PERTURBATION, INTERPOLATION, AND MAXIMAL REGULARITY

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ABSTRACT. We prove perturbation theorems for sectoriality and R–sectoriality in Banach spaces, which yield results on perturbation of generators of analytic semigroups and on perturbation of maximal L^p–regularity. For a given sectorial or R–sectorial operator A in a Banach space X we give conditions on intermediate spaces Z and W such that, for an operator $S : Z \to W$ of small norm, the perturbed operator $A + S$ is again sectorial or R–sectorial, respectively. These conditions are obtained by factorising the perturbation as $S = -BC$, where B acts on an auxiliary Banach space Y and C maps into Y. Our results extend previous work on perturbations in the scale of fractional domain spaces associated with A and allow for a greater flexibility in choosing intermediate spaces for the action of perturbation operators. At the end we illustrate our results with several examples, in particular with an application to a "rough" boundary value problem.

1. Introduction

Perturbation theorems are of fundamental importance in the applications of semigroup theory to, e.g., partial differential equations, transport equations, delay equations, population dynamics, or control theory (cf., e.g., [1, 9]). In particular, when studying parabolic problems, there is a natural interest in perturbation results for generators of analytic semigroups and on maximal L^p-regularity, which in turn can be obtained from more general results for sectorial and R-sectorial operators.

The standard example of such a perturbation theorem is the following. Given that $-A$ generates an analytic semigroup, then also $-A + S$ does so, whenever $S : \mathcal{D}(A) \to X$ is a linear operator that is A–small. A different class of admissible perturbations S is obtained by making use of the so-called *extrapolation space* $R_{-1} := (X, \|(1 + A)^{-1} \cdot \|)^\sim$ (see Section 2). If the semigroup generated by $-A$ is bounded analytic, and the operator $S : X \to R_{-1}$ has sufficiently small norm, then also the semigroup generated by $-(A + S)$ is bounded analytic. Here by $A + S := (A_{-1}+S)_X$ we mean the part in X of $A_{-1}+S$, where A_{-1} is the extrapolated version of A in R_{-1}.

As a kind of intermediate cases to the two examples above, in [16] perturbations $S : D_\theta \to R_{\theta-1}$ in the scale of fractional domain spaces have been studied. Here we let $\theta \in (0,1)$, $D_\theta := (D(A^\theta), \|(1 + A)^\theta \cdot \|)$, and $R_{\theta-1} := (X, \|(1 + A)^{\theta-1} \cdot \|)^\sim$ (see end of Section 2). The results in [16] may be viewed as a Banach space version of the well-known method of form-bounded perturbations in Hilbert space.

In this paper we will extend the results from [16] to a more general setting, allowing a greater variety of perturbing operators. Our considerations are based on a key observation (Lemma 3.3) which was inspired by the study of feedback in infinite-dimensional linear control systems. This basic idea, i.e., the underlying representation of the resolvent of the perturbed operator, is very simple (cf. Section 3). It uses assumptions on the composition of resolvents of A with two additional

operators C and B, which are assumed to act linearly $E \to Y$ and $Y \to G$, respectively. Here Y is an auxiliary Banach space and E, G are intermediate spaces (without topology) in a grid of extrapolation spaces associated with A. In linear systems theory, B would be a control operator and C an observation operator. The perturbation S is then given as $S = -BC$ (a very simple feedback situation is presented in Example (7.4)). In order to give a precise meaning to the spaces E and G mentioned above, we prefer to give a detailed presentation (cf. Section 2) of the extrapolation spaces associated with a sectorial operator (which differs in some aspects from, e.g., [1, 9]). The general setup for perturbations (Section 3) is a natural analogue of the usual setup for an A–small perturbation S where A is *non-invertible*. Of course, the additional assumption $0 \in \rho(A)$ leads to simplifications, but recent progress has shown the significance of using scales of *homogeneous* spaces (e.g., in [17], [13], implicitly already in [16]). In particular, we introduce homogeneous counterparts $\dot{D}$ and $\dot{R}$ of $D := D_1$ and R_{-1}, respectively, and require $\dot{D} \cap X \subset E \subset \dot{D} + X$ and $X \cap \dot{R} \subset G \subset X + \dot{R}$. Moreover, we study the case of sectorial operators A which are injective but not densely defined. This requires a modification in the above definition of extrapolation spaces.

The main new feature of our work is that the estimates in the hypothesis of Lemma 3.3 can be characterised by means of real interpolation spaces (cf. Section 4). On the one hand, this leads to manageable criteria on perturbations via a detailed study of the intermediate spaces $D_1 \subset Z \subset X$ and $X \subset W \subset R_{-1}$, which we may allow for a norm-small continuous action of the perturbation operator $S : Z \to W$ in order to obtain (R–)sectoriality of the perturbed operator $A + S$. These can be checked in concrete situations (cf. our main results in section 5). Our study includes real and complex interpolation spaces for the pairs $(\dot{D}, X)$ and $(X, \dot{R})$, but the method also yields new proofs for known results mentioned above. On the other hand, the estimates are preserved for the perturbed operator $A + S$, and the characterisations in Section 4 lead to persistence results on certain real interpolation spaces, e.g., $(X, D(A))_{\sigma,q} = (X, D(A + S))_{\sigma,q}$ for all $\sigma \in (0, \theta)$ and $q \in [1, \infty]$ if the perturbation S acts $D_\theta \to R_{\theta-1}$ and $\theta \in (0, 1)$. These results are new even for the perturbations studied in [16] and may also serve as justification of our approach. We remark that all that is known on persistence of fractional domains $D(A^\sigma) = D((A + S)^\sigma)$ or complex interpolation spaces $[X, D(A)]_\sigma = [X, D((A + S)^\sigma)]_\sigma$ is much more involved and only valid under additional assumptions on the operator A (cf. [28], [13]).

We present our main results on perturbation of sectorial operators in Section 5 (Theorems 5.1, 5.3 and 5.5): If $\theta \in (0, 1)$ and $(\dot{D}, X)_{\theta,1} \subseteq Z \subseteq (\dot{D}, X)_{\theta,\infty}$, $W := (A_{-1}(Z), \|(A_{-1})^{-1} \cdot \|_Z)$ and the part A_Z of A_{-1} in Z is sectorial, then sectoriality persists under norm-small perturbations $Z \to W$ (Theorem 5.3). If A_Z is densely defined and the perturbation is compact then a suitable translate of the perturbed operator is sectorial again (Theorem 5.5). In particular, we may take as Z domains of fractional powers, real or complex interpolation spaces (Remarks 5.6).

Furthermore, we study in this paper perturbation of R–sectorial operators. In Section 6 we recall the definition of maximal L^p-regularity and the connection with R-sectoriality. An (almost obvious) modification of our key observation (Lemma 3.3) leads to an analog for R-sectoriality (Proposition 6.5). It turns out that the R–boundedness assumptions in Proposition 6.5 cannot be characterised by interpolation spaces in a way similar to what is done in Section 4. This is due to the fact that the role of the space Y is now more significant (Remark 6.7 and Example 6.13). Nevertheless, we obtain several intermediate spaces Z and W such that, for a given R-sectorial operator A, a continuous action $S : Z \to W$ with small norm leads to R-sectoriality of $(A_{-1}+S)_X$ (Theorems 6.11 and 6.12). In the final Section 7 we present examples that illustrate our results and show how they extend existing

perturbation theorems, e.g., those of [16]. We point out that, for R–sectoriality, Proposition 6.5 may still be applied in some situations which are not covered by Theorems 6.11 and 6.12 (Example 6.13), and we give an application to "rough" boundary value problems (Example 7.12).

We conclude this introduction with the following two remarks. First, in this paper we concentrate on perturbations with small norm in the sense of $\|Bx\| \leq \eta\|Ax\|$, η small. This is the interesting case, since perturbations B satisfying $\|Bx\| \leq \eta\|Ax\| + b\|x\|$, with η being small, also satisfy the first condition for $\nu + A$ in place of A for ν large. Second, it is well known (cf. [6]) that in a reflexive space, compact perturbations may be studied by resorting to perturbations of small norm. Since our main interest in Section 6 are UMD spaces (which are reflexive), we do not state a theorem on compact perturbations for R–sectoriality.

Notations: If X and Y are Banach spaces, we write $X \hookrightarrow Y$ if $X \subset Y$ and the inclusion is continuous. The set of all bounded operators from X to Y shall be denoted by $B(X, Y)$, for the norm in $B(X, Y)$ we write $\|\cdot\|_{X \to Y}$. For a linear operator A on a Banach space X, we denote by $\mathcal{D}(A)$ its domain, by $\mathcal{R}(A)$ its range, and by $\mathcal{N}(A)$ its kernel. If Z is a subspace of X, then A_Z shall denote the part of A in Z, i.e. A restricted to the set $\{x \in Z \cap \mathcal{D}(A) \mid Ax \in Z\}$. The resolvent set of a linear operator A on X is denoted by $\varrho(A)$ and its spectrum by $\sigma(A)$. For $0 < \omega \leq \pi$ we denote by

$$S_\omega := \{z = re^{i\varphi} \mid r > 0, \ |\varphi| < \omega\}$$

the open sector of angle 2ω, symmetric about the positive real axis. In addition we define $S_0 := (0, \infty)$.

2. Extrapolation Spaces for Sectorial Operators

In this section we construct what is called *extrapolation spaces* associated with an injective sectorial operator. The relevance of extrapolation scales in the study of evolution equations is well known (cf., e.g., [1, 9]). Here, we recast the frame in order to include homogeneous spaces that we need for our perturbation results.

Let $0 \leq \omega < \pi$. A (possibly unbounded) operator A on a Banach space X, is called *sectorial of type* ω if $\sigma(A) \subset \overline{S_\omega}$ and for each $\nu \in (\omega, \pi]$ one has an estimate

$$\|\lambda(\lambda + A)^{-1}\|_{X \to X} \leq \alpha \quad (\lambda \in S_{\pi-\nu}) \tag{1}$$

for some constant $\alpha = \alpha(A, \nu) < \infty$. The minimum of all such $\omega \in [0, \pi)$ is called the *sectoriality angle* of A. An operator A which is sectorial of some type, is simply called a *sectorial operator*. In the literature, sectorial operators are sometimes called *non-negative*, sometimes *pseudo-sectorial*. One can consult [19] for elementary properties of sectorial operators. It is well-known (see [3, Thm 3.8]) that for a densely defined sectorial operator one has $\mathcal{N}(A) \cap \overline{\mathcal{R}(A)} = 0$, whence a densely defined sectorial operator (in our definition) with dense range is always injective. Moreover, if X is a reflexive Banach space, then the domain $\mathcal{D}(A)$ of a sectorial operator A is automatically dense in X.

Let A be an injective sectorial operator. Then the operator A^{-1} with domain $\mathcal{D}(A^{-1}) = \mathcal{R}(A)$ is also sectorial (even of the same type), and one has the formula

$$\lambda(\lambda + A^{-1})^{-1} = I - \frac{1}{\lambda}\left(\frac{1}{\lambda} + A\right)^{-1} \qquad (\lambda \in S_{\pi-\omega}).$$

We can consider the following natural spaces:

- $D := D_1 := \mathcal{D}(A)$ with norm $\|x\|_D := \|(1 + A)x\|$.
- $R := R_1 := \mathcal{R}(A)$ with norm $\|x\|_R := \|(1 + A^{-1})x\|$.
- $D \cap R := \mathcal{D}(A) \cap \mathcal{R}(A)$ with the norm $\|x\|_{D \cap R} = \|(2 + A + A^{-1})x\|$.
- $D_2 := \mathcal{D}(A^2)$ with the norm $\|x\|_{D_2} = \|(1 + A)^2 x\|$.

- $R_2 := \mathcal{R}(A^2) = \mathcal{D}(A^{-2})$ with the norm $\|x\|_{R_2} = \|(1 + A^{-1})^2 x\|$.

This amounts to the following picture:
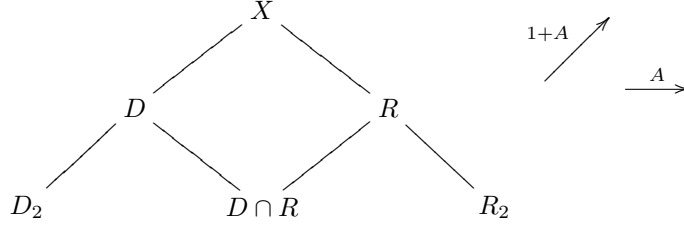
Here a downward meeting of two lines means intersection and an upward meeting of two lines means sum of the spaces. E.g., $X = D + R$ and $D = D_2 + (D \cap R)$. The operator $A + 1$ acts as an isometric isomorphism in the $\nearrow$-direction, i.e. $D_2 \xrightarrow{A+1} D \xrightarrow{A+1} X$ or $D \cap R \xrightarrow{A+1} R$. (One can replace $A + 1$ by $A + \lambda$ for each $\lambda > 0$, but then the isomorphisms cease to be isometric.) Furthermore, the operator A acts as an isometric isomorphism in the $\rightarrow$-direction, e.g., $D \xrightarrow{A} R$ or $D_2 \xrightarrow{A} D \cap R$.

We now enlarge this picture by some "extrapolation spaces" embedding X into a new space X_{-1}. In order to do so, we note that the operator $T := A(1 + A)^{-2} : X \longrightarrow D \cap R$ is an isometric isomorphism. Hence, by abstract nonsense, we can construct a Banach space X_{-1} and an embedding $\iota : X \longrightarrow X_{-1}$ together with an isometric isomorphism $T_{-1} : X_{-1} \longrightarrow X$ making the diagram

commute. Without restriction we can assume that $X \subset X_{-1}$ and ι is in fact the inclusion mapping. This implies that T can be regarded the restriction of T_{-1} to X. Within X_{-1} we will consider the following spaces:

- $D_{-1} := [T_{-1}]^{-1}(D)$ with the norm $\|u\|_{D_{-1}} = \|T_{-1}u\|_D$.
- $R_{-1} := [T_{-1}]^{-1}(R)$ with the norm $\|u\|_{R_{-1}} = \|T_{-1}u\|_R$.

Thus, the triple (X_{-1}, D_{-1}, R_{-1}) is just the "pullback" of (X, D, R) under T_{-1}. We therefore can define an operator $A_{-1} : D_{-1} \longrightarrow R_{-1}$ by

$$A_{-1} := [T_{-1}]^{-1}AT_{-1} : \quad D_{-1} \xrightarrow{T_{-1}} D \xrightarrow{A} R \xrightarrow{[T_{-1}]^{-1}} R_{-1}$$

Since T_{-1} restricts to T within X and $T : X \longrightarrow D \cap R$ is an isomorphism, we have $X = D_{-1} \cap R_{-1}$. We thus arrive at the following situation:
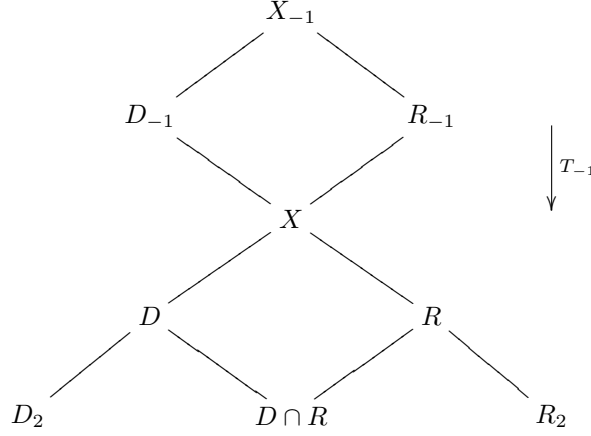
Lemma 2.1. *Consider the operator A_{-1} as an operator in X_{-1}, with $\mathcal{D}(A_{-1}) = D_{-1}$ and $\mathcal{R}(A_{-1}) = R_{-1}$. Then A_{-1} is a sectorial operator isometrically similar to A. Since $D \subset X \subset D_{-1}$ one can restrict A_{-1} to D and obtains $A_{-1}\big|_D = A$. Moreover, one has $T_{-1} = A_{-1}(1 + A_{-1})^{-2}$ and $(1 + A_{-1}) : X \longrightarrow R_{-1}$ is an isometric isomorphism.*

Proof. The proof is easy. □

We call the space R_{-1} the *first extrapolation space* with respect to the operator A. (Observe that often, e.g. in [9], the first extrapolation space is denoted by X_{-1}, and that we reserved this notation for a different space.) Similarly, D_{-1} is the first extrapolation space with respect to A^{-1}.

