<m\leqslant N}\alpha_n\overline{\alpha_m}(h_n,h_m)_H\\ &\leqslant \phi(0)\sum_{n\in\mathbb{Z}}|\alpha_n|^2+2\sum_{n<m}|\alpha_n||\alpha_m|\phi(n-m)\\ &= \phi(0)\sum_{n\in\mathbb{Z}}|\alpha_n|^2+2\sum_{j\geqslant 1}\phi(j)\sum_{n\in\mathbb{Z}}|\alpha_n|\,|\alpha_{n+j}|\end{aligned}$$

$$\leqslant \quad \Big(\phi(0) + 2\sum_{j\geqslant 1}\phi(j)\Big)\sum_{n\in\mathbb{Z}}|\alpha_n|^2,$$

where the last estimate follows from the Cauchy-Schwarz inequality. □

As a special case we have the following example, which will be needed in the proof of Theorem 4.8.

Example 4.4. Let $\alpha \in (0, \frac{1}{2}]$, $r > 0$, and $\vartheta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Let $\mu_n = r2^n e^{i\vartheta}$, $n \in \mathbb{Z}$, and let

$$f_n(s) := \mu_n^\alpha e^{-\mu_n s}, \quad s \in \mathbb{R}_+. \tag{4.1}$$

Then $(f_n)_{n\in\mathbb{Z}}$ is a Hilbert sequence in $L^2(\mathbb{R}_+)$. Indeed, for $n, m \in \mathbb{Z}$, $n \neq m$, we have

$$|(f_n, f_m)| \leqslant \int_0^\infty r^{2\alpha}2^{\alpha(n+m)}e^{-r(2^n+2^m)s\cos\vartheta}\,ds = \frac{r^{2\alpha-1}2^{\alpha(n+m)}}{(2^n+2^m)\cos\vartheta}.$$

Since $\alpha \in (0, \frac{1}{2}]$ we have

$$\frac{r^{2\alpha-1}2^{\alpha(n+m)}}{(2^n+2^m)\cos\vartheta} \leqslant \frac{r^{2\alpha-1}}{\cos\vartheta}\frac{2^{\alpha(n+m)}}{2^{\max(n,m)}} \leqslant \frac{r^{2\alpha-1}}{\cos\vartheta}2^{-\alpha|n-m|},$$

and Proposition 4.3 applies. Notice that for $\alpha = \frac{1}{2}$, the obtained Hilbert constant estimate is bounded by $C/\cos\vartheta$, where C is a universal constant.

From now on, H is again a separable infinite-dimensional Hilbert space. The main abstract result of this section reads as follows.

For an operator $\Phi \in \gamma(L^2(\mathbb{R}_+; H), E)$ and a function $f \in L^2(\mathbb{R}_+)$ we define the operator $f(\Phi) \in \gamma(H, E)$ by

$$f(\Phi)h := \Phi(f \otimes h), \quad h \in H.$$

Below we shall apply this definition to the functions $f(t) = e^{-\lambda t}$ with $\operatorname{Re}\lambda > 0$ to in order to define the 'Laplace transform' of Φ.

Theorem 4.5. *Let $(f_i)_{i\in I}$ be a Hilbert sequence in $L^2(\mathbb{R}_+)$ with Hilbert constant C. Then for all $\Phi \in \gamma(L^2(\mathbb{R}_+; H), E)$ the family*

$$\mathscr{T} := \{f_i(\Phi) : \ i \in I\}$$

is uniformy γ–radonifying and we have

$$\|\mathscr{T}\|_{\text{unif-}\gamma} \leqslant C\|\Phi\|_{\gamma(L^2(\mathbb{R}_+;H),E)}.$$

