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α-ADMISSIBILITY OF OBSERVATION AND CONTROL OPERATORS

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Abstract. If $T(t) = e^{-tA}$ is a bounded strongly continuous semigroup on some Banach space X, and if $C\colon D(A^m) \to Y$ is a continuous mapping valued in some Banach space Y, we say that C is α-admissible if it satisfies an estimate of the form $\int_0^\infty t^\alpha \|CT(t)x\|^2\, dt \le M^2\|x\|^2$. This extends the usual notion of admissibility, which corresponds to $\alpha = 0$. In the case when $T(t)$ is a bounded analytic semigroup and A has a 'square function estimate', the second named author showed the validity of the so-called Weiss conjecture: C is admissible if and only if $\{t^{\frac{1}{2}} C(t+A)^{-1} \,:\, t > 0\}$ is a bounded set. In this paper, we extend that characterisation to our new setting. We show (under the same conditions on $T(t)$ and A) that α-admissibility is equivalent to an appropriate resolvent estimate.

1. Introduction

Let X be a Banach space, and let $-A$ be the generator of a bounded strongly continuous semigroup $T(t)$ on X. Let Y be another Banach space and let $C\colon D(A) \to Y$ be a linear operator defined on the domain of A. Assume that C is continuous with respect to the norm $\|x\|_1 = \|(I_X + A)x\|_X$ on $D(A)$. By definition, C is admissible for A if there is a constant $M > 0$ such that

$$\int_0^\infty \big\|CT(t)x\big\|_Y^2\, dt \le M^2\|x\|_X^2 \tag{1}$$

for any $x \in D(A)$. The problem of whether an operator C is admissible for A has received much attention recently. For a wide information on this topic, we refer

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the reader to the excellent survey [4] and the references therein. A key observation due to Weiss [18] is that if C is admissible for A, then there is a constant $K > 0$ such that

$$\big(-\mathrm{Re}(\lambda)\big)^{\frac{1}{2}}\|C(\lambda - A)^{-1}\| \leq K \tag{2}$$

for any complex number λ with $\mathrm{Re}(\lambda) < 0$. To find conditions on X, Y or A that ensure that the converse implication '(2) $\Rightarrow$ (1)' holds true is one of the most important questions in the area. Weiss observed very early that this converse is wrong on general Banach spaces. However, the question whether '(2) $\Rightarrow$ (1)' when X and Y are Hilbert spaces remained open for a while under the name of 'Weiss conjecture'. This conjecture was disproved by Jacob, Partington and Pott [5] in the case when $X = Y = \ell^2$, and by Jacob and Zwart [6] in the case when $X = \ell^2$ and $Y = \mathbb{C}$.

In this paper we introduce a generalisation of admissibility that we call α-admissibility. An operator C will be α-admissible (for $\alpha > -1$) if it satisfies an estimate

$$\int_0^\infty t^\alpha \big\|CT(t)x\big\|_X^2\, dt \leq M^2\|x\|^2. \tag{3}$$

Here the observation operator C may be defined only on the domain $D(A^m)$ of a power of A (see Sections 2 and 3 for precise definitions). In this context, there is a natural analogue of (2) which is implied by α-admissibility (see Lemma 3.3) and again, the main question is to find conditions which ensure that the converse holds true.

Condition (3) is quite natural in the case when $T(t)$ is a bounded analytic semigroup, and we will mainly focus on that case. In [11], the second named author showed that if $T(t)$ is bounded analytic, then $A^{\frac{1}{2}}$ is admissible for A if and only if for any Y and any $C\colon D(A) \to Y$, the two conditions (2) and (1) are equivalent. Our main result, namely Theorem 4.2, is a generalisation of that result to α-admissibility. We will also consider the dual situation, that is, α-admissible control operators. As in [11], we will make use of the H^∞ functional calculus ([1, 13]), which is briefly explained in the next section.

2. Preliminaries on semigroups and H^∞ functional calculus

We will use standard notation and results on semigroups that the reader can easily find in e.g. [15] or [2]. Let $-A$ be the generator of a bounded strongly continuous semigroup $T(t)$ on X. We let $\varrho(A)$ denote the resolvent set of A, and for any $\lambda \in \varrho(A)$, we let $R(\lambda, A) = (\lambda - A)^{-1}$ denote the associated resolvent operator. Let I_X denote the identity on X. We are going to use the notions of interpolation and extrapolation spaces in the sense of [2, Section II.5]. For any integer $m \geq 1$, the interpolation space X_m is the domain $D(A^m)$ of the mth power of A, equipped with the norm $\|x\|_m = \|(I_X + A)^m x\|$. We set $X_0 = X$. Then for any $m \geq 1$, the extrapolation space X_{-m} is the completion of X for the norm $\|x\|_{-m} = \|R(-1, A)^m x\| = \|(I_X + A)^{-m} x\|$. The restriction or extension of $T(t)$ to one of these spaces X_m (for $m \in \mathbb{Z}$) is denoted by $T_m(t)$.