Let us now introduce the following spaces:

- $\dot{D} := [T_{-1}]^{-1}(D_2)$ with norm $\|u\|_{\dot{D}} = \|T_{-1}u\|_{D_2}$.
- $\dot{R} := [T_{-1}]^{-1}(R_2)$ with norm $\|u\|_{\dot{R}} = \|T_{-1}u\|_{R_2}$.

Since $D_2 \subset D$ and $R_2 \subset R$ we have $\dot{D} \subset D_{-1}$ and $\dot{R} \subset R_{-1}$.

Lemma 2.2. *The following statements hold:*

a) *The operators $\dot{D} \xrightarrow{A_{-1}} X \xrightarrow{A_{-1}} \dot{R}$ are isometric isomorphisms.*

b) *$D \subset \dot{D}$ and $\|x\|_{\dot{D}} = \|Ax\|$ for $x \in D$.*

c) *$D = \dot{D} \cap X$ and $\dot{D} + X = D_{-1}$.*

Analogous results hold for the triple $(R_{-1}, X, \dot{R})$. Here, $R \subset \dot{R}$ and $\|x\|_{\dot{R}} = \|A^{-1}x\|$ for $x \in R$. Moreover, $\dot{R} = A_{-1}(X)$. The spaces $\dot{D}, \dot{R}$ are called the *homogeneous spaces* associated to the operator A. We can depict the situation as follows:

$$
\begin{array}{c}
X_{-1} \\
D_{-1} \qquad R_{-1} \\
\dot{D} \qquad X \qquad \dot{R} \\
D \qquad R \\
D_2 \qquad D \cap R \qquad R_2
\end{array}
$$

$$
\begin{array}{c}
A_{-1}+1 \nearrow \\
\xrightarrow{\;A_{-1}\;} \\
\downarrow T_{-1}
\end{array}
$$
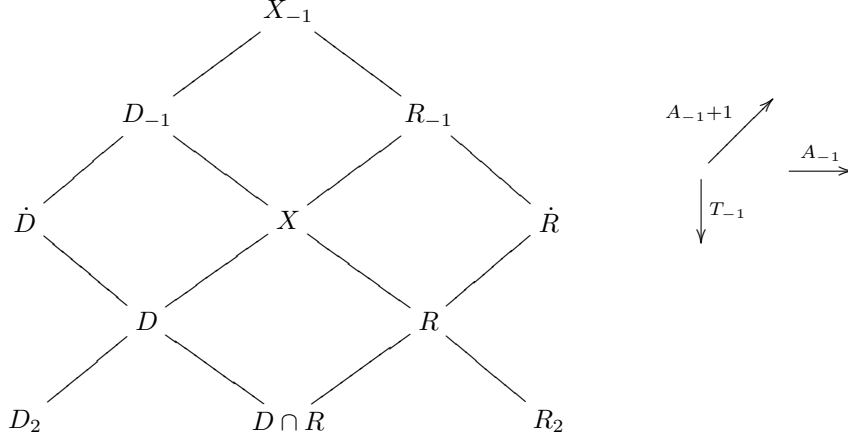
If $\mathcal{D}(A)$ is dense in X, then all inclusions in the $\nearrow$-direction are dense. If $\mathcal{R}(A)$ is dense, then all inclusions in the $\searrow$-direction are dense.

Within this array of spaces we can define inhomogeneous and homogeneous *fractional domain spaces*. These arise naturally when we consider the fractional powers $[A_{-1}]^{\theta} = [T_{-1}]^{-1}A^{\theta}T_{-1}$ of the operator A_{-1} for $0 < \theta < 1$. We obtain the spaces

- $D_{\theta} := \mathcal{D}(A^{\theta})$ with norm $\|u\|_{D_{\theta}} = \|(1+A)^{\theta}u\|$,
- $R_{\theta} := \mathcal{R}(A^{\theta}) = \mathcal{D}(A^{-\theta})$ with norm $\|u\|_{D_{\theta}} = \|(1+A^{-1})^{\theta}u\|$
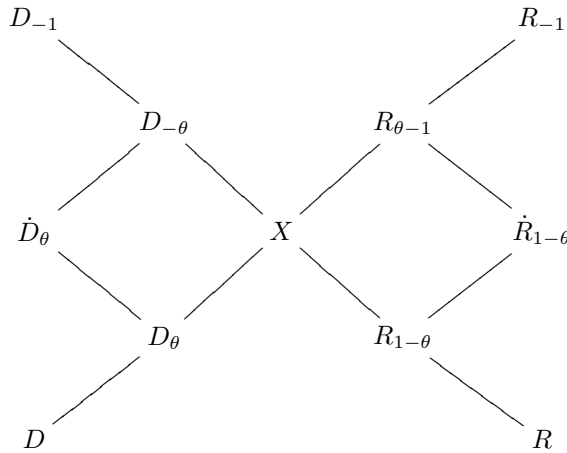
(inhomogeneous), and

- $\dot{D}_{\theta} := [A_{-1}]^{-\theta}(X)$ with norm $\|u\|_{\dot{D}_{\theta}} = \|[A_{-1}]^{\theta}u\|_X$,
- $\dot{R}_{\theta} := [A_{-1}]^{\theta}(X)$ with norm $\|u\|_{\dot{D}_{\theta}} = \|[A_{-1}]^{-\theta}u\|_X$

(homogeneous). Note that, e.g., $\dot{D}_{\theta} \cap X = D_{\theta}$, and that $[A_{-1}]^{\theta}$ acts isometrically in the horizontal direction (cf. the picture below).

Finally, we introduce for $0 < \theta < 1$ the extrapolated fractional domain spaces

- $R_{\theta-1} := (1+A_{-1})D_{\theta}$ with norm
$$
\|u\|_{R_{\theta-1}} := \|(1+A_{-1})^{-1}u\|_{D_{\theta}} = \|(1+A_{-1})^{\theta-1}u\|_X,
$$
- $D_{-\theta} := (1+A_{-1}^{-1})R_{1-\theta}$ with norm
$$
\|u\|_{D_{-\theta}} := \|(1+A_{-1}^{-1})^{-1}u\|_{R_{1-\theta}} = \|(1+A_{-1}^{-1})^{-\theta}u\|_X.
$$

We end up with the following picture.

$$
\begin{array}{c}
D_{-1} \qquad\qquad\qquad R_{-1} \\
D_{-\theta} \qquad R_{\theta-1} \\
\dot{D}_{\theta} \qquad X \qquad \dot{R}_{1-\theta} \\
D_{\theta} \qquad R_{1-\theta} \\
D \qquad\qquad\qquad R
\end{array}
$$

In the case that A is densely defined, the space $R_{\theta-1}$ can be obtained by completing the space X with respect to the norm $\|[1+A]^{\theta-1}u\|_X$. If A has dense range, the homogeneous spaces $\dot{D}_{\theta}$ can be obtained by completing the space D with respect to the homogeneous norm $\|A^{\theta}u\|_X$. This is the situation considered in [16].

More about extrapolation spaces can be found in [12, Appendix].

3. A Setting for Perturbation

We start with an easy lemma from linear algebra which may be well known. We omit its elementary proof.

Lemma 3.1. *Let V, W be vector spaces and $S : V \longrightarrow W$, $T : W \longrightarrow V$ linear mappings such that $I - ST : W \longrightarrow W$ is invertible. Then also $I - TS : V \longrightarrow V$ is invertible with*

$$(I - TS)^{-1} = I + T(I - ST)^{-1}S.$$

Let us now describe the setup of our main lemma. Let A be an injective sectorial operator on the Banach space X. We choose a constant $\alpha \geq 0$ such that (1) holds (for $\nu = \pi$, to begin with). Assume that we are given a vector space E (no topology!) with the following properties:

a) $D \subset E \subset D_{-1}$, and

b) E is invariant under $(\lambda + A_{-1})^{-1}$ for all $\lambda > 0$.

We form the spaces $F := A_{-1}E$ and $\tilde{E} := E + F$. By (1), F is intermediate between R and R_{-1}. We then have $E \cap F \subset D_{-1} \cap R_{-1} = X$. Moreover,

$$E \xrightarrow{(\lambda + A_{-1})} E + F = \tilde{E} \quad \text{and} \quad E \cap F \xrightarrow{(\lambda + A_{-1})} F$$

are algebraic isomorphisms. We obtain the following diagram:
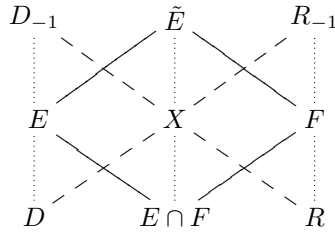
We use different types of lines for the inclusions in order to indicate that spaces where different types of lines meet are *not* necessarily the sum (the intersection, respectively.) of the involved other spaces. This is to say that it may well be that $D \neq E \cap X \neq E \cap F \neq X \cap F \neq D$, $X \neq D_{-1} \cap \tilde{E} \neq E$, $X + F \neq \tilde{E}$ and so on.

Suppose now in addition that we are given a Banach space Y and linear operators

$$C : E \longrightarrow Y, \quad B : Y \longrightarrow \tilde{E} \cap R_{-1}$$

such that for some $0 \leq \eta < 1$ we have

$$\left\| C(\lambda + A_{-1})^{-1}B \right\|_{Y \to Y} \leq \eta \quad (\lambda > 0). \tag{2}$$

(Note that B has to map Y to $\tilde{E}$ in order that $C(\lambda + A_{-1})^{-1}B : Y \to Y$ is meaningful.) We will use the abbreviation $\tilde{P} := A_{-1} - BC$, which is an operator defined on $\mathcal{D}(\tilde{P}) := E$.

Lemma 3.2. *In the situation described above, the operator $\lambda + \tilde{P} : E \longrightarrow \tilde{E}$ is an algebraic isomorphism for each $\lambda > 0$, and its inverse is given by*

$$R_\lambda := \left\{ I + (\lambda + A_{-1})^{-1} B \left[I - C(\lambda + A_{-1})^{-1}B \right]^{-1} C \right\} (\lambda + A_{-1})^{-1}. \tag{3}$$

If one in addition has $B : Y \longrightarrow F$ and $\|C[A_{-1}]^{-1}B\|_{Y \to Y} < 1$ then the operator $\tilde{P} : E \longrightarrow F$ is an algebraic isomorphism with inverse

$$\tilde{P}^{-1} = \left\{ I + (A_{-1})^{-1} B \left[I - C(A_{-1})^{-1}B \right]^{-1} C \right\} (A_{-1})^{-1}.$$

Proof. Let us write for short $V_\lambda := C\,(\lambda + A_{-1})^{-1}B$. Then, by the standard Neumann series argument the operator $I - V_\lambda = I - C(\lambda + A_{-1})^{-1}B : Y \longrightarrow Y$ is invertible with its inverse satisfying

$$\left\|(I - V_\lambda)^{-1}\right\|_{Y \to Y} \le \frac{1}{1 - \eta}. \tag{4}$$

Lemma 3.1 applied to the data $V := E, W = Y, S = C$, and $T = (\lambda + A_{-1})^{-1}B$, yields that the operator $I - (\lambda + A_{-1})^{-1}BC : E \longrightarrow E$ is invertible. We now write $\lambda + (A_{-1} - BC) = (\lambda + A_{-1})(I - (\lambda + A_{-1})^{-1}BC)$. The second factor is an algebraic isomorphism on E, the first is an algebraic isomorphism from E to $\tilde{E}$, whence the product is an algebraic isomorphism from E onto $\tilde{E}$. The inverse is computed by taking inverses separately and reversing the order. To compute the inverse of the second factor, we use again Lemma 3.1. This proves the first statement.
The proof of the second statement is quite similar. All we have to do is to take $\lambda = 0$ and F instead of $\tilde{E}$ in the above considerations. $\square$

We illustrate the whole situation with a picture:
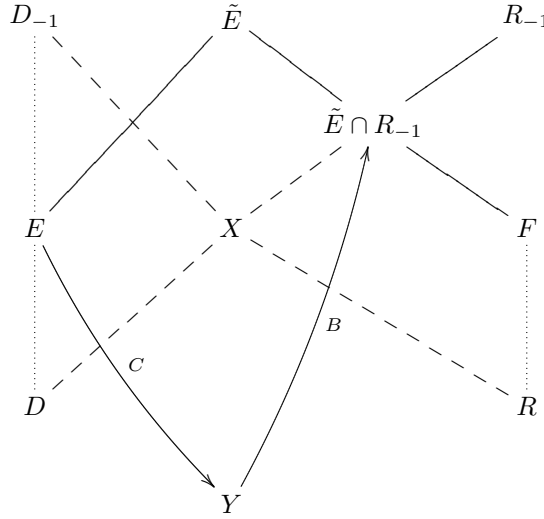
Let us now consider the operator

$$P := (A_{-1} - BC)_X, \quad \mathcal{D}(P) := \{x \in X \cap E \mid \tilde{P}x \in X\}$$

which can be seen as an additive perturbation of the original operator A. Observe that since B maps into $R_{-1} \cap \tilde{E}$, the space X is left invariant by each operator $R_\lambda = (\lambda + \tilde{P})^{-1}$, $\lambda > 0$. Hence,

$$\lambda + P : \mathcal{D}(P) \longrightarrow X$$

is invertible with $(\lambda + P)^{-1} = R_\lambda\big|_X$ for each $\lambda > 0$. We are going to describe conditions which ensure that P is in fact a sectorial operator on X.

Lemma 3.3. *Let A be an injective sectorial operator on the Banach space X satisfying (1), and let $\mathcal{D}(A) \subset E \subset D_{-1}$ be a vector subspace invariant under the family $\left((\lambda + A_{-1})^{-1}\right)_{\lambda > 0}$. Suppose a Banach space Y and linear mappings $C : E \longrightarrow Y$, $B : Y \longrightarrow \tilde{E} \cap R_{-1}$ are given such that (2) holds for some $0 \le \eta < 1$. Assume furthermore that there exists $\theta \in [0, 1]$ and constants $\beta, \gamma \ge 0$ such that*

$$\left\|\lambda^{1-\theta}C(\lambda + A)^{-1}\right\|_{X \to Y} \le \gamma \qquad and \tag{5}$$

$$\left\|\lambda^\theta(\lambda + A_{-1})^{-1}B\right\|_{Y \to X} \le \beta \tag{6}$$

for all $\lambda > 0$. Then the operator

$$P := (A_{-1} - BC)\big|_X$$

is a sectorial operator on X. If A is densely defined, so is P.

Proof. By what we have said above, clearly $\lambda + P$ is invertible and its inverse is given by $(\lambda + P)^{-1} = R_\lambda\big|_X$. To prove sectoriality we have to estimate λR_λ on X. In order to do this, we use the representation (3) and write

$$\lambda R_\lambda\big|_X = \quad \lambda(\lambda + A)^{-1} +$$
$$\left[\lambda^\theta(\lambda + A_{-1})^{-1}B\right]\left[I - C(\lambda + A_{-1})^{-1}B\right]^{-1}\left[\lambda^{1-\theta}C(\lambda + A)^{-1}\right]$$

which yields $\sup_{\lambda > 0}\|\lambda R_\lambda\|_{X \to X} \leq \alpha + \frac{\beta\gamma}{1-\eta}$.