Proof. Fix an orthonormal basis $(h_k)_{k\geqslant 1}$ in H and let $(i_k)_{k\geqslant 1}$ be an arbitrary sequence in I. Put $J := \{i \in I : \ i_k = i \text{ for some } k \in K\}$. For each $i \in J$, put $K(i) := \{k \geqslant 1 : \ i_k = i\}$. Fix a Gaussian sequence $(\gamma_k)_{k\geqslant 1}$ on a probability space $(\Omega, \mathbb{P})$, as well as a doubly indexed Gaussian sequence $(\gamma'_{ik})_{i\in I,\, k\geqslant 1}$ on another probability space $(\Omega', \mathbb{P}')$. We have

$$\begin{aligned}\mathbb{E}\Big\|\sum_{k\geqslant 1}\gamma_k f_{i_k}(\Phi)h_k\Big\|^2 &= \mathbb{E}'\Big\|\sum_{k\geqslant 1}\gamma'_{i_k k}f_{i_k}(\Phi)h_k\Big\|^2 \\ &= \mathbb{E}'\Big\|\sum_{i\in J}\sum_{k\in K(i)}\gamma'_{ik}f_i(\Phi)h_k\Big\|^2 \leqslant \mathbb{E}'\Big\|\sum_{i\in I}\sum_{k\geqslant 1}\gamma'_{ik}f_i(\Phi)h_k\Big\|^2.\end{aligned}$$

To prove convergence of the double sum on the right-hand side we note that that the sequence $(f_i \otimes h_k)_{i\in I,\, k\geqslant 1}$ is a Hilbert sequence in $L^2(\mathbb{R}_+; H)$ with Hilbert constant

C. Indeed, this follows from

$$\begin{aligned}\Big\|\sum_{i\in I}\sum_{k\geqslant 1}\alpha_{ik} f_i\otimes h_k\Big\|^2_{L^2(\mathbb{R}_+;H)} &= \int_0^\infty \Big\|\sum_{i\in I}\sum_{k\geqslant 1}\alpha_{ik} f_i(t)h_k\Big\|^2_H\,dt\\ &= \int_0^\infty \sum_{k\geqslant 1}\Big|\sum_{i\in I}\alpha_{ik} f_i(t)\Big|^2\,dt\\ &= \sum_{k\geqslant 1}\Big\|\sum_{i\in I}\alpha_{ik} f_i\Big\|^2_{L^2(\mathbb{R}_+)}\\ &\leqslant C^2\sum_{i\in I}\sum_{k\geqslant 1}|\alpha_{ik}|^2.\end{aligned}$$

Hence by Proposition 4.1,

$$\mathbb{E}\Big\|\sum_{k\geqslant 1}\gamma_k f_{i_k}(\Phi)h_k\Big\|^2 \leqslant C^2\|\Phi\|^2_{\gamma(L^2(\mathbb{R}_+;H),E)}.$$

□

We shall present three applications of this result.

The *Laplace transform* of an operator $\Phi\in\gamma(L^2(\mathbb{R}_+;H),E)$ is the function $\widehat{\Phi}:\mathbb{C}_+\to\gamma(H,E)$ defined by

$$\widehat{\Phi}(\lambda)h := e_\lambda(\Phi)h = \Phi(e_\lambda\otimes h),\quad h\in H,$$

where $\mathbb{C}_+ := \{\mathrm{Re}\,\lambda>0\}$ and $e_\lambda(t):=e^{-\lambda t}$ for $t\in\mathbb{R}_+$ and $\lambda\in\mathbb{C}_+$.

An operator $\Phi\in\gamma(L^2(\mathbb{R}_+;H),E)$ is said to be *H-strongly L^1-representable* if for all $h\in H$ there exists a function $\phi_h\in L^1(\mathbb{R}_+;E)\cap L^2(\mathbb{R}_+;E)$ such that for all $f\in L^2(\mathbb{R}_+)$ we have

$$\Phi(f\otimes h) = \int_0^\infty f(t)\phi_h(t)\,dt.$$

Under this assumption we have

$$\widehat{\Phi}(\lambda)h = \widehat{\phi_h}(\lambda),\quad \mathrm{Re}\,\lambda>0.$$

Theorem 4.6 (γ–Riemann-Lebesgue lemma)**.** *If $\Phi\in\gamma(L^2(\mathbb{R}_+;H),E)$ is H-strongly L^1-representable, then*

$$\lim_{n\to\infty}\|\widehat{\Phi}(\lambda_n)\|_{\gamma(H,E)} = 0$$

for any sequence $(\lambda_n)_{n\geqslant 1}$ in $\mathbb{C}_+$ such that $(e_{\lambda_n})_{n\geqslant 1}$ is a Hilbert sequence.

