ADMISSIBILITY AND CONTROLLABILITY OF DIAGONAL VOLTERRA EQUATIONS WITH SCALAR INPUTS

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Abstract. This article studies Volterra evolution equations from the point of view of control theory, in the case that the generator of the underlying semigroup has a Riesz basis of eigenvectors. Conditions for admissibility of the system's control operator are given in terms of the Carleson embedding properties of certain discrete measures. Moreover, exact and null controllability are expressed in terms of a new interpolation question for analytic functions, providing a generalization of results known to hold for the standard Cauchy problem. The results are illustrated by examples involving heat conduction with memory.

1. Introduction

Consider the evolution equation

$$x(t) = x_0 + \int_0^t a(t-s)Ax(s)\,ds + \int_0^t Bu(s)\,ds, \qquad t \geq 0. \tag{1}$$

Here we assume that A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X, $a \in L^1_{loc}(0,\infty)$ is real-valued and of at most exponential growth, and the control operator $B \in \mathcal{L}(U, D(A^*)^*)$, where U is another Hilbert space. It is further assumed that the uncontrolled system

$$x(t) = x_0 + \int_0^t a(t-s)Ax(s)\,ds \qquad t \geq 0 \tag{2}$$

is well-posed, which is equivalent to the existence of a unique family of bounded linear operators $(S(t))_{t\geq 0}$ on X, such that

(a) $S(0) = I$ and $(S(t))_{t\geq 0}$ is strongly continuous on $\mathbb{R}_+$.
(b) $S(t)$ commutes with A, which means $S(t)(D(A)) \subset D(A)$ for all $t \geq 0$, and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$.
(c) For all $x \in D(A)$ and all $t \geq 0$ the resolvent equations hold:

$$S(t)x = x + \int_0^t a(t-s)AS(s)x\,ds. \tag{3}$$

The family of bounded linear operators $(S(t))_{t\geq 0}$ is called the *resolvent* or *solution family* for (2). We refer to the monograph by Prüss [24] for more about resolvents. In particular, if we assume further that the resolvent $(S(t))_{t\geq 0}$ is exponentially

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bounded, say $\|S(t)\| \leq Me^{\omega t}$ for $t \geq 0$, then the Laplace transform of $S(\cdot)x_0$ is well-defined and satisfies

$$\widehat{S}(\lambda)x_0 = \frac{1}{\lambda}(I - \widehat{a}(\lambda)A)^{-1}x_0 \qquad (\operatorname{Re}\lambda > \omega)$$

(here the hat denotes Laplace transform). The assumption of an exponential growth of the resolvent is indeed a restriction of generality: in contrast with semigroups or cosine families, resolvents may grow super-exponentially in time even if the kernel a is integrable and of class C^∞ (see [5] for more details).

Notice that by adding $\omega \cdot a * x$ on both sides of equation (2) we obtain an equation of the same form where x is replaced by $v = x + \omega \cdot a * x$, A is replaced by $A + \omega$, and a by the solution r of $r + \omega \cdot a * r = a$. Indeed,

$$\begin{aligned} v &= x_0 + a * (A+\omega)x = x_0 + [r + \omega \cdot a * r] * (A+\omega)x \\ &= x_0 + r * (A+\omega)x + \omega \cdot a * r * (A+\omega)x \\ &= x_0 + r * (A+\omega)(x + \omega \cdot a * x) = x_0 + r * (A+\omega)v. \end{aligned}$$

This transformation shows that without loss of generality we may assume A to generate a uniformly exponentially stable semigroup. We notice that $1/\widehat{r}(\lambda) = [1/\widehat{a}(\lambda)] + \omega$ in this case.

Example 1.1. (a) Consider the standard kernel $a(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$ for $\beta \in (0,2)$ given in [24, Example 2.1]. We have $\widehat{a}(\lambda) = \lambda^{-\beta}$. In our main result in Theorem 3.10 we consider a class of kernels that admit upper and lower estimates against this standard kernel.

(b) Another important class of kernels is that given by [24, Example 2.2]:

$$a(t) = \int_0^\infty e^{-st}\, d\alpha(s), \tag{4}$$

or

$$\widehat{a}(\lambda) = \int_0^\infty \frac{1}{\lambda + s}\, d\alpha(s),$$

where α is a non-decreasing function on $[0,\infty)$ such that

$$\int_1^\infty d\alpha(s)/s < \infty.$$

In Section 3.2 we give a result on controllability of special cases of such kernels.

(c) Let $a(t) = \int_0^\infty \frac{t^{\rho-1}}{\Gamma(\rho)}\, d\rho$ as considered in [24, Example 2.3]. We then have $\widehat{a}(\lambda) = 1/\log(\lambda)$. In Theorem 3.10 we obtain a sufficient criterion for admissibility in this case.

The mild solution of (1) is formally given by the variation of constants formula

$$x(t) = S(t)x_0 + (S * Bu)(t), \qquad t \geq 0,$$

which is actually the classical solution if $B \in \mathcal{L}(U,X)$, $x_0 \in D(A)$ and u sufficiently smooth. In general however, B is not a bounded operator from U to X and so an additional assumption on B will be needed to ensure $x(t) \in X$ for every $x_0 \in X$ and every $u \in L^2(0,\infty;U)$.

In Section 2 we introduce the idea of admissibility for a control operator, and explain some of its properties. We also present the more familiar theme of controllability, which will also be studied in this paper. The main results of the paper are contained in Section 3, where we specialise to diagonal systems, and derive conditions for admissibility and controllability of such systems, presented in terms of Carleson embedding and interpolation properties. Finally, in Section 4, we illustrate the ideas of this paper with examples involving heat conduction with memory.

2. Admissibility and Controllability

Since the resolvent for (2) commutes with the operator A, it can be easily seen that the resolvent operator $(S(t))_{t\geq 0}$ can be restricted/extended to a resolvent operator on $D(A)/D(A^*)^*$. We denote the restriction/extension again by $(S(t))_{t\geq 0}$. Similarly, the operator A can be extended/restricted to a generator of a C_0-semigroup on $D(A)/D(A^*)^*$, again denoted by A.

Definition 2.1. Let $B \in \mathcal{L}(U, D(A^*)^*)$. Then B is called *admissible for* $(S(t))_{t\geq 0}$ if there exists a constant $M > 0$ such that

$$\|(S * Bu)(t)\|_X \leq M\|u\|_{L^2(0,\infty;\,U)}, \qquad u \in L^2(0,\infty;\,U), \qquad t \geq 0. \tag{5}$$

Remark 2.2. $B \in \mathcal{L}(U, D(A^*)^*)$ is admissible if and only if there exists a constant $M > 0$ such that

$$\left\| \int_0^\infty S(t)Bu(t)\,dt \right\|_X \leq M\|u\|_{L^2(0,\infty;\,U)}. \tag{6}$$

for all $u \in L^2(0,\infty;\,U)$ with compact support.