We now give a brief account on sectorial operators and H^∞ functional calculus. We refer the reader e.g. to [13, 1, 10, 7] for details and complements. Given any $0 < \theta \leq \pi$, we let $S(\theta)$ be the open sector of all $z \in \mathbb{C}\backslash\{0\}$ such that $\mathrm{Arg}(z) \in (-\theta, \theta)$. Then we let Γ_θ be the boundary of $S(\theta)$, oriented counterclockwise. The set of all bounded holomorphic functions f on $S(\theta)$ is denoted by $H^\infty(S(\theta))$. This is a Banach algebra for the norm $\|f\|_\theta = \sup\{|f(z)| : z \in S(\theta)\}$. We let $H_0^\infty(S(\theta))$ be the subalgebra of all $f \in H^\infty(S(\theta))$ for which there exist positive numbers $\delta > 0, \epsilon > 0$ such that $|f(z)| = O(|z|^{-\delta})$ at ∞, and $|f(z)| = O(|z|^\epsilon)$ at 0.

Let $0 < \omega < \pi$. A densely defined operator $(A, D(A))$ on X is called sectorial of type ω if its spectrum is contained in the closure of $S(\omega)$, and if for any $\theta \in (\omega, \pi)$, there is a constant C_θ such that

$$\|zR(z, A)\| \leq C_\theta, \qquad z \notin \overline{S(\theta)}.$$

It is clear that if $-A$ generates a bounded strongly continuous semigroup, then A is sectorial of type $\frac{\pi}{2}$. Furthermore, $-A$ generates a bounded analytic semigroup if and only if A is sectorial of type $\omega < \frac{\pi}{2}$.

Assume that A is a sectorial operator of type ω. Let $\theta \in (\omega, \pi)$ and let $f \in H_0^\infty(S(\theta))$. We set

$$f(A) := \tfrac{1}{2\pi i} \int_\Gamma f(z) R(z, A)\, dz,$$

where $\Gamma = \Gamma_\gamma$ for some $\gamma \in (\omega, \theta)$. Then $f(A)$ is well defined and belongs to $B(X)$, and its definition does not depend on the choice of γ. Moreover the mapping $f \mapsto f(A)$ is an algebra homomorphism on $H_0^\infty(S(\theta))$.

Definition 2.1. Let A be a sectorial operator of type $\omega \in (0, \pi)$ on a Banach space X and let $\theta \in (\omega, \pi)$. Then A admits a bounded $H^\infty(S(\theta))$ calculus if there is a constant $K \geq 0$ such that

$$\|f(A)\| \leq K \|f\|_\theta, \qquad f \in H_0^\infty(S(\theta)).$$

If a sectorial operator A has a dense range, then it is also 1-1 by [1, Theorem 3.8]. In that case, there is a natural way to define a closed, possibly unbounded operator $f(A)$ for any $f \in H^\infty(S(\theta))$. Furthermore it is shown in [13, 1] that A admits a bounded $H^\infty(S(\theta))$ calculus in the above sense if and only if $f(A)$ is bounded for any $f \in H^\infty(S(\theta))$.

Lemma 2.2. *Let A be a sectorial operator with a dense range. Then for any integer $k \geq 1$, the operator $A^k(I_X + A)^{-(k+1)}$ has a dense range.*

PROOF. This is a well-known fact. Indeed, $nA(I_X + nA)^{-1} \to I_X$ and $n(n + A)^{-1} \to I_X$ pointwise when $n \to \infty$. Hence for a fixed integer $k \geq 1$, the sequence

$$\Delta_n = n^{k+1} A^k (n + A)^{-1} (I_X + nA)^{-k}$$

converges pointwise to I_X. Moreover the ranges of Δ_n and $A^k(I_X + A)^{-(k+1)}$ coincide for any $n \geq 1$. Thus for any $x \in X$, $(\Delta_n(x))_{n\geq 1}$ is a sequence in the range of $A^k(I_X + A)^{-(k+1)}$ converging to x. □

Square functions associated to sectorial operators play a key role in our paper. If A is sectorial of type ω and if F is a non-zero function belonging to $H_0^\infty(S(\theta))$ for some $\theta \in (\omega, \pi)$, we set

$$\|x\|_F = \left(\int_0^\infty \left\| F(tA)x \right\|_X^2 \frac{dt}{t} \right)^{\frac{1}{2}}, \qquad x \in X.$$

Note that $\|x\|_F$ may be equal to $+\infty$. These square functions were introduced by McIntosh in [13], see also [14]. The following was proved by McIntosh and Yagi in the case when X is a Hilbert space. Its proof extends verbatim to the Banach space case.

Theorem 2.3. ([14, Theorem 5]) *Let A be a sectorial operator of type ω on a Banach space X, and assume that A has dense range. Let $F, G \in H_0^\infty(S(\theta)) \setminus \{0\}$, where $\theta > \omega$. Then there exist two positive constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1 \|x\|_G \leq \|x\|_F \leq c_2 \|x\|_G, \qquad x \in X.$$

This leads to the following

Definition 2.4. Let A be as in Theorem 2.3, and let $F \in H_0^\infty(S(\theta)) \setminus \{0\}$, where $\theta > \omega$. We say that A has a square function estimate if there is a constant $c > 0$ such that

$$\|x\|_F \leq c \|x\|_X, \qquad x \in X.$$

By Theorem 2.3, this definition does not depend on F.

If X is a Hilbert space, then A has a bounded $H^\infty(S(\theta))$ calculus if and only if A and A^* admit square function estimates in the above sense [13, Section 8]. Note however, that the situation is quite different on non-Hilbertian Banach spaces. We will come back to this question in Remark 4.4 (1) below.