Proof. Let $(\lambda_n)_{n\geqslant 1}$ be a sequence in $\mathbb{C}_+$ as stated. By Theorem 4.5, the family $\{\widehat{\Phi}(\lambda_n):\ n\geqslant 1\}$ is uniformly γ–radonifying. Moreover, for all $h\in H$ we have

$$\lim_{|\lambda|\to\infty}\widehat{\Phi}(\lambda)h = \lim_{|\lambda|\to\infty}\widehat{\phi_h}(\lambda) = 0$$

by the Riemann-Lebesgue lemma. Consequently, for every $h\in H$ the set $\{\widehat{\Phi}(\lambda)h:\ \lambda\in\mathbb{C}_+\}$ is relatively compact in E. Theorem 3.3 then shows that $\{\widehat{\Phi}(\lambda_n):\ n\geqslant 1\}$ is relatively compact in $\gamma(H,E)$. Therefore, $\lim_{n\to\infty}\widehat{\Phi}(\lambda_n)h=0$ implies $\lim_{n\to\infty}\widehat{\Phi}(\lambda_n)=0$ in $\gamma(H,E)$. □

In particular we obtain that if $\Phi\in\gamma(L^2(\mathbb{R}_+;H),E)$ is H-strongly L^1-representable, then for all $b>0$ we have

$$\lim_{|s|\to\infty}\|\widehat{\Phi}(b+is)\|_{\gamma(H,E)} = 0.$$

In the next two applications of Theorem 4.5 we consider the uniform γ–radonification of Laplace transforms in right half-planes and sectors, respectively.

Let $S = \{\lambda \in \mathbb{C} :\ 0 < \operatorname{Re}\lambda < 1\}$. If $N : \overline{S} \to \mathscr{L}(E,F)$ is strongly continuous and bounded on $\overline{S}$ and harmonic on S, then by the Poisson formula for the strip [50], cf. also [38], we have, for $\lambda = \alpha + i\beta$ with $0 < \alpha < 1$ and $\beta \in \mathbb{R}$,

$$N(\lambda)x = \sum_{j=0,1} \int_{-\infty}^{\infty} P_j(\alpha, \beta - t) N(j + it)x\, dt, \qquad x \in E, \tag{4.2}$$

where

$$P_j(\alpha, s) = \frac{1}{\pi} \frac{e^{\pi s}\sin(\pi\alpha)}{\sin^2(\pi\alpha) + (\cos(\pi\alpha) - (-1)^j e^{\pi s})^2}.$$

Theorem 4.7 (Uniform γ–radonification in half-planes)**.** *Let $\Phi \in \gamma(L^2(\mathbb{R}_+; H), E)$ be given. For all $b > 0$ the family*

$$\mathscr{T}_b^{\Phi} := \big\{\widehat{\Phi}(\lambda) :\ \operatorname{Re}\lambda \geqslant b\big\}$$

is uniformly γ–radonifying in $\gamma(H,E)$ and

$$\|\mathscr{T}_b^{\Phi}\|_{\text{unif-}\gamma} \leqslant \frac{C}{\sqrt{b}} \|\Phi\|_{\gamma(L^2(\mathbb{R}_+;H),E)},$$

where C is a universal constant.

Proof. By Example 4.2 and a substitution (cf. [15, Theorem 3.1]), for $\sigma \in [\frac12 b, \frac32 b]$ and $\rho \in [0,1)$ fixed, the sequence $(g_n)_{n\in\mathbb{Z}}$ given by

$$g_n(t) := e^{-\sigma t + i(n+\rho)bt}, \quad t \in \mathbb{R}_+,$$

is a Hilbert sequence with Hilbert constant $\leqslant \frac{C}{\sqrt{b}}$, where $C = \sqrt{\frac{2\pi e^{2\pi}}{e^{2\pi}-1}}$. Consequently, Theorem 4.5 shows the uniform γ–radonification of the set $\{\widehat{\Phi}(\sigma + i(n{+}\rho)b) :\ n \in \mathbb{Z}\}$ with constant $\leqslant \frac{C}{\sqrt{b}}\|\Phi\|_{\gamma(L^2(\mathbb{R}_+;H),E)}$.