Assume now that A is densely defined. Then the density of $\mathcal{D}(P)$ in X is, by sectoriality of P and $\overline{D} = X$, equivalent to $\lim_{\lambda \to \infty} \lambda R_\lambda x = x$ in X for each $x \in D$. Choose $x \in D$. In case $0 < \theta \leq 1$ we have

$$\lambda R_\lambda x = \lambda(\lambda + A)^{-1}x +$$
$$\left[\lambda^\theta(\lambda + A_{-1})^{-1}B\right]\left[I - C(\lambda + A_{-1})^{-1}B\right]^{-1}\lambda^{-\theta}\,C\lambda(\lambda + A)^{-1}x$$

Note that $\lambda(\lambda + A)^{-1}x \to x$ even in D, and that $C\big|_D : D \longrightarrow Y$ is bounded, by (5). Hence the second summand tends to 0 in X as $\lambda \to \infty$ and we are done. In case $\theta = 0$ a longer argument is needed. In this case one obtains the inequality $\|Cx\| \leq \gamma\|x\|$ for all $x \in D$ (see the proof of Proposition 4.5 below). Since D is dense, C has a unique extension to a bounded operator $\tilde{C} : X \to Y$ such that $\|\tilde{C}\|_{X \to Y} \leq \gamma$. Next observe that the whole setup and also the operator P remains unchanged if we replace C by $\tilde{C}$ and E by X. So we can assume $E = X, C = \tilde{C}$. Then $\lambda + A_1 - BC : X \longrightarrow R_{-1}$ is invertible, its inverse being the operator R_λ given by (3). From that formula one can deduce that $\|\lambda R_\lambda\|_{R_{-1} \to R_{-1}}$ stays bounded. In fact, $\left\|\lambda(\lambda + A_{-1})^{-1}\right\|_{R_{-1} \to X}$ is bounded for small λ whereas $\left\|(\lambda + A_{-1})^{-1}\right\|_{R_{-1} \to X}$ is bounded for large λ. All this implies just that $\tilde{P} := A_{-1} - BC$ is a sectorial operator in R_{-1} with domain X. But then $\mathcal{D}(P) = \mathcal{D}(\tilde{P}^2)$ is dense in $X = \mathcal{D}(\tilde{P})$. $\qquad\square$

Lemma 3.4. *Let the situation of Lemma 3.3 be given. If A has dense range and $0 \leq \theta < 1$, then also P has dense range. This is also true for $\theta = 1$ if X is reflexive. If A is invertible, then also P is and its inverse is given by*

$$P^{-1} = \{I + [A_{-1}]^{-1}B(I - V_0)^{-1}C\}A^{-1}.$$

If $B : Y \longrightarrow F$ and $\|C[A_{-1}]^{-1}B\|_{Y \to Y} < 1$, then P is injective.

Proof. Assume that A has dense range, i.e., $\overline{R} = X$, and that $0 \leq \theta < 1$. Since P is sectorial and A has dense range, it suffices to show that $\lambda R_\lambda Ay \to 0$ as $\lambda \searrow 0$ for each $y \in \mathcal{D}(A)$ (see [19, Prop.1.1.3]). Given such y we write

$$\lambda R_\lambda Ay = \lambda(\lambda + A)^{-1}Ay +$$
$$\left[\lambda^\theta(\lambda + A_{-1})^{-1}B\right]\left[I - C(\lambda + A_{-1})^{-1}B\right]^{-1}\lambda^{1-\theta}\,C(\lambda + A)^{-1}Ay.$$

Now $(\lambda + A)^{-1}Ay$ is bounded in D and since $C : D \to Y$ is bounded we see that $\lambda R_\lambda Ay \to 0$ in X as $\lambda \to 0$.

Let $\theta = 1$ but assume that X is reflexive. (Note that A must be densely defined in this case.) Take $y \in Y$. By reflexivity of X, the bounded sequence $(n(n + A_{-1})^{-1}By)_{n \in \mathbb{N}}$ in X must have a subsequence weakly convergent in X. However, $n(n + A_{-1})^{-1}By \to By$ in R_{-1}, since A is densely defined. Therefore, $By \in X$. The Closed Graph Theorem now shows that $B : Y \to X$ is bounded. On the other hand, condition (5) implies that $C : \dot{D} \longrightarrow Y$ is bounded on D (see Corollary 4.8

below). This means that $CA^{-1} : X \to Y$ is bounded on R, and since R is dense in X, it extends to a bounded operator from X to Y. Now

$$C(\lambda + A_{-1})^{-1}B = [CA^{-1}] [A(\lambda + A)^{-1}]B$$

and by the density of R in X one has $A(\lambda + A)^{-1}By \to By$ in X as $\lambda \to 0$ for every $y \in Y$. Condition (2) now shows that $\|CA^{-1}B\|_{Y \to Y} \leq \eta < 1$. Applying Lemma 3.1 yields that $I - BCA^{-1}$ is an invertible operator on X. However, it is clear that $P = (A_{-1} - BC)\big|_X = A - BC = (I - B[CA^{-1}])A$, and since A has dense range, also P must have dense range.

Now suppose that A is invertible, i.e., $D \subset E \subset X \subset F = \tilde{E}$. We first ensure that (2) still holds for $\lambda = 0$. Now $C[\lambda + A_{-1}]^{-1}B - C[A_{-1}]^{-1}B = -\lambda\, C(\lambda + A)^{-1}[A_{-1}]^{-1}B$, whence

$$\big\| C(\lambda + A_{-1})^{-1}B - C[A_{-1}]^{-1}B \big\|_{Y \to Y} \leq$$
$$\lambda\, \|C\|_{D \to Y} \big\|(\lambda + A)^{-1}\big\|_{X \to D} \big\|[A_{-1}]^{-1}B\big\|_{Y \to X}.$$

However, this goes to zero as $\lambda \searrow 0$ since $\|(\lambda + A)^{-1}\|_{X \to D} = \|A(\lambda + A)^{-1}\|_{X \to X} \leq \alpha + 1$. Now we can apply Lemma 3.2 to conclude that $\tilde{P} : E \longrightarrow F$ is bijective. Obviously, X is invariant under $\tilde{P}^{-1}$.

Finally, assume that B maps Y to F and $\|C[A_{-1}]^{-1}B\|_{Y \to Y} < 1$. Then one can apply again Lemma 3.2 to conclude that $\tilde{P} : E \longrightarrow F$ is bijective. It follows that P is injective.   □

Remark 3.5. We do not know if in the case $\theta = 1$ always P has dense range if A has dense range.

Up to now, we only cared for sectoriality without considering the precise angle. The next lemma fills this gap.

Lemma 3.6. *Let $\omega \in (0, \pi]$ and let A be sectorial of type $< \omega$. Assume in the situation of Lemma 3.3 that*

$$\begin{align} \|\lambda(\lambda + A)^{-1}\| &\leq \alpha, \quad \lambda \in S_{\pi - \omega}, &(7)\\ \|C(\lambda + A_{-1})^{-1}B\| &\leq \eta, \quad \lambda \in S_{\pi - \omega} &(8) \end{align}$$

for some $0 \leq \eta < 1$ and that E is invariant under $((\lambda + A_{-1})^{-1})_{\lambda \in S_{\pi - \omega}}$. If equations (5) and (6) hold, then there are β', γ' only depending on α, β, and γ such that

$$\begin{align} \big\|\lambda^{1-\theta} C(\lambda + A)^{-1}\big\|_{X \to Y} &\leq \gamma', \quad \lambda \in S_{\pi - \omega} \quad and &(9)\\ \big\|\lambda^{\theta}(\lambda + A_{-1})^{-1}B\big\|_{Y \to X} &\leq \beta', \quad \lambda \in S_{\pi - \omega}. &(10) \end{align}$$

Moreover, the estimates in Corollary 3.8 hold for $\lambda \in S_{\pi - \omega}$ with (β', γ') in place of (β, γ). In particular, P is sectorial of type $< \omega$.

Note that $\omega = \pi$ gives back Lemma 3.3, thanks to our definition of S_0.

Proof. One can now reduce the statement to Lemma 3.3 in replacing A by $e^{\pm(\pi - \omega')i}A$ for all $\omega' \in (\omega, \pi]$. The only thing left is to prove existence of β' and γ' with (9) and (10). But this follows easily from the resolvent identity. Namely,

$$|\lambda|^{1-\theta} C(\lambda + A)^{-1} - |\lambda|^{1-\theta} C(|\lambda| + A)^{-1} = |\lambda|^{1-\theta} C(|\lambda| + A)^{-1} \left[\frac{|\lambda|}{\lambda} - 1 \right] [\lambda(\lambda + A)^{-1}]$$

for $\lambda \in S_{\pi - \omega}$. Hence one can choose $\gamma' := \gamma + 2\gamma\alpha$. A similar argument deals with the expression involving the operator B.

So far these consideration imply that $e^{\pm(\pi - \omega')i}P$ is sectorial for all $\omega' \in (\omega, \pi]$, but with a uniform sectoriality constant. Then a well-known Taylor-series argument shows that the type of P must even be strictly less than ω.   □

Remark 3.7. In the case $\omega < {}^\pi/_2$, Lemma 3.6 yields that $-P$ generates a bounded analytic semigroup in X, provided $-A$ does.

We now make sure that the whole setup for A is reproduced for P. Note that $(\lambda + P)^{-1}$ was just the part of the isomorphism $R_\lambda : E_{-1} \longrightarrow E$ in X.

Corollary 3.8. *In the situation of Lemma 3.6 one has the estimates*

$$\left\|\lambda(\lambda + P)^{-1}\right\|_{X \to X} \le \alpha + \frac{\beta'\gamma'}{1 - \eta}$$

$$\left\|C\,\lambda^{1-\theta}(\lambda + P)^{-1}\right\|_{X \to Y} \le \frac{\gamma'}{1 - \eta}$$

$$\left\|\lambda^\theta\,R_\lambda\,B\right\|_{Y \to X} \le \frac{\beta'}{1 - \eta}$$

$$\left\|C\,R_\lambda\,B\right\|_{Y \to Y} \le \frac{\eta}{1 - \eta}$$

for $\lambda \in S_{\pi-\omega}$.

Proof. Just apply the representation (3) and the original estimates. $\square$

4. Characterisation of the Crucial Conditions by Means of Interpolation Spaces

We are going to characterise the conditions (5) and (6) in terms of interpolation spaces. For this we need a result in abstract interpolation theory from [11].

Proposition 4.1. *Let (X, Y) be any Banach couple. Then the identities*

a) $(X + Y, Y)_{\theta,p} \cap X = (X, Y)_{\theta,p} \cap X = (X, X \cap Y)_{\theta,p}$,

b) $(X + Y, Y)_{\theta,p} \cap (X + Y, X)_{1-\theta,p} = (X, Y)_{\theta,p}$, *and*

c) $(X, X \cap Y)_{\theta,p} + (Y, X \cap Y)_{1-\theta,p} = (X, Y)_{\theta,p}$

hold for all $p \in [1, \infty], \theta \in [0, 1]$.

We now turn to the characterisation of the growth conditions as announced. Let A be any sectorial operator on a Banach space X. On the space $\mathcal{D}(A)$ we will always use the norm $\|x\|_{\mathcal{D}(A)} := \|x\| + \|Ax\|$. Of course, in the notation of Section 2 we have $\mathcal{D}(A) = D$ with equivalent norms, but we stick to the notation "$\mathcal{D}(A)$" for the sake of readability. If A is injective we will occasionally use the space $\dot{D}$ as introduced in Section 2. The following result is mentioned in a special case in [27, Remark 3.3].

Proposition 4.2. *Let A be a sectorial operator on the Banach space X, let $B : Y \longrightarrow X$ be a bounded operator, where Y is another Banach space, and let $\theta \in [0, 1]$. Then the equivalences*

$$\mathcal{R}(B) \subset (X, \mathcal{D}(A))_{\theta,\infty} \iff \sup_{\lambda > 1}\left\|\lambda^\theta(\lambda + A)^{-1}B\right\|_{Y \to \mathcal{D}(A)} < \infty$$

$$\iff \sup_{\lambda > 1}\left\|\lambda^\theta(\lambda + A)^{-1}B\right\|_{Y \to \dot{D}} < \infty$$

hold, with the latter applying only when A is injective. Furthermore, if A is injective, then also the equivalences

$$\mathcal{R}(B) \subset (X, \mathcal{R}(A))_{1-\theta,\infty} \iff \sup_{0 < \lambda < 1}\left\|\lambda^\theta(\lambda + A)^{-1}B\right\|_{Y \to X} < \infty$$

$$\iff \sup_{0 < \lambda < 1}\left\|\lambda^\theta(\lambda + A)^{-1}B\right\|_{Y \to \mathcal{D}(A)} < \infty$$

hold true.

Proof. First we observe that, by the sectoriality of A, the second biimplications in both (a) and (b) hold trivially. The assertion in (a) follows immediately from the (well-known) characterisation

$$(X, \mathcal{D}(A))_{\theta,\infty} = \{x \in X \mid \sup_{\lambda > 1} \|\lambda^\theta A(\lambda + A)^{-1}x\| < \infty\} \tag{11}$$

(see [18]) and the Closed Graph theorem. In order to prove (b) we write $\mu = \frac{1}{\lambda}$ and obtain

$$\lambda^\theta(\lambda + A)^{-1} = \mu^{1-\theta}\frac{1}{\mu}\left(\frac{1}{\mu} + A\right)^{-1} = \mu^{1-\theta}A^{-1}(\mu + A^{-1})^{-1}.$$

We use again (11) with A replaced by A^{-1} and the Closed Graph theorem to finish the proof. $\square$

Remark 4.3. Given a bounded operator $B : Y \to R_{-1}$, the infimum of all $\theta > 0$ for which $\mathcal{R}(B) \subset (X, R_{-1})_{\theta,\infty}$ is equal to the notion of *degree of unboundedness* of B defined in [22].

Combining what we know with Proposition 4.1 we obtain the following result.

Corollary 4.4. *Let A be an injective sectorial operator on the Banach space X, let $B : Y \longrightarrow X$ be a bounded operator, where Y is another Banach space, and let $\theta \in [0,1]$. Then the condition*

$$\sup_{0 < \lambda < \infty} \|\lambda^\theta(\lambda + A)^{-1}B\|_{Y \to \mathcal{D}(A)} < \infty$$

is equivalent to $\mathcal{R}(B) \subset (\mathcal{D}(A), \mathcal{R}(A))_{1-\theta,\infty}$.

Proof. Combining both conditions from Proposition 4.2 yields

$$\mathcal{R}(B) \subset (X, \mathcal{D}(A))_{\theta,\infty} \cap (X, \mathcal{R}(A))_{1-\theta,\infty}.$$

Now observe that $X = \mathcal{D}(A) + \mathcal{R}(A)$ and apply b) of Proposition 4.1. $\square$

Let us turn to the "dual" conditions.

Proposition 4.5. *Let A be a sectorial operator on the Banach space X, let $C : \mathcal{D}(A) \longrightarrow Y$ a bounded operator, where Y is a second Banach space, and let $\theta \in [0,1]$. Then the condition*

$$\sup_{\lambda > 1} \|\lambda^{1-\theta}C(\lambda + A)^{-1}\|_{X \to Y} < \infty \tag{12}$$

is trivially satisfied if $\theta = 1$. For $0 < \theta < 1$ it is equivalent to the boundedness of

$$C : (X, \mathcal{D}(A))_{\theta,1} \longrightarrow Y$$

on the space D. In case $\theta = 0$ it is equivalent to the fact that C (defined on $D = \mathcal{D}(A)$) is bounded for the norm $\| \cdot \|_X$.