3. α-Admissibility

Let $T(t)$ be a bounded strongly continuous semigroup on X, with generator equal to $-A$. For simplicity we will assume throughout this section that A has a dense range, so that Theorem 2.3 and Definition 2.4 apply to A.

By definition, an observation operator (for A) is a linear map $C\colon X_m = D(A^m) \to Y$ defined on the domain of A^m for some $m \geq 1$, which is continuous when X_m is equipped with its norm $\| \ \|_m$. Here Y is an arbitrary Banach space. For any $x \in D(A_m)$, the function $t \mapsto CT(t)x$ is continuous from $(0, \infty)$ into Y. This allows to define the integral in the next definition.

Definition 3.1. Let $C\colon X_m \to Y$ be an observation operator for A, and let $\alpha > -1$. We say that C is *α-admissible* for A, if there is a constant $M > 0$, such that

$$\text{For all } x \in X_m, \qquad \int_0^\infty t^\alpha \|CT(t)x\|_Y^2 \, dt \leq M^2 \|x\|_X^2.$$

Of course, admissibility of order 0 corresponds to the usual admissibility. To explain the motivation for this generalised form of admissibility, it is instructive to have a look at the analytic case. Assume that A is sectorial of type $< \frac{\pi}{2}$. Let φ_0 be defined by $\varphi_0(z) = z^{1/2} e^{-z}$. Then $\varphi_0 \in H_0^\infty(S(\theta))$ for any $\theta < \frac{\pi}{2}$. As was observed in [11], we have

$$\int_0^\infty \|A^{1/2} T(t)x\|_X^2 \, dt = \int_0^\infty \|(tA)^{1/2} T(t)x\|_X^2 \, \tfrac{dt}{t} = \int_0^\infty \|\varphi_0(tA)x\|_X^2 \, \tfrac{dt}{t}.$$

Thus $A^{1/2}$ is admissible for A if and only if A has a square function estimate. Likewise, for any $\alpha > -1$, we let

$$\varphi_\alpha(z) = z^{\frac{1+\alpha}{2}} e^{-z}.$$

Then $\varphi_\alpha \in H_0^\infty(S(\theta))$ for any $\theta < \frac{\pi}{2}$, and

$$\int_0^\infty t^\alpha \|A^{\frac{1+\alpha}{2}} T(t)x\|_X^2 \, dt = \int_0^\infty \|\varphi_\alpha(tA)x\|_X^2 \, \tfrac{dt}{t}.$$

This yields the following

Lemma 3.2. *If A is sectorial of type $< \frac{\pi}{2}$, then $A^{\frac{1+\alpha}{2}}$ is α-admissible for A if and only if A has a square function estimate.*

Note that according to Theorem 2.3, $A^{1/2}$ is admissible for A if and only if $A^{\frac{1+\alpha}{2}}$ is α-admissible for A. The following is an analogue of the Weiss necessary condition.

Lemma 3.3. *Let $C\colon X_m \to Y$ be an observation operator for some $m \geq 1$. Let $\alpha > -1$ and $\beta > -1$ be two real numbers such that $k = \frac{\alpha+\beta}{2}$ is a nonnegative integer. If C is α-admissible for A, then there exists a constant $K > 0$ such that*

$$\big\|(-\mathrm{Re}(\lambda))^{\frac{1+\beta}{2}}\, CR(\lambda, A)^{k+1}\big\| \leq K, \qquad \lambda \in \mathbb{C},\ \mathrm{Re}(\lambda) < 0.$$

Proof. Let M be the constant appearing in Definition 3.1. We start from the fact that for any $\lambda \in \mathbb{C}$ with negative real part and for any $x \in X$, we have

$$R(\lambda, A)^{1+k}x = \frac{(-1)^{k+1}}{k!} \int_0^\infty t^k e^{\lambda t} T(t)x\, dt\,.$$

If $x \in X_m = D(A^m)$, then $R(\lambda, A)^{1+k}x \in X_m$. Furthermore $t \mapsto T(t)x$ is continuous with values in X_m. Since C is continuous on X_m, we deduce that

$$CR(\lambda, A)^{1+k}x = \frac{(-1)^{k+1}}{k!} \int_0^\infty t^k e^{\lambda t} CT(t)x\, dt\,.$$

Hence by Cauchy-Schwarz, we have

$$\begin{aligned}
\|CR(\lambda, A)^{1+k}x\| &\leq \frac{1}{k!} \int_0^\infty t^{\frac{\alpha}{2}} t^{\frac{\beta}{2}}\, e^{\mathrm{Re}(\lambda)t}\, \|CT(t)x\|\, dt \\
&\leq \frac{1}{k!} \left(\int_0^\infty t^\alpha \|CT(t)x\|^2\, dt\right)^{\frac{1}{2}} \left(\int_0^\infty t^\beta e^{2\mathrm{Re}(\lambda)t}\, dt\right)^{\frac{1}{2}} \\
&\leq \frac{M\|x\|}{k!} \Big(\frac{1}{-2\mathrm{Re}(\lambda)}\Big)^{\frac{1+\beta}{2}} \Gamma(1+\beta)^{\frac{1}{2}}\,,
\end{aligned}$$

where Γ is the usual Gamma function. This shows our result. □

4. Main result

We now come to the main result characterizing α-admissibility in the analytic case. We will prove a generalization of [11, Theorem 4.1] which says that (2) implies (1) if A is sectorial of type $< \frac{\pi}{2}$ and has a square function estimate. We start with a technical result on holomorphic functions. The next statement in the case $k = 1$ appears in [9]. We are grateful to Nigel Kalton who showed us a proof of the general case.