Let $(h_k)_{k\geqslant1}$ be an orthonormal basis of H and let $(\lambda_k)_{k\geqslant1}$ be a sequence on the line $\{\operatorname{Re}\lambda = b\}$, say $\lambda_k = b + i(n_k + \rho_k)b$ with $n_k \in \mathbb{Z}$ and $0 \leqslant \rho_k < 1$.

Fix indices $1 \leqslant M \leqslant N$. Following the argument of [38, Theorem 4.3], the Poisson integral formula (4.2) can be used with $N(\lambda) = \widehat{\Phi}((\frac12 + \lambda)b)$ to estimate

$$\begin{aligned}
&\Big(\mathbb{E}\Big\| \sum_{k=M}^{N} \gamma_k\, \widehat{\Phi}(\lambda_k) h_k\Big\|\Big)^{\frac12} \\
&= \Big\| \sum_{j=0,1}\sum_{k=M}^{N} \gamma_k \int_{-\infty}^{\infty} P_j(\tfrac12, n_k + \rho_k - t)\widehat{\Phi}((\tfrac12 + j)b + itb) h_k\, dt\Big\|_{L^2(\Omega;E)} \\
&\leqslant \sum_{j=0,1}\int_{-\infty}^{\infty} \Big\| \sum_{k=M}^{N} \gamma_k\, P_j(\tfrac12, \rho_k - \tau)\widehat{\Phi}((\tfrac12 + j)b + i(n_k+\tau)b) h_k\Big\|_{L^2(\Omega;E)} d\tau \\
&\leqslant \sum_{j=0,1}\int_{-\infty}^{\infty} \sup_{\rho\in[0,1)} P_j(\tfrac12, \rho - \tau)\, \Big\| \sum_{k=M}^{N} \gamma_k\, \widehat{\Phi}((\tfrac12 + j)b + i(n_k+\tau)b) h_k\Big\|_{L^2(\Omega;E)} d\tau.
\end{aligned}$$

In the last estimate we used the contraction principle. For fixed $\tau \in \mathbb{R}$ we have

$$\lim_{M,N\to\infty} \Big\| \sum_{k=M}^{N} \gamma_k\, \widehat{\Phi}((\tfrac12 + j)b + i(n_k+\tau)b) h_k\Big\|_{L^2(\Omega;E)} = 0$$

since $\big\{\widehat{\Phi}((\frac12+j)b+i(n+\tau)b):\ n\in\mathbb{Z}\big\}$ is uniformly γ–radonifying, and

$$\begin{aligned}\sup_{\rho\in[0,1)} P_j(\tfrac12,\rho-\tau)\Big\|\sum_{k=M}^{N}\gamma_k\,\widehat{\Phi}((\tfrac12+j)b+i(n_k+\tau)b)h_k\Big\|_{L^2(\Omega;E)}\\ \leqslant \frac{C}{\sqrt{b}}\Big(\sup_{\rho\in[0,1)} P_j(\tfrac12,\rho-\tau)\Big)\|\Phi\|_{\gamma(L^2(\mathbb{R}_+;H),E)}.\end{aligned}$$

Since the right-hand side is an integrable function of τ we may apply dominated convergence to conclude that

$$\lim_{M,N\to\infty}\Big(\mathbb{E}\Big\|\sum_{k=M}^{N}\gamma_k\,\widehat{\Phi}(\lambda_k)h_k\Big\|\Big)^{\frac12}=0.$$

This shows that $\{\widehat{\Phi}(\lambda):\ \operatorname{Re}\lambda=b\}$ is uniformly γ–radonifying. Moreover, taking $M=1$ and letting $N\to\infty$ in the above estimates, we obtain the bound

$$\Big(\mathbb{E}\Big\|\sum_{k\geqslant1}\gamma_k\,\widehat{\Phi}(\lambda_k)h_k\Big\|\Big)^{\frac12}\leqslant 2\sup_{j=0,1}\frac{C}{\sqrt{b}}\Big(\int_{-\infty}^{\infty}\sup_{\rho\in[0,1)} P_j(\tfrac12,\rho-\tau)\,d\tau\Big)\|\Phi\|_{\gamma(L^2(\mathbb{R}_+;H),E)}.$$