Proof. Let B be admissible and assume that $u \in L^2(0,\infty;\,U)$ has compact support, say $[a,b]$. Define $\widetilde{u}(s) := u(b-s)$ when $b-s \in [a,b]$ and zero otherwise. Then,

$$\left\| \int_0^\infty S(s)Bu(s)\,ds \right\| = \left\|(S * B\widetilde{u})(b)\right\| \leq M\left\|\widetilde{u}\right\|_{L^2(0,\infty;\,U)} = M\left\|u\right\|_{L^2(0,\infty;\,U)}.$$

Conversely, let $t \geq 0$ and $u \in L^2(0,\infty;\,U)$. Define $v_t(s) = u(t-s)$ for $s \in [0,t]$ and zero otherwise. Then (6) implies

$$\|(S * Bu)(t)\| = \left\| \int_0^\infty S(s)Bv_t(s)\,ds \right\| \leq M\|v_t\|_{L^2(0,t;\,U)} \leq M\|u\|_{L^2(0,\infty;\,U)},$$

whence B is admissible. $\square$

Admissibility of the operator B guarantees that the operator

$$\mathcal{B}_\infty : \{u \in L^2(0,\infty;\,U) \mid u \text{ has compact support}\} \to X,$$

given by,

$$\mathcal{B}_\infty u := \int_0^\infty S(s)Bu(s)\,ds, \tag{7}$$

possesses a unique extension to a linear, bounded operator from $L^2(0,\infty;\,U)$ to X. We denote this extension again by $\mathcal{B}_\infty$. If the solution family is exponentially stable, then formula (7) holds for every $u \in L^2(0,\infty;\,U)$.

There is also the notion of admissibility of an observation operator $C \in \mathcal{L}(D(A), Y)$, where Y is another Hilbert space, guaranteeing that the output y, where

$$y(t) = Cx(t), \qquad t \geq 0,$$

lies in L^2. For infinite-time admissibility, the following is the most natural definition.

Definition 2.3. The operator C is called an *admissible observation operator* for the uncontrolled system (2), if there exists a constant $M > 0$ such that

$$\|y(\cdot)\|_{L^2(0,\infty;Y)} = \|CS(\cdot)x_0\|_{L^2(0,\infty;Y)} \leq M\|x_0\|, \qquad x_0 \in D(A).$$

The operator C is called a *finite-time admissible observation operator* for (2), if there exist constants $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|y(\cdot)\| = \|CS(\cdot)x_0\|_{L^2(0,t;Y)} \leq Me^{\omega t}\|x_0\|, \qquad x_0 \in D(A), t > 0.$$

Notice that the dual operator $\mathcal{B}_\infty^*$ is given by $x^* \mapsto B^*S(\cdot)^*x^*$. Therefore, there is a natural duality between admissibility of control operators and admissibility of observation operators, that is, $B \in \mathcal{L}(U, D(A^*)^*)$ is an admissible control operator if and only if $B^* \in \mathcal{L}(D(A^*), U^*)$ is an admissible observation operator. This is explained in detail in [15, Section 4]. For more on admissibility for the Cauchy problem (i.e., $a \equiv 1$), we refer to the survey [14].

We shall also be interested in obtaining conditions for exact controllability of the system (1). Accordingly, we make the following definitions.

Definition 2.4. The system (1) is said to be *exactly controllable*, if every state can be achieved by a suitable control, i.e., if $R(\mathcal{B}_\infty) \supseteq X$.

It is said to be *null-controllable in time* $\tau > 0$ if $R(\mathcal{B}_\infty) \supseteq R(S(\tau))$.

For a recent discussion of these properties in the context of the Cauchy problem, we refer to [16].

3. Admissible and Controllable Diagonal Systems

From now on we assume that A is the infinitesimal generator of an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ on H with a sequence of normalised eigenvectors $\{\phi_n\}_{n\in\mathbb{N}}$ forming a Riesz basis for H, with associated eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$, that is,

$$A\phi_n = \lambda_n\phi_n, \qquad n \in \mathbb{N}.$$

Let $\mathcal{S}(\theta)$ be the open sector of angle 2θ symmetric about the positive real axis. Recall that the condition $-\lambda_n \in \mathcal{S}(\pi/2)$ for all $n \in \mathbb{N}$ is necessary for A to generate a bounded semigroup and that $-\lambda_n \in \mathcal{S}(\theta)$ with $\theta < \pi/2$ is equivalent to A generating a bounded *analytic* semigroup.

Since $(T(t))_{t\geq 0}$ is assumed to be exponentially stable we have $\sup_{n\in\mathbb{N}} \operatorname{Re} \lambda_n < 0$. Let ψ_n be an eigenvector of A^* corresponding to the eigenvalue $\overline{\lambda_n}$. Without loss of generality we can assume that $\langle \phi_n, \psi_n \rangle = 1$. Then the sequence $\{\psi_n\}_{n\in\mathbb{N}}$ forms a Riesz basis of H and every $x \in H$ can be written as

$$x = \sum_{n\in\mathbb{N}} \langle x, \psi_n \rangle \phi_n = \sum_{n\in\mathbb{N}} \langle x, \phi_n \rangle \psi_n.$$

Note that the Volterra system is also diagonal, in the sense that there exist functions c_n such that $S(t)\phi_n = c_n(t)\phi_n$; indeed

$$\widehat{S}(\lambda)\phi_n = \sigma(\lambda, -\lambda_n)\phi_n,$$

where

$$\sigma(\lambda,\mu) = \frac{1}{\lambda(1+\mu\widehat{a}(\lambda))}, \qquad \operatorname{Re}\mu, \operatorname{Re}\lambda > 0,$$

so the Laplace transform of c_n is $\sigma(\cdot, -\lambda_n)$.

For example, if $\widehat{a}(\lambda) = \xi/(\lambda+s)$ with $\xi > 0$ and $s \geq 0$, the simplest case of (4), then

$$\sigma(\lambda, -\lambda_n) = \frac{\lambda+s}{\lambda(\lambda+s-\lambda_n\xi)} = \frac{s}{\lambda(s-\lambda_n\xi)} - \frac{\lambda_n\xi}{(\lambda-\lambda_n\xi)(\lambda+s-\lambda_n\xi)},$$

and hence

$$c_n(t) = \frac{s}{s-\lambda_n\xi} - \frac{\lambda_n\xi}{s-\lambda_n\xi}\exp(\lambda_n\xi - s)t. \tag{8}$$

3.1. **Admissibility.** In [17, Theorem 4.3] the following result was established for the case of a one-dimensional observation, e.g. a point evaluation (i.e., letting $Y = \mathbb{C}$).

Theorem 3.1. *Let $a = 1 + 1 * k$ with $k \in W^{1,2}(0,\infty)$. Then C is a finite-time admissible observation operator for (2) if and only if there are constants $M > 0$ and $\omega \in \mathbb{R}$ such that*

$$\sum_{n=1}^{\infty} \frac{|C\phi_n|^2}{|\lambda|^2|1-\widehat{a}(\lambda)\lambda_n|^2} \leq \frac{M}{\operatorname{Re}\lambda - \omega}, \qquad \operatorname{Re}\lambda > \omega.$$

A similar statement does not hold for infinite-time admissibility, see [17, Example 5.1].

One may rewrite the theorem by duality for the controlled systems under consideration; however, the kernels given in Example 1.1 do not satisfy the requirements of the above result. This observation is a primal motivation for the present article.

For the control of distributed parameter systems, the study of one-dimensional inputs may seem a severe restriction of generality. However, as explained in [11, Rem. 2.4] for the Cauchy case, in the case of (finite) n-dimensional input spaces, admissibility is equivalent to the simultaneous admissibility of n one-dimensional systems. Moreover, the following proposition shows that a one-dimensional criterion leads immediately to a sufficient criterion for admissibility of control operators $B : U \to X$ for infinite-dimensional input spaces. This observation is of great practical value since the sufficient condition of admissibility in Theorem 3.10 can be verified rather easily. Similar results for the Cauchy problem (i.e., $a \equiv 1$) are well known in the literature, see e.g. [11, 12], or [9] for the case that $X = \ell_q$ and $U = \ell_p$. Our proposition generalises directly [27, Proposition 5.3.7].