Proof. We choose α such that (1) holds for $\nu = \pi$. Let us start with the case $0 < \theta < 1$. If $C : (X, \mathcal{D}(A))_{\theta,1} \longrightarrow X$ is bounded, we can estimate

$$\|t^{1-\theta}C(t + A)^{-1}x\| \leq ct^{1-\theta}\|(t + A)^{-1}x\|_{(X,\mathcal{D}(A))_{\theta,1}}$$

$$\leq ct^{1-\theta}\|(t + A)^{-1}x\| + ct^{1-\theta}\int_0^\infty \left\|s^\theta A(s + A)^{-1}(t + A)^{-1}x\right\|\frac{ds}{s}$$

$$\leq ct^{-\theta}\alpha + c\int_0^\infty \left\|s^{\theta-1}A^{1-\theta}(1 + \frac{1}{s}A)^{-1}t^{-\theta}A^\theta(1 + \frac{1}{t}A)^{-1}x\right\|\frac{ds}{s}$$

$$= ct^{-\theta}\alpha + c\int_0^\infty \left\|\varphi_{1/s}(A)\psi_{1/t}(A)x\right\|\frac{ds}{s}$$

for $x \in X$, $t > 0$ and some constant c, where we have used the notation $\varphi(z) := z^{1-\theta}/1 + z$ and $\psi(z) := z^\theta/1 + z$. Using the functional calculus via the Cauchy integral as in [20] one sees that the second summand is uniformly bounded in $t > 0$.

Assume now that (12) holds, again under the assumption $0 < \theta < 1$. We want to show the boundedness of $C : (X, \mathcal{D}(A))_{\theta,1} \longrightarrow Y$, and since $\mathcal{D}(A) = \mathcal{D}(A+1)$, we can assume without restriction that A is invertible and

$$c' := \sup_{t>0} \|t^{1-\theta} C(t+A)^{-1}\| < \infty.$$

From general interpolation theory (see, e.g. [18, Prop. 1.2.12]) we know that $\mathcal{D}(A^2)$ is dense in $(X, \mathcal{D}(A))_{\theta,1}$. Let $\tau(z) := \frac{z}{(1+z)^2}$ and $c := \int_0^\infty \tau(s^{-1}) \frac{ds}{s} > 0$. Then the theory of functional calculus yields $x = c^{-1} \int_0^\infty \tau(s^{-1}A)x \frac{ds}{s}$ for all $x \in \mathcal{D}(A)$. So, for $x \in \mathcal{D}(A^2)$ this integral converges in $\mathcal{D}(A)$, whence

$$\begin{aligned}
\|Cx\| &\leq c^{-1} \int_0^\infty \|C\tau(s^{-1}A)x\| \frac{ds}{s} \\
&= c^{-1} \int_0^\infty \|C(1+s^{-1}A)^{-1}s^{-1}A(1+s^{-1}A)^{-1}x\| \frac{ds}{s} \\
&= c^{-1} \int_0^\infty \|s^{1-\theta}C(s+A)^{-1}s^\theta A(s+A)^{-1}x\| \frac{ds}{s} \\
&\leq c'c^{-1} \int_0^\infty \|s^\theta A(s+A)^{-1}x\| \frac{ds}{s} \leq c'c^{-1} \|x\|_{(X,\mathcal{D}(A))_{\theta,1}}
\end{aligned}$$

for $x \in \mathcal{D}(A^2)$. Since $\mathcal{D}(A^2)$ is dense in $(X, \mathcal{D}(A))_{\theta,1}$, we obtain the desired result.

We are left to deal with the cases $\theta = 0, 1$. It is easily seen that condition (12) with $\theta = 1$ is equivalent to $\|C\|_{\mathcal{D}(A) \to Y} < \infty$. So let $\theta = 0$. If C is bounded for the norm $\|\cdot\|_X$ then clearly (12) holds, by sectoriality of A. For the converse assume $\|\lambda C(\lambda + A)^{-1}x\|_Y \leq \gamma\|x\|$ for each $x \in X$ and $\lambda > 1$. Let $x \in \mathcal{D}(A)$. Then

$$\|Cx\| = \|\lambda C(\lambda + A)^{-1}x\| + \|C(\lambda + A)^{-1}Ax\| \leq \gamma\|x\| + \frac{1}{\lambda}\|Ax\|$$

for all $\lambda > 1$. Letting $\lambda \to \infty$ yields $\|Cx\| \leq \gamma\|x\|$ for all $x \in \mathcal{D}(A)$. $\qquad\square$

Proposition 4.6. *Let A be an injective sectorial operator on the Banach space X, let $C : \mathcal{D}(A) \longrightarrow Y$ a bounded operator, where Y a second Banach space, and let $\theta \in [0,1]$. Then the condition*

$$\sup_{0<\lambda<1} \|\lambda^{1-\theta}C(\lambda + A)^{-1}\|_{X \to Y} \tag{13}$$

is trivially satisfied for $\theta = 0$. In case $0 < \theta < 1$ it is equivalent to the boundedness of

$$CA^{-1} : (X, \mathcal{R}(A))_{1-\theta,1} \longrightarrow Y$$

on $R = \mathcal{R}(A)$. In case $\theta = 1$, it is equivalent to the boundedness of $C : \dot{D} \longrightarrow Y$ on $D = \mathcal{D}(A)$.

Proof. Assume $0 < \theta < 1$. Writing $\mu = \frac{1}{\lambda}$ yields $\lambda^{1-\theta}C(\lambda + A)^{-1} = \lambda^{-\theta}C\lambda(\lambda + A)^{-1} = \mu^\theta C[I - \mu(\mu + A^{-1})^{-1}]^{-1} = \mu^\theta[CA^{-1}](\mu + A^{-1})^{-1}$. Now one can apply Proposition 4.5 with A replaced by A^{-1} and θ replaced by $1 - \theta$.

Let us look at the remaining choices for θ. The case $\theta = 0$ being easy, we assume $\theta = 1$. If $C : \dot{D} \longrightarrow Y$ is bounded, one has $\|C(\lambda + A)^{-1}x\|_Y \leq \|C\|_{\dot{D} \to Y}\|A(\lambda + A)^{-1}x\| \leq \|C\|_{\dot{D} \to Y}(\alpha + 1)$, hence (13). Conversely, suppose $\|C(\lambda + A)^{-1}x\| \leq \gamma\|x\|$ for all $x \in X$ and all $0 < \lambda < 1$. Then

$$\|Cx\| \leq \lambda\|C(\lambda + A)^{-1}x\| + \|C(\lambda + A)^{-1}Ax\| \leq \lambda\gamma\|x\| + \gamma\|Ax\|$$

for $x \in \mathcal{D}(A)$. Hence, if we let $\lambda \to 0$ we obtain $\|Cx\| \leq \gamma\|x\|_{\dot{D}}$. $\qquad\square$

Again, we combine the last two propositions with the abstract interpolation result above.

Corollary 4.7. *Let A be an injective sectorial operator on the Banach space X, let $C : \mathcal{D}(A) \longrightarrow Y$ be a bounded operator, where Y is another Banach space, and let $\theta \in [0, 1]$. Then the condition*

$$\sup_{0 < \lambda < \infty} \|\lambda^{1-\theta} C(\lambda + A)^{-1}\|_{X \to Y} < \infty$$

is equivalent to the boundedness of

$$\begin{cases} C : X \longrightarrow Y, & \text{if } \theta = 0 \\ C : (X, \dot{D})_{\theta,1} \longrightarrow Y, & \text{if } 0 < \theta < 1 \\ C : \dot{D} \longrightarrow Y, & \text{if } \theta = 1 \end{cases}$$

on $D = \mathcal{D}(A)$.

Proof. In the cases $\theta = 0, 1$ this is just a combination of Propositions 4.5 and 4.6. Assume $0 < \theta < 1$. Since A is injective, it induces a topological isomorphism

$$A : (\dot{D}, \mathcal{D}(A)) \longrightarrow (X, \mathcal{R}(A))$$

of Banach couples. Hence the boundedness of $CA^{-1} : (X, \mathcal{R}(A))_{1-\theta,1} \longrightarrow Y$ is then equivalent to the boundedness of $C : (\dot{D}, \mathcal{D}(A))_{1-\theta,1} \longrightarrow Y$. This yields the boundedness of $C : (X, \mathcal{D}(A))_{\theta,1} + (\dot{D}, \mathcal{D}(A))_{1-\theta,1} \longrightarrow Y$ as a characterisation. However, part c) of Proposition 4.1 applies and we obtain the identity $(X, \mathcal{D}(A))_{\theta,1} + (\dot{D}, \mathcal{D}(A))_{1-\theta,1} = (X, \dot{D})_{\theta,1}$ which finishes the proof. $\qquad\square$

Let us apply the foregoing to the situation described in Section 3. We will utilise the following auxiliary notation:

$$(\dot{D}, X)^{\bullet}_{\theta,1} := \begin{cases} \dot{D} & \text{if } \theta = 0 \\ (\dot{D}, X)_{\theta,1} & \text{if } 0 < \theta < 1 \\ X & \text{if } \theta = 1 \end{cases}$$

Corollary 4.8. *Let A be an injective sectorial operator on a Banach space X. Let Y be another Banach space, $C : D \longrightarrow Y$ and $B : Y \longrightarrow R_{-1}$ linear mappings, and $\theta \in [0, 1]$. Then we have the following equivalences:*

    (a) $\sup_{\lambda > 0} \|\lambda^{\theta}(\lambda + A_{-1})^{-1} B\|_{Y \to X} < \infty$ *if, and only if $B : Y \longrightarrow R_{-1}$ is bounded and $\mathcal{R}(B) \subset (X, \dot{R})_{1-\theta,\infty}$.*

    (b) $\sup_{\lambda > 0} \|\lambda^{1-\theta} C(\lambda + A)^{-1}\|_{X \to Y} < \infty$ *is equivalent to the boundedness of*

$$C : (\dot{D}, X)^{\bullet}_{1-\theta,1} \longrightarrow Y$$

    on the space D.

Proof. The second assertion is just Corollary 4.7. For the first, apply Corollary 4.4 to the operator A_{-1} on the space R_{-1} with domain X. $\qquad\square$

5. Main Theorems

In this section we combine the results from Section 3 with the characterisations obtained in Section 4. This leads to our main results on perturbation of sectorial operators: Theorems 5.1 and 5.3 cover perturbations for which a suitable norm is small, and Theorem 5.5 studies compact perturbations. The latter is an extension of [6, Thm. 1] which covers the case $\theta = 1$. At the end (Remarks 5.6) we give some concrete examples of intermediate spaces to which our results apply.

Theorem 5.1. *Let $\omega \in (0, \pi]$ and let A be an injective sectorial operator of type $< \omega$ on a Banach space X. Let $\theta \in [0, 1]$ and let $Z, W \hookrightarrow X_{-1}$ be Banach spaces with*

$$(\dot{D}, X)^{\bullet}_{1-\theta,1} \subset Z \quad \text{and} \quad W \subset (X, \dot{R})_{1-\theta,\infty}.$$

Moreover assume that $[A_{-1}]^{-1} : W \longrightarrow Z$ continuously. Assume either one of the following two conditions.

(a) A_{-1} *restricts to a sectorial operator of type* $< \omega$ *on* Z.

(b) A_{-1} *restricts to a sectorial operator of type* $< \omega$ *on* W.

Let Y *be another Banach space and* $B : Y \to W$, $C : Z \to Y$ *be bounded operators. Then, if* $\|C\|_{Z \to Y} \cdot \|B\|_{Y \to W}$ *is sufficiently small, the operator* $\tilde{P} = A_{-1} - BC$ *(defined originally on* $Z \cap D_{-1}$*) restricts to an injective sectorial operator* P *of type* $< \omega$ *on* X. *Furthermore,* P *is densely defined/invertible if* A *is. If* A *has dense range and* $\theta \neq 1$ *or* X *is reflexive, then also* P *has dense range.*

Proof. Assume Hypothesis (a). We want to apply the main Lemma 3.3 (Lemma 3.6, resp.) with $E := Z \cap D_{-1}$. Clearly, $D \subset E \subset D_{-1}$. Furthermore, since Z is invariant under each operator $(\lambda + A_{-1})^{-1}$ by assumption, also E is. Again by assumption, $[A_{-1}]^{-1}$ maps W continuously into Z, but since $W \subset R_{-1}$, actually $[A_{-1}]^{-1}(W) \subset E$. In particular one has $W \subset F := A_{-1}(E) \subset \tilde{E} \cap R_{-1}$. Hence we have indeed $B : Y \longrightarrow \tilde{E} \cap R_{-1}$ as required. Corollary 4.8 shows that the growth conditions (6) and (5) are satisfied for certain constants β, γ. Therefore, we are left to show the estimate (8)

$$\|C(\lambda + A_{-1})^{-1}B\|_{Y \to Y} \leq \|C\|_{Z \to Y} \|A_{-1}(\lambda + A_{-1})^{-1}\|_{Z \to Z}$$
$$\cdot \|[A_{-1}]^{-1}\|_{W \to Z} \|B\|_{Y \to W}$$

for $\lambda \in S_{\pi - \omega}$ and since $\alpha_Z := \sup_{\lambda \in S_{\pi - \omega}} \|A_{-1}(\lambda + A_{-1})^{-1}\|_{Z \to Z} < \infty$ by assumption, we obtain (8) for some $0 < \eta < 1$ whenever $\varepsilon := \|C\|_{Z \to Y} \cdot \|B\|_{Y \to W}$ is small. The estimate $\|C[A_{-1}]^{-1}B\| \leq \varepsilon \|[A_{-1}]^{-1}\|_{W \to Z} < 1$ shows that also the last statement of Lemma 3.4 is applicable which implies that the operator P is in fact injective. The remaining statements also follow from Lemma 3.4.

Assume Hypothesis (b). Since we can replace W by $W + R$ without changing anything, we can assume $R \subset W$. Define $E := [A_{-1}]^{-1}(W)$. Then $D \subset E \subset Z \cap D_{-1} \subset D_{-1}$ as required and $F := A_{-1}E = W$. Moreover, E is $(\lambda + A_{-1})^{-1}$-invariant, since W is (by assumption). Again, Corollary 4.8 shows that the growth conditions (6) and (5) are satisfied for certain constants β, γ. The remaining part of the proof is similar as in the first case.

Note that in both cases one has to check that the part of $\tilde{P} = A_{-1} - BC$ (defined on $Z \cap D_{-1}$) in X is the same as the part of $\tilde{P}$ (defined on E) in X. $\qquad\square$

Remark 5.2. Let us shortly look at the situation when A is invertible. In this case $X_{-1} = R_{-1} = \dot{R}$, $D_{-1} = X = R$ and $\dot{D} = D = D \cap R$. Given the hypotheses of Theorem 5.1 and replacing (now without restriction) Z by $Z \cap X$ we obtain the picture

$$\mathcal{D}(A) - - - (X, \mathcal{D}(A))^{\bullet}_{\theta,1} - - - Z \overset{[A_{-1}]^{-1}}{- - -} X - - - W - - - (X_{-1}, X)_{\theta,\infty} - - - X_{-1}$$

$$C \searrow \qquad \nearrow B$$
$$Y$$

where the dashed horizontal lines indicate inclusions. The conclusion of Theorem 5.1 is that $A_{-1} - BC$ restricts to an invertible sectorial operator on X.

We recall (cf. Introduction) that in case $\theta \in (0, 1)$ it is not known if in our general situation one can hope for persistence of fractional domain spaces or complex interpolation spaces for the perturbed operator. It is therefore most remarkable, that certain real interpolation spaces between X and the homogeneous spaces $\dot{D}$ respectively. $\dot{R}$ (which we will be denoted by $\dot{D}(A)$ and $\dot{R}(A)$ for obvious reasons) coincide with those spaces between X and the homogeneous spaces given with respect to P (that will be denoted by $\dot{D}(P)$ and $\dot{R}(P)$).