This proves that $\{\widehat{\Phi}(\lambda):\ \operatorname{Re}\lambda=b\}$ is uniformly γ–radonifying with constant $\leqslant C'/\sqrt{b}\|\Phi\|_{\gamma(L^2(\mathbb{R}_+;H),E)}$, where C' is universal. By Proposition 2.7, $\{\widehat{\Phi}(\lambda):\ \operatorname{Re}\lambda\geqslant b\}$ is then uniformly γ–radonifying with at most twice this constant. ☐

Combining this theorem with Corollary 2.13 we recover [38, Theorem 3.4], which asserts that the Laplace transform of Φ is R–bounded on $\{\operatorname{Re}\lambda\geqslant b\}$ for all $b>0$, with an R–bound of order $O(\frac{1}{\sqrt{b}})$ as $b\downarrow 0$. In view of Example 3.5, Theorem 4.7 represents a genuine strengthening of this result.

Next we turn to the uniform γ–radonification of Laplace transforms in sectors. Before we can state and prove our main result in this direction, Theorem 4.8, we introduce some notations.

For $0<\vartheta<\pi$ and $0<r<R$ we define

$$S_\vartheta:=\{z\in\mathbb{C}:\ z\neq0,\ |\arg z|<\vartheta\},$$

where the argument is taken in $(-\pi,\pi)$.

Theorem 4.8 (Uniform γ–radonification in sectors)**.** *Let $\Phi\in\gamma(L^2(\mathbb{R}_+;H),E)$ be given. For all $0<\vartheta<\frac{\pi}{2}$ the family*

$$\mathscr{T}_\vartheta^\Phi:=\{\sqrt{\lambda}\,\widehat{\Phi}(\lambda):\ \lambda\in S_\vartheta\}$$

is uniformly γ–radonifying in $\gamma(H,E)$ and

$$\|\mathscr{T}_\vartheta^\Phi\|_{\text{unif-}\gamma}\leqslant\frac{C'}{\cos\vartheta}\|\Phi\|_{\gamma(L^2(\mathbb{R}_+;H),E)},$$

where C' is a universal constant.

Proof. The proof follows the lines of Theorem 4.7, the difference being that instead of using Example 4.2 we now use Example 4.4.

Fix $\vartheta<\theta<\frac{\pi}{2}$ such that $\cos\theta\geqslant\frac12\cos\vartheta$. One obtains that for any $r>0$ fixed, the sequences $(f_n^+)_{n\in\mathbb{Z}}$ and $(f_n^-)_{n\in\mathbb{Z}}$ given by

$$f_n^\pm(t)=\sqrt{\mu_n^\pm}e^{-\mu_n^\pm t}$$

with

$$\mu_n^\pm=r2^ne^{\pm i\theta}$$

are Hilbert sequences whose Hilbert constants are bounded by $C/\cos\theta$, where C is a universal constant. Hence, arguing along the lines of Theorem 4.5, we obtain that the sequence $(\sqrt{\mu_n}\widehat{\Phi}(\mu_n))_{n\in\mathbb{Z}}$ is uniformly γ–radonifying, with bound $C/\cos\theta$.

By a Poisson transform argument (e.g., by using the logarithm to conformally map sectors to strips and then using the argument of Theorem 4.7), we obtain that $\sqrt{\lambda}\widehat{\Phi}(\lambda)$ is uniformly γ–radonifying on the sector S_ϑ, with a bound $C'/\cos\theta \leqslant 2C'/\cos\vartheta$, where C' is another universal constant. □

