THE STOCHASTIC WEISS CONJECTURE FOR BOUNDED ANALYTIC SEMIGROUPS

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ABSTRACT. Suppose $-A$ admits a bounded H^∞-calculus of angle less than $\pi/2$ on a Banach space E which has Pisier's property (α), let B be a bounded linear operator from a Hilbert space H into the extrapolation space E_{-1} of E with respect to A, and let W_H denote an H-cylindrical Brownian motion. Let $\gamma(H,E)$ denote the space of all γ-radonifying operators from H to E. We prove that the following assertions are equivalent:

(a) the stochastic Cauchy problem $dU(t) = AU(t)\,dt + B\,dW_H(t)$ admits an invariant measure on E;

(b) $(-A)^{-1/2}B \in \gamma(H,E)$;

(c) the Gaussian sum $\sum_{n\in\mathbb{Z}} \gamma_n 2^{n/2} R(2^n, A)B$ converges in $\gamma(H,E)$ in probability.

This solves the stochastic Weiss conjecture of [7].

1. Introduction

Let A be the generator of a strongly continuous bounded analytic semigroup $S = (S(t))_{t\geqslant 0}$ on a Banach space E, let F be another Banach space, and let $C : \mathsf{D}(A) \to F$ be a bounded operator. If there exists a constant $M \geqslant 0$ such that

$$\int_0^\infty \|CS(t)x\|_F^2\,dt \leqslant M^2\|x\|_E^2, \quad \forall x \in \mathsf{D}(A),$$

an easy Laplace transform argument shows that

$$\sup_{\lambda>0} \lambda^{1/2}\|CR(\lambda,A)\|_{\mathscr{L}(E)} \leqslant M.$$

Here, as usual, $R(\lambda,A) = (\lambda - A)^{-1}$ denotes the resolvent of A at λ.

The celebrated *Weiss conjecture* in linear systems theory is the assertion that the converse also holds. It was solved affirmatively for normal operators A acting on a Hilbert space by Weiss [24], for generators of analytic Hilbert space contraction semigroups with $F = \mathbb{C}$ by Jacob and Partington [9], and subsequently for operators admitting a bounded H^∞-calculus of angle $< \pi/2$ acting on an L^p-space, $1 < p < \infty$, by Le Merdy [16, 17]. Counterexamples to the general statement were found by Jacob, Partington and Pott [10], Zwart, Jacob, and Staffans [25], and Jacob and Zwart [11].

Whereas the Weiss conjecture is concerned with *observation operators*, in the context of stochastic evolution equations it is natural to consider a 'dual' version of the conjecture in terms of *control operators*. To be more precise, we consider the following situation. Let $W_H = (W_H(t))_{t\in[0,T]}$ be a cylindrical Brownian motion in

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a Hilbert space H and let $B \in \mathscr{L}(H, E_{-1})$ be a bounded linear operator. Here, E_{-1} denotes the extrapolation space of E with respect to A (see Subsection 2.5). The *stochastic Weiss conjecture*, proposed recently in [7], is the assertion that, under suitable assumptions on the linear operator A, the existence of an invariant measure for the linear stochastic Cauchy problem

$$(\mathrm{SCP})_{(A,B)} \qquad \begin{cases} dU(t) = AU(t)\,dt + B\,dW_H(t), & t \in [0,T], \\ U(0) = 0, \end{cases}$$

is equivalent to an appropriate condition on the operator-valued function $\lambda \mapsto \lambda^{1/2} R(\lambda, A)B$. This conjecture is justified by the observation (cf. Proposition 2.4 below) that an invariant measure exists if and only if $t \mapsto S(t)B$ defines an element of the space $\gamma(L^2(\mathbb{R}_+; H), E)$ (see Subsection 2.3 for the definition of this space).

In the paper just cited, an affirmative solution was given in the case where A and B are simultaneously diagonalisable. The aim of this article is to prove the stochastic Weiss conjecture for the class of operators admitting a bounded H^∞-calculus of angle $< \pi/2$. Denoting by $S(E)$ the class of all sectorial operators $-A$ on E of angle $< \pi/2$ that are injective and have dense range, our main result reads as follows.

Theorem 1.1. *Let E have property (α) and assume that $-A \in S(E)$ admits a bounded H^∞–calculus of angle $< \pi/2$ on E. Let $B : H \to E_{-1}$ be a bounded operator. Then the following assertions are equivalent:*

(a) $(\mathrm{SCP})_{(A,B)}$ *admits an invariant measure on E;*
(b) $(-A)^{-1/2} B \in \gamma(H, E)$;
(c) $\lambda \mapsto \lambda^{1/2} R(\lambda, A)B$ *defines an element in $\gamma(L^2(\mathbb{R}_+, \frac{d\lambda}{\lambda}; H), E)$;*
(d) *for all $\lambda > 0$ we have $R(\lambda, A)B \in \gamma(H, E)$ and the Gaussian sum*

$$\sum_{n\in\mathbb{Z}} \gamma_n 2^{n/2} R(2^n, A)B$$

converges in $\gamma(H, E)$ in probability (equivalently, in $L^p(\Omega; \gamma(H, E))$ for some (all) $1 \leqslant p < \infty$).

Since B maps into the extrapolation space E_{-1}, some care has to be taken in giving a rigorous interpretations of these assertions. The details will be explained below.

In the special case when E is a Hilbert space and H is a separable Hilbert space with orthonormal basis $(h_k)_{k\geqslant 1}$, condition (a) is equivalent to

$$\sum_{k=1}^{\infty} \int_0^\infty \|S(t)Bh_k\|^2\,dt < \infty, \tag{1.1}$$

and condition (d) reduces to

$$\sum_{k=1}^{\infty} \sum_{n\in\mathbb{Z}} 2^n \|R(2^n, A)Bh_k\|^2 < \infty. \tag{1.2}$$

Compared to the Weiss conjecture, we see that a uniform boundedness condition on $\lambda^{1/2} R(\lambda, A)B$ gets replaced by a (dyadic) square summability condition along $(h_k)_{k\geqslant 1}$ in (1.2); this is consistent with the square summability condition along $(h_k)_{k\geqslant 1}$ in (1.1).

All spaces are real. When we use spectral arguments, we turn to the complexifications without further notice.

2. Preliminaries

In this section we collect some notations and results that will be used in the proof of Theorem 1.1.

2.1. **Property** (α)**.** A *Rademacher sequence* is a sequence of independent random variables taking the values ± 1 with probability $1/2$. Let $(r'_j)_{j=1}^\infty$ and $(r''_k)_{k=1}^\infty$ be Rademacher sequences on probability spaces $(\Omega', \mathbb{P}')$ and $(\Omega'', \mathbb{P}'')$, and let $(r_{jk})_{j,k=1}^\infty$ be a doubly indexed Rademacher sequence on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. It is important to observe that the sequence $(r'_j r''_k)_{j,k=1}^\infty$ is not a Rademacher sequence. By standard randomisation techniques one proves (see, e.g., [21]):

Proposition 2.1. *For a Banach space E the following assertions are equivalent:*

(1) *there exists a constant $C \geqslant 0$ such that for all finite sequences $(a_{jk})_{j,k=1}^n$ in $\mathbb{R}$ and $(x_{jk})_{j,k=1}^n$ in E we have*
$$\mathbb{E}'\mathbb{E}''\Big\| \sum_{j,k=1}^n a_{jk} r'_j r''_k x_{jk}\Big\|^2 \leqslant C^2 \big(\max_{1\leqslant j,k \leqslant n} |a_{jk}|\big)^2 \mathbb{E}'\mathbb{E}''\Big\| \sum_{j,k=1}^n r'_j r''_k x_{jk}\Big\|^2;$$

(2) *there exists a constant $C \geqslant 0$ such that for all finite sequences $(x_{jk})_{j,k=1}^n$ in E we have*
$$\frac{1}{C^2}\mathbb{E}\Big\| \sum_{j,k=1}^n r_{jk} x_{jk}\Big\|^2 \leqslant \mathbb{E}'\mathbb{E}''\Big\| \sum_{j,k=1}^n r'_j r''_k x_{jk}\Big\|^2 \leqslant C^2 \mathbb{E}\Big\| \sum_{j,k=1}^n r_{jk} x_{jk}\Big\|^2.$$

A Banach space E is said to have *property* (α) if it satisfies the above equivalent conditions. Examples of spaces having this property are Hilbert spaces and the spaces $L^p(\mu)$ with $1 \leqslant p < \infty$. Property (α) was introduced by Pisier [22], who proved that a Banach lattice has property (α) if and only if it has finite cotype. In particular, the space c_0 fails property (α).