Theorem 5.3. *Let $\omega \in (0, \pi]$ and let A be an injective sectorial operator of type $< \omega$ on a Banach space X. Let $\theta \in [0, 1]$ and let Z be a Banach space with*

$$(\dot{D}, X)^{\bullet}_{1-\theta,1} \hookrightarrow Z \hookrightarrow (\dot{D}, X)_{1-\theta,\infty}. \tag{14}$$

Assume further that A_{-1} restricts to a sectorial operator of type $< \omega$ on Z. Then for each $T \in B(Z)$ with sufficiently small norm the operator $\tilde{P} := A_{-1}(1 - T)$ (defined on Z) restricts to an injective sectorial operator P of type $< \omega$ on X. If A is densely defined/invertible, so is P. If A has dense range and $\theta \neq 1$ (or X is reflexive) then also P has dense range. Moreover, if $\theta \in (0, 1)$, then

$$(X, \dot{D}(A))_{\sigma,q} = (X, \dot{D}(P))_{\sigma,q} \quad and \quad (\dot{R}(A), X)_{\tau,q} = (\dot{R}(P), X)_{\tau,q}$$

for every $\sigma \in (0, \theta)$, $\tau \in (\theta, 1)$ and $q \in [1, \infty]$.

Proof. For the first assertion choose $W = A_{-1}(Z)$, $Y = Z$, $C = T$ and $B = A_{-1}$ and apply Theorem 5.1.

For the second assertion let $\theta \in (0, 1)$. In the sequel we make use of reiteration for real interpolation spaces and refer e.g. to [24, Section 1.10].

First, set $C := \text{Id}_Z$, $Y := Z$ and $B := -A_{-1}T$. Then the assumptions (5) and (6) of Lemma 3.3 (and Corollary 3.8) are satisfied by an application of Corollary 4.8 to equation (14). Thus we obtain the uniform boundedness of $\|\lambda^{1-\theta}(\lambda+P)^{-1}\|_{X \to Z}$, which, again by Corollary 4.8, yields

$$(X, \dot{D}(P))_{\theta,1} \hookrightarrow Z. \tag{15}$$

Since $\widetilde{P}(X, \dot{D}(P))_{\theta,1} = (\dot{R}(P), X)_{\theta,1}$ (cf. 2.2 (a)) we obtain $(\dot{R}(P), X)_{\theta,1} \hookrightarrow \widetilde{P}(Z)$. Notice that $\widetilde{P}(Z) \hookrightarrow W$ since $\widetilde{P}(A_{-1})^{-1} = A_{-1}(I + T)(A_{-1})^{-1}$ is bounded from $W \to W$ and surjective if $\|T\| < 1$. So,

$$\| \cdot \|_W \leq c \|A_{-1}\widetilde{P}^{-1} \cdot \|_W = \|\widetilde{P}^{-1} \cdot \|_Z.$$

Summing up, we obtain

$$(\dot{R}(P), X)_{\theta,1} \hookrightarrow W. \tag{16}$$

Now we set $C := -A_{-1}T$, $Y := W$ and $B := \text{Id}_W$. Observe, that since $(\dot{R}(A), X)_{\theta,\infty} = A_{-1}(X, \dot{D}(A))_{\theta,\infty}$,

$$(\dot{R}(A), X)_{\theta,1} \hookrightarrow W \hookrightarrow (\dot{R}(A), X)_{\theta,\infty}. \tag{17}$$

Additionally, $A_{-1}T : (X, \dot{D}(A))_{\theta,1} \to W$. Applying Corollary 4.8 to these embeddings shows the assumptions (5) and (6) of Lemma 3.3 (and Corollary 3.8) to hold true. Consequently, $\|\lambda^\theta R_\lambda\|_{W \to X}$ is uniformly bounded, and Corollary 4.8 gives us

$$W \hookrightarrow (\dot{R}(P), X)_{\theta,\infty}. \tag{18}$$

So, $P^{-1}(W) \hookrightarrow (X, \dot{D}(P))_{\theta,\infty}$. But $\|\widetilde{P} \cdot \|_W = \|A_{-1}(\text{Id}+T) \cdot \|_W = \|(\text{Id}+T) \cdot \|_Z \leq c\| \cdot \|_Z$, whence

$$Z \hookrightarrow (X, \dot{D}(P))_{\theta,\infty}. \tag{19}$$

Let $\sigma \in (0, \theta)$ and $q \in [1, \infty]$. Set $\nu := \sigma/\theta \in (0, 1)$. Then,

$$\begin{aligned}
(X, \dot{D}(P))_{\sigma,q} &= \left(X, (X, \dot{D}(P))_{\theta,1}\right)_{\nu,q} = (X, Z)_{\nu,q} \\
&= \left(X, (X, \dot{D}(A))_{\theta,1}\right)_{\nu,q} = (X, \dot{D}(A))_{\sigma,q}
\end{aligned}$$

by reiteration using (14), (15) and (19). On the other hand, if $\tau \in (\theta, 1)$, set $\nu := (\tau-\theta)/(1-\theta) \in (0, 1)$. Then, $\tau = (1 - \nu)\theta + \nu$ and therefore

$$\begin{aligned}
(\dot{R}(P), X)_{\tau,q} &= \left((\dot{R}(P), X)_{\theta,\infty}, X\right)_{\nu,q} = (W, X)_{\nu,q} \\
&= \left((\dot{R}(A), X)_{\theta,\infty}, X\right)_{\nu,q} = (\dot{R}(A), X)_{\tau,q}
\end{aligned}$$

by reiteration using (16), (17) and (18). This finishes the proof. $\qquad\square$

Remark 5.4. Assume the hypotheses of the last theorem. If one defines $W :=$ $A_{-1}(Z)$ with the induced norm, then bounded operators $S : Z \longrightarrow W$ correspond to operators $[A_{-1}]^{-1}T$ with $T \in B(Z)$. Therefore, Theorem 5.3 has an equivalent formulation as a perturbation result for additive perturbations $S : Z \to W$ of the operator A_{-1}.

In the following result we can forget about extrapolation spaces.

Theorem 5.5. *Let* $\omega \in (0, \pi]$ *and let* A *be a sectorial operator of type* $< \omega$ *on a Banach space* X. *Let* $\theta \in [0, 1]$ *and let* Z *be a Banach space with*

$$(X, \mathcal{D}(A))_{\theta,1}^{\bullet} \subset Z \subset (X, \mathcal{D}(A))_{\theta,\infty}. \tag{20}$$

Assume that A *restricts to a densely defined sectorial operator of type* $< \omega$ *on* Z. *Let* $T : Z \longrightarrow Z$ *be a compact operator. Then, for* $\nu > 0$ *large enough the operator* $P := \nu + A(1 + T)$ *with its natural domain* $\mathcal{D}(P) = \{z \in Z \mid (1 + T)z \in \mathcal{D}(A)\}$ *is an invertible, sectorial operator of type* $< \omega$ *on* X *which is densely defined if* A *is. If* $\theta \in (0, 1)$, *then* $(X, D(A))_{\sigma,q} = (X, D(P))_{\sigma,q}$ *for every* $\sigma \in (0, \theta)$ *and* $q \in [1, \infty]$.

Proof. We will apply Theorem 5.1 with A replaced by $\tilde{A} := A + \nu$ where $\nu > 0$ has to be chosen appropriately. We let $W := (\nu + A_{-1})Z$ (with the induced norm), $Y = Z, C = I$, and $B := A_{-1}T$. Since $\tilde{A}$ is sectorial and invertible, almost all conditions of Theorem 5.1 are satisfied. We only have to make sure that $c_\nu := \|C\|_{Z \to Y} \cdot \|B\|_{Y \to \tilde{Z}} = \|(\nu + A_{-1})^{-1}A_{-1}T\|_{Z \to Z} = \|A_{-1}(\nu + A_{-1})^{-1}T\|_{Z \to Z}$ is "small enough". From the proof of Theorem 5.1 we see that this means that $c_\nu < \left[(\alpha_\nu + 1)\|(\nu + A_{-1})^{-1}\|_{W \to Z}\right]^{-1}$, where α_ν is the sectoriality constant of the operator $(\nu + A)$ within Z. This constant is obviously bounded in ν. But $\|(\nu + A_{-1})^{-1}\|_{W \to Z} = 1$ by definition. So we need to know that $\|A(\nu + A)^{-1}T\|_{Z \to Z} \to 0$ as $\nu \to \infty$. Now, since $A\big|_Z$ is densely defined, $A(\nu + A)^{-1} \to 0$ strongly in Z, whence also uniformly on compacts. By compactness of T, this implies $\|A(\nu + A)^{-1}T\|_{Z \to Z} \to 0$.

Up to now we know that $\tilde{P} := A_{-1}(1 + T)$, defined on Z, restricts to an invertible sectorial operator P on X. However, it is easy to see that P indeed has the domain $\{x \in Z \mid (1 + T)x \in \mathcal{D}(A)\}$. The last assertion follows immediately from Theorem 5.3. □

We conclude this section with a list of spaces Z, to which our Theorems 5.3 and 5.5 may be applied.

Remarks 5.6.

1) If $\theta = 1$ we may take $Z = \dot{D}$ in Theorem 5.3 and $Z = \mathcal{D}(A)$ in Theorem 5.5. Then A_Z is sectorial and the assertion is well known (cf. [9, Thm. III.2.10], [6, Thm. 1]). If $\theta = 0$, we may always take $Z = X$. Also in this case the assertions are known.

2) If $\theta \in (0, 1)$, then we may take as Z the homogeneous fractional domain space $\dot{D}_\theta$ (see the end of Section 2) in Theorem 5.3, and we may take $Z := D_\theta = \mathcal{D}(A^\theta)$ in Theorem 5.5. In case A is densely defined, such perturbations have been studied in [16] (cf. Rem. 17 there). Note that in both cases A_Z is (even isometrically) similar to A and therefore sectorial of the same type as A.

3) Let $\theta \in (0, 1)$ and $\mathcal{F}$ be an interpolation functor of type θ (cf. [24, 1.2.2]). Then it is well known that $Z := \mathcal{F}(\dot{D}, X)$ satisfies the assumption of Theorem 5.3 and that $Z := \mathcal{F}(\mathcal{D}(A), X)$ satisfies the assumption (20) of Theorem 5.5 (cf. [2, 3.9.1]). Sectoriality of A_Z in Z is obtained by interpolation. Examples for interpolation spaces $\mathcal{F}(X, Y)$ with respect to interpolation functors $\mathcal{F}$ of type $\theta \in (0, 1)$ are
   - real interpolation spaces $(X, Y)_{\theta,q}$ where $q \in [1, \infty]$ (cf. [24, 1.3.2], [2, 3.4]),

- complex interpolation spaces $[X, Y]_\theta$ (cf. [24, 1.9.2], [2, 4.1]),

If we exclude the case $q=\infty$ in these examples, then A_Z is densely defined in Z if A is densely defined in X.

6. Maximal Regularity

In this section we present perturbation results for maximal L^p-regularity and for R-sectoriality. Again, the considerations in Section 3 are crucial but the assumptions on boundedness in norm are replaced by corresponding R-boundedness assumptions (Proposition 6.5). The characterisations of Section 4, however, have only a partial counterpart for these R-boundedness conditions (cf. Lemma 6.9). Besides Proposition 6.5, the main results in this section are Theorems 6.11 and Theorem 6.12.

We briefly survey maximal L^p-regularity and its connection with R-sectoriality. Let $-A$ be the generator of a bounded analytic semigroup $T(\cdot)$ on a Banach space X. It is well known that the Cauchy problem

$$x'(t) + Ax(t) = f(t), \quad t \geq 0, \quad x(0) = 0 \tag{21}$$

has a unique mild solution $x \in L^p_{\mathrm{loc}}(\mathbb{R}_+, X)$ for every $f \in L^p(\mathbb{R}_+, X)$, $p \in (1, \infty)$. The operator A is said to have *maximal L^p–regularity on* $[0, \infty)$, if for every $f \in L^p([0, \infty), X)$, the unique solution $x = T * f$ to (21) is almost everywhere differentiable, has values in $\mathcal{D}(A)$ almost everywhere, and there is a constant $C > 0$ with

$$\|x'\|_{L^p([0,\infty),X)} + \|Ax\|_{L^p([0,\infty),X)} \leq C\|f\|_{L^p([0,\infty),X)}.$$

In UMD-spaces X, the property of maximal L^p-regularity has been characterised by WEIS [26], see Theorem 6.4 below. We first recall some notations:

Definition 6.1. Let X, Y be Banach spaces and $\mathcal{T}$ be a set of operators in $B(X, Y)$. The set $\mathcal{T}$ is called R–*bounded* if there is a constant C such that for all $T_1, \ldots, T_m \in \mathcal{T}$ and $x_1, \ldots, x_m \in X$

$$\left(\mathbb{E} \left\| \sum_{n=1}^m r_n T_n x_n \right\|_Y^2 \right)^{1/2} \leq C \left(\mathbb{E} \left\| \sum_{n=1}^m r_n x_n \right\|_X^2 \right)^{1/2} \tag{22}$$

where (r_n) is a sequence of independent symmetric $\{1, -1\}$-valued random variables, e.g., the Rademacher functions $r_n(t) := \operatorname{sgn} \sin(2^n \pi t)$ on $[0, 1]$. The infimum of all constants C, for which (22) holds is called the R–*bound* of the set $\mathcal{T}$, and is denoted by $[\![\, \mathcal{T} \,]\!]^{\mathcal{R}}_{X \to Y}$.

Remarks 6.2. 1) R–boundedness of $\mathcal{T}$ in $B(X)$ implies uniform boundedness of $\mathcal{T}$ in $B(X)$, but the converse holds only in Hilbert spaces. However, if X has cotype 2 and Y has type 2, then R–boundedness and boundedness of $\mathcal{T}$ in $B(X, Y)$ are equivalent. These geometric conditions on X and Y cannot be weakened, as Example 6.13 shows.

2) Let us introduce the space

$$\mathrm{Rad}(X) := \overline{\mathrm{span}}\{\sum_{k=1}^n r_k x_k \,\big|\, n \in \mathbb{N}, x_1, \ldots, x_n \in X\} \subset L^2([0, 1], X).$$

Then R–boundedness of a set $\mathcal{T} \subseteq B(X, Y)$ may be rephrased as follows: There is a constant $C > 0$ such that any diagonal operator $\tilde{T}$,

$$\tilde{T} : \sum_{k=1}^n r_k x_k \mapsto \sum_{k=1}^n r_k T_k x_k$$

where $n \in \mathbb{N}$, $T_1, \ldots, T_n \in \mathcal{T}$, is bounded from $\mathrm{Rad}(X)$ to $\mathrm{Rad}(Y)$ with a norm of at most C (see, e.g., [17, Sect.2]).

3) If $X = L^p(\Omega)$, $p \in (1, \infty)$ then by Khintchine's and Kahane's inequalities (cf. [7, Chap. 11]) we have

$$\left(\mathbb{E} \left\| \sum_{j=1}^{m} r_j f_j \right\|_p^q \right)^{1/q} \sim \left\| \left(\sum_{j=1}^{m} |f_j|^2 \right)^{1/2} \right\|_p \tag{23}$$

for all $q \in [1, \infty)$ which implies that $\mathrm{Rad}(L^p)$ is isomorphic to $L^p(l_2)$ via $\sum r_j f_j \mapsto (f_j)$, see [17, 2.9]. In particular, the definition of R–boundedness above does not change when replacing the 2 in $(\mathbb{E}\| \ldots \|^2)^{1/2}$ by any $q \in [1, \infty)$.

Definition 6.3. Let $\omega \in [0, \pi)$. A sectorial operator A of type ω is called R–*sectorial* of type $\omega_R \in [\omega, \pi)$, if for all $\nu \in (\omega_R, \pi]$ the set $\{\lambda(\lambda + A)^{-1} \mid \lambda \in S_{\pi - \nu}\} \subset B(X)$ is R–bounded.

We refer to [26] and [17] for more details. Now WEIS' characterisation reads as follows:

Theorem 6.4 ([26, Thm. 4.2]). *Let X be a UMD–space and $-A$ the generator of a bounded analytic semigroup on X. Then A has maximal L^p–regularity for one (and thus all) $p \in (1, \infty)$ if and only if A is R–sectorial of type $\omega_R < {}^\pi/_2$.*

Consequently, perturbation theorems for R–sectorial operators yield results on perturbation of maximal L^p–regularity. We now present an R–sectoriality version of our key lemma (Lemma 3.3 in Section 3) where we also include the assertions of Lemma 3.4, an analog of Corollary 3.8, and Lemma 3.6.