Let $U = X = \ell_2$. Let $B : U \to X_{-1}$ be linear and bounded. Then there are functionals $\varphi_n \in (\ell_2)^* = \ell_2$ such that $(Bu)_n = \langle u, \varphi_n\rangle$.

Proposition 3.2. *Let $X = U = \ell_2$. Let (φ_n) be a sequence of elements in U^* and consider the scalar sequence b defined by $b_n = \|\varphi_n\|$ and let the operator B be defined by $(Bu)_n = \langle u, \varphi_n\rangle$.*

If $b \in \mathcal{D}(A^*)^*$ is an admissible input element for the resolvent family $(S(t))_{t\geq 0}$, then B is bounded from U to $\mathcal{D}(A^*)^*$ and B is an admissible control operator for the resolvent family $(S(t))_{t\geq 0}$ as well.

Proof. The elementary estimate $|\langle u, \varphi_n\rangle| \leq \|u\|\,\|\varphi_n\|$ implies that B is linear and bounded from U to $\mathcal{D}(A^*)^*$. Now let $u \in L^2(0,\infty;U) = L^2(0,\infty;\ell_2)$ have compact support and let $u_j(\cdot)$ denote its coordinate functions. Then

$$\begin{aligned}
\left\|\int_0^t S(t-s)Bu(s)\,ds\right\|_{\ell_2}^2 &= \sum_{n=1}^{\infty}\left|\int_0^t c_n(t-s)\langle u(s),\varphi_n\rangle\,ds\right|^2 \\
&= \sum_{n=1}^{\infty}\left|\left\langle\int_0^t c_n(t-s)u(s)\,ds,\varphi_n\right\rangle\right|^2 \\
&\leq \sum_{n=1}^{\infty}\|\varphi_n\|^2\left\|\int_0^t c_n(t-s)u(s)\,ds\right\|_U^2 \\
&= \sum_{n=1}^{\infty}\|\varphi_n\|^2\sum_{j=1}^{\infty}\left|\int_0^t c_n(t-s)u_j(s)\,ds\right|^2 \\
&= \sum_{j=1}^{\infty}\left[\sum_{n=1}^{\infty}\left|\int_0^t c_n(t-s)\|\varphi_n\|u_j(s)\,ds\right|^2\right] \\
&= \sum_{j=1}^{\infty}\left\|\int_0^t S(t-s)bu_j(s)\,ds\right\|^2.
\end{aligned}$$

By assumption, b is an admissible input element, and so

$$\left\|\int_0^t S(t-s)Bu(s)\,ds\right\|_{\ell_2}^2 \leq C\sum_{j=1}^{\infty}\|u_j\|_{L^2}^2 = C\,\|u\|_{L^2(0,\infty;\ell_2)}^2.$$

□

After this consideration we focus on the case that $U = \mathbb{C}$, i.e., that the input space is one-dimensional and let $B \in \mathcal{L}(U, D(A^*)^*)$. Then we may write $B = \sum_{n\in\mathbb{N}} b_n\phi_n$, where $\{b_n|\lambda_n|^{-1}\} \in \ell_2$. Further, we assume that the solution family is exponentially stable.

We have for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$

$$\begin{aligned}
\mathcal{B}_\infty(e^{-\lambda\cdot}) &= \int_0^\infty S(t)Be^{-\lambda t}\,dt \\
&= \sum_{n=1}^{\infty} b_n\left(\int_0^\infty c_n(t)e^{-\lambda t}dt\right)\phi_n \\
&= \sum_{n=1}^{\infty} b_n\sigma(\lambda,-\lambda_n)\phi_n \\
&= \sum_{n=1}^{\infty} b_n\frac{1}{\lambda-\lambda\widehat{a}(\lambda)\lambda_n}\phi_n.
\end{aligned} \tag{9}$$

This implies

$$\left\|\mathcal{B}_\infty(e^{-\lambda\cdot})\right\|_H^2 = \sum_{n=1}^\infty \frac{|b_n|^2}{|\lambda|^2|1-\widehat{a}(\lambda)\lambda_n|^2}.$$

Denote by k_λ the reproducing kernel $k_\lambda(z) := \frac{1}{z+\overline{\lambda}}$. We arrive at the following result.

Proposition 3.3. *The following is a necessary condition for admissibility of a rank-one control operator B: There exists a constant $M > 0$ such that*

$$\sum_{n=1}^\infty \frac{|b_n|^2}{|\lambda|^2|1-\widehat{a}(\lambda)\lambda_n|^2} \leq \frac{M}{\operatorname{Re}\lambda}, \qquad \operatorname{Re}\lambda > 0,$$

or, letting $\nu := \sum_n |b_n|^2\delta_{-\overline{\lambda_n}}$ where δ_z denotes the Dirac mass in z, equivalently

$$\|k_{\frac{1}{\widehat{a}(\lambda)}}\|^2_{L^2(\mathbb{C}_+,\nu)} \leq M\frac{|\lambda\widehat{a}(\lambda)|^2}{\operatorname{Re}\lambda}, \qquad \operatorname{Re}\lambda > 0. \tag{10}$$

Example 3.4. (a) For the particular choice of $\widehat{a}(\lambda) = \lambda^{-\beta}$, $\beta \in (0,2)$, the necessary condition of the proposition reads

$$\left\|k_{\lambda^\beta}\right\|^2_{L^2(\mathbb{C}_+,\nu)} \leq M\frac{|\lambda|^{2-2\beta}}{\operatorname{Re}\lambda}, \qquad \operatorname{Re}\lambda > 0. \tag{11}$$

(b) In case $\widehat{a}(\lambda) = \int_0^\infty \frac{1}{\lambda+s}\,d\alpha(s)$, the above necessary condition reads

$$\left\|k_{\frac{1}{\widehat{a}(\lambda)}}\right\|^2_{L^2(\mathbb{C}_+,\nu)} \leq M\frac{1}{\operatorname{Re}\lambda}\left|\int_0^\infty \frac{\lambda}{\lambda+s}\,d\alpha(s)\right|^2$$

(c) If $\widehat{a}(\lambda) = \frac{1}{\log\lambda}$, the necessary condition reads

$$\left\|k_{\log(\lambda)}\right\|^2_{L^2(\mathbb{C}_+,\nu)} \leq M\frac{|\lambda|^2}{\operatorname{Re}\lambda|\log\lambda|^2}$$

There is a strong link between admissibility and Carleson measures in the Cauchy case $a(t) \equiv 1$, as first observed in [13]. For Volterra systems, we shall establish a similar connection.

Definition 3.5. For $\gamma > 0$, a measure μ on $\mathbb{C}_+$ is an *embedding γ–Carleson measure* if for one (and hence all) $q \in (1,\infty)$ satisfying $\gamma q > 1$ there is an absolute constant M_q such that $\|f\|_{L^q(\mathbb{C}_+,\mu)} \leq M_q\|f\|_{H^{\gamma q}(\mathbb{C})}$ for all $f \in H^{\gamma q}(\mathbb{C})$.

We begin with the case $\gamma \in (0,1]$. Here, a measure μ on $\mathbb{C}$ is γ–Carleson if, and only if there is an absolute constant C such that

$$\mu(Q_h)^\gamma \leq C\,h \tag{12}$$

for every Carleson square Q_h of side h. In case $\gamma = 1$ this characterisation is a celebrated result of Carleson [2, 3], and the extension to $\gamma < 1$ is due to Duren [7]. Measures μ that satisfy (12) are called *geometric γ–Carleson measures.* In the case $\gamma \leq 1$ in which the embedding and geometric γ–Carleson coincides we simply speak of γ–Carleson measures.