2.2. γ**-Boundedness.** A family $\mathscr{T} \subseteq \mathscr{L}(E,F)$ is called γ-*bounded* if there exists a constant $C \geqslant 0$ such that for all finite sequences $(T_n)_{n=1}^N$ in $\mathscr{T}$ and $(x_n)_{n=1}^N$ in E we have
$$\mathbb{E}\Big\| \sum_{n=1}^N \gamma_n T_n x_n\Big\|^2 \leqslant C^2 \mathbb{E}\Big\| \sum_{n=1}^N \gamma_n x_n\Big\|^2.$$
The least admissible constant in this inequality is called the γ-*bound* of $\mathscr{T}$.

By letting N=1 it is seen that γ-bounded families are uniformly bounded. For Hilbert spaces E and F, the notions of uniform boundedness and γ-boundedness are equivalent. For detailed expositions on γ-boundedness and the closely related notion of R-boundedness, as well as for references to the extensive literature we refer the reader to [2, 4, 15, 23].

2.3. γ**-Radonifying operators.** Let $\mathscr{H}$ be a Hilbert space and E a Banach space. For a finite rank operator $T : \mathscr{H} \to E$ of the form
$$T = \sum_{n=1}^N h_n \otimes x_n,$$
where $(h_n)_{n=1}^N$ is an orthonormal sequence in $\mathscr{H}$ and $(x_n)_{n=1}^N$ is a sequence in E, we define
$$\|T\|_{\gamma(\mathscr{H},E)} := \Big\| \sum_{n=1}^N \gamma_n x_n\Big\|_{L^2(\Omega;E)}. \tag{2.1}$$
Here, $(\gamma_n)_{n=1}^N$ is a sequence of independent standard Gaussian random variables on a probability space $(\Omega, \mathbb{P})$. The Banach space $\gamma(\mathscr{H}, E)$ is defined as the completion of the linear space of finite rank operators with respect to this norm.

The following *γ-Fatou lemma* holds (see [13, 18]). Suppose $(T_n)_{n=1}^\infty$ is a bounded sequence in $\gamma(\mathscr{H}, E)$ and $T \in \mathscr{L}(\mathscr{H}, E)$ is an operator such that

$$\lim_{n\to\infty} \langle T_n h, x^*\rangle = \langle Th, x^*\rangle, \quad \forall h \in \mathscr{H},\ x^* \in E^*.$$

Then, if E does not contain a closed subspace isomorphic to c_0, we have $T \in \gamma(\mathscr{H}, E)$ and

$$\|T\|_{\gamma(\mathscr{H},E)} \leqslant \liminf_{n\to\infty} \|T_n\|_{\gamma(\mathscr{H},E)}. \tag{2.2}$$

The *Kalton–Weis extension theorem* [13, Proposition 4.4] (see also [18]) asserts that if $T : H_1 \to H_2$ is a bounded linear operator, then the tensor extension $T : H_1 \otimes E \to H_2 \otimes E$,

$$T(h \otimes x) := Th \otimes x$$

extends to a bounded operator (with the same norm) from $\gamma(H_1, E)$ to $\gamma(H_2, E)$.

The *Kalton–Weis multiplier theorem* [13, Proposition 4.11] (see [18] for the formulation given here) asserts that if (X, μ) is a σ-finite measure space, E and F are Banach spaces with F not containing a closed subspace isomorphic to c_0, and if $M : X \to \mathscr{L}(E, F)$ is measurable with respect to the strong operator topology and has γ-bounded range, then the mapping

$$(\mathbf{1}_B \otimes h) \otimes x \mapsto (\mathbf{1}_B \otimes h) \otimes Mx$$

has a unique extension to a bounded linear operator from $\gamma(L^2(X, \mu; H), E)$ into $\gamma(L^2(X, \mu; H), F)$ (with norm equal to the γ-bound of the range of M).

Below we shall use (see [21]) that a Banach space E has property (α) if and only if, whenever $\mathscr{H}_0$ and $\mathscr{H}_1$ are nonzero Hilbert spaces, the mapping $(h_0 \otimes h_1) \otimes x \mapsto h_0 \otimes (h_1 \otimes x)$ extends to an isomorphism of Banach spaces

$$\gamma(\mathscr{H}_0 \widehat{\otimes} \mathscr{H}_1, E) \simeq \gamma(\mathscr{H}_0, \gamma(\mathscr{H}_1, E)).$$

Here, $\mathscr{H}_0 \widehat{\otimes} \mathscr{H}_1$ denotes the Hilbert space completion of the algebraic tensor product $\mathscr{H}_0 \otimes \mathscr{H}_1$. We will be particularly interested in the case $\mathscr{H}_0 = L^2(\mathbb{R}_+, \frac{dt}{t})$, in which case the above isomorphism then takes the form

$$\gamma(L^2(\mathbb{R}_+, \tfrac{dt}{t}; H), E) \simeq \gamma(L^2(\mathbb{R}_+, \tfrac{dt}{t}), \gamma(H, E)).$$

2.4. **Stochastic integration.** Let H be a Hilbert space and let $(\Omega, \mathbb{P})$ be a probability space. A *cylindrical Brownian motion in* H is a mapping $W_H : L^2(\mathbb{R}_+; H) \to L^2(\Omega)$ such that $W_H f$ is a centred Gaussian random variable for all $f \in L^2(\mathbb{R}_+; H)$ and

$$\mathbb{E}(W_H f \cdot W_H g) = [f, g]_{L^2(\mathbb{R}_+;H)}$$

for all $f, g \in L^2(\mathbb{R}_+; H)$. Such a mapping is linear and bounded.

A function $\Phi : \mathbb{R}_+ \to \mathscr{L}(H, E)$ is said to be *stochastically integrable* with respect to W_H if it is scalarly square integrable, i.e., for all $x^* \in E^*$ the function $\Phi^* x^* : t \mapsto \Phi^*(t)x^*$ belongs to $L^2(\mathbb{R}_+; H)$, and for all Borel sets $B \subseteq \mathbb{R}_+$ there exists a random variable $X_B \in L^2(\Omega; E)$ such that

$$\int_B \Phi^* x^*\, dW_H := W_H(\mathbf{1}_B \Phi^* x^*) = \langle X_B, x^*\rangle, \quad \forall x^* \in E^*.$$

In that case we define

$$\int_B \Phi\, dW_H := X_B.$$

The following result was proved in [19].

Proposition 2.2. *A scalarly square integrable function* $\Phi : \mathbb{R}_+ \to \mathscr{L}(H, E)$ *is stochastically integrable with respect to* W_H *if and only if there exists an operator* $R \in \gamma(L^2(\mathbb{R}_+; H), E)$ *such that* $R^* x^* = \Phi^* x^*$ *in* $L^2(\mathbb{R}_+; H)$ *for all* $x^* \in E^*$.

2.5. **Existence, uniqueness and invariant measures.** Let A be the generator of a strongly continuous semigroup $S = (S(t))_{t\geqslant 0}$ on a Banach space E. We define $E_{-1} := (E \times E)/\mathscr{G}(A)$, where $\mathscr{G}(A) = \{(x, Ax) : \ x \in \mathsf{D}(A)\}$ is the graph of A. The mapping

$$i_{-1} : x \mapsto (0, x) + \mathscr{G}(A)$$

defines a dense embedding i_{-1} from E into E_{-1}. We shall always identify E with it image $i_{-1}(E)$ in E_{-1}.

The operator A extends to a bounded operator A_{-1} from E into E_{-1} by defining

$$A_{-1}x := (-x, 0) + \mathscr{G}(A).$$

To see that this indeed gives an extension of A, note that for $x \in \mathsf{D}(A)$ we have

$$i_{-1}Ax = (0, Ax) + \mathscr{G}(A) = (-x, 0) + \mathscr{G}(A) = A_{-1}x.$$

It is easy to see that the operator A_{-1}, which is densely defined and closed as a linear operator in E_{-1} with domain $\mathsf{D}(A_{-1}) = E$, generates a strongly continuous semigroup $S_{-1} = (S_{-1}(t))_{t\geqslant 0}$ on E_{-1} which satisfies $S_{-1}(t)i_{-1}x = i_{-1}S(t)x$ for all $x \in E$ and $t \geqslant 0$.