Lemma 4.1. *Let $\sigma \in (0,\pi)$, let $\varphi \in H_0^\infty(S(\sigma))$, and let $k \geq 1$ be an integer. There exist a function $f \in H_0^\infty(S(\sigma))$ and a constant $a \in \mathbb{C}$ such that*

$$\varphi(z) = z^k\, f^{(k)}(z) \,+\, a\,\frac{z^k}{(1+z)^{k+1}}\,, \qquad z \in S(\sigma). \tag{4}$$

Furthermore, if $\delta, \epsilon \in (0,1)$ are positive numbers such that

$$|\varphi(z)| = O(|z|^{-\delta}) \quad \text{at } \infty \qquad \text{and} \qquad |\varphi(z)| = O(|z|^{\epsilon}) \quad \text{at } 0, \tag{5}$$

then f can be chosen so that we also have $|f(z)| = O(|z|^{-\delta})$ at ∞, and $|f(z)| = O(|z|^{\epsilon})$ at 0.

Proof. We start with a general integration principle, stated as a

Claim: If $g : S(\sigma) \to \mathbb{C}$ is a holomorphic function such that $|g(z)| = O(|z|^{-r})$ at ∞ for some $r > 1$, there exists a (necessarily unique) holomorphic function $G\colon S(\sigma) \to \mathbb{C}$ such that $G' = g$ and $|G(z)| = O(|z|^{-r+1})$ at ∞. If further $|g(z)| = O(|z|^{-s})$ at 0 for some $s > 1$, then we have $|G(z)| = O(|z|^{-s+1})$ at 0. Moreover if g is bounded away from 0 (i.e. $\{g(z) \,:\, z \in S(\sigma),\, |z| \geq \eta\}$ is bounded for any $\eta > 0$), then G also is bounded away from 0.

Indeed, note that since $|g(z)| = O(|z|^{-r})$ at ∞, with $r > 1$, one can define

$$G(z) = -\int_z^\infty g(\lambda)\, d\lambda\,, \qquad z \in S(\sigma),$$

this integral being defined on any reasonable contour. For example, if $z = |z|e^{i\sigma'}$ with $|\sigma'| < \sigma$, we can write

$$G(z) = -e^{i\sigma'} \int_{|z|}^\infty g(te^{i\sigma'})\, dt\,.$$

Clearly G is holomorphic and we have $G' = g$. Moreover if $|g(z)| \leq K|z|^{-r}$ for $|z|$ large enough, then we find that

$$|G(z)| \leq \int_{|z|}^\infty Kt^{-r}\, dt = \frac{K}{r-1}|z|^{-r+1}.$$

This proves the estimate at infinity. Then the assertion on boundedness away from 0 is clear. For the estimate at 0, note that we have

$$|g(\lambda)| \leq K \left\{ \begin{array}{ll} |\lambda|^{-r} & \text{if } |\lambda| \geq 1 \\ |\lambda|^{-s} & \text{if } |\lambda| \in (0,1) \end{array} \right.$$

for some $K > 0$. Hence we have an estimate $|G(z)| \leq c|z|^{1-s} + d$, for $|z| < 1$. Since $s > 1$, we deduce that $G(z) = O(|s|^{-s+1})$ at zero as expected.

We now prove our lemma. Let φ in $H_0^\infty(S(\sigma))$, and let $\delta \in (0,1)$ and $\epsilon \in (0,1)$ such that (5) holds true. We apply the above claim with the function $g(z) = \frac{\varphi(z)}{z^k}$ and the exponent $r = k+\delta$. We let $G_{k-1} = -\int_{\bullet}^{\infty} \frac{\varphi(\lambda)}{\lambda^k}\, d\lambda$ denote the associated function. Then we have $|G_{k-1}(z)| = O(|z|^{-k+1-\delta})$ at infinity. We set $G_k = g$ for convenience. Next (if $k \geq 2$), we can use our claim repeatedly to define by induction holomorphic functions $G_{k-2}, \ldots, G_0$ such that $G'_p = G_{p+1}$ for any $0 \leq p \leq k-2$, and $|G_p(z)| = O(|z|^{-p-\delta})$ at infinity. Thus we obtain a holomorphic function $G_0 : S(\sigma) \to \mathbb{C}$ such that

$$G_0^{(k)}(z) = G_k(z) = \frac{\varphi(z)}{z^k}, \tag{6}$$

and

$$|G_0(z)| = O(|z|^{-\delta}) \quad \text{at} \quad \infty. \tag{7}$$

Moreover G_0 is bounded away from 0. For the behaviour at zero, we can also use the claim repeatedly, using the fact that $\epsilon < 1$. We obtain that

$$|G_1(z)| = 0(|z|^{\epsilon-1}) \quad \text{at} \quad 0. \tag{8}$$

Since $\epsilon < 1$, this implies that G_1 is integrable on $(0,\infty)$. We can therefore define a constant