Remark 3.6. We shall require several times the following easy calculation, where we set $\operatorname{Re}\lambda = \xi > 0$, and make the substitution $y = \xi t$:

$$\|k_\lambda\|^p_{H^p} \quad = \quad \int_{-\infty}^\infty \frac{dy}{(y^2+\xi^2)^{p/2}}$$

$$= \int_{-\infty}^{\infty} \frac{\xi\, dt}{\xi^p (t^2+1)^{p/2}} = C_p^p\, \xi^{1-p}, \quad \text{say.}$$

That is, $\|k_\lambda\|_{H^p} = C_p(\operatorname{Re}\lambda)^{-1/p'}$, where C_p is a constant depending only on p, and p' is the conjugate index to p.

It is possible to use reproducing kernels as test functions for the geometric γ–Carleson property.

Lemma 3.7. *Assume that for $p, q \in (1,\infty)$ there exists a constant $M > 0$ such that*

$$\|k_z\|_{L^q(\mathbb{R}^2_+,\mu)} \le M \|k_z\|_{H^p(\mathbb{R}^2_+)} \tag{13}$$

for all $z \in \mathbb{C}_+$. Then μ is geometric γ–Carleson for $\gamma = p/q$. If the support of μ is contained in a sector $\mathcal{S}(\theta)$ with $\theta < \pi/2$, the conclusion is true when (13) merely holds for all $z > 0$.

Proof. The proof is a modification of standard arguments that can be seen, for example in [23, Lec. VII].

By Remark 3.6, $\|k_\lambda\|_{H^p} = C_p(\operatorname{Re}(\lambda))^{-1/p'}$. For $\omega \in \mathbb{R}$ and $r > 0$, let $\lambda = i\omega + r$. Consider the Carleson square $Q_{\omega,r}$ with centre ω and length r. Then the triangle inequality yields $|k_\lambda(z)| \ge 1/r$ for all $z \in Q_{\omega,r}$ and therefore,

$$\begin{aligned}\mu(Q_{\omega,r}) &= \int_{Q_{\omega,r}} d\mu \le r^q \int_{Q_{\omega,r}} \left|k_\lambda(z)\right|^q d\mu \\ &\le M r^q \left\|k_\lambda(z)\right\|^q_{H^p} = M C_p r^{q-q/p'} = C\, r^{1/\alpha}\end{aligned}$$

This shows that μ is geometric γ–Carleson. If μ has support in a sector, and if $Q_{\omega,h}$ is a Carleson square that intersects the sector $\mathcal{S}(\theta)$, then for $x + iy \in Q_{\omega,h} \cap \mathcal{S}(\theta)$ we have $0 \le x \le h$ and $|y| \le h \tan\theta$. Thus $Q_{\omega,h} \cap \mathcal{S}(\theta) \subset Q_{0,h\sec\theta}$ which justifies testing with kernels on the real line. □

Remark 3.8. The lemma asserts in particular that for $0 < \gamma \le 1$, the 'reproducing kernel thesis' holds, i.e., if the estimate from Definition 3.5 holds for the reproducing kernel functions k_λ, $\lambda \in \mathbb{C}_+$, the measure μ is embedding γ–Carleson. This implication fails in the case $\gamma > 1$. Indeed, in this case the conditions for a regular Borel measure μ on $\mathbb{C}_+$ to be embedding γ–Carleson is strictly stronger than to be geometric γ–Carleson, see e.g. [26] for a concrete counterexample.

The following necessary and sufficient condition for being embedding γ–Carleson in the case $\gamma > 1$ can be found in [28], see also [20, Thm. C]. Let S_μ denote the *balayage* of μ,

$$S_\mu(i\omega) = \int_{\mathbb{C}_+} p_z(i\omega) d\mu(z),$$

where

$$p_z(i\omega) = \pi^{-1} \frac{x}{x^2 + (y-\omega)^2}$$

denotes the Poisson kernel for $z = x + iy$ on $i\mathbb{R}$. Then μ is embedding γ–Carleson if and only if $S_\mu \in L^{\gamma'}(i\mathbb{R})$ where γ' is the conjugate exponent to γ.

A similar characterisation is possible via the Fefferman–Stein maximal function $\psi_\mu = \sup_{x \in Q} \frac{1}{h}\mu(Q)$ associated with the measure μ, see [28]. The arguments of the preceding lemma show that if the measure μ is geometric β–Carleson and supported in a sector $\mathcal{S}(\theta)$ with $\theta < \frac{\pi}{2}$, the Carleson square length $h \geq c|x|$ and so $\psi_\mu \in L^{\beta',\infty}$ where β' is the conjugate exponent of β. Consequently, one obtains from the Marcinkiewicz interpolation theorem that if μ satisfies (12) with γ equals β_1 and β_2 then μ is embedding γ–Carleson for all $\gamma \in (\beta_1, \beta_2)$ (see also [9]).

We now introduce the machinery of frames in order to analyse admissibility.

Definition 3.9. Let H be a separable Hilbert space and suppose a sequence $(f_n)_{n\geq 1}$ is given. Then $(f_n)_{n\geq 1}$ is called a *frame* if there exist constants $B > A > 0$ such that

$$A \left\|\varphi\right\|_H^2 \leq \sum_{n=1}^{\infty} \left|\langle \varphi, f_n \rangle_H\right|^2 \leq B \left\|\varphi\right\|_H^2$$

for all $\varphi \in H$.

We recall some basic facts from [4, Chapter 3]. If $(f_n)_{n\geq 1}$ is a frame, then the so-called *frame operator* $F : H \to \ell_2$, given by $(F\varphi)_n = \langle \varphi, f_n \rangle$ is clearly bounded. From the very definition of F it follows that F^*F is bounded and invertible and it can be shown the elements $\widetilde{f_n} = (F^*F)^{-1} f_n$ form another (so-called *dual*) frame satisfying

$$B^{-1} \left\|\varphi\right\|_H^2 \leq \sum_{n=1}^{\infty} \left|\langle \varphi, \widetilde{f_n} \rangle_H\right|^2 \leq A^{-1} \left\|\varphi\right\|_H^2,$$

together with $\varphi = \sum_n f_n \langle \varphi, \widetilde{f_n} \rangle$ for $\varphi \in H$ (see e.g. [4, Proposition 3.2.3]). In particular, we may always find a decomposition $\varphi = \sum c_n f_n$ satisfying the 'Besselian' estimate

$$\left\|(c_n)\right\|_{\ell_2}^2 \leq A^{-1} \left\| \sum_{n=1}^{\infty} c_n f_n \right\|_H^2$$

This elementary property will be used in the proof below. Recall (see [24, Def. I.3.2]) that the kernel a is *sectorial of angle* $\theta < \pi/2$ if $|\arg \widehat{a}(\lambda)| \leq \theta$ for all λ with $\operatorname{Re} \lambda > 0$, and that the kernel a is *1-regular* if there is a constant $c > 0$ such that $|\lambda \widehat{a}'(\lambda)| \leq c|\widehat{a}(\lambda)|$ for all λ with $\operatorname{Re} \lambda > 0$ (see [24, Def. I.3.3]).