For a bounded operator $B : H \to E_{-1}$ we are interested in E-valued solutions to the stochastic evolution equation $(\mathrm{SCP})_{(A,B)}$. To formulate this problem rigorously, we first consider the problem $(\mathrm{SCP})_{(A_{-1},B)}$ in E_{-1}:

$$(\mathrm{SCP})_{(A_{-1},B)} \qquad \begin{cases} dU_{-1}(t) = A_{-1}U_{-1}(t)\,dt + B\,dW_H(t), & t \in [0,T], \\ U_{-1}(0) = 0. \end{cases}$$

Here, as always, W_H is a cylindrical Brownian motion in H, and we adopt the standard notation $W_H(t)h := W_H(\mathbf{1}_{(0,t)} \otimes h)$.

An E-valued process $U = (U(t))_{t\in[0,T]}$ is called a *weak solution* of $(\mathrm{SCP})_{(A,B)}$ if the E_{-1}-valued process $i_{-1}U = (i_{-1}U(t))_{t\in[0,T]}$ is a weak solution of $(\mathrm{SCP})_{(A_{-1},B)}$, i.e., for all $x_{-1}^* \in \mathsf{D}(A_{-1}^*)$ the function $t \mapsto \langle i_{-1}U(t), A_{-1}^*x_{-1}^*\rangle$ is integrable almost surely and if for each $t \in [0,T]$ we have, almost surely,

$$\langle i_{-1}U(t), x_{-1}^*\rangle = \int_0^t \langle i_{-1}U(s), A_{-1}^*x_{-1}^*\rangle\,ds + W_H(t)B^*x_{-1}^*.$$

An E-valued process U is called a *mild solution* of $(\mathrm{SCP})_{(A,B)}$ if the E_{-1}-valued process $i_{-1}U$ is a mild solution of $(\mathrm{SCP})_{(A_{-1},B)}$, i.e., if the function $t \mapsto S_{-1}(t)B$ is stochastically integrable in E_{-1} with respect to W_H and if for each $t \in [0,T]$ we have, almost surely,

$$i_{-1}U(t) = \int_0^t S_{-1}(t-s)B\,dW_H(s). \tag{2.3}$$

The following proposition is an extension of the main result of [19] (where the case $B \in \mathscr{L}(H,E)$ was considered).

Proposition 2.3. *Under the above assumptions, for an E-valued process U the following assertions are equivalent:*

(a) *U is weak solution of* $(\mathrm{SCP})_{(A,B)}$;
(b) *U is mild solution of* $(\mathrm{SCP})_{(A,B)}$;
(c) *there exists an operator $R_T \in \gamma(L^2(0,T;H),E)$ such that for all $x_{-1}^* \in E_{-1}^*$*

$$R_T^*(i_{-1}^*x_{-1}^*) = B^*S_{-1}^*(\cdot)x_{-1}^* \quad \text{in } L^2(0,T;H). \tag{2.4}$$

Proof. Let us prove the equivalence (b)⇔(c), because this is what we need in the sequel. The proof of (a)⇔(b) is left to the reader.

(b)⇒(c): By assumption there is a strongly measurable random variable $U(T) : \Omega \to E$ such that in E_{-1} we have

$$i_{-1}U(T) = \int_0^T S_{-1}(T-s)B\,dW_H(s).$$

For all $x^*_{-1} \in E^*_{-1}$, the random variable $\langle U(T), i^*_{-1}x^*_{-1}\rangle$ is Gaussian. Since $F := \{i^*_{-1}x^*_{-1} : \; x^*_{-1} \in E^*_{-1}\}$ is weak*-dense in E^* and the range of $U(T)$ is separable up to a null set, from [1, Corollary 1.3] it follows that $\langle U(T), x^*\rangle$ is Gaussian for all $x^* \in E^*$, i.e., $U(T)$ is Gaussian distributed.

By the results of [19] the operator $R_{-1,T} : L^2(0,T;H) \to E_{-1}$, defined by

$$R_{-1,T}f = \int_0^T S_{-1}(T-s)Bf(s)\,ds,$$

belongs to $\gamma(L^2(0,T;H), E_{-1})$. Define the linear operator $R^*_T : F \to L^2(0,T;H)$ by

$$R^*_T i^*_{-1}x^*_{-1} := R^*_{-1,T}x^*_{-1}.$$

Then,

$$\begin{aligned} \|R^*_T i^*_{-1}x^*_{-1}\|^2_{L^2(0,T;H)} &= \|R^*_{-1,T}x^*_{-1}\|^2_{L^2(0,T;H)} \\ &= \int_0^T \|B^*S^*_{-1}(T-s)x^*_{-1}\|^2_H\,ds \\ &= \mathbb{E}\Big|\int_0^T B^*S^*_{-1}(T-s)x^*_{-1}\,dW_H(s)\Big|^2_H \\ &= \mathbb{E}\langle U(T), i^*_{-1}x^*_{-1}\rangle^2 = \|i^*_T i^*_{-1}x^*_{-1}\|^2_{\mathscr{H}_T}, \end{aligned} \tag{2.5}$$

where i_T is the canonical inclusion mapping of the reproducing kernel Hilbert space $\mathscr{H}_T$, associated with the Gaussian random variable $U(T)$, into E. This shows that R^*_T is well-defined and bounded on F. At this point we would like to use a density argument to infer that R^*_T extends to a bounded operator from E^* into $L^2(0,T;H)$ which satisfies

$$\|R^*_T x^*\|^2_{L^2(0,T;H)} = \|i^*_T x^*\|^2_{\mathscr{H}_T}, \quad \forall x^* \in E^*. \tag{2.6}$$

However, this will not work, since F is only weak*-dense in E^*. The correct way to proceed is as follows. The injectivity of $i_{-1} \circ i_T$ implies that $i^*_T \circ i^*_{-1}$ has weak*-dense range in $\mathscr{H}_T$. As $\mathscr{H}_T$ is reflexive, this range is weakly dense and therefore, by the Hahn-Banach theorem, it is dense. Fixing an arbitrary $x^* \in E^*$, we may choose a sequence $(x^*_{-1,n})_{n\geqslant 1}$ in E^*_{-1} such that $i^*_T i^*_{-1}x^*_{-1,n} \to i^*_T x^*$ in $\mathscr{H}_T$. By (2.5) the sequence $(R^*_T i^*_{-1}x^*_{-1,n})_{n\geqslant 1}$ is Cauchy in $L^2(0,T;H)$ and converges to some $f_{x^*} \in L^2(0,T;H)$. It is routine to check that f_{x^*} is independent of the approximating sequence. Thus we may extend the R^*_T to E^* by putting

$$R^*_T x^* := f_{x^*}.$$

Clearly, for this extended operator the identity (2.6) is obtained.

We claim that its adjoint $R^{**}_T : L^2(0,T;H) \to E^{**}$ actually takes values in E, and that this operator is the one we are looking for.

First, for $f = \mathbf{1}_{(a,b)} \otimes h$ and $x^* \in E^*$ of the form $x^* = i^*_{-1}x^*_{-1}$ we have

$$\begin{aligned} \langle x^*, R^{**}_T f\rangle &= [R^*_T i^*_{-1}x^*_{-1}, f]_{L^2(0,T;H)} \\ &= \int_a^b \langle S_{-1}(T-s)Bh, x^*_{-1}\rangle\,ds = \langle i_{-1}y, x^*_{-1}\rangle = \langle y, x^*\rangle, \end{aligned}$$

where $y = \int_a^b S_{-1}(T-s)Bh\,ds$ belongs to $\mathsf{D}(A_{-1}) = E$. It follows that R^{**}_T maps the dense subspace of all H-valued step functions into E, and therefore it maps all of $L^2(0,T;H)$ into E.

Viewing $R_T := R_T^{**}$ as an operator from $L^2(0,T;H)$ to E, we finally note that the identity (2.6) exhibits $R_T \circ R_T^* = i_T \circ i_T^*$ as the covariance operator of the E-valued Gaussian random variable $U(T)$. This means that R_T is γ-radonifying as an operator from $L^2(0,T;H)$ to E (see, e.g., [18]).