$$c := -\int_0^\infty G_1(t)\, dt\,.$$

Then we set

$$f(z) := G_0(z) - \frac{c}{1+z}, \qquad z \in S(\sigma).$$

This obviously defines a holomorphic function, which is bounded away from 0. It readily follows from (6) that (4) is satisfied, with $a = c(-1)^k k!$. Thus it remains to check that f belongs to $H_0^\infty(S(\sigma))$ and has the desired estimates at ∞ and 0. On the one hand, (7) ensures that $|f(z)| = O(|z|^{-\delta})$ at infinity (here we use the fact that $\delta < 1$). On the other hand, we see using holomorphy that

$$c - G_0(z) = -\int_0^z G_1(\lambda)\, d\lambda, \qquad z \in S(\sigma).$$

Hence arguing as in the claim, we deduce from (8) that $|G_0(z) - c| = O(|z|^\epsilon)$ at zero. Now writing

$$f(z) = (G_0(z) - c) + c\,\frac{z}{z+1},$$

we deduce that we also have $|f(z)| = O(|z|^\epsilon)$ at zero. $\square$

Throughout the rest of this paper, we let $T(t)$ be a bounded strongly continuous semigroup on X, we let $-A$ denote its generator, and we assume as in Section 3 that A has a dense range.

Theorem 4.2. *Let A be a sectorial operator of type $\omega < \frac{\pi}{2}$ on X which has a square function estimate. Let $C\colon X_m \to Y$ be an observation operator for some $m \geq 1$. Let $\alpha > -1$ and let $\beta \in (-1, 3)$ such that $k = \frac{\alpha+\beta}{2}$ is a nonnegative integer. Then C is α-admissible for A if (and only if) there is a constant $K > 0$ such that*

$$t^{\frac{1+\beta}{2}} \left\|CR(-t, A)^{k+1}\right\| \leq K, \qquad t > 0. \tag{9}$$

Remark 4.3. Let $\alpha > -1$ and $\beta > -1$ be such that $k = \frac{\alpha+\beta}{2}$ is a nonnegative integer, and assume that (9) holds true for some $K > 0$. Let $\beta' = \beta + 2$ and $k' = k + 1$, so that $k' = \frac{\alpha+\beta'}{2}$. Since A is sectorial, the set $\{tR(-t, A) : t > 0\}$ is bounded, hence for any $t > 0$, we have

$$t^{\frac{1+\beta'}{2}} \left\|CR(-t, A)^{k'+1}\right\| \leq t^{\frac{1+\beta}{2}} \left\|CR(-t, A)^{k+1}\right\| \left\|tR(-t, A)\right\| \leq K',$$

for some $K' > 0$. Thus (9) holds true with (β', k', K') instead of (β, k, K).

Proof. (Of Theorem 4.2) The 'only if' part clearly follows from Lemma 3.3, so we only have to prove the 'if' part. Thus we assume throughout that (9) holds true for C. We may assume that $m = k + 1$, so that we actually have $C\colon X_{k+1} \to Y$. Indeed, if we had $m > k + 1$, then (9) ensures that we can extend C to a continuous operator on X_{k+1}. We also assume that $k \geq 1$, the special case $k = 0$ being treated at the end of this proof.

We will use the (unbounded) operator A^{-1}, densely defined on the range of A. We set $F_k(z) := z^k e^{-z}$. Then for any $x \in X_{k+1}$ and any $t > 0$, we have

$$t^{\frac{\alpha}{2}} CT(t)x = t^{\frac{\alpha}{2}-k} CA^{-k} F_k(tA)x. \tag{10}$$

Let $\epsilon \in (0, 1)$ and consider the decomposition $F_k(z) = \varphi(z)\psi(z)$ where

$$\varphi(z) = z^{\epsilon}(1 + z)^{-1}, \qquad \text{and} \qquad \psi(z) = z^{k-\epsilon}(1 + z)e^{-z}. \tag{11}$$

The precise value of $\epsilon \in (0, 1)$ will be decided later (it will actually depend on β). Note that $\psi \in H_0^\infty(S(\theta))$ for any $\theta < \frac{\pi}{2}$, whereas $\varphi \in H_0^\infty(S(\sigma))$ for any $\sigma < \pi$. By (10), we have

$$\int_0^\infty t^\alpha \left\|CT(t)x\right\|_Y^2 dt \leq \int_0^\infty \left\|t^{\frac{\alpha+1}{2}-k} CA^{-k}\varphi(tA)\right\|^2 \left\|\psi(tA)x\right\|_X^2 \frac{dt}{t}. \tag{12}$$

We fix $\sigma \in (\omega, \pi)$ and apply Lemma 4.1 to φ, with $\delta = 1-\epsilon$. We let $f \in H_0^\infty(S(\sigma))$ denote the corresponding function satisfying equation (4). Note that according to that equation, $z \mapsto z^k f^{(k)}(z)$ belongs to $H_0^\infty(S(\sigma))$. Let $\Gamma = \Gamma_\gamma$ for some $\gamma \in (\omega, \sigma)$. Our aim is to show the representation formula

$$CA^{-k}[z^k f^{(k)}(z)](tA)x = \tfrac{k!}{2\pi i}\int_\Gamma f(\lambda) t^k\, CR(\lambda, tA)^{k+1} x\, d\lambda\,. \tag{13}$$