Proposition 6.5. *Let $\omega \in (0, \pi]$, and let A be an injective R-sectorial operator of type $< \omega$ on the Banach space X satisfying*

$$\llbracket \lambda(\lambda + A)^{-1} \mid \lambda \in S_{\pi - \omega} \rrbracket_{X \to X}^{\mathcal{R}} \leq \alpha_R. \tag{24}$$

Let $\mathcal{D}(A) \subset E \subset D_{-1}$ be a vector subspace invariant under the family of operators $\left((\lambda + A_{-1})^{-1}\right)_{\lambda \in S_{\pi - \omega}}$. Suppose a Banach space Y and linear mappings $C : E \longrightarrow Y$, $B : Y \longrightarrow \tilde{E} \cap R_{-1}$ are given such that

$$\llbracket C(\lambda + A_{-1})^{-1} B \mid \lambda \in S_{\pi - \omega} \rrbracket_{Y \to Y}^{\mathcal{R}} \leq \eta_R < 1. \tag{25}$$

Assume furthermore that there exists $\theta \in [0, 1]$ and constants $\beta_R, \gamma_R \geq 0$ such that

$$\llbracket C\lambda^{1-\theta}(\lambda + A)^{-1} \mid \lambda > 0 \rrbracket_{X \to Y}^{\mathcal{R}} \leq \gamma_R, \tag{26}$$

$$\llbracket \lambda^{\theta}(\lambda + A_{-1})^{-1} B \mid \lambda > 0 \rrbracket_{Y \to X}^{\mathcal{R}} \leq \beta_R. \tag{27}$$

Then there are β_R', γ_R' only depending on β_R, γ_R and α_R such that

$$\llbracket \lambda^{1-\theta} C(\lambda + A)^{-1} \mid \lambda \in S_{\pi - \omega} \rrbracket_{X \to Y}^{\mathcal{R}} \leq \gamma_R', \tag{28}$$

$$\llbracket \lambda^{\theta}(\lambda + A_{-1})^{-1} B \mid \lambda \in S_{\pi - \omega} \rrbracket_{Y \to X}^{\mathcal{R}} \leq \beta_R', \tag{29}$$

and the operator

$$P := (A_{-1} - BC)\big|_X$$

is a sectorial operator and satisfies

$$\llbracket \lambda(\lambda + P)^{-1} \mid \lambda \in S_{\pi - \omega} \rrbracket_{X \to X}^{\mathcal{R}} \leq \alpha_R + \frac{\beta_R' \gamma_R'}{1 - \eta_R}. \tag{30}$$

In particular, P is R-sectorial of type $< \omega$. If A is densely defined/invertible, then P is densely defined/invertible. If A has dense range and $\theta \neq 1$ (or X is reflexive), then P has dense range. If $B : Y \longrightarrow F$ and $\|C[A_{-1}]^{-1} B\|_{Y \to Y} < 1$, then P is injective. The estimate (28) holds for P and $\frac{\gamma_R'}{1 - \eta_R}$ in place of A and γ_R', and the estimates (29) and (25) hold for R_λ, $\frac{\beta_R'}{1 - \eta_R}$, $\frac{\eta_R}{1 - \eta_R}$ in place of $(\lambda + A_{-1})^{-1}$, β_R', η_R, where R_λ is as in Lemma 3.2.

Proof. We only have to observe that under our assumptions in the proof of Lemma 3.3 and Lemma 3.6 the given representations also yield R–boundedness. Recall that the power series expansion of the resolvent allows to obtain (30) with an angle $> \pi - \omega$ and a larger constant on the right hand side. $\qquad\square$

An application of WEIS' Theorem 6.4 now yields the following perturbation result for maximal L^p-regularity.

Corollary 6.6. *Let X be a UMD–space and A be an injective operator having maximal L^p-regularity in X. Assume that E, Y, B, and C are as in Proposition 6.5 where $\omega \le {}^\pi\!/_2$. Then P has maximal regularity in X.*

Remark 6.7. For an investigation of conditions (26) and (27) in the style of Section 4 we recall Remark 6.2 and Corollary 4.8. Let $C : D \to Y$ and $B : Y \to R_{-1}$ be bounded. If X has cotype 2 and Y has type 2, then (26) holds if and only if $C : (\dot{D}, X)^\bullet_{1-\theta,1} \to Y$ is bounded on D. If, conversely, Y has cotype 2 and X has type 2 then (27) holds if and only if $\mathcal{R}(B) \subseteq (X, \dot{R})_{1-\theta,\infty}$. But recall that a Banach space Z with type and cotype 2 is isomorphic to a Hilbert space (cf. [7, Cor. 12.20]).

In general, the characterisation of these conditions via interpolation spaces does not extend to the R–case, as Example 6.13 shows. To find appropriate spaces Z and W that allow an application of Proposition 6.5, the following fact shall turn out useful.

Remark 6.8. Let (X, Y) be an interpolation couple and $\mathcal{F}$ an interpolation functor. If X and Y are B–convex, or, equivalently, if X and Y have nontrivial type, then $\mathrm{Rad}(X)$ and $\mathrm{Rad}(Y)$ are complemented in $L^2([0,1], X)$ and $L^2([0,1], Y)$ respectively ([7]). By [24, 1.2.4] we thus obtain $\mathcal{F}(\mathrm{Rad}(X), \mathrm{Rad}(Y)) = \mathrm{Rad}(\mathcal{F}(X, Y))$ (cf. [13, Prop. 3.7]). Consequently, if a set $\mathcal{T}$ of linear operators is R-bounded in both $B(X)$ and $B(Y)$, and if X and Y are B-convex, then $\mathcal{T}$ is also R-bounded in $\mathcal{F}(X, Y)$.

In order to be able to use Remark 6.8, we now assume that X is B–convex. Let A be an injective R–sectorial operator in X, and the question is, for which intermediate spaces Z and W with $D \subseteq Z \subseteq \dot{D} + X$ and $R \subseteq W \subseteq X + \dot{R}$ we have

$$\left[\!\left[\lambda^{1-\theta}(\lambda+A)^{-1} \mid \lambda > 0 \right]\!\right]^{\mathcal{R}}_{X \to Z} < \infty, \tag{31}$$

$$\left[\!\left[\lambda^{\theta}(\lambda+A_{-1})^{-1} \mid \lambda > 0 \right]\!\right]^{\mathcal{R}}_{W \to X} < \infty \tag{32}$$

and that the restriction A_Z of A_{-1} to Z is R-sectorial in Z.
Let us, for simplicity, restrict to the case $W := (A_{-1}(Z), \|A_{-1}^{-1} \cdot\|_Z)$. Hence the part of A_{-1} in W is R-sectorial in W if and only if A_Z is R-sectorial in Z. Moreover, (32) holds if and only if

$$\left[\!\left[\lambda^{\theta} A(\lambda+A_{-1})^{-1} \mid \lambda > 0 \right]\!\right]^{\mathcal{R}}_{Z \to X} < \infty. \tag{33}$$

In the following two lemmas we give three examples of intermediate spaces Z for which (31) and (32) hold.

Lemma 6.9.   (a) *If $\theta \in (0,1)$, then for $Z := \dot{D}_\theta$ the conditions (31) and (33) hold, and the operator A_Z is R-sectorial in Z.*
   (b) *Let $\theta \in (0,1)$ and $Z := [X, \dot{D}]_\theta$. Then A_Z is R-sectorial in Z. Moreover, (31) and (33) hold.*

Proof. (a) Use the fact that for any $s \in (0,1)$ the set $\{\lambda^s A^{1-s}(\lambda+A)^{-1} \mid \lambda > 0\}$ is R–bounded in $B(X)$ (cf. [16, Lem.10]). R-sectoriality of A_Z is due to similarity.
(b) R-sectoriality of A_Z holds by Remark 6.8. (31) can be seen by Stein interpolation (cf. [13, 17]) and Remark 6.8: choose a dense sequence λ_j in $(0, \infty)$ and consider the function $z \mapsto (\lambda_j^{1-z}(\lambda_j + A)^{-1})$ where the operator $(x_j)_j \mapsto (\lambda_j^{1-z}(\lambda_j + A)^{-1} x_j)_j$ acts $\mathrm{Rad}(X) \to \mathrm{Rad}(X)$ for $\mathrm{Re}\, z = 0$ and $\mathrm{Rad}(X) \to \mathrm{Rad}(\dot{D})$ for $\mathrm{Re}\, z = 1$. For (33)

one has to consider the function $z \mapsto (\lambda_j^z A (\lambda_j + A)^{-1})$, acting $\mathrm{Rad}(X) \to \mathrm{Rad}(X)$ for $\mathrm{Re}\, z = 0$ and $\mathrm{Rad}(\dot{D}) \to \mathrm{Rad}(X)$ for $\mathrm{Re}\, z = 1$. $\qquad\square$

Of course, Remark 6.8 yields R-sectoriality of A_Z in all spaces $Z = \mathcal{F}(\dot{D}, X)$ where $\mathcal{F}$ is an interpolation functor, in particular in all real interpolation spaces $(\dot{D}, X)_{\theta,p}$. However, (31) and (33) are false, in general, as Example 6.13 below shows. These conditions hold only for a *certain* real interpolation functor, and only for an even smaller class of spaces X. Although the argument could be made to work in several other Banach lattices, we restrict to L^p-spaces. Let us emphasise that Example 6.13 also shows that the index 2 in the assertion of the following lemma is optimal for real interpolation.

Lemma 6.10. *If* $X = L^p(\Omega)$, $p \in (1, \infty)$ *and* $\theta \in (0, 1)$ *then* (31) *and* (33) *hold for the real interpolation space* $Z := (X, \dot{D})_{\theta, 2}$.

Proof. Again, we choose a dense sequence (λ_n) in $(0, \infty)$. From the R-boundedness in $B(X)$ of the set $\mathcal{T} := \{\lambda(\lambda + A)^{-1} \mid \lambda \in (0, \infty)\}$ it follows readily that $\mathcal{S} := \{(\lambda + A)^{-1} \mid \lambda \in (0, \infty)\}$ is R–bounded in $B(X, \dot{D})$. Using again the R–boundedness of $\mathcal{T}$ we obtain the estimate

$$\left\| \left(\sum_j |\lambda_j (\lambda_j + A)^{-1} f_j|^2 \right)^{1/2} \right\|_p \leq M \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_p,$$

or, equivalently

$$\left\| ((\lambda_j + A)^{-1} f_j) \right\|_{\mathrm{Rad}(X)} \leq C \left\| \left(\sum_j \left| \frac{f_j}{\lambda_j} \right|^2 \right)^{1/2} \right\|_p.$$

Similarly we obtain an estimate

$$\left\| ((\lambda_j + A)^{-1} f_j) \right\|_{\mathrm{Rad}(\dot{D})} \leq K \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_p$$

from R-boundedness of $\mathcal{S}$. Now exploiting that L^p is B–convex for $1 < p < \infty$ and thus $\left(\mathrm{Rad}(X), \mathrm{Rad}(\dot{D}) \right)_{\theta, 2} = \mathrm{Rad}\left(X, \dot{D} \right)_{\theta, 2}$ (cf. Remark 6.8), we obtain the desired estimate

$$\left\| ((\lambda_j + A)^{-1} f_j) \right\|_{\mathrm{Rad}(X, \dot{D})_{\theta,2}} \leq c \left\| \left(\sum_j \left| \frac{f_j}{\lambda_j^{1-\theta}} \right|^2 \right)^{1/2} \right\|_p$$

applying

$$\left(L^p(l_2(\lambda_j^{-1})), L^p(l_2) \right)_{\theta, 2} = L^p\left(\left(l_2(\lambda_j^{-1}), l_2 \right)_{\theta, 2} \right) = L^p\left(l_2(\lambda_j^{\theta-1}) \right),$$

cf. [24, Thm 1.18.4, Thm. 1.18.5]. As before, we may use Remark 6.8 to obtain R–sectoriality of A_Z in $Z = (X, \dot{D})_{\theta, 2}$. $\qquad\square$

Proposition 6.5 in connection with Lemma 6.9 yields the main results of this section.

Theorem 6.11. *Let* X *be a* B–*convex Banach space,* A *an injective* R–*sectorial operator of type* $< \omega$ *in* X. *Let* $\theta \in (0, 1)$ *and* $Z := \dot{D}_\theta$ *or* $Z := [X, \dot{D}]_\theta$.

   (a) *If* $T \in B(Z)$ *has small norm, then* $P := A(1 + T)$ *with natural domain is a densely defined* R–*sectorial operator in* X *of type* $< \omega$.
   (b) *If* $W := (A_{-1}(Z), \|(A_{-1})^{-1} \cdot \|_Z)$, *i.e.* $W = R_{\theta-1}$ *or* $W = [\dot{R}, X]_\theta$ *respectively and* $S : Z \to W$ *has small norm then* $P := (A_{-1} + S)_X$ *is a densely defined* R–*sectorial operator in* X *of type* $< \omega$.

When we combine Proposition 6.5 with Lemma 6.10 we obtain the following.

Theorem 6.12. *Let $X = L^p(\Omega)$, $p \in (1, \infty)$ and A be an R–sectorial operator of type $< \omega$ in X and $\theta \in (0, 1)$. Then the assertions (a) and (b) in Theorem 6.11 hold for $Z := (X, \dot{D})_{\theta,2}$ and $W := (\dot{R}, X)_{\theta,2}$.*

We close this section with the example which we already have referred to several times.

Example 6.13. Let $X = L^2(\mathbb{R})$ and $A = -\Delta$. Then $D = D(A) = W_2^2(\mathbb{R})$ is a Bessel potential space and $\dot{D} = \dot{H}_2^2(\mathbb{R})$ is a Riesz potential space. From now on we drop the $\mathbb{R}$ in notation. For $\theta = \frac{1}{2}$ and $q \in [1, \infty]$ we obtain $(X, D)_{\frac{1}{2},q} = B_{2q}^1$, an (inhomogeneous) Besov space, and $(X, \dot{D})_{\frac{1}{2},q} = \dot{B}_{2q}^1$, a homogeneous Besov space (see [25, 5.1.3]). We now show that $\{\lambda^{\frac{1}{2}}(\lambda+A)^{-1} \mid \lambda > 0\}$ is not R-bounded in $B(L^2, \dot{B}_{2q}^1)$ if $1 \leq q < 2$. In fact, for the arguments we shall give it makes no difference if we take B_{2q}^1 or $\dot{B}_{2q}^1$, since the support of the Fourier transform of the functions we consider in the sequel is uniformly bounded away from 0.

We fix $q \in [1, 2)$. In order to write down the norm of the Besov space we fix a C^∞-function ψ satisfying $\psi(\xi) = 1$ for $|\xi| \leq 1$ and $\psi(\xi) = 0$ for $|\xi| \geq \frac{3}{2}$, and we set $\varphi(\xi) := \psi(\xi) - \psi(\frac{\xi}{2})$. Then $\varphi \in C^\infty$ satisfies $\varphi(\xi) = 1$ for $|\xi| \in [3/2, 2]$ and $\varphi(\xi) = 0$ for $|\xi| \notin [1, 3]$. We set $\varphi_j(\xi) := \varphi(2^{-j}\xi)$ for $\xi \in \mathbb{R}$ and $j \in \mathbb{N}_0$. Observe that $\varphi_j(\xi) = 1$ for $\xi \in [3 \cdot 2^{j-1}, 2^{j+1}] =: I_j$ for any $j \in \mathbb{N}_0$. For a function $g \in L^2$ whose Fourier transform $\widehat{g}$ has compact support $\operatorname{supp} \widehat{g} \subset [3/2, \infty)$ the norm in B_{2q}^1 and in $\dot{B}_{2q}^1$ is equivalent to $(\sum_{j=0}^\infty 2^{qj} \|\xi \mapsto \varphi_j(\xi)\widehat{g}\|_2^q)^{1/q}$. Note that we used Plancherel's identity (in L^2) here, and that the sum is actually finite.