(c)⇒(b): We follow the ideas of [19]. We have $L^2(0,T;H) = \mathsf{N}(R_T) \oplus \overline{\mathsf{R}(R_T^*)}$. By the general theory of γ-radonifying operators, $G := \overline{\mathsf{R}(R_T^*)}$ is separable (see [18]). By a Gram-Schmidt argument we may select a sequence $(x_{-1,n}^*)_{n\geqslant 1}$ in E_{-1}^* such that $(g_n)_{n\geqslant 1} := (R_T^* i_{-1}^* x_{-1,n}^*)_{n\geqslant 1}$ is an orthonormal basis for G. Then the Gaussian random variables

$$\gamma_n := \int_0^T B^* S_{-1}^*(T-s) x_{-1,n}^* \, dW_H(s)$$

are independent and normalised. Since R_T is γ-radonifying, the E-valued random variable

$$U(T) := \sum_{n\geqslant 1} \gamma_n R_T g_n$$

is well-defined, and it is easy to check that it satisfies (2.3) with t replaced by T. By well-known routine arguments, this is enough to assure that $(\text{SCP})_{(A,B)}$ has a mild solution U in E. □

Suppose now that the problem $(\text{SCP})_{(A_{-1},B)}$ admits a mild solution U_{-1} in E_{-1} and let $\mu_{-1,t}$ denote the distribution of the random variable $U_{-1}(t)$. The weak limit $\mu_{-1,\infty}$ of these measures, if it exists, is called the (minimal) *invariant measure* associated with $(\text{SCP})_{(A_{-1},B)}$. Thus, by definition, the invariant measure, if it exists, is the unique Radon probability measure on E_{-1} which satisfies

$$\int_{E_{-1}} f \, d\mu_{-1,\infty} = \lim_{t\to\infty} \int_{E_{-1}} f \, d\mu_{-1,t}, \quad \forall f \in C_{\mathrm{b}}(E_{-1}).$$

For an explanation of this terminology and a more systematic approach we refer the reader to [3]. This references deals with Hilbert spaces E; extensions of the linear theory to the Banach space setting were presented in [6, 20].

A Radon probability measure μ on E is an *invariant measure* for $(\text{SCP})_{(A,B)}$ if the image measure $i_{-1}(\mu)$ on E_{-1} is an invariant measure for $(\text{SCP})_{(A_{-1},B)}$. Extending a result from [20] (where the case $B \in \mathscr{L}(H,E)$ was considered) we have the following result. A proof is obtained along the same line of reasoning as in the previous proposition and is left as an exercise to the reader.

Proposition 2.4. *Under the above assumptions, for a Radon probability measure μ on E the following assertions are equivalent:*

(a) $(\text{SCP})_{(A,B)}$ *admits an invariant measure;*
(b) *there exists an operator $R_\infty \in \gamma(L^2(\mathbb{R}_+;H),E)$ such that for all $x_{-1}^* \in E_{-1}^*$*

$$R_\infty^*(i_{-1}^* x_{-1}^*) = B^* S_{-1}^*(\cdot) x_{-1}^* \quad \text{in } L^2(\mathbb{R}_+;H). \tag{2.7}$$

Formally, (2.4) and (2.7) express that the operators R_T and R_∞ are integral operators with kernels $S(\cdot)B$. Strictly speaking this makes no sense, since B maps into E_{-1} rather than into E. It will be convenient, however, to refer to R_T and R_∞ as the operators 'associated with $S(\cdot)B$' and we shall do so in the sequel without further warning.

2.6. **Sectorial operators and H^∞-calculus.** For $\theta \in (0,\pi)$ let

$$\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} : \ |\arg(z)| < \theta\}$$

denote the open sector of angle θ. A densely defined closed linear operator $-A$ in a Banach space E is called *sectorial (of angle $\theta \in (0,\pi)$)* if the spectrum of $-A$ is contained in $\overline{\Sigma_\theta}$ and

$$\sup_{z \notin \overline{\Sigma_\theta}} \|z\,(z+A)^{-1}\| < \infty.$$

The infimum of all $\theta \in (0,\pi)$ such that $-A$ is sectorial of angle θ is called the *angle of sectoriality* of $-A$.

It is well known (see [5, Theorem II.4.6]) that $-A$ is sectorial of angle less than $\pi/2$ if and only if A generates a strongly continuous bounded analytic semigroup on E.

Following [14] we denote by $S(E)$ the set of all densely defined, closed, injective operators in E that are sectorial of angle less than $\pi/2$ and have dense range. The injectivity and dense range conditions are not very restrictive: if A is a sectorial operator on a reflexive Banach space E, then we have the direct sum decomposition

$$E = \mathsf{N}(A) \oplus \overline{\mathsf{R}(A)}$$

in terms of the null space and closure of the range of A. In that case, the part of A in $\overline{\mathsf{R}(A)}$ is sectorial and satisfies the additional injectivity and dense range conditions.

Let $-A \in S(E)$ be sectorial of angle $\theta \in (0,\pi/2)$ and fix $\eta \in (\theta,\pi/2)$. We denote by $H_0^\infty(\Sigma_\eta)$ the linear space of all bounded analytic functions $f : \Sigma_\eta \to \mathbb{C}$ with some power type decay at zero and infinity, i.e., for which there exists an $\varepsilon > 0$ such that

$$|f(z)| \leqslant C|z|^\varepsilon/(1+|z|)^{2\varepsilon}, \quad \forall z \in \Sigma_\eta.$$

For such functions we may define a bounded operator

$$f(-A) = \frac{1}{2\pi i}\int_{\partial\Sigma_{\eta'}} f(z)(z+A)^{-1}\,dz,$$

with $\eta' \in (\theta,\eta)$. The operator $-A$ is said to have a bounded H^∞-calculus if there exists a constant C, independent of f, such that

$$\|f(-A)\| \leqslant C\|f\|_\infty, \quad \forall f \in H_0^\infty(\Sigma_\eta).$$

The infimum of all admissible η is called the *angle* of the H^∞-calculus of $-A$.

Examples of operators A for which $-A$ has a bounded H^∞-calculus of angle less than $\pi/2$ are generators of strongly continuous analytic contraction semigroups on Hilbert spaces and second order elliptic operators on L^p-spaces whose coefficients satisfy mild regularity assumptions. We refer to [4, 8, 15] for more details and examples.

If $-A \in S(E)$ has a bounded H^∞-calculus, the mapping $f \mapsto f(-A)$ extends (uniquely, in some natural sense discussed in [15]) to a bounded algebra homomorphism from $H^\infty(\Sigma_\eta)$ into $\mathscr{L}(E)$ of norm at most C. A proof the following result can be found in [15].

Proposition 2.5. *Suppose that $-A \in S(E)$ admits a bounded H^∞-calculus of angle $\eta < \pi/2$ and let $\eta < \eta' < \pi/2$. Then $-A$ is* γ-sectorial *of any angle $\eta < \eta' < \pi/2$, i.e., the family*

$$\{z\,(z+A)^{-1} :\; z \notin \overline{\Sigma_{\eta'}}\}$$

is γ-bounded. If, in addition, E has property (α), then the family

$$\{f(-A) :\; f \in H^\infty(\Sigma_{\eta'}),\; \|f\|_\infty \leqslant 1\}$$

is γ-bounded.