We let $\Gamma' = \Gamma_{\gamma'}$ for some $\gamma' \in (\gamma, \sigma)$. Then we have

$$\begin{aligned}
[z^k f^{(k)}(z)](tA) &= \tfrac{1}{2\pi i}\int_\Gamma \lambda^k f^{(k)}(\lambda) R(\lambda, tA)\, d\lambda \\
&= \tfrac{1}{2\pi i}\int_\Gamma \Big[\tfrac{k!}{2\pi i}\int_{\Gamma'} \frac{1}{(\zeta-\lambda)^{k+1}}\, f(\zeta)\, d\zeta\Big]\, \lambda^k R(\lambda, tA)\, d\lambda \\
&= \tfrac{1}{2\pi i}\int_{\Gamma'} f(\zeta)\, \Big[\tfrac{k!}{2\pi i}\int_\Gamma \frac{\lambda^k}{(\zeta-\lambda)^{k+1}} R(\lambda, tA)\, d\lambda\Big]\, d\zeta \\
&= \tfrac{k!}{2\pi i}\int_\Gamma f(\zeta)(tA)^k R(\zeta, tA)^{k+1}\, d\zeta.
\end{aligned}$$

The use of Fubini's Theorem in this chain of equalities is justified by the fact that since $R(\lambda, tA) = t^{-1}R(t^{-1}\lambda, A)$, for some appropriate constants $K_0, K_1 > 0$, we have

$$\begin{aligned}
\int_\Gamma\int_{\Gamma'} \frac{|\lambda|^k |f(\zeta)|}{|\zeta-\lambda|^{k+1}}\, \|R(\lambda, tA)\|\, |d\zeta||d\lambda| &\leq\ K_0 \int_{\Gamma'} |f(\zeta)| \int_\Gamma \frac{|\lambda|^{k-1}}{|\zeta-\lambda|^{k+1}}\, |d\lambda|\ |d\zeta| \\
&\overset{\lambda=|\zeta|\mu}{\leq}\ K_1 K_0 \int_{\Gamma'} |f(\zeta)| \big|\tfrac{d\zeta}{\zeta}\big|\ < \infty\,.
\end{aligned}$$

The above calculation shows that equation (13) holds true if x belongs to the range of $A^k(I_X + A)^{-k-1}$. The latter is dense by Lemma 2.2. Hence to deduce equation (13) for any $x \in X$, it therefore suffices to show that

$$\int_\Gamma |f(\lambda)|\, \|CR(\lambda, tA)^{k+1}\|\, |d\lambda|\ < \infty\,. \tag{14}$$

Let $\lambda \in \mathbb{C}^*$ with $|\mathrm{Arg}(\lambda)| \geq \gamma$ and let $s > 0$. By the resolvent equation, we have

$$R(\lambda, A) = R(-s, A)\big(I_X - (\lambda + s)R(\lambda, A)\big).$$

Hence

$$CR(\lambda, A)^{k+1} = CR(-s, A)^{k+1}\big(I_X - (\lambda + s)R(\lambda, A)\big)^{k+1}.$$

Since A is sectorial of type $< \gamma$, the set $\{(\lambda + |\lambda|)R(\lambda, A)\ :\ \lambda \in \Gamma\}$ is bounded. Hence applying the above identity with $s = |\lambda|$, and our assumption (9), we obtain that there exists a constant $K_\gamma > 0$ such that

$$|\lambda|^{\frac{1+\beta}{2}}\, \big\|CR(\lambda, A)^{k+1}\big\| \leq K_\gamma, \qquad |\mathrm{Arg}(\lambda)| \geq \gamma. \tag{15}$$

As already remarked, $R(\lambda, tA) = t^{-1}R(t^{-1}\lambda, A)$; hence we deduce that (14) holds true provided that

$$\int_\Gamma |\lambda|^{-\frac{1+\beta}{2}}\, |f(\lambda)|\, d\lambda \,<\infty\,.$$

Now recall that by Lemma 4.1, we have $|f(\lambda)| = O(|\lambda|^\epsilon)$ at 0 and $|f(\lambda)| = O(|\lambda|^{\epsilon-1})$ at ∞. Hence the above integral is finite provided that we both have $\frac{1+\beta}{2} - \epsilon < 1$ and $\frac{1+\beta}{2} + (1-\epsilon) > 1$, or equivalently, that we have

$$\epsilon < \tfrac{1+\beta}{2} < 1 + \epsilon.$$

This tells us how to choose ϵ. By assumption, $\beta \in (-1, 3)$, hence $\frac{1+\beta}{2} \in (0, 2)$. Thus we can certainly find $\epsilon \in (0, 1)$ satisfying the above double inequality. Then we have proved (14), and hence (13) for any $x \in X$. In turn, that equation and the above calculation imply that there exists a constant $M > 0$ such that

$$\big\| t^{\frac{\alpha+1}{2}-k} CA^{-k}[z^k f^{(k)}(z)](tA)\big\| \leq M, \qquad t > 0.$$

Let $t > 0$. Since $\frac{\alpha+1}{2} - (k+1) = -\,\frac{1+\beta}{2}$, we have by (9)

$$\big\|t^{\frac{\alpha+1}{2}} C(1+tA)^{-k-1}\big\| = \big\|t^{-\,\frac{1+\beta}{2}} C\big(t^{-1} + A\big)^{-k-1}\big\| \leq K.$$