The symbol of the operator $\lambda^{\frac{1}{2}}(\lambda+A)^{-1}$ is $\lambda^{\frac{1}{2}}(\lambda+\xi^2)^{-1}$. We take $\mu_k = \lambda_k^{\frac{1}{2}} = 2^k$ and let $\rho_k(\xi) := 2^k(4^k+\xi^2)^{-1}$ for $k \in \mathbb{N}_0$. Finally we define $g_k \in L^2$ by $\widehat{g_k} := |I_k|^{-\frac{1}{2}} \mathbb{1}_{I_k}$ for $k \in \mathbb{N}_0$. We fix $n \in \mathbb{N}$ and calculate

$$
\left(\mathbb{E} \left\| \sum_{k=0}^{n-1} \varepsilon_k \mu_k (\mu_k^2+A)^{-1} g_k \right\|_{B_{21}^1}^q \right)^{1/q} \sim \left(\mathbb{E} \sum_{j=0}^\infty 2^{qj} \left\| \sum_{k=1}^{n-1} \varepsilon_k \varphi_j \rho_k \widehat{g_k} \right\|_2^q \right)^{1/q}
$$

$$
= \left(\sum_{j=0}^\infty \mathbb{E} \left\| \sum_{k=1}^{n-1} \varepsilon_k 2^j \varphi_j \rho_k \widehat{g_k} \right\|_2^q \right)^{1/q}
$$

$$
\sim \left(\sum_{j=0}^\infty \left(\sum_{k=0}^{n-1} \left\| 2^j \varphi_j \rho_k \widehat{g_k} \right\|_2^2 \right)^{q/2} \right)^{1/q},
$$

where we used the Khintchine-Kahane equivalence (23) in the last step. Observe that by construction $\varphi_j \widehat{g_k} = \delta_{jk} \widehat{g_k}$ which leads to a considerable simplification of the last term, namely to

$$
\sim \left(\sum_{j=0}^{n-1} \left\| 2^j \rho_j \widehat{g_j} \right\|_2^q \right)^{1/q}.
$$

But, substituting $\xi = 2^j \eta$, we obtain

$$
\left\| 2^j \rho_j \widehat{g_j} \right\|_2^2 = |I_j|^{-1} \int_{I_j} \left(\frac{4^j}{4^j + \xi^2} \right)^2 d\xi = 2 \int_{3/2}^2 (1+\eta^2)^{-2} d\eta =: c^2 > 0,
$$

which is independent of $j \in \mathbb{N}_0$. Hence we arrive at

$$
\left(\mathbb{E} \left\| \sum_{k=0}^{n-1} \varepsilon_k \mu_k (\mu_k^2+A)^{-1} g_k \right\|_{B_{2q}^1}^q \right)^{1/q} \sim n^{1/q}, \quad n \in \mathbb{N}.
$$

On the other hand we have

$$
\left(\mathbb{E} \left\| \sum_{k=0}^{n-1} \varepsilon_k g_k \right\|_2^q \right)^{1/q} \sim \left(\sum_{k=0}^{n-1} \|\widehat{g_k}\|_2^2 \right)^{1/2} = n^{1/2}, \quad n \in \mathbb{N}.
$$

Letting $n \to \infty$ we see that there is no constant $C \in (0, \infty)$ satisfying

$$\left(\mathbb{E} \left\| \sum_{k=0}^{n-1} \varepsilon_k \mu_k (\mu_k^2 + A)^{-1} g_k \right\|_{B_{2q}^1}^q \right)^{1/q} \leq C \mathbb{E} \left\| \sum_{k=0}^{n-1} \varepsilon_k g_k \right\|_2, \quad n \in \mathbb{N},$$

and R-boundedness of $\{\mu_k (\mu_k^2 + A)^{-1} \mid k \in \mathbb{N}_0\}$ in $B(L^2, B_{2q}^1)$ and $B(L^2, \dot{B}_{2q}^1)$ fails.

7. Applications and Examples

In this section we illustrate our results. We start with an application to the case of $A = \varepsilon - \Delta$, on $X = L^p(\mathbb{R}^n)$ where $\varepsilon > 0$, $n \in \mathbb{N}$ and $1 < p < \infty$ are fixed. Since all function spaces in the first two examples are spaces on $\mathbb{R}^n$ we shall omit $\mathbb{R}^n$ in notation temporarily.

Example 7.1. Let $A = \varepsilon - \Delta$ where $\varepsilon > 0$. Then $D(A)$ is the usual Sobolev space $W_p^2(\mathbb{R}^n)$ which equals the Bessel potential space $H_p^2(\mathbb{R}^n)$. We have $0 \in \rho(A)$, $D = H_p^2 = \dot{D}$, $R = X = L_p = D_{-1}$, and $X_{-1} = R_{-1} = \dot{R} = H_p^{-2}$. For the domains of the fractional powers we obtain

$$H_p^{2\theta} = \begin{cases} D_\theta, & 0 < \theta < 1, \\ R_\theta, & -1 < \theta < 0. \end{cases}$$

These spaces coincide with the complex interpolation spaces $[X, D]_\theta = H_p^{2\theta}$ and $[R_{-1}, X]_\theta = H_p^{2(\theta-1)}$ for $\theta \in (0, 1)$. By real interpolation we obtain Besov spaces: $(X, D)_{\theta, q} = B_{p,q}^{2\theta}$ and $(R_{-1}, X)_{\theta, q} = B_{p,q}^{2(\theta-1)}$ for $\theta \in (0, 1)$ and $q \in [1, \infty]$. This means that we may apply Theorem 5.3 to operators $T : H_p^{2\theta} \to H_p^{2\theta}$ in the scale of Bessel potential spaces, or via Remark 5.4 to perturbations $S : H_p^{2\theta} \to H_p^{2(\theta-1)}$ (which are already covered by [16]) but also to perturbations $T : B_{p,q}^{2\theta} \to B_{p,q}^{2\theta}$ and $S : B_{p,q}^{2\theta} \to B_{p,q}^{2(\theta-1)}$, respectively, acting in the scale of Besov spaces (which are not covered by [16]). For an application of Theorem 5.5 we need that A_Z is densely defined in Z which means that we have to exclude the case $q = \infty$ in the Besov scale (but instead of $B_{p,\infty}^{2\theta}$ we may use the closure of C_c^∞ in $B_{p,\infty}^{2\theta}$, which is denoted by $\mathring{B}_{p,\infty}^{2\theta}$ in [24]).

In the next example the operator A is not invertible.

Example 7.2. Let $p \in (1, \infty)$ and $A = -\Delta$ on $L_p(\mathbb{R}^n)$. Again, $D(A) = D = H_p^2$. The operator A is injective, densely defined, has dense range, but $0 \in \sigma(A)$. We still have $R_{-1} = H_p^{-2}$, but now $\dot{D} = \dot{H}_p^2 = \dot{D}$ and $\dot{R} = \dot{H}_p^{-2}$ are homogeneous or Riesz potential spaces. As for the other spaces in our diagrams, we have

$$X_{-1} = \{\varphi \mid \mathcal{F}^{-1}(|\xi|^2 (1 + |\xi|^2)^{-2} \hat{\varphi}(\xi)) \in L_p\} = H_p^{-2} + \dot{H}_p^2$$
$$D_{-1} = \{\varphi \mid \mathcal{F}^{-1}(|\xi|^2 (1 + |\xi|^2)^{-1} \hat{\varphi}(\xi)) \in L_p\} = L_p + \dot{H}_p^2.$$

For the homogeneous fractional scale we obtain

$$\dot{H}_p^{2\theta} = \begin{cases} \dot{D}_\theta, & 0 < \theta < 1, \\ \dot{R}_\theta, & -1 < \theta < 0. \end{cases}$$

These spaces coincide with the complex interpolation spaces $[X, \dot{D}]_\theta = \dot{H}_p^{2\theta}$ and $[\dot{R}, X]_\theta = \dot{H}_p^{2(\theta-1)}$ for $\theta \in (0, 1)$. By real interpolation we obtain homogeneous Besov spaces: $(X, \dot{D})_{\theta, q} = \dot{B}_{p,q}^{2\theta}$ and $(\dot{R}, X)_{\theta, q} = \dot{B}_{p,q}^{2(\theta-1)}$ for $\theta \in (0, 1)$ and $q \in [1, \infty]$. This means that we may apply Theorem 5.3 (via Remark 5.4) to perturbations $S : \dot{H}_p^{2\theta} \to \dot{H}_p^{2(\theta-1)}$ in the scale of Riesz potential spaces (which are already

covered by [16]) but also to perturbations $S : \dot{B}^{2\theta}_{p,q} \to \dot{B}^{2(\theta-1)}_{p,q}$ in the scale of homogeneous Besov spaces (which are not covered by [16]). Mutatis mutandis, the remarks on the case $q = \infty$ in the previous example apply also here.

Now we present an example where A is invertible but not densely defined.

Example 7.3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^2$. In $X := C(\overline{\Omega})$, consider $A := -\Delta$ with Dirichlet boundary conditions, i.e., take as domain

$$D = \mathcal{D}(A) = \{u \in X : u \in \bigcap_{1 < p < \infty} W^2_{p,\text{loc}}(\Omega), \Delta u \in X, u|_{\partial\Omega} = 0\}.$$

Then A is sectorial and invertible, in particular $\dot{D} = D$, but D is not dense in X. It is well known that, for $\theta \in (0, \tfrac{1}{2})$, we have

$$Z_\theta := (X, D)_{\theta,\infty} = \{u \in C^{2\theta}(\overline{\Omega}) : u|_{\partial\Omega} = 0\}.$$

Now let $\gamma \in (0,1)$ and $g \in C^\gamma(\overline{\Omega})$ with small norm. Then the multiplication operator $T_g : f \mapsto g \cdot f$ acts $Z_\theta \to Z_\theta$ with small norm for $\theta \in (0, \tfrac{\gamma}{2})$. We conclude by Theorem 5.3 that $P := A_{-1}(1 + T_g)$ with natural domain is sectorial in X and satisfies, for $\theta \in (0, \tfrac{\gamma}{2})$,

$$(X, \mathcal{D}(P))_{\theta,\infty} = \{u \in C^{2\theta}(\overline{\Omega}) : u|_{\partial\Omega} = 0\}.$$

Observe that P is a realisation of $-\Delta(1 + g)$, but in general $\mathcal{D}(P) \neq \mathcal{D}(A)$, thus $\mathcal{D}(P)$ is not accessible.

Example 7.4. Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with a smooth boundary $\partial\Omega$ and $0 \in \Omega$ (take, e.g., the open unit ball in $\mathbb{R}^n$). We are interested in controlling a heat equation in Ω via an interior point observation at 0 and an autonomous boundary feedback of Neumann type. We consider the problem in the following form

$$\begin{cases} \partial_t u(t,x) - \Delta_x u(t,x) &= 0, & (t,x) \in (0,\infty) \times \Omega, \\ \frac{\partial}{\partial\nu} u(t,x) &= u(t,0)g(x), & (t,x) \in (0,\infty) \times \partial\Omega, \\ u(0,x) &= u_0(x), & x \in \Omega. \end{cases} \tag{34}$$

Here $\nu : \partial\Omega \to \mathbb{R}^n$ denotes the outer normal unit vector, and g is a continuous function on the boundary. We assume that g is continued inside Ω in a reasonable way and consider g as a function in $C(\overline{\Omega})$.

Problem (34) is governed by the operator A which is $-\Delta$ restricted to the domain given by $\frac{\partial}{\partial\nu} u|_{\partial\Omega} = u(0)g$. Viewing this operator as a perturbation of the Neumann Laplacian A_0 and denoting the extrapolated version of A_0 by $A_{0,-1}$ we are lead to $A = (A_{0,-1} + A_{0,-1}g \otimes \delta_0)|_X = (A_{0,-1}(I + T_g))|_X$, where $T_g := g \otimes \delta_0$. We discuss realisations in several state spaces X and shall see that there is a close relation to additional regularity properties of g we have to require.

If we take $X = C(\overline{\Omega})$ then $\delta_0 : u \mapsto u(0)$ is continuous on X,

$$D(A_0) = \{u \in \bigcap_{1 < p < \infty} W^2_p(\Omega) : \Delta u \in X, \frac{\partial}{\partial\nu} u|_{\partial\Omega} = 0\},$$

and A_0 is a densely defined sectorial operator, but not injective. For any $g \in C(\overline{\Omega})$ the operator $T_g = A_{0,-1}g \otimes \delta_0$ acts $X \to X$ as a compact operator. Hence Theorem 5.5 applies, and A generates an analytic semigroup in X.

Now we take $X = L^p(\Omega)$ where $n/2 < p < \infty$. Then $D(A_0) = \{u \in W^2_p(\Omega) : \frac{\partial}{\partial\nu} u|_{\partial\Omega} = 0\}$, and A_0 is a densely defined sectorial operator in X, but not injective. By [24, Thm 2.8.1] we see that the Besov space $Z := B^{n/p}_{p,1}(\Omega)$ embeds into $C(\overline{\Omega})$, and that this space is the largest in the class of Besov spaces or spaces $H^{2\theta}_p(\Omega)$, $\theta \in [0,1]$, enjoying this property. It is also the largest space in this class on which δ_0 is continuous (cf. [21, Ex. 2, p.50]). Hence, for $g \in Z$, the operator $T_g : Z \to Z$ is compact. We note that $Z \subset H^{n/p}_p(\Omega)$, the latter being equal to $D((1+A_0)^{n/(2p)})$ for

$n/p < 1 - 1/p$, i.e., for $p > n + 1$ ([23]). Hence, for such p, the space $Z = B_{p,1}^{n/p}(\Omega)$ equals $(X, D(A_0))_{n/(2p),1}$, and Theorem 5.5 shows that A generates an analytic semigroup.

Observe that the condition $g \in C(\overline{\Omega})$ that was sufficient in the state space $C(\overline{\Omega})$ has to be restricted to $g \in B_{p,1}^{n/p}(\Omega)$ in the state space $L^p(\Omega)$, $p > n + 1$. We compare this to what can be said by applying the results from [16]. If $\gamma \in (0,1)$ and $p > \frac{n+1}{1-\gamma}$ then $D((1 + A_0)^{(\gamma+n/p)/2}) = H_p^{\gamma+n/p}(\Omega)$, and the results from [16] allow to study the case $g \in H_p^{\gamma+n/p}(\Omega)$. Thus we would have to impose further restrictions on p and on g depending on the additional regularity $\gamma > 0$ we have to require. In order to point out the gap we remark that $g \in H_p^{\gamma+n/p}(\Omega)$ implies $g|_{\partial\Omega} \in C^\gamma(\partial\Omega)$. Hence it is clear that perturbation in the scale of fractional domains *cannot* be applied if g is *not Hölder continuous* on $\partial\Omega$, but one can still apply our results if $g|_{\partial\Omega} \in B_{p,1}^{(n-1)/p}(\partial\Omega)$.

In the next example we study elliptic operators with coefficients in weak Lebesgue spaces.

Example 7.5. Let $m, j \in \mathbb{N}$ with $2m \geq j$, let $n > j$, and consider $A := (-\Delta)^m$ and $b \in L^{n/j,\infty}(\mathbb{R}^n)$. Assume that $\alpha \in \mathbb{N}_0^n$ is a multi-index of length $|\alpha| = k = 2m - j$. We shall give a sense to $A + b\partial^\alpha$ in $L^p(\mathbb{R}^n)$ where $p \in (1,\infty)$ and $\|b\|_{n/j,\infty}$ is small (with smallness depending on p). By Hölder's inequality for weak Lebesgue spaces $f \mapsto bf$ is bounded $L^q \to L^{r,\infty}$ whenever $r^{-1} = q^{-1} + j/n$ and $r \in (1, n/j)$, $q \in (n/(n-j), \infty)$. We fix such r and q for the moment and clearly have that $b\partial^\alpha : \dot{H}_q^k \to L^{r,\infty}$ is bounded. We denote by $\dot{H}_{q,\infty}^k$ the spaces obtained from the scale $(\dot{H}_{\tilde{q}}^k)_{\tilde{q}\in(1,\infty)}$ by real interpolation $(\cdot,\cdot)_{\theta,\infty}$, e.g., $\dot{H}_{r,\infty}^{2m}$ is $\dot{D}(A)$ for the realisation of A in the space $L^{r,\infty}$.