2.7. **Rademacher interpolation.** If $-A$ is a sectorial operator on E, then for $\theta \in \mathbb{R}$ we may define the Banach space $\dot{E}_\theta$ as the completion of $\mathsf{D}((-A)^\theta)$ with respect to the norm

$$\|x\|_{\dot{E}_\theta} := \|(-A)^\theta x\|.$$

Note that $(-A)^\theta$ extends uniquely to an isomorphism from $\dot{E}_\theta$ onto E; with some abuse of notation this extension will also be denoted by $(-A)^\theta$. In particular, $\dot{E}_{-1}$ is the completion of the range $\mathsf{R}(A)$ with respect to the norm

$$\|Ax\|_{\dot{E}_{-1}} := \|x\|.$$

Note that

$$E + \dot{E}_{-1} = i_{-1}E_{-1} \tag{2.8}$$

with equivalent norms. For the reader's convenience we include the short proof. We trivially have $E \hookrightarrow E_{-1}$, and the embedding $\dot{E}_{-1} \hookrightarrow E_{-1}$ is a consequence of the fact that for all $x \in \mathsf{D}((-A)^{-1}) = \mathsf{R}(A)$, say $x = Ay$, we have

$$\|x\|_{E_{-1}} \leqslant C\|(I-A)^{-1}x\| = C\|(I-A)^{-1}A\|\|y\| = C\|(I-A)^{-1}A\|\|x\|_{\dot{E}_{-1}}.$$

It follows that $E + \dot{E}_{-1} \hookrightarrow E_{-1}$ with continuous inclusion. Since $I - A$ is surjective from E onto E_{-1}, every $x \in E_{-1}$ is of the form $x = y - Ay$ for some $y \in E$, which implies that $x \in E + \dot{E}_{-1}$. It follows that the inclusion $E + \dot{E}_{-1} \hookrightarrow E_{-1}$ is surjective, and the claim now follows from the open mapping theorem.

Let (X_0, X_1) be an interpolation couple of Banach spaces. Let $(r_n)_{n\in\mathbb{Z}}$ be a Rademacher sequence on a probability space $(\Omega, \mathbb{P})$. For $0 < \theta < 1$ the *Rademacher interpolation space* $\langle X_0, X_1\rangle_\theta$ consists of all $x \in X_0 + X_1$ which can be represented as a sum

$$x = \sum_{n\in\mathbb{Z}} x_n, \quad x_n \in X_0 \cap X_1, \tag{2.9}$$

convergent in $X_0 + X_1$, such that

$$\mathscr{C}_0((x_n)_{n\in\mathbb{Z}}) := \sup_{N\geqslant 0} \mathbb{E}\Big(\Big\|\sum_{n=-N}^{N} r_n 2^{-n\theta} x_n\Big\|_{X_0}^2\Big)^{1/2} < \infty,$$

$$\mathscr{C}_1((x_n)_{n\in\mathbb{Z}}) := \sup_{N\geqslant 0} \mathbb{E}\Big(\Big\|\sum_{n=-N}^{N} r_n 2^{n(1-\theta)} x_n\Big\|_{X_1}^2\Big)^{1/2} < \infty.$$

The norm of an element $x \in \langle X_0, X_1\rangle_\theta$ is defined as

$$\|x\|_{\langle X_0,X_1\rangle_\theta} := \inf\ \Big(\max\big\{\mathscr{C}_0((x_n)_{n\in\mathbb{Z}}),\, \mathscr{C}_1((x_n)_{n\in\mathbb{Z}})\big\}\Big),$$

where the infimum extends over all representations (2.9). This interpolation method was introduced by Kalton, Kunstmann and Weis, who proved that if $-A$ admits a bounded H^∞–calculus (of any angle $< \pi$), then for all $0 < \theta < 1$ and real numbers $\alpha < \beta$ one has

$$\langle \dot{E}_\alpha, \dot{E}_\beta\rangle_\theta = \dot{E}_{(1-\theta)\alpha+\theta\beta}$$

with equivalent norms [12, Theorem 7.4]. Applying this to the induced operator $I \otimes A$ on $L^2(\Omega; E)$, defined by $(I \otimes A)(f \otimes x) := f \otimes Ax$ for $f \in L^2(\Omega)$ and vectors $x \in \mathscr{D}(A)$, we obtain the following vector-valued extension of this result:

Proposition 2.6. *If $-A \in S(E)$ admits a bounded H^∞–calculus, then*

$$\langle L^2(\Omega; \dot{E}_\alpha), L^2(\Omega; \dot{E}_\beta)\rangle_\theta = L^2(\Omega; \dot{E}_{(1-\theta)\alpha+\theta\beta}).$$

3. Proof of Theorem 1.1

We begin with a useful observation.

Lemma 3.1. *Let A generate a strongly continuous semigroup on E and suppose that the equivalent conditions of Proposition 2.4 be satisfied. Then for all $\lambda \in \varrho(A)$ there exists an operator $\widehat{S}(\lambda)B \in \gamma(H,E)$ such that*

$$i_{-1} \circ \widehat{S}(\lambda)B = R(\lambda, A_{-1}) \circ B.$$

Proof. It suffices to prove this for one $\lambda \in \varrho(A)$; then, by the resolvent identity, this holds for all $\lambda \in \varrho(A)$.

Fix an arbitrary $\lambda > \omega_0(S_{-1})$, the exponential growth bound of $(S_{-1}(t))_{t\geqslant 0}$. By assumption there exists an operator $R_\infty \in \gamma(L^2(\mathbb{R}_+;H),E)$ such that for all $x_{-1}^* \in E_{-1}^*$ we have $R_\infty^*(i_{-1}^* x_{-1}^*) = B^* S_{-1}^*(\cdot) x_{-1}^*$ in $L^2(\mathbb{R}_+;H)$. The operator $\widehat{S}(\lambda)B : H \to E$ given by

$$\widehat{S}(\lambda)Bh := R_\infty(e^{-\lambda\cdot} \otimes h)$$

is γ-radonifying and satisfies, for all $x_{-1}^* \in E_{-1}^*$,

$$\langle i_{-1}\widehat{S}(\lambda)Bh, x_{-1}^*\rangle = \int_0^\infty e^{-\lambda t}\langle S_{-1}(t)Bh, x_{-1}^*\rangle\,dt = \langle R(\lambda, A_{-1})Bh, x_{-1}^*\rangle.$$

Hence by the Hahn-Banach theorem, $\widehat{S}(\lambda)B$ satisfies the desired identity. □

If the semigroup generated by A is analytic, then $R(\lambda, A_{-1})$ maps E_{-1} into $\mathsf{D}(A_{-1}) = E$ and therefore we may interpret $R(\lambda, A_{-1})B$ as an operator from H to E. By the injectivity of i_{-1} this operator equals $\widehat{S}(\lambda)B$. From now on we simply write

$$R(\lambda, A)B := \widehat{S}(\lambda)B$$

to denote this operator.

Proposition 3.2. *Suppose that $-A \in S(E)$ has a bounded H^∞-calculus of angle $\omega < \pi/2$ on a Banach space E with property (α). Then for all $B \in \mathscr{L}(H, E_{-1})$ and $\theta \in (\omega, \pi)$ the following assertions are equivalent:*

(a) *$B \in \gamma(H, \dot{E}_{-1/2})$;*
(b) *$t \mapsto \phi(-tA)B$ belongs to $\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}; H), \dot{E}_{-1/2})$ for all $\phi \in H_0^\infty(\Sigma_\theta)$;*
(c) *$t \mapsto \psi(-tA)B$ belongs to $\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}; H), \dot{E}_{-1/2})$, with $\psi(z) = z^{1/2}/(1+z)^{3/2}$.*

In this situation, for any two $\phi, \tilde{\phi} \in H_0^\infty(\Sigma_\theta)$ satisfying

$$\int_0^\infty \phi(t)\frac{dt}{t} = \int_0^\infty \tilde{\phi}(t)\frac{dt}{t} = 1$$

we have an equivalence of norms

$$\|t \mapsto \phi(-tA)B\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t};H),\dot{E}_{-1/2})} \eqsim \|t \mapsto \tilde{\phi}(-tA)B\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t};H),\dot{E}_{-1/2})} \tag{3.1}$$

with implied constants independent of ϕ and $\tilde{\phi}$.

Proof. We shall prove the implications (a) ⇒ (b) ⇒ (c) ⇒ (a).

(a) ⇒ (b): This follows from [13, Theorem 7.2 and Remark 7.3(2)] and [21, Theorem 5.3].