However, by Lemma 4.1 we have

$$t^{\frac{\alpha+1}{2}-k} CA^{-k}\varphi(tA) = t^{\frac{\alpha+1}{2}-k} CA^{-k}[z^k f^{(k)}(z)](tA) \ +\ a\, t^{\frac{\alpha+1}{2}} C(1+tA)^{-k-1},$$

hence we have proved that

$$\big\| t^{\frac{\alpha+1}{2}-k} CA^{-k}\varphi(tA)\big\| \leq M' := M + |a|K.$$

It therefore follows from (12) that

$$\int_0^\infty t^\alpha \big\|CT(t)x\big\|^2 dt \leq M'^2 \int_0^\infty \big\|\psi(tA)x\big\|^2 \tfrac{dt}{t}\,.$$

Since A has a square function estimate, this implies that C is α-admissible for A.

It remains to prove the theorem when $k = 0$. In this case, $\beta = -\alpha$, hence $\beta < 1$. Let $\beta' = \beta + 2$. According to Remark 4.3, the set $\{t^{\frac{1+\beta'}{2}} CR(-t, A)^2\}$ is bounded. Since $\beta' < 3$, our theorem in the case $k = 1$ ensures that C is indeed α-admissible for A. $\square$

Remark 4.4.

(1) If X is a Banach space of cotype 2, and if A has a bounded $H^\infty(S(\theta))$ calculus for some $\theta < \pi$, then A has a square function estimate. This is shown in the proof of [11, Theorem 4.2], to which we refer for further explanations. We merely recall that any L^p-space with $p \leq 2$ has cotype 2. Conversely an infinite

dimensional L^p-space with $p > 2$ is not of cotype 2. It turns out that for any $2 < p < \infty$, the Laplacian $A = -\Delta$ on $L^p(\mathbb{R}^n)$ admits a bounded $H^\infty(S(\theta))$ calculus for any $\theta > 0$, but does not have a square function estimate (see [1, Section 6]).

(2) To apply Theorem 4.2, we are facing the following (simple) question: given a real number $\alpha > -1$, what are the numbers $\beta \in (-1, 3)$ such that $k = \frac{\alpha+\beta}{2}$ is an integer? If α in an odd integer, there is exactly one possible value, namely $\beta = 1$. Otherwise, there are exactly two possible values, let us call them $\beta \in (-1, 1)$ and $\beta' = \beta + 2 \in (1, 3)$. In that case, Theorem 4.2 has two variants, the first one with β and $k = \frac{\alpha+\beta}{2}$, the second one with β' and $k' = \frac{\alpha+\beta'}{2}$. According to Remark 4.3, the second variant is the strongest.

Consider for example the case $\alpha = 0$. Then our two couples are $(\beta, k) = (0, 0)$ and $(\beta', k') = (1, 1)$. If we apply Theorem 4.2 with the latter couple, we obtain the following strengthening of [11, Theorem 4.1]: If A has a square function estimate and if C is an observation operator, then C is admissible for A if (and only if) the set $\{t^{\frac{3}{2}} CR(-t, A)^2 \,:\, t > 0\}$ is bounded.

Remark 4.5. In this remark, we will show that the assumption that A has a square function estimate in Theorem 4.2 cannot be omitted.

Let A be a sectorial operator of type $< \frac{\pi}{2}$, and let $0 \leq s < 1$. Then there is a constant $K_s \geq 0$ such that

$$t^s \|A^{1-s} R(-t, A)\| \leq K_s, \qquad t > 0.$$

Indeed, this follows from [7, Proposition 4.2] and its proof. Let $\alpha > -1$ and $\beta \in (-1, 3)$ be such that $k = \frac{\alpha+\beta}{2}$ is a nonnegative integer. Then let $n \geq 0$ be an integer, and $0 \leq s < 1$ such that $\frac{1+\beta}{2} = n + s$. Then $k \geq n$, and

$$\tfrac{1+\alpha}{2} = (1 - s) + (k - n).$$

Hence we have

$$t^{\frac{1+\beta}{2}} A^{\frac{1+\alpha}{2}} R(-t, A)^{k+1} = t^s A^{1-s} R(-t, A) \left[t^n R(-t, A)^n\right] \left[A^{k-n} R(-t, A)^{k-n}\right]$$

for any $t > 0$. Since A is sectorial, the two sets

$$\{t^n R(-t, A)^n \,:\, t > 0\} \quad \text{and} \quad \{A^{k-n} R(-t, A)^{k-n} \,:\, t > 0\}$$

are bounded. Therefore, there is a constant $K > 0$ such that

$$t^{\frac{1+\beta}{2}} \left\|A^{\frac{1+\alpha}{2}} R(-t, A)^{k+1}\right\| \leq K, \qquad t > 0.$$

Thus we have proved that $A^{\frac{1+\alpha}{2}}$ satisfies (9). Consequently, if the conclusion of Theorem 4.2 holds true, then $A^{\frac{1+\alpha}{2}}$ has to be α-admissible. Hence A must have a square function estimate by Lemma 3.2.