5. The stochastic Weiss conjecture

Let A be the generator of a C_0–semigroup $S = (S(t))_{t\geqslant 0}$ on a Banach space E and let $s(A)$, $s_0(A)$, and $\omega_0(A)$ denote the spectral bound, the abscissa of uniform boundedness of the resolvent, and growth bound of A, respectively:

$$\begin{aligned} s(A) &= \sup\{\operatorname{Re}\lambda : \ \lambda\in\sigma(A)\},\\ s_0(A) &= \inf\{\omega > \sigma(A) : \sup_{\operatorname{Re}\lambda\geqslant\omega} \|R(\lambda,A)\| < \infty\},\\ \omega_0(A) &= \inf\Big\{\omega\in\mathbb{R} : \ \|S(t)\| \leqslant Me^{\omega t} \text{ for some } M\geqslant 1 \text{ and all } t\geqslant 0\Big\}. \end{aligned}$$

Here, $R(\lambda,A) := (\lambda - A)^{-1}$. Recall that $-\infty \leqslant s(A) \leqslant s_0(A) \leqslant \omega_0(A) < \infty$.

It is shown in [36] that the linear stochastic Cauchy problem

$$(\mathrm{SCP})_{(A,B)} \qquad \begin{cases} dU(t) = AU(t)\,dt + B\,dW_H(t), & t\in[0,T],\\ U(0) = u_0, \end{cases}$$

where $(W_H(t))_{t\in[0,T]}$ is an H–cylindrical Wiener process and $B\in\mathscr{B}(H,E)$ is a bounded operator, has a solution if and only if for some (all) $t>0$ the $\mathscr{B}(H,E)$-valued function $S(\cdot)B$ *represents* an element of $\gamma(L^2(0,t;H),E)$, in the sense that the integral operator

$$f\mapsto \int_0^t S(s)Bf(s)\,ds, \quad f\in L^2(0,t;H),$$

belongs to $\gamma(L^2(0,t;H),E)$. In this situation the solution is unique up to modification. For the precise notion of 'solution' as well as other unexplained terminology we refer to [36].

As an application of Theorem 4.6 we obtain the following necessary condition for the existence of solutions to the problem $(\mathrm{SCP})_{(A,B)}$.

Theorem 5.1. *If the problem* $(\mathrm{SCP})_{(A,B)}$ *has a solution, then for all* $b > s_0(A)$ *we have*

$$\lim_{|s|\to\infty} \|R(b+is,A)B\|_{\gamma(H,E)} = 0.$$

Proof. By [38, Proposition 4.5], for $b>\omega_0(A)$ the integrable $\mathscr{B}(H,E)$-valued function $t\mapsto e^{-bt}S(t)B$ represents an element of $\gamma(L^2(\mathbb{R}_+;H),E)$. Hence for $b>\omega_0(A)$ the assertion is an immediate consequence of Theorem 4.6. For $b>s_0(A)$ the result then follows by a standard resolvent identity argument. □

Note that we did not assume that $B\in\gamma(H,E)$. Indeed, in many examples the problem $(\mathrm{SCP})_{(A,B)}$ admits a solution without such an assumption on B. For operators $B\in\gamma(H,E)$ the theorem is trivial, since then we may apply the Riemann-Lebesgue lemma in the space $L^1(\mathbb{R}_+;\gamma(H,E))$.

We recall the fact, proved in [38, Proposition 4.4], that the problem $(\mathrm{SCP})_{(A,B)}$ admits an invariant measure if and only if the function $t\mapsto S(t)B$ represents an element of $\gamma(L^2(\mathbb{R}_+;H),E)$. In this situation the mapping $\lambda\mapsto R(\lambda,A)B$ extends to an analytic $\gamma(H,E)$-valued function on $\mathbb{C}_+$; this extension is given by $\lambda\mapsto\widehat{\Phi}(\lambda)$, where $\Phi(t) := S(t)B$. With a slight abuse of notation we shall write $R(\lambda, A)B$ for this extension, keeping in mind that this notation is formal; indeed, examples can be given where A has spectrum in the open right-half plane.

As an application of Theorem 4.7 we obtain the following necessary conditions for the existence of an invariant measure for the problem $(\mathrm{SCP})_{(A,B)}$.