(b) ⇒ (c): This is trivial, as ψ belongs to $H_0^\infty(\Sigma_\theta)$ for all $\theta < \pi$;

(c) ⇒ (a): Let $(r_j)_{j\geqslant 1}$ be a Rademacher sequence on a probability space $(\Omega, \mathbb{P})$ and let $(h_j)_{j=1}^k$ be an orthonormal system in H. Using that $\psi \in H_0^\infty(\Sigma_\theta)$, from [8, Theorem 5.2.6] we obtain

$$\begin{aligned}\sum_{j=1}^{k} r_j B h_j &\eqsim \sum_{j=1}^{k} r_j \int_0^\infty (-tA)^{3/2}(1-tA)^{-3} B h_j \,\frac{dt}{t}\\ &= \sum_{j=1}^{k}\sum_{n\in\mathbb{Z}} r_j \int_{2^n}^{2^{n+1}} (-tA)^{3/2}(1-tA)^{-3} B h_j \,\frac{dt}{t}\end{aligned}$$

with convergence in $L^2(\Omega; E_{-1}) = L^2(\Omega; \dot E_{-1}) + L^2(\Omega; E)$ (cf. (2.8)). Defining the vectors $x_n \in L^2(\Omega; E) \cap L^2(\Omega; \dot E_{-1})$ by

$$x_n := \sum_{j=1}^{k} r_j \int_{2^n}^{2^{n+1}} (-tA)^{3/2}(1-tA)^{-3} B h_j \,\frac{dt}{t}$$

and setting $m_N(t) = (2^{-n}t)^{1/2}$ for $t \in [2^n, 2^{n+1})$, $n = -N, \dots, N$, and $m_N(t) = 0$ for $t \notin [2^{-N}, 2^{N+1})$, we obtain (relative to the spaces $X_0 = L^2(\Omega; \dot E_{-1})$ and $X_1 = L^2(\Omega; E)$)

$$\begin{aligned}&\mathscr{C}_0((x_n)_{n\in\mathbb{Z}})^2\\ &= \sup_{N\geqslant 1} \tilde{\mathbb{E}}\Big\| \sum_{j=1}^{k}\sum_{n=-N}^{N} r_j \tilde r_n 2^{-n/2} \int_{2^n}^{2^{n+1}} (-tA)^{3/2}(1-tA)^{-3} B h_j \,\frac{dt}{t}\Big\|^2_{L^2(\Omega;\dot E_{-1})}\\ &= \sup_{N\geqslant 1} \tilde{\mathbb{E}}\Big\| \sum_{j=1}^{k}\sum_{n=-N}^{N} r_j \tilde r_n \int_{2^n}^{2^{n+1}} (2^{-n}t)^{1/2}(-tA)(1-tA)^{-3} B h_j \,\frac{dt}{t}\Big\|^2_{L^2(\Omega;\dot E_{-1/2})}\\ &= \sup_{N\geqslant 1} \tilde{\mathbb{E}}\Big\| \sum_{j=1}^{k}\sum_{n=-N}^{N} r_j \tilde r_n \int_0^\infty m_N(t)(-tA)(1-tA)^{-3}\mathbf{1}_{(2^n,2^{n+1})}(t) B h_j \,\frac{dt}{t}\Big\|^2_{L^2(\Omega;\dot E_{-1/2})}\\ &\eqsim \sup_{N\geqslant 1} \mathbb{E}'\Big\| \sum_{j=1}^{k}\sum_{n=-N}^{N} r'_{jn} \int_0^\infty m_N(t)(-tA)(1-tA)^{-3}\mathbf{1}_{(2^n,2^{n+1})}(t) B h_j \,\frac{dt}{t}\Big\|^2_{\dot E_{-1/2}}.\end{aligned}$$

In the last step, property (α) was used to pass from double Rademacher sums (on $(\Omega,\mathbb{P})\times(\tilde\Omega,\tilde{\mathbb{P}})$) to doubly indexed Rademacher sums (on some other probability space $(\Omega',\mathbb{P}')$). Now, estimating Rademacher sums in terms of Gaussian sums we have

$$\begin{aligned}&\mathscr{C}_0((x_n)_{n\in\mathbb{Z}})^2\\ &\quad\eqsim \sup_{N\geqslant 1} \mathbb{E}'\Big\| \sum_{j=1}^{k}\sum_{n=-N}^{N} \gamma'_{jn} \int_0^\infty m_N(t)(-tA)(1-tA)^{-3}\mathbf{1}_{(2^n,2^{n+1})}(t) B h_j \,\frac{dt}{t}\Big\|^2_{\dot E_{-1/2}}\end{aligned}$$

Since the functions $\mathbf{1}_{(2^n,2^{n+1})} \otimes h_j$ in $L^2(\mathbb{R}_+, \frac{dt}{t}; H)$ are orthonormal (up to the numerical constant $(\ln 2)^{1/2}$), one may estimate the above right-hand side by

$$\lesssim \sup_{N\geqslant 1} \|t \mapsto m_N(t)\phi(-tA)B\|^2_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t};H),\dot E_{-1/2})}$$

where $\phi \in H^\infty_0(\Sigma_\theta)$ is given by $\phi(z) = z/(1+z)^3$. Finally, using the Kalton–Weis γ-multiplier theorem and the γ-boundedness of the operators $(-tA)^{1/2}(1-tA)^{-3/2}$, $t > 0$, (which follows from Proposition 2.5) we conclude that

$$\begin{aligned}\mathscr{C}_0((x_n)_{n\in\mathbb{Z}})^2 &\lesssim \|t \mapsto \phi(-tA)B\|^2_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t};H),\dot E_{-1/2})}\\ &\lesssim \|t \mapsto \psi(-tA)B\|^2_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t};H),\dot E_{-1/2})}\end{aligned}$$

with $\psi(z) = z^{1/2}/(1+z)^{3/2}$.

Similarly,

$$
\begin{aligned}
&\mathscr{C}_1((x_n)_{n\in\mathbb{Z}})^2 \\
&= \sup_{N\geqslant 1} \tilde{\mathbb{E}}\Big\| \sum_{j=1}^{k}\sum_{n=-N}^{N} r_j\tilde{r}_n 2^{n/2}\int_{2^n}^{2^{n+1}} (-tA)^{3/2}(1-tA)^{-3}Bh_j\,\frac{dt}{t}\Big\|^2_{L^2(\Omega;E)} \\
&= \sup_{N\geqslant 1} \tilde{\mathbb{E}}\Big\| \sum_{j=1}^{k}\sum_{n=-N}^{N} r_j\tilde{r}_n \\
&\qquad\times\int_0^\infty (2^{-n}t)^{-1/2}(-tA)^2(1-tA)^{-3}\mathbf{1}_{(2^n,2^{n+1})}(t)Bh_j\,\frac{dt}{t}\Big\|^2_{L^2(\Omega;\dot{E}_{-1/2})} \\
&\lesssim_E \|t\mapsto\tilde{\phi}(-tA)B\|^2_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t};H),\dot{E}_{-1/2})} \\
&\lesssim_E \|t\mapsto\psi(-tA)B\|^2_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t};H),\dot{E}_{-1/2})}
\end{aligned}
$$

with $\tilde{\phi}(z)=z^2/(1+z)^3$ and $\psi(z)=z^{1/2}/(1+z)^{3/2}$ as before.

By Proposition 2.6 and estimating Gaussian sums by Rademacher sums, this proves that

$$
\begin{aligned}
\Big\|\sum_{j=1}^k \gamma_j Bh_j\Big\|_{L^2(\Omega;\dot{E}_{-1/2})} &\eqsim_E \Big\|\sum_{j=1}^k r_j Bh_j\Big\|_{L^2(\Omega;\dot{E}_{-1/2})} \\
&\lesssim_E \|t\mapsto\psi(-tA)B\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t};H),\dot{E}_{-1/2})}.
\end{aligned}
$$

Taking the supremum over all finite orthonormal systems in H and using that E has property (α) and therefore does not contain an isomorphic copy of c_0, we obtain (using a theorem of Hoffmann-Jørgensen and Kwapień, see [18, Theorem 4.3]) that B is γ-radonifying as an operator from H into $\dot{E}_{-1/2}$ and

$$\|B\|_{\gamma(H,\dot{E}_{-1/2})} \lesssim \|t\mapsto\psi(-tA)B\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t};H),\dot{E}_{-1/2})}.$$

We have now proved the equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c). It remains to check that these equivalent conditions imply the norm equivalence (3.1). Let μ be the centred Gaussian measure on $\dot{E}_{-1/2}$ associated with the γ-radonifying operator $B\in\gamma(H,\dot{E}_{-1/2})$. Suppose $\phi,\tilde{\phi}\in H^\infty_0(\Sigma_\theta)$ are nonzero functions. By [21, Theorems 5.2, 5.3], assertion (a) implies

$$
\begin{aligned}
\|t\mapsto\phi(-tA)B\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t};H),\dot{E}_{-1/2})} &\eqsim \int_{\dot{E}_{1/2}} \|t\mapsto\phi(-tA)x\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),\dot{E}_{-1/2})}\,d\mu(x) \\
&\overset{(1)}{\eqsim} \int_{\dot{E}_{1/2}} \|t\mapsto\tilde{\phi}(-tA)x\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),\dot{E}_{-1/2})}\,d\mu(x) \\
&\eqsim \|t\mapsto\tilde{\phi}(-tA)B\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t};H),\dot{E}_{-1/2})}.
\end{aligned}
$$