We let $Z := \dot{H}_{q,\infty}^k$, $W := L^{r,\infty}$. We know that $A^{-1} : L^{\tilde{r}} \to \dot{H}_{\tilde{r}}^{2m}$ and, by Sobolev embedding, $\dot{H}_{\tilde{r}}^{2m} \hookrightarrow \dot{H}_{\tilde{q}}^k$ whenever $\tilde{r} \in (1, n/j)$ and $\tilde{q}^{-1} = \tilde{r}^{-1} - j/n$ (recall $2m = k + j$). Hence $A^{-1} : L^{\tilde{r}} \to \dot{H}_{\tilde{q}}^k$ is bounded for such $\tilde{r}, \tilde{q}$. Real interpolation $(\cdot,\cdot)_{\theta,\infty}$ with a suitable $\theta \in (0,1)$ yields that $A^{-1} : L^{r,\infty} \to \dot{H}_{q,\infty}^k$ is bounded. We also obtain by interpolation that A is sectorial in $L^{r,\infty} = Z$ and in $\dot{H}_{q,\infty}^k = W$. Taking now $X = L^p$ (with p still unspecified) and $\theta \in (0,1)$ we have

$$(X, \dot{D}_1)_{\theta,1} = (L^p, \dot{H}_p^{2m})_{\theta,1} = \dot{B}_{p,1}^{2m\theta}$$

and

$$(\dot{R}_1, X)_{\theta,\infty} = (\dot{H}_p^{-2m}, L^p)_{\theta,\infty} = \dot{B}_{p,\infty}^{-2m(1-\theta)}.$$

All we have to verify in order to apply Theorem 5.1 are, for some $\theta \in (0,1)$, the inclusions

$$\dot{B}_{p,1}^{2m\theta} \hookrightarrow \dot{H}_{q,\infty}^k \quad \text{and} \quad L^{r,\infty} \hookrightarrow \dot{B}_{p,\infty}^{-2m(1-\theta)}.$$

For $q^{-1} - k/n = p^{-1} - (2m\theta)/n$ we have

$$\dot{B}_{p,1}^{2m\theta} \subset \dot{H}_p^{2m\theta} \subset \dot{H}_q^k \subset \dot{H}_{q,\infty}^k,$$

which is the first inclusion. By the following lemma the second inclusion holds for $p > r$ and $r^{-1} = p^{-1} + 2m(1-\theta)/n$. Recall $r^{-1} = q^{-1} + j/n$, $2m = k + j$, and observe that, for $p^{-1} \in (q^{-1}, r^{-1})$ fixed, both conditions on θ coincide and imply $\theta \in (0,1)$.

We conclude that, choosing r and q appropriately, we can apply Theorem 5.1 in any L^p, $p \in (1,\infty)$, if $\|b\|_{n/j,\infty}$ is sufficiently small.

Lemma 7.6. *Let $r, p \in (1,\infty)$ and $s > 0$ such that $r^{-1} = p^{-1} + s/n$. Then*

$$L^{r,\infty}(\mathbb{R}^n) \hookrightarrow \dot{B}_{p,\infty}^{-s}(\mathbb{R}^n).$$

Proof. Let $s > 0$. For all $\tilde{r}, \tilde{p} \in (1, \infty)$ satisfying $\tilde{r}^{-1} = \tilde{p}^{-1} + s/n$ we have by Sobolev embedding $\dot{H}^s_{\tilde{p}'} \hookrightarrow L^{\tilde{r}'}$, and – a forteriori – $\dot{B}^s_{\tilde{p}',1} \hookrightarrow L^{\tilde{r}'}$. Dualisation of these embeddings yields $L^{\tilde{r}} \hookrightarrow \dot{B}^{-s}_{\tilde{p},\infty}$, and we obtain the assertion from these embeddings by real interpolation $(\cdot, \cdot)_{\theta,\infty}$ with a suitable $\theta \in (0, 1)$. $\qquad\square$

Remark 7.7. The arguments of Example (7.5) may be adapted to work for operators on domains. We remark that, for $m = 1$ on $\mathbb{R}^n$, it was shown in [14] that a potential V with small norm in $L^{n/2,\infty}$ gives rise to a form bounded perturbation in L^2, i.e., multiplication by V acts $H^1_2 \to H^{-1}_2$. The arguments used there are more involved, and it does not seem to be clear if they apply in more general situations.

Remark 7.8. Concerning applications of the results in Section 6, the situation is a bit more complicated. Let $X = L^p(\mathbb{R}^n)$ where $p \in (1, \infty)$ and let $\theta \in (0, 1)$. On the one hand, we may apply Theorem 6.11 to perturbations $S : H^{2\theta}_p \to H^{2(\theta-1)}_p$ of $A = \varepsilon - \Delta$ (which are already covered by [16]) or Theorem 6.12 to perturbations $S : B^{2\theta}_{p,2} \to B^{2(\theta-1)}_{p,2}$ of $A = \varepsilon - \Delta$ (which are not covered by [16]). Example 6.13 shows that in general we cannot take arbitrary perturbations in other Besov spaces. On the other hand, Proposition 6.5 can be applied to *certain* perturbations in other scales. As a concrete illustration of this phenomenon we study perturbations of rank one, i.e. perturbations that may be factored through $Y = \mathbb{C}$.

Example 7.9. Let $p \in (1, \infty)$, $\varepsilon > 0$, and consider again $A = \varepsilon - \Delta$ on $X = L^p(\mathbb{R}^n)$. We fix $\theta \in (0, 1)$. The smallest space Z admissible in Theorem 5.1 is $B^{2\theta}_{p,1}$ and the largest space W is $B^{2(\theta-1)}_{p,\infty}$. Hence we consider $\psi \in B^{-2\theta}_{p',\infty} = (B^{2\theta}_{p,1})'$ and $\varphi \in B^{2(\theta-1)}_{p,\infty}$ and the perturbation $S := \psi \otimes \varphi : x \mapsto \psi(x)\varphi$. By Theorem 5.5, a translate of the perturbed operator $(A_{-1} + S)_X$ is sectorial if $\psi \otimes \varphi : B^{2\theta}_{p,q} \to B^{2(\theta-1)}_{p,q}$ and $q \in [1, \infty)$. By Theorem 6.11, a translate of this operator is R–sectorial if $\psi \otimes \varphi : H^{2\theta}_p \to H^{2(\theta-1)}_p$, and, by Theorem 6.12 a translate is R-sectorial if $\psi \otimes \varphi : B^{2\theta}_{p,2} \to B^{2(\theta-1)}_{p,2}$. Now $S = BC$ where $C = \psi : (X, D)_{\theta,1} \to \mathbb{C}, x \mapsto \psi(x)$ and $B =: \mathrm{Id}_{\mathbb{C}} \otimes \varphi : \mathbb{C} \to (R_{-1}, X)_{\theta,\infty}$. For a direct application of Proposition 6.5 we need that the sets $\tau_\psi := \{\psi\lambda^{1-\theta}(\lambda + A)^{-1} \mid \lambda > 0\}$ and $\tau_\varphi := \{\lambda^\theta(\lambda + A_{-1})^{-1}\varphi \mid \lambda > 0\}$ are R-bounded subsets of $B(X, \mathbb{C}) = X' = L^{p'}$ and $B(\mathbb{C}, X) = X = L^p$, respectively. Clearly, the situation in the case $p < 2$ is dual to the situation for $p > 2$, and we only study the latter.

If $p > 2$, then by Remark 6.2 the set τ_φ is R–bounded if τ_φ is bounded, i.e., if $\varphi \in (R_{-1}, X)_{\theta,\infty} = B^{2(\theta-1)}_{p,\infty}$. R–Boundedness of τ_ψ, however, means

$$\Big(\sum_j |\lambda_j^{1-\theta}\psi((\lambda_j + A)^{-1}f_j)|^2\Big)^{1/2} \le C\Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_p \tag{35}$$

for all choices of $\lambda_j > 0$, $f_j \in L^p$. Since

$$\lambda\frac{d}{d\lambda}(\psi\lambda^{1-\theta}(\lambda + A)^{-1}) = (1-\theta)\psi(\lambda^{1-\theta}(\lambda + A)^{-1}) - \psi(\lambda^{1-\theta}(\lambda + A)^{-1})(\lambda(\lambda + A)^{-1})$$

and A is R-sectorial, it is by [26, Rem.3.5(a)] sufficient to have (35) for $\lambda_j = a2^j$, $j \in \mathbb{Z}$, with a constant C independent of $a \in [1, 2]$. In the dual formulation this means

$$\Big\|\Big(\sum_j |\alpha_j(a2^j)^{1-\theta}(a2^j + A_{-1})^{-1}\psi|^2\Big)^{1/2}\Big\|_{p'} \le C\Big(\sum_j |\alpha_j|^2\Big)^{1/2} \tag{36}$$

for all choices of scalars α_j with C independent of $a \in [1, 2]$. In this form, the condition may be seen to hold if ψ belongs to the Triebel-Lizorkin space $F^{-2\theta}_{p',\infty} = (F^{2\theta}_{p,1})'$ (we refer to [24] for definition and properties of these spaces, recall that $F^s_{p,2} = H^s_p$ for $p \in (1, \infty)$ and $s \in \mathbb{R}$).

Now observe $B_{p,1}^s \subset F_{p,q}^s \subset F_{p,\infty}^s \subset B_{p,\infty}^s$ for all $s \in \mathbb{R}$ and $q \in (1, \infty)$. By the following lemma, the operator $A = \varepsilon - \Delta$ is R-sectorial in all spaces $F_{p,q}^s$, $p, q \in (1, \infty)$, $s \in \mathbb{R}$. It is also densely defined in these spaces. Hence Proposition 6.5 applies to $S = \psi \otimes \varphi : F_{p,q}^{2\theta} \to F_{p,q}^{2(\theta-1)}$ for all $q \in (1, \infty)$.

Lemma 7.10. *Let $p, q \in (1, \infty)$. Then $-\Delta$ is R-sectorial of type 0 in all spaces $F_{p,q}^s$, $s \in \mathbb{R}$.*

Proof. Since $F_{p,q}^s \to F_{p,q}^0$, $f \mapsto (1-\Delta)^{s/2}f$ is an isomorphism commuting with $-\Delta$ it is sufficient to study the case $s = 0$. By $f \mapsto (\varphi_j(D)f)$ an isometrical isomorphism is given from $F_{p,q}^0(\mathbb{R}^n)$ onto a closed subspace of $L^p(\mathbb{R}^n, l_q)$. Since $\varphi_j(D)$ commutes with $(\lambda - \Delta)^{-1}$ we only have to show that the set

$$\mathcal{T} := \{\lambda(\lambda - \Delta)^{-1}\mathrm{Id}_{l_q} \mid \lambda \in S_\omega\}$$

is R–bounded in $B(L^p(\mathbb{R}^n, l_q))$ for any $\omega > (0, \pi)$. To this aim we write

$$\lambda(\lambda - \Delta)^{-1}(f_j) = \mathcal{F}^{-1}\big(\xi \mapsto \lambda(\lambda + |\xi|^2)^{-1}\big(\hat{f}_j(\xi)\big)\big),$$

where $\mathcal{F}$ denotes the Fourier transform on $L^p(\mathbb{R}^n, l_q)$. In other words, $M_\lambda := \xi \mapsto \lambda(\lambda + |\xi|^2)^{-1}\mathrm{Id}_{l_q}$ is the symbol of $\lambda(\lambda - \Delta)^{-1}\mathrm{Id}_{l_q}$. For any multi-index $\alpha \in \mathbb{N}_0^n$ the set

$$\{|\xi|^{|\alpha|}\partial_\xi^\alpha \lambda(\lambda + |\xi|^2)^{-1} \mid \xi \in \mathbb{R}^n \backslash \{0\}, \lambda \in S_\omega\}$$

is bounded (in $\mathbb{C}$) whence

$$\{|\xi|^{|\alpha|}\partial_\xi^\alpha M_\lambda(\xi) \mid \xi \in \mathbb{R}^n \backslash \{0\}, \lambda \in S_\omega\}$$

is R–bounded in $B(l_q)$. Now R–boundedness of $\mathcal{T}$ in $B(L^p(\mathbb{R}^n), l_q)$ follows from the R–version of Mikhlin's multiplier theorem in [10, Thm. 3.2]. $\square$

Remark 7.11. Mikhlin's theorem for scalar-valued multipliers on $L^p(\mathbb{R}^n, l_q)$ may likewise be used to show that $-\Delta$ has a bounded $H^\infty(S_\omega)$–calculus in $L^p(\mathbb{R}^n, l_q)$ for any $\omega > 0$.

In Remark 7.8 and Example 7.9 we have $D(A^\theta) = [X, \mathcal{D}(A)]_\theta$ for $\theta \in (0, 1)$, i.e., both pairs (Z, W) in Theorem 6.11 coincide. This property is well-known to hold if A has bounded imaginary powers (BIP) in X (cf. [24, Thm. 1.15.3]). However, equality fails in general if A does not have BIP. In concrete situations this means that fractional domains $D(A^\theta)$ may not be accessible, even if the spaces $[X, \mathcal{D}(A)]_\theta$, which can be obtained from the domain $D(A)$, are known. We illustrate this by a "rough" boundary value problem.

Example 7.12. Fix $p \in (1, \infty)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^\infty$ (for simplicity) and $m \in \mathbb{N}$. Let $A := \sum_{|\alpha| \leq 2m} a_\alpha(x)\partial^\alpha + \nu$ where $a_\alpha \in C(\overline{\Omega})$ be an elliptic operator of order $2m$ and $B_1, \ldots, B_m$ be differential operators with C^∞-coefficients (for simplicity) on the boundary of orders $0 \leq m_1 \leq \ldots \leq m_m \leq 2m-1$ that satisfy the Lopatinskij-Shapiro (or Agranovich-Vishik) condition of parameter-ellipticity (cf., e.g., [8, p.112] or [5, part II] for the definition). If $\nu \geq 0$ is sufficiently large then R–sectoriality of $A_B := A|_{D(A_B)}$, $D(A_B) = \{u \in W_p^{2m}(\Omega) \mid B_j u = 0, j = 1, \ldots, m\}$ in $X = L^p(\Omega)$ was shown in [5], whereas the results in [4] imply that A_B has BIP if $a_\alpha \in C^\gamma(\overline{\Omega})$ for some $\gamma > 0$ and all $|\alpha| = 2m$. Hence, if the coefficients a_α, $|\alpha| = 2m$, are *not* Hölder-continuous then Theorem 6.11 still covers perturbations in the complex interpolation scale, but it is not clear if the results in [16] on perturbations in the domain scale of fractional powers of A_B can be applied to such perturbations. Observe that we have

$$[L^p(\Omega), D(A_B)]_\theta =$$

$$\{u \in H_p^{2m\theta}(\Omega) \mid B_j u = 0 \text{ for all } j \text{ such that } 2m\theta > m_j + 1/p\}$$

for $\theta \in (0, 1)$ (see [23]).

We give a concrete example: Let $m = 1$ and $m_1 = 1$ which means that we have an oblique derivative problem for a second order operator. For $\theta = \frac{1}{2}$ we obtain $[L^p(\Omega), \mathcal{D}(A_B)]_{\frac{1}{2}} = H_p^1(\Omega) = W_p^1(\Omega)$. If the coefficients of A are merely bounded uniformly continuous and $T : W_p^1(\Omega) \to W_p^1(\Omega)$ is an operator of small norm, then a suitable translate of $(A_B)_{-1}(1 + T)$ is again R-sectorial and has maximal L^q–regularity for all $q \in (1, \infty)$.

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