Theorem 3.10. *For $U = \mathbb{C}$ consider the control operator $B \in \mathcal{L}(U, D(A^*)^*)$ and the associated measure $\mu := \sum |b_n|^2 \delta_{-\lambda_n}$ to the system (A, B).*

(a) *Suppose that $-\lambda_n \in \mathcal{S}(\theta)$ for all $n \in \mathbb{N}$ and some $\theta < \pi/2$, that the kernel a satisfies $\widehat{a}((0,\infty)) = (0,\infty)$ and $|\widehat{a}(\lambda)| \leq C\,|\lambda|^{-\beta}$ for some $C, \beta > 0$ and every $\lambda > 0$.*

Then μ being geometric β–Carleson is necessary for admissibility of B.

(b) *Suppose that the kernel a is 1-regular, sectorial of angle $\theta < \pi/2$ and that $|\widehat{a}(\lambda)| \geq c\,|\lambda|^{-\beta}$ for some constants $c > 0$ and $\beta > 1/2$ and every $\lambda > 0$. Let $\beta_2 > \beta > \beta_1 > \max\{1/2, \beta/3\}$.*

Then μ being embedding β_1 and β_2–Carleson is sufficient for admissibility of B.

Proof. (a) Let B be an admissible control operator. It follows from Remark 3.6 on letting $1/p + 1/p' = 1$, that we have

$$\|k_{\frac{1}{\widehat{a}(\lambda)}}\|_{H^p} = C_p\big(\operatorname{Re} \tfrac{1}{\widehat{a}(\lambda)}\big)^{-1/p'} \tag{14}$$

and hence we obtain $\|k_{\frac{1}{\widehat{a}(\lambda)}}\|_{H^p} \geq C_p\, |\widehat{a}(\lambda)|^{1/p'}$ for $\operatorname{Re}\lambda > 0$. Using condition (10) we have for $\lambda > 0$

$$\|k_{\frac{1}{\widehat{a}(\lambda)}}\|_{L^2(\mathbb{C}_+,\nu)} \leq \sqrt{M}\,\frac{|\lambda\widehat{a}(\lambda)|}{\lambda^{1/2}} \leq \sqrt{M}C^{1/p}\,|\lambda|^{1/2-\beta/p}\,|\widehat{a}(\lambda)|^{1/p'} \leq M'\,\|k_{\frac{1}{\widehat{a}(\lambda)}}\|_{H^p},$$

where $p = 2\beta$. It follows from Lemma 3.7 that μ is a geometric β–Carleson measure, which proves the first assertion.

(b) We now assume that μ is embedding β_1 and β_2–Carleson and let $p_1 = 2\beta_1$, $p_2 = 2\beta_2$. Moreover, let $u_\lambda(t) = 2(\operatorname{Re}\lambda)^{3/2}te^{-\lambda t}$ and $\mu_{j,k} := 2^{-j} + ik2^{-j}$ for $j,k \in \mathbb{Z}$. Then, $\|u_\lambda\|_{L^2(0,\infty)} = 1$ for all λ with $\operatorname{Re}\lambda > 0$ and in [6] it is shown that the system $(u_{\mu_{j,k}})_{j,k\in\mathbb{Z}}$ is a frame for $L^2(0,\infty)$. An easy calculation shows that

$$B_\infty(u_\lambda) = 2(\operatorname{Re}\lambda)^{3/2}\sum_{n=1}^{\infty} b_n\frac{1-\widehat{a}(\lambda)\lambda_n - \lambda\widehat{a}'(\lambda)\lambda_n}{(\lambda-\lambda\widehat{a}(\lambda)\lambda_n)^2}\phi_n.$$

We further define

$$g_\lambda(s) := 2(\operatorname{Re}\lambda)^{3/2}\,\frac{1+\widehat{a}(\lambda)s+\lambda\widehat{a}'(\lambda)s}{(\lambda+\lambda\widehat{a}(\lambda)s)^2}.$$

Using the 1-regularity of the kernel a, there exists a constant $C_1 > 0$ such that

$$\begin{aligned}\left|\frac{1+\widehat{a}(\lambda)s+\lambda\widehat{a}'(\lambda)s}{(1+\widehat{a}(\lambda)s)^2}\right| &= \left|\frac{\lambda\widehat{a}'(\lambda)+\widehat{a}(\lambda)}{\widehat{a}(\lambda)}\,\frac{1}{1+\widehat{a}(\lambda)s} - \frac{\lambda\widehat{a}'(\lambda)}{\widehat{a}(\lambda)}\,\frac{1}{(1+\widehat{a}(\lambda)s)^2}\right|\\ &\leq C_1\max\left\{\frac{1}{|1+\widehat{a}(\lambda)s|},\frac{1}{|1+\widehat{a}(\lambda)s|^2}\right\}.\end{aligned}$$

Moreover, 1-regularity of the kernel implies $|\widehat{a}(\lambda)| \sim |\widehat{a}(|\lambda|)|$ up to a positive constant, see [24, Lemma 8.1]. Thus $|\widehat{a}(\lambda)| \geq c'\,|\lambda|^{-\beta}$ for all $\lambda \in \mathbb{C}_+$. This estimate, together with sectoriality of the kernel implies for $p \in \{p_1, p_2\}$

$$\begin{aligned}&\|g_\lambda\|_{H^p(\mathbb{C}_+)}\\ &\leq C_1\frac{(\operatorname{Re}\lambda)^{3/2}}{|\lambda|^2}\max\left\{\left\|\frac{1}{1+\widehat{a}(\lambda)s}\right\|_{H^p(\mathbb{C}_+)},\left\|\frac{1}{(1+\widehat{a}(\lambda)s)^2}\right\|_{H^p(\mathbb{C}_+)}\right\}\\ &\overset{(14)}{=} C_2\frac{(\operatorname{Re}\lambda)^{3/2}}{|\lambda|^2}\max\left\{\frac{1}{|\widehat{a}(\lambda)|}\left(\operatorname{Re}\frac{1}{\widehat{a}(\lambda)}\right)^{-1+1/p},\frac{1}{|\widehat{a}(\lambda)|^2}\left(\operatorname{Re}\frac{1}{\widehat{a}(\lambda)}\right)^{-2+1/p}\right\}\\ &= C_2\frac{(\operatorname{Re}\lambda)^{3/2}}{|\lambda|^2}\max\left\{\frac{(\operatorname{Re}\widehat{a}(\lambda))^{-1+1/p}}{|\widehat{a}(\lambda)|^{-1+2/p}},\frac{(\operatorname{Re}\widehat{a}(\lambda))^{-2+1/p}}{|\widehat{a}(\lambda)|^{-2+2/p}}\right\}\\ &\leq C_3\frac{(\operatorname{Re}\lambda)^{3/2}}{|\lambda|^2}|\widehat{a}(\lambda)|^{-1/p} \leq \quad C_4\frac{(\operatorname{Re}\lambda)^{3/2}}{|\lambda|^2}|\lambda|^{\beta/p},\end{aligned}$$

for positive constants C_1, C_2, C_3 and C_4. Thus,

$$\left\|g_{\mu_{j,k}}\right\|^2_{H^p(\mathbb{C}_+)} \leq C_4^2\,\frac{2^{j(1-2\beta/p)}}{(1+k^2)^{2-\beta/p}},$$

and

$$M_1 := \left(\sum_{j\geq 0,k\in\mathbb{Z}}\left\|g_{\mu_{j,k}}\right\|^2_{H^{p_1}}\right)^{1/2} + \left(\sum_{j<0,k\in\mathbb{Z}}\left\|g_{\mu_{j,k}}\right\|^2_{H^{p_2}}\right)^{1/2} < \infty.$$