Here, step (1) follows from [13, Proposition 7.7]. The implied constants are independent of ϕ and $\tilde{\phi}$ under the normalisation as stated in the proposition. □

Remark 3.3. The only step in the proof where we made use of the boundedness of the functional calculus is the Rademacher interpolation argument. For all other parts, γ-sectoriality of angle less than $\pi/2$ is sufficient. However, one actually needs only the continuous embedding

$$\langle L^2(\Omega;E), L^2(\Omega;\dot{E}_{-1})\rangle_{1/2} \hookrightarrow L^2(\Omega;\dot{E}_{-1/2})$$

instead of an equality. As in Proposition 2.6 this boils down to having the embedding for the underlying Banach spaces $\langle E,\dot{E}_{-1}\rangle_{1/2}\hookrightarrow\dot{E}_{-1/2}$. An inspection of the proof of [12, Theorems 4.1 and 7.4] shows that the latter embedding does not require the full power of the boundedness of the functional calculus but merely a (discrete dyadic) square function estimate of the form

$$\sup_{\epsilon_k=\pm 1}\Big\|\sum_k \epsilon_k \varphi(2^k A^\sharp)x\Big\| \lesssim \|x\|$$

for some $\varphi \in H_0^\infty(\Sigma_\theta)$ for $\theta \in (0,\pi)$, where $A^\sharp$ denotes the part of A^* in $E^\sharp = \overline{\mathsf{D}(A^*)} \cap \overline{\mathsf{R}(A^*)}$ (the closures are taken in the strong topology of E^*). These 'dual' square function estimates match the hypothesis in Le Merdy's theorem on the Weiss conjecture [16, Theorem 4.1] in the sense that Le Merdy treats observation operators and requires upper square function estimates for A whereas we treat control operators and therefore need 'dual' square function estimates. The construction of $A^\sharp$ instead of A^* is needed when non-reflexive Banach spaces are concerned. On reflexive spaces one has $A^\sharp = A^*$, and the explained duality with Le Merdy's result is more apparent.

In the next lemma, $\widehat{f}$ denotes the Laplace transform of a function f.

Lemma 3.4 (Laplace transforms). *For all $f \in L^2(\mathbb{R}_+, \frac{dt}{t}; H)$, the function $Lf(t) := t\widehat{f}(t)$ belongs to $L^2(\mathbb{R}_+, \frac{dt}{t}; H)$ and*

$$\|Lf\|_{L^2(\mathbb{R}_+, \frac{dt}{t}; H)} \leqslant \|f\|_{L^2(\mathbb{R}_+, \frac{dt}{t}; H)}.$$

Proof. By the Cauchy-Schwarz inequality,

$$\begin{aligned}\int_0^\infty t^2 \|\widehat{f}(t)\|_H^2 \,\frac{dt}{t} &= \int_0^\infty \Big\|\int_0^\infty f(s)\, te^{-st}\,ds\Big\|_H^2 \,\frac{dt}{t} \\ &\leqslant \int_0^\infty \int_0^\infty \|f(s)\|_H^2\, te^{-st}\,ds\,\frac{dt}{t} \\ &= \int_0^\infty \int_0^\infty \|f(s)\|_H^2\, e^{-st}\,dt\,ds = \int_0^\infty \|f(s)\|_H^2\,\frac{ds}{s}. \qquad \square\end{aligned}$$

As a consequence, the mapping $L : f \mapsto Lf$ is a contraction on $L^2(\mathbb{R}_+, \frac{dt}{t}; H)$. By the Kalton–Weis extension theorem, L extends to a linear contraction on the space $\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}; H), E)$, for any Banach space E.

Proof of the equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) *of Theorem 1.1.* (a) $\Rightarrow$ (b): By assumption, $t \mapsto S(t)B$ belongs to $\gamma(L^2(\mathbb{R}_+; H), E)$. It follows that $t \mapsto \eta(-tA)B$ belongs to $\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}; H), \dot{E}_{-1/2})$, with $\eta(z) = z^{1/2}\exp(-z)$. The Laplace transform of $t \mapsto (tz)^{1/2}\exp(-tz)$ equals $\lambda \mapsto \frac12\sqrt{\pi} z^{1/2}(\lambda+z)^{-3/2}$. Hence, by [15, Lemma 9.12] or by using the Phillips calculus (see [8]),

$$\tfrac12\sqrt{\pi}(-A)^{1/2}(\lambda - A)^{-3/2}B = \int_0^\infty e^{-\lambda t}(-tA)^{1/2}S(t)B\,dt,$$

or, equivalently,

$$\tfrac12\sqrt{\pi}(-A/\lambda)^{1/2}(1 - A/\lambda)^{-3/2}B = \lambda\int_0^\infty e^{-\lambda t}\eta(-tA)B\,dt.$$

By Lemma 3.4 and the remark following it, we obtain that $\lambda \mapsto (-A/\lambda)^{1/2}(1 - A/\lambda)^{-3/2}B$ belongs to $\gamma(L^2(\mathbb{R}_+, \frac{d\lambda}{\lambda}; H), \dot{E}_{-1/2})$. Upon substituting $1/\lambda = \mu$ we find that $\mu \mapsto \psi(-\mu A)B$ belongs to $\gamma(L^2(\mathbb{R}_+, \frac{d\mu}{\mu}; H), \dot{E}_{-1/2})$ with $\psi(z) = z^{1/2}/(1+z)^{3/2}$. Now (b) follows as an application of Proposition 3.2.

(b) $\Rightarrow$ (c): From Proposition 3.2 we get that $t \mapsto (-tA)^{1/2}(1-tA)^{-1}B$ belongs to $\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}; H), \dot{E}_{-1/2})$, or equivalently, that $t \mapsto t^{1/2}(1-tA)^{-1}B$ belongs to $\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}; H), E)$. Substituting $t = 1/s$ we obtain that $s \mapsto s^{1/2}(s - A)^{-1}B$ belongs to $\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}; H), E)$.

(c) ⇒ (b): By substituting $t = 1/s$ the assumption implies that $s \mapsto s^{1/2}(1 - sA)^{-1}B$ belongs to $\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}; H), E)$, or equivalently, that $s \mapsto (-sA)^{1/2}(1 - sA)^{-1}B$ belongs to $\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}; H), \dot{E}_{-1/2})$. Then by the γ-multiplier lemma (using that the operators $(1-sA)^{-1/2}$, $s > 0$, are γ-bounded by Proposition 2.5), we obtain that assumption (c) of Proposition 3.2 is satisfied.

(b) ⇒ (a): By Proposition 3.2, $t \mapsto (-tA)^{1/2}\exp(tA)B = (-tA)^{1/2}S(t)B$ belongs to $\gamma(L^2(\mathbb{R}_+, \frac{dt}{t}; H), \dot{E}_{-1/2})$. This is equivalent to saying that $t \mapsto S(t)B$ belongs to $\gamma(L^2(\mathbb{R}_+; H), E)$. □

For the proofs of the implications (b) ⇒ (d) ⇒ (c) we need some further preparations.

An interval in $\mathbb{R}_+$ will be called *dyadic* (with respect to the measure $\frac{dt}{t}$) if it is of the form $[2^{k/2^M}, 2^{(k+1)/2^M})$ with $M \in \mathbb{N}$ and $k \in \mathbb{Z}$.

Lemma 3.5. *Let $-A \in S(E)$ be γ-sectorial and let $I_1, \dots, I_N$ be dyadic intervals. For any choice of the numbers $s_n, t_n \in I_n$ we have the equivalence*

$$\Big\| \sum_{n\in F} \gamma_n s_n^{1/2} R(s_n, A)B \Big\|_{L^2(\Omega;\gamma(H,E))} \eqsim \Big\| \sum_{n\in F} \gamma_n t_n^{1/2} R(t_n, A)B \Big\|_{L^2(\Omega;\gamma(H,E))}$$

with constants independent of the finite subset $F \subseteq \mathbb{Z}$, the intervals I_n, and the choice of s_n, t_n.