5. Final Remarks

Let $T(t)$ and A be as in Section 3, and assume that X is reflexive. We define a control operator (for A) to be a bounded linear map $B\colon U \to X_{-m}$, where U is a Banach space and $m \geq 1$ is an integer. Let $\alpha > -1$ be a real number. We say that B is α-admissible for A if there is a constant $M > 0$ such that

$$\int_0^\infty \left|\left\langle t^{\frac{\alpha}{2}} T_{-m}(t)Bu(t), \eta\right\rangle\right| dt \,\leq M\|u\|_{L^2(\mathbb{R}_+,U)} \,\|\eta\|_{X^*} \tag{16}$$

for any $u \in L^2(\mathbb{R}_+, U)$ and any $\eta \in (X_{-m})^*$. Since X is reflexive, $(X_{-m})^* \subset X^*$ is a dense subspace. Hence if B is α-admissible, the functional

$$\eta \mapsto \int_0^\infty t^{\frac{\alpha}{2}} \left\langle T_{-m}(t)Bu(t), \eta\right\rangle dt$$

uniquely extends to an element of $X^{**} = X$. If we let $\int_0^\infty t^{\frac{\alpha}{2}} T_{-m}(t)Bu(t)\, dt$ denote this element (which is a Pettis integral), then (16) yields

$$\left\|\int_0^\infty t^{\frac{\alpha}{2}} T_{-m}(t)Bu(t)\, dt\right\|_X \,\leq\, M\, \|u\|_{L^2(\mathbb{R}_+,U)}.$$

Since X is reflexive, $-A^*$ is the generator of the dual semigroup $T(t)^*$ on X^*. For $l \in \mathbb{N}$ let $(X^*)_l$ denote the interpolation space associated to A^*. For any $m \geq 1$, let $-A_{-m}$ denote the generator of $T_{-m}(t)$ on X_{-m}. There is an isomorphism $\Psi_m\colon (X^*)_m \to (X_{-m})^*$ given by

$$\langle x, \Psi_m(\eta)\rangle = \langle (I_{X_{-m}} + A_{-m})^{-m}x, (I_X + A^*)^m\eta\rangle, \qquad x \in X_{-m},\, \eta \in (X^*)_m.$$

According to that duality, we may regard $C = B^*\colon (X_{-m})^* \to U^*$ as an observation operator for A^*. Then it is not hard to check that B is α-admissible if and only if B^* is α-admissible. Thus Theorem 4.2 implies the following result.

Theorem 5.1. *Let A be a sectorial operator of type $\omega < \frac{\pi}{2}$ on a reflexive Banach space X, and assume that A^* has a square function estimate. Let $B\colon U \to X_{-m}$ be a control operator for some $m \geq 1$. Let $\alpha > -1$ and let $\beta \in (-1, 3)$ such that $k = \frac{\alpha+\beta}{2}$ is a nonnegative integer. Then B is α-admissible for A if (and only if) there is a constant $K > 0$ such that*

$$t^{\frac{1+\beta}{2}} \left\|R(-t, A_{-m})^{k+1}B\right\| \leq K, \qquad t > 0.$$

Our last remark is that there is another way to define square functions on non Hilbertian Banach spaces, which leads to an alternative framework for α-admissibility. This theory will be developed in [3]. Here we will only outline the principle ideas. We let $I = (0,\infty)$ be equipped with the measure $\frac{dt}{t}$. Let us first consider the case when $X = L^p(\Omega)$, for some $1 < p < \infty$. The square function $\|x\|_F$ can be defined to be the norm of $F(tA)x$ in $L^p(\Omega, L^2(I))$ (instead of its norm in $L^2(I, L^p(\Omega))$). In this context, a square function estimate is therefore an inequality of the form

$$\Big\| \Big(\int_0^\infty \big|F(tA)x\big|^2 \, \tfrac{dt}{t} \Big)^{\frac{1}{2}} \Big\|_{L^p} \, \leq \, c\|x\|.$$

The theory of H^∞ functional calculus shows that perhaps these square functions are more natural if $p \neq 2$. Indeed, the existence of a bounded H^∞ calculus implies such a square function estimate, see [1]. Then given an operator $C\colon X_m \to L^q(\Omega')$, $1 \leq q < \infty$, one can define α-admissibility by demanding an estimate of the form

$$\Big\| \Big(\int_0^\infty t^\alpha \, |CT(t)x|^2 \, dt \Big)^{\frac{1}{2}} \Big\|_{L^q} \, \leq M \, \|x\|. \tag{17}$$

In this parallel setting, analogues of Theorems 4.2 and 5.1 can be shown. Indeed, the 0-admissibility of C in the sense of (17) was already treated in [12]. It should be mentioned that in this context another notion of boundedness for sets of the form $\{t^{\frac{1+\beta}{2}} CR(-t,A)^{k+1}\}$ naturally enters the game: Rademacher–, or R–boundedness (see i.e. [17]). On general Banach spaces, R–boundedness is stronger than uniform boundedness, but these notions coincide in the Hilbert space setting.

In [8] and [9] it is shown that the norms on $L^p(\Omega, L^2(I))$ have a generalisation to arbitrary Banach spaces X instead of $L^p(\Omega)$, using so-called Gaussian structures. In this context it is possible (see [3]) to extend the two characterisation theorems 4.2 and 5.1 to arbitrary Banach spaces X, under a simple geometric condition on the control and observation spaces U and Y, namely Pisier's property (α) (see [16]), which holds i.e. for Hilbert spaces and L^q–spaces with $q \in [1,\infty)$.

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