Let u be a finite linear combination of the functions $(u_{\mu_{j,k}})_{j,k\in\mathbb{Z}}$ and let $\alpha_{j,k} = \langle u, \widetilde{u}_{j,k}\rangle$. By the Besselian property of the coefficients $\alpha_{j,k}$ we have

$$\sum_{j,k} |\alpha_{j,k}|^2 \le M_2 \left\|u\right\|^2_{L^2(0,\infty)} \tag{15}$$

for some constant $M_2 > 0$, independent of u. This implies

$$\begin{aligned}
&\|B_\infty(u)\|_H \\
&= \Big(\sum_n |b_n|^2 \Big|\sum_{j,k\in\mathbb{Z}} \alpha_{j,k} g_{\mu_{j,k}}(-\lambda_n)\Big|^2\Big)^{1/2} \\
&= \Big(\int_{\mathbb{C}_+} \Big|\sum_{j,k\in\mathbb{Z}} \alpha_{j,k} g_{\mu_{j,k}}(s)\Big|^2 d\mu(s)\Big)^{1/2} \\
&\le \Big(\int_{\mathbb{C}_+} \Big|\sum_{j\ge 0,k\in\mathbb{Z}} \alpha_{j,k} g_{\mu_{j,k}}(s)\Big|^2 d\mu(s)\Big)^{1/2} + \Big(\int_{\mathbb{C}_+} \Big|\sum_{j<0,k\in\mathbb{Z}} \alpha_{j,k} g_{\mu_{j,k}}(s)\Big|^2 d\mu(s)\Big)^{1/2} \\
&\le M_3\Big(\Big\|\sum_{j\ge 0,k\in\mathbb{Z}} \alpha_{j,k} g_{\mu_{j,k}}\Big\|_{H^{p_1}(\mathbb{C}_+)} + \Big\|\sum_{j<0,k\in\mathbb{Z}} \alpha_{j,k} g_{\mu_{j,k}}\Big\|_{H^{p_2}(\mathbb{C}_+)}\Big)
\end{aligned}$$

since μ is embedding β_1 and β_2–Carleson. Now since $p_1, p_2 \ge 1$, the Minkowski and Cauchy–Schwarz inequalities together with (15) yield

$$\begin{aligned}
&\le M_2 M_3 \|u\|_{L^2(0,\infty)}\Big(\Big(\sum_{j\ge 0,k\in\mathbb{Z}} \|g_{\mu_{j,k}}\|^2_{H^{p_1}}\Big)^{1/2} + \Big(\sum_{j<0,k\in\mathbb{Z}} \|g_{\mu_{j,k}}\|^2_{H^{p_2}}\Big)^{1/2}\Big) \\
&= M_1 M_2 M_3 \|u\|_{L^2(0,\infty)},
\end{aligned}$$

and the proof is done. □

Remark 3.11. (a) If the spectrum of A is contained in a sector $\mathcal{S}(\theta)$ with $\theta < \pi/2$, the assumption of μ being embedding β–Carleson can be weakened to geometric β–Carleson in the second part of the Theorem by using Remark 3.8.

(b) The sectoriality of a with angle θ already implies an estimate on the growth $|\widehat{a}(\lambda)|$ in the half plane. Indeed, as explained in Monniaux–Prüss [22, Proposition 1],

$$\log \widehat{a}(\lambda) = k_0 + \tfrac{i}{\pi}\int_{-\infty}^{\infty} \Big[\frac{1-ir\lambda}{\lambda - ir}\Big] \arg(\widehat{a}(ir))\frac{dr}{1+r^2}$$

by a Poisson formula applied to $\arg(\widehat{a}(\lambda))$. Here, k_0 is a suitable real constant. Considering real $\lambda > 1$ the authors infer a growth bound $|\widehat{a}(\lambda)| \ge c|\lambda|^{-\alpha}$ with $\alpha = \frac{\pi}{2\theta}$. Combining with [24, Lemma 8.1] extends the estimate to $|\lambda| > 1$. Inside the unit ball this estimate is not true in general. Let e.g. $a(t) = t^{-\frac{1}{2}}e^{-t}$. Then a is a sectorial kernel of type $\frac{\pi}{4}$ but $\widehat{a}(\lambda) = \big(\frac{\pi}{1+z}\big)^{\frac{1}{2}}$ attains a finite non-zero limit at the origin.

Assume that the sectoriality angle satisfies $\theta \le \frac{\pi}{2\beta}$. Then the growth condition on $\widehat{a}$ in the second part of Theorem 3.10 at infinity is automatic. It remains however a non-trivial condition on $\widehat{a}$ in the origin.

3.2. **Controllability.** We are now ready to use the techniques of interpolation to give conditions for controllability of the Volterra system (1). Again we assume that the solution family is exponentially stable.

Lemma 3.12. *The following formula for* $\mathcal{B}_\infty$ *holds.*

$$\mathcal{B}_\infty u = \frac{1}{2\pi i}\sum_{n=1}^{\infty} b_n \int_{i\mathbb{R}} \frac{\widehat{u}(\lambda)}{\lambda_n - (1/\widehat{a}(-\lambda))}\frac{1}{\lambda\widehat{a}(-\lambda)}\, d\lambda\, \phi_n.$$

Proof.

$$\begin{aligned}\mathcal{B}_\infty u &= \int_0^\infty S(t)Bu(t)\,dt \\ &= \sum_{n=1}^{\infty} b_n \int_0^\infty c_n(t)u(t)\,dt\,\phi_n \\ &= \frac{1}{2\pi}\sum_{n=1}^{\infty} b_n \langle \widehat{u}, \widehat{c_n}\rangle_{H^2(\mathbb{C}_+)}\,\phi_n,\end{aligned}$$

by Plancherel's theorem. Thus, for suitably small $\delta > 0$, we have

$$\begin{aligned}\mathcal{B}_\infty u &= \frac{1}{2\pi}\sum_{n=1}^{\infty} b_n \int_{-\infty}^{\infty} \widehat{u}(\delta + i\omega)\overline{\widehat{c_n}(i\omega+\delta)}\,d\omega\,\phi_n \\ &= \frac{1}{2\pi}\sum_{n=1}^{\infty} b_n \int_{-\infty}^{\infty} \widehat{u}(\delta + i\omega)\sigma(-i\omega-\delta, -\lambda_n)\,d\omega\,\phi_n \\ &= \frac{1}{2\pi}\sum_{n=1}^{\infty} b_n \int_{-\infty}^{\infty} \frac{\widehat{u}(\delta + i\omega)}{-\delta - i\omega + (\delta+i\omega)\widehat{a}(-\delta-i\omega)\lambda_n}\,d\omega\,\phi_n \\ &= \frac{1}{2\pi i}\sum_{n=1}^{\infty} b_n \int_{\delta+i\mathbb{R}} \frac{\widehat{u}(\lambda)}{\lambda_n - (1/\widehat{a}(-\lambda))}\frac{1}{\lambda\widehat{a}(-\lambda)}\,d\lambda\,\phi_n.\end{aligned}$$

□

Consider now the kernels defined in (4). For example, we may take $\widehat{a}(\lambda) = \xi/(\lambda+s)$, where $\xi \in \mathbb{R}$ and $s \geq 0$. We then obtain

$$\mathcal{B}_\infty u = \frac{1}{2\pi i}\sum_{n=1}^{\infty} b_n \int_{\delta+i\mathbb{R}} \frac{\widehat{u}(\lambda)(-\lambda+s)}{\lambda(\lambda_n\xi + (\lambda - s))}\,d\lambda\,\phi_n,$$

which can be calculated using the residue formula as

$$\mathcal{B}_\infty u = \sum_{\substack{n=1 \\ \mathrm{Re}(s-\lambda_n\xi)>0}}^{\infty} b_n \frac{\widehat{u}(s-\lambda_n\xi)\lambda_n\xi}{\lambda_n\xi - s}\,\phi_n. \tag{16}$$

The surjectivity of $\mathcal{B}_\infty$ reduces to an interpolation problem of the type analysed in McPhail [21] (the case $s = 0$ and $\xi = 1$ being applied to controllability questions in [16]).