Theorem 5.2. *If the problem* $(\mathrm{SCP})_{(A,B)}$ *admits an invariant measure, then for all* $0 < \vartheta < \frac{\pi}{2}$ *the family*

$$\mathscr{T}_\vartheta := \big\{\sqrt{\lambda}R(\lambda, A)B : \ \lambda \in S_\vartheta\big\}$$

is uniformly γ–radonifying and we have

$$\|\mathscr{T}_\vartheta\|_{\text{unif-}\gamma} \leqslant \frac{C_{A,B}}{\cos\vartheta},$$

where $C_{A,B}$ is a constant depending only on A and B.

We conjecture that the following converse of this theorem holds.

Conjecture 5.3 (Stochastic Weiss conjecture)**.** *Let E be a Banach space with finite cotype and assume that the operator $-A$ is injective and sectorial of angle $< \frac{\pi}{2}$ on E and admits a bounded H^∞–calculus. The following assertions are equivalent:*

(a) *the stochastic Cauchy problem* $(\mathrm{SCP})_{(A,B)}$ *admits an invariant measure;*
(b) *The operator* $(-A)^{-\frac{1}{2}}B$ *is γ–radonifying;*
(c) *the set* $\big\{\sqrt{\lambda}R(\lambda, A)B : \ \lambda \in S_\vartheta\big\}$ *is uniformly γ–radonifying for some/all* $0 < \vartheta < \frac{\pi}{2}$.

The implication (a)⇒(c) follows from Theorem 5.2, and the implication (b)⇒(a) can be proved as follows. Since $(-A)^{-\frac{1}{2}}B \in \gamma(H, E)$ we may apply [8, Theorem 6.2] to obtain that for all $t > 0$ the function $S(\cdot)B = (-A)^{\frac{1}{2}}S(\cdot)((-A)^{-\frac{1}{2}}B)$ belongs to $\gamma(L^2(0, t; H), E)$, with a uniform bound $\sup_{t>0} \|S(\cdot)B\|_{\gamma(L^2(0,t;H),E)} < \infty$. Since E has finite cotype, E does not contain a copy of c_0 and the theorem of Hoffmann-Jørgensen and Kwapień implies that $S(\cdot)B \in \gamma(L^2(\mathbb{R}_+; H), E)$.

Thus the implication that remains to be proved is (c)⇒(b). A direct proof of (c)⇒(a) would also be of interest, as it would show the equivalence of (a) and (c). By standard H^∞-functional methods it is easy to prove that (c) implies the weaker result that $(-A)^{-\alpha}S(\cdot)B$ is in $\gamma(L^2(\mathbb{R}_+; H), E)$ for any $\alpha > 0$.

Following Weiss [47, Note, page 369], we offer 100 euro for a positive or negative resolution of these problems. A consequence of Theorem 5.2 is that the conjecture is true for bounded and invertible operators A (although this is not of great practical value) as well as certain other cases, for instance when A and B diagonalise simultaneously. To see the latter, suppose there is an orthonormal basis $(h_k)_{k\geqslant1}$ in H and a sequence $(x_k)_{k\geqslant1}$ in E such that

$$Bh_k = \beta_k x_k, \quad Ax_k = -\lambda_k x_k,$$

with $\lambda_k > 0$ for all $k \geqslant 1$. Taking $t_k = \lambda_k$ and assuming the uniform γ–radonification of the set $\big\{\sqrt{\lambda}R(\lambda, A)B : \ \lambda > 0\big\}$, we obtain convergence in E of the sum

$$\begin{aligned}\mathbb{E}\Big\|\sum_{k=1}^\infty \gamma_k(-A)^{-\frac{1}{2}}Bh_k\Big\|^2 &= \mathbb{E}\Big\|\sum_{k=1}^\infty \gamma_k\lambda_k^{-\frac{1}{2}}\beta_k x_k\Big\|^2 \\ &= 4\mathbb{E}\Big\|\sum_{k=1}^\infty \gamma_k\frac{t_k^{\frac{1}{2}}\beta_k}{t_k+\lambda_k}x_k\Big\|^2 = 4\,\mathbb{E}\Big\|\sum_{k=1}^\infty \gamma_k t_k^{\frac{1}{2}}R(t_k, A)Bh_k\Big\|^2.\end{aligned}$$

Consequently, $A^{-\frac{1}{2}}B$ is γ–radonifying.

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