We may use McPhail's theorem as expressed in the half-plane version in [16]. Namely, given (s_n) distinct points in $\mathbb{C}_+$ and (ν_n) non-zero complex numbers, one can find a solution in $H^2(\mathbb{C}_+)$ to $F(s_n) = \nu_n x_n$ for every $(x_n) \in \ell_2$, if and only if

$\nu = \sum_{n=1}^{\infty} \frac{(\operatorname{Re} s_n)^2 |\nu_n|^2}{\varepsilon_n^2} \delta_{s_n}$ is a Carleson measure, where $\varepsilon_n = \prod_{k \neq n} \left| \frac{s_n - s_k}{s_n + \bar{s}_k} \right|$.

The following result therefore generalises part of Theorem 3.1 of [16].

Theorem 3.13. *In the case $\widehat{a}(\lambda) = \xi/(\lambda + s)$, where $\xi \in \mathbb{R}$ and $s \geq 0$, exact controllability is equivalent to the property that*

$$\sum_{n=1}^{\infty} \frac{|\operatorname{Re}(s - \lambda_n \xi)|^2 |\lambda_n \xi - s|^2}{\varepsilon_n^2 |b_n|^2 |\lambda_n \xi|^2} \delta_{\lambda_n \xi - s}$$

should be a Carleson measure, where

$$\varepsilon_n = \prod_{k \neq n} \left| \frac{\xi(\lambda_n - \lambda_k)}{2s - \xi(\lambda_n + \overline{\lambda}_k)} \right|.$$

Likewise, we may obtain conditions for null controllability in time τ. The following result reduces to part of Theorem 2.1 of [16] in the case $s = 0$, $\xi = 1$.

Theorem 3.14. *In the case $\widehat{a}(\lambda) = \xi/(\lambda + s)$, where $\xi \in \mathbb{R}$ and $s \geq 0$, null controllability in time $\tau > 0$ is equivalent to the property that*

$$\sum_{n=1}^{\infty} \frac{|\operatorname{Re}(s - \lambda_n \xi)|^2 |\lambda_n \xi - s|^2 |c_n(\tau)|^2}{\varepsilon_n^2 |b_n|^2 |\lambda_n \xi|^2} \delta_{\lambda_n \xi - s}$$

should be a Carleson measure, where

$$\varepsilon_n = \prod_{k \neq n} \left| \frac{\xi(\lambda_n - \lambda_k)}{2s - \xi(\lambda_n + \overline{\lambda}_k)} \right|$$

and c_n is given by (8).

Proof. This follows on observing that the interpolation problem to be solved now has the form $\mathcal{B}_\infty u = \sum_{n=1}^{\infty} c_n(\tau) x_n \phi_n$ where (x_n) in ℓ_2 is arbitrary, and where $\mathcal{B}_\infty u$ is given in (16). □

For higher-order rational functions, the interpolation problems that arise are more complicated and will repay future investigation. We now outline some of the issues involved. For functions h and ϕ we define the weighted composition operator $C_{h,\phi}$ by

$$(C_{h,\phi} \widehat{u})(\lambda) = h(\lambda) \widehat{u}(\phi(\lambda)).$$

We assume that $1/\widehat{a}(-\cdot)$ maps a piecewise smooth curve Γ bijectively onto $i\mathbb{R}$. Let ϕ denote the inverse function of $1/\widehat{a}(-\cdot)$ mapping $i\mathbb{R}$ onto Γ. Assuming that there are no singularities of the integrand below between $i\mathbb{R}$ and Γ, so that (17) and (18) are equivalent, we have

$$\mathcal{B}_\infty u = \frac{1}{2\pi i} \sum_{n=1}^{\infty} b_n \int_{i\mathbb{R}} \frac{\widehat{u}(\lambda)}{\lambda_n - (1/\widehat{a}(-\lambda))} \frac{1}{\lambda \widehat{a}(-\lambda)} \, d\lambda \, \phi_n \tag{17}$$

$$= \frac{1}{2\pi i} \sum_{n=1}^{\infty} b_n \int_{\Gamma} \frac{\widehat{u}(\lambda)}{\lambda_n - (1/\widehat{a}(-\lambda))} \frac{1}{\lambda \widehat{a}(-\lambda)} \, d\lambda \, \phi_n \tag{18}$$

$$= \frac{1}{2\pi i} \sum_{n=1}^{\infty} b_n \int_{i\mathbb{R}} \frac{\widehat{u}(\phi(z))}{\lambda_n - z} \frac{z \phi'(z)}{\phi(z)} \, dz \, \phi_n$$

$$= \sum_{n=1}^{\infty} b_n \langle P_{H^2} C_{\frac{z\phi'(z)}{\phi(z)},\phi} u, k_{\lambda_n}\rangle \, \phi_n.$$

Example 3.15. Let $\widehat{a}(\lambda) = \lambda^{-1/2}$, i.e., $\phi(z) = z^2$. In this situation we obtain

$$\begin{aligned} \mathcal{B}_\infty u &= \tfrac{1}{\pi i} \sum_{n=1}^{\infty} b_n \int_{i\mathbb{R}} \frac{\widehat{u}(z^2)}{\lambda_n - z}\, dz \, \phi_n, \\ &= \sum_{n=1}^{\infty} 2 b_n P_{H^2}(C_\phi u)(\lambda_n)\phi_n. \end{aligned}$$

Now, if $v \in H^2(\mathbb{C}_+)$ and $v(s) = O(s^{-2})$ as $|s| \to \infty$, then it follows by an easy estimate of $\int_{-\infty}^{\infty} |v((x+iy)^{1/2})|^2 \, dy$ that the function $u : s \mapsto v(s^{1/2})$ lies in $H^2(\mathbb{C}_+)$, and thus for such functions v we have

$$B_\infty u = \sum_{n=1}^{\infty} 2 b_n v(\lambda_n) \phi_n.$$

Thus exact controllability is linked to the condition that for all sequences $(x_n) \in \ell_2$ there is a function $u \in H^2(\mathbb{C}_+)$ with $b_n u(\lambda_n^{1/2}) = x_n$, which can once more be expressed in terms of Carleson measures.

4. Examples

The monograph of Prüss [24] contains numerous examples of Volterra systems to which Theorem 3.10 can be applied. Here we study one particular example.

Consider a simplified problem of heat conduction with memory in a bounded domain $\Omega \subseteq \mathbb{R}^d$. The uncontrolled situation has been studied by Zacher [31]. Integrating his equations from zero to t one obtains

$$x(t) = x_0 + \int_0^t a(t-s)\Delta x(s)\, ds + \int_0^t Bu(s)\, ds, \qquad t \geq 0 \tag{19}$$

with some boundary conditions (to be specified later) for the unknown temperature x. Here, the kernel is given by $a(t) = t^\alpha$ where $\alpha \in [0, 1)$ is a material parameter and $B : U \to D(A^*)^*$ is the control operator. Notice that the case $\alpha = 0$ corresponds to the classical heat equation and that the (excluded) parameter $\alpha = 1$ would correspond to a wave equation.

Dirichlet boundary conditions. Consider (19) under the Dirichlet boundary condition $x|_{\partial\Omega} = 0$. The problem then reads as (1) where A is the Dirichlet Laplacian. It is well known that A is self-adjoint and has compact resolvent and thus generates a diagonal semigroup. Notice that $\widehat{a}(\lambda) = \Gamma(1+\alpha)\lambda^{-1-\alpha}$ satisfies the growth conditions of Theorem 3.10 (here Γ denotes the Gamma function). Moreover, a is evidently k–regular for all $k \in \mathbb{N}$. From

$$H(\lambda) = (I - \widehat{a}(\lambda)A)^{-1}/\lambda = \tfrac{1}{\lambda}\, \lambda^{\alpha+1}\big(\lambda^{\alpha+1} - \Gamma(\alpha+1)A\big)^{-1}$$

and the sectoriality of A (actually with arbitrary small angle) one concludes finally that the equation is parabolic in the sense of [24, Definition I.3.1]. Therefore, [24, Theorem I.3.1] assures the existence of a bounded resolvent family $(S(t))_{t\geq 0}$ that is even $C^\infty((0,\infty), B(X))$.

We study a rank one control $B : \mathbb{C} \to D(A^*)^*$, i.e., $B = (b_n)$ via Theorem 3.10 in the most easy case of a one-dimensional rod of length one, say $[0, 1]$. Then

$\lambda_n = -n^2\pi^2$ for $n \in \mathbb{N}_{\geq 1}$. In virtue of the preceding remark it sufficient to verify that μ is geometric β_1 and β_2–Carleson. Let b_n satisfy $|b_n| \leq Cn^\delta$ for some $\delta > 0$. Notice that $\lfloor\sqrt{h}/\pi\rfloor = 0$ when $h \in [0, \pi^2)$. We may thus restrict ourselves to $h > \pi^2$ in the following estimate:

$$\mu(Q_h) \leq C \sum_{j=1}^{\lfloor\sqrt{h}/\pi\rfloor} j^{2\delta} \leq C \int_0^{\lfloor\sqrt{h}/\pi\rfloor} x^{2\delta}\,dx = \frac{C}{1+2\delta}\left(\lfloor\sqrt{h}/\pi\rfloor\right)^{1+2\delta}.$$

Since $h > 1$ it is sufficient to establish the estimate (12) for the maximum of β_1 and β_2 that may be chosen arbitrarily near to $\beta = 1+\alpha$. Using the trivial inequality $\lfloor x\rfloor \leq x$, one concludes that for $\alpha \in [0,1)$ given, all elements $b = (b_n)$ that satisfy

$$|b_n| \leq Cn^\delta \qquad \text{with} \qquad \delta < \tfrac{1}{2}\,\tfrac{1-\alpha}{1+\alpha}$$

are admissible. Let $X = \ell_2$ and let X_θ denote the fractional domain space of X. Since A is boundedly invertible, we have $b \in X_{-\theta}$ if, and only if

$$\sum_{n=1}^{\infty} \left|\frac{b_n}{\lambda_n^\theta}\right|^2 < \infty$$

which implies $|b_n| \leq Cn^{2\theta-\frac{1}{2}}$. Consequently, all elements $b \in X_{-\theta}$ with $\theta < \frac{1}{2}\frac{1}{1+\alpha}$ are admissible. This result matches well with the case $\alpha = 0$ of the classical heat equation, where it is well known that $\theta < 1/2$ is sufficient for admissibility – a result that fails for $\theta = 1/2$ (see [30]).

Neumann boundary conditions. Next, we study (19) under a Neumann boundary condition $\frac{\partial}{\partial\nu}x = 0$. To this end, let $\Omega \subset \mathbb{R}^d$ be a bounded domain that admits a bi-Lipschitz map from Ω with constant L onto the unit ball in $\mathbb{R}^n$. Let A denote the negative Neumann Laplacian $-\Delta_N$ on Ω. The eigenvalues of A can be arranged according to their multiplicities as

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \ldots$$

As in the one-dimensional case with Dirichlet boundary conditions, the uncontrolled problem is parabolic and the existence of a bounded solution family $(S(t))_{t\geq 0}$ is assured by [24, Theorem I.3.1]. It is known (see e.g. [19]) that the eigenvalues of Δ_N satisfy an estimate

$$C_{|\Omega|,d}\big(1 - C_{d,L}\, n^{-1/d}\big)\, n^{2/d} \leq \mu_n \leq C'_{d,L}\, n^{2/d}. \tag{20}$$

where the subscripts refer to the dependencies of the constant on the dimension d, the bi-Lipschitz constant L and the volume $|\Omega|$ respectively. Let ϕ_n, $n \geq 0$ be a basis of eigenvectors of Δ_N, normalised in a suitable Hilbert function space X, and let $B : U \to D(A^*)^*$ be bounded, where U is another Hilbert space. Then

$$Bu = \sum_{n=1}^{\infty} \phi_n \langle u, f_n\rangle_U$$

with $f_n = B^*\phi_n \in U^*$. Combining Theorem 3.10 and Proposition 3.2 yields that B is admissible provided that the measure

$$\nu = \sum_{n=1}^{\infty} \delta_{\mu_n} \|f_n\|_U^2$$

is geometric β_1 and β_2–Carleson for suitable β_i close to $\beta = 1+\alpha$. Notice that we did not put any weight at $\mu_0 = 0$ since this would destroy the β–Carleson property for all $\beta > 0$. Since the support of ν does not intersect with a small ball around the origin, it is again sufficient to establish estimate (12) for the maximum of β_1 and β_2; moreover, we may restrict to large μ_n, i.e., we may assume without loss of generality that $\frac{1}{c} n^{2/d} \le \mu_n \le c\, n^{2/d}$ for some $c > 0$. Then, essentially the same calculation as in the example of the one-dimensional rod above yields that if $\|f_n\|_U \le Cn^\delta$ for all $n \in \mathbb{N}$, B is an admissible control operator provided that $\delta < \frac{1}{2} \frac{2/d - 1 - \alpha}{1+\alpha}$.

In the linear case, multiplying a boundary control system of the form $x'(t) + \Delta x(t) = 0$, $\frac{\partial}{\partial \nu} x = u$ with a test function and integrating by parts shows that it fits into the abstract setting $x'(t) + Ax(t) = Bu(t)$ where $A = \Delta_N$ and B is the adjoint operator of the Dirichlet trace, see e.g. [1]. One needs therefore to estimate the boundary traces of the Neumann eigenvectors. The choice of the Hilbert function spaces X (in the domain Ω) and U (on its boundary) plays an important rôle.

Take e.g. $X = H^1(\Omega)$ and $U = L^2(\partial\Omega)$ and assume that Ω has a C^1 boundary. Together with $\mu_n \ge c\, n^{2/d}$ for large n and the boundedness of the Dirichlet trace from $H^{1/2}(\Omega)$ to $L^2(\partial\Omega)$, it readily follows in the case $d = 1$ and $d = 2$ that for $\alpha \in [0, 1)$ the boundary control B is admissible. For $d \ge 3$ the problem can be analysed by passing to a higher-order Sobolev space. For smoother domains, it is possible to use more sophisticated estimates for the Dirichlet trace operator, such as those given by Tataru [25].

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