Proof. First note that, since I_n is dyadic, $|s_n^{1/2} \pm t_n^{1/2}| \leqslant 4\max\{s_n^{1/2}, t_n^{1/2}\}$.

We have, using the resolvent identity, the γ-boundedness of the operators $tR(t, A)$ for $t > 0$, and the contraction principle,

$$\begin{aligned}
&\Big\| \sum_{n\in F} \gamma_n (s_n^{1/2} R(s_n, A) - t_n^{1/2} R(t_n, A))B \Big\|_{L^2(\Omega;\gamma(H,E))} \\
&\quad\leqslant \Big\| \sum_{n\in F} \gamma_n \frac{t_n - s_n}{t_n^{1/2} s_n^{1/2}} s_n R(s_n, A) t_n^{1/2} R(t_n, A)B \Big\|_{L^2(\Omega;\gamma(H,E))} \\
&\qquad + \Big\| \sum_{n\in F} \gamma_n \frac{s_n^{1/2} - t_n^{1/2}}{t_n^{1/2}} t_n^{1/2} R(t_n, A)B \Big\|_{L^2(\Omega;\gamma(H,E))} \\
&\quad\lesssim \Big\| \sum_{n\in F} \gamma_n t_n^{1/2} R(t_n, A)B \Big\|_{L^2(\Omega;\gamma(H,E))}.
\end{aligned}$$

By the triangle inequality in $L^2(\Omega; \gamma(H, E))$ it then follows that

$$\Big\| \sum_{n\in F} \gamma_n s_n^{1/2} R(s_n, A)B \Big\|_{L^2(\Omega;\gamma(H,E))} \lesssim \Big\| \sum_{n\in F} \gamma_n t_n^{1/2} R(t_n, A)B \Big\|_{L^2(\Omega;\gamma(H,E))}.$$

The converse inequality is obtained by reversing the roles of s_n and t_n. □

Lemma 3.6. *Let $f : \Sigma_\theta \to H$ be a bounded analytic function and suppose that, for some $0 < \eta < \theta$, the functions $t \mapsto f(e^{\pm i\eta}t)$ belong to $L^2(\mathbb{R}_+, \frac{dt}{t}; H)$. Then*

$$\sum_{n\in\mathbb{Z}} \|f(2^n)\|_H^2 < \infty.$$

Proof. Since f is continuous we may suppose that H is separable. By expanding the values of f with respect to an orthonormal basis in H, it suffices to prove the lemma for the case H equals the scalar field.

By considering $g(z) = f(\exp(z))$, we may reformulate the problem on the strip $S_\theta = \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \theta\}$. The objective is then to show that if the restriction of a bounded analytic function g on S_θ to the lines $\mathrm{Im}\, z = \pm\eta$ belongs to $L^2(\mathbb{R})$, then

$\sum_{n\in\mathbb{Z}} |g(n\ln 2)|^2 < \infty$. The proof of this uses the following standard technique. By the Poisson formula for the strip we have

$$\sup_{|\zeta|<\eta} \big\| g|_{\{\operatorname{Im} z=\zeta\}}\big\|_2 < \infty$$

and therefore $g|_{S_\eta} \in L^2(S_\eta)$. For $0<\delta<\eta$ consider the discs

$$Q_n = \{z\in\mathbb{C} : |z-n\ln 2| < \delta\}, \qquad n\in\mathbb{Z},$$

centred around $n\in\mathbb{Z}$. Taking δ small enough, the functions $\phi_n = |Q_n|^{-1/2}\mathbf{1}_{Q_n}$ have disjoint support and are hence orthonormal in $L^2(S_\eta)$. By the mean value theorem we obtain

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |g(n\ln 2)|^2 &= \sum_{n\in\mathbb{Z}} \Big|\frac{1}{|Q_n|}\int_{Q_n} g(x+iy)\,dx\,dy\Big|^2\\ &= \frac{1}{\pi\delta^2}\sum_{n\in\mathbb{Z}}\Big|\int_{S_\eta} g(x+iy)\phi_n(x+iy)\,dx\,dy\Big|^2\\ &\leqslant \frac{1}{\pi\delta^2}\big\|g|_{S_\eta}\big\|^2_{L^2(S_\eta)}.\end{aligned}$$

□

This lemma can be restated as saying that the mapping $f\mapsto (f(2^n))_{n\in\mathbb{Z}}$ is bounded from the weighted Hardy space $H^2(\Sigma_\eta,\mu;H)$ to $\ell^2(H)$, where μ is the image on the sector Σ_η of the Lebesgue measure on the strip S_η under the exponential mapping; note that Lebesgue measure on horizontal lines in the strip S_η is mapped to the measure dt/t on rays emanating from the origin in the sector Σ_η.

By the Kalton–Weis extension theorem, this mapping extends to a bounded operator from $\gamma(H^2(\Sigma_\eta,\mu;H),E)$ to $\gamma(\ell^2(H),E)$, for any Banach space E. This is what will be needed below.

End of the proof of Theorem 1.1. We shall now prove the remaining implications (b) ⇒ (d) ⇒ (c).

We begin with the proof of (b) ⇒ (d). First of all, Lemma 3.1 implies that $R(t,A)B \in \gamma(H,E)$ for all $t>0$. By the implication (b) ⇒ (c) applied to the operators $e^{\pm i\theta}A$ for a sufficiently small $\theta>0$ we find that the functions

$$t\mapsto t^{1/2}R(t,e^{\pm i\theta}A)B = e^{\mp i\theta}\,t^{1/2}R(te^{\mp i\theta},A)B$$

belong to $\gamma(L^2(\mathbb{R}_+,\frac{dt}{t};H),E)$. By Lemma 3.6 and the remark following it, we obtain that the sequence $(2^{n/2}R(2^n,A)B)_{n\in\mathbb{Z}}$ belongs to $\gamma(\ell^2(H),E)$. But this is the same as saying that (d) holds.

We turn to the proof of (d) ⇒ (c). Let $S^{(M)}_{nm}$ denote the average of $t^{1/2}R(t,A)$ (with respect to dt/t) over the dyadic interval $I^{(M)}_{nm} = [2^{n+m2^{-M}}, 2^{n+(m+1)2^{-M}})$. Let $t^{(M)}_{nm} = 2^{n+m2^{-M}}$ be the left endpoint of the interval $I^{(M)}_{nm}$. Then

$$\begin{aligned}S^{(M)}_{nm} &= \fint_{I^{(M)}_{nm}} t^{1/2}R(t,A)B\frac{dt}{t}\\ &= \fint_{I^{(M)}_{nm}} t^{1/2}\big(R(t,A)(t^{(M)}_{nm}-A)\big)R(t^{(M)}_{nm},A)B\frac{dt}{t}\\ &= \Big(\fint_{I^{(M)}_{nm}} \frac{t^{1/2}}{(t^{(M)}_{nm})^{1/2}}\big(\frac{t^{(M)}_{nm}}{t}\cdot tR(t,A) - AR(t,A)\big)\frac{dt}{t}\Big)\circ[(t^{(M)}_{nm})^{1/2}R(t^{(M)}_{nm},A)B]\\ &=: U^{(M)}_{nm}\circ[(t^{(M)}_{nm})^{1/2}R(t^{(M)}_{nm},A)B].\end{aligned}$$

Since $t/t^{(M)}_{nm}\in[1,2]$ on $I^{(M)}_{nm}$, the operators $U^{(M)}_{nm}$ belong (up to a constant) to the closure of the absolute convex hull of $\{AR(t,A),\, tR(t,A) :\ t>0\}$. By γ-sectoriality of A (which follows from Proposition 2.5) this family is γ-bounded.

Fix a finite set $F \subseteq \mathbb{Z}$. Then,

$$\begin{aligned}
&\Big\| \sum_{n\in F} \sum_{m=0}^{2^M-1} \mathbf{1}_{I_{nm}^{(M)}} \otimes S_{nm}^{(M)} B \Big\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t};H),E)} \\
&\overset{(1)}{\eqsim} \Big\| \sum_{n\in F} \sum_{m=0}^{2^M-1} \mathbf{1}_{I_{nm}^{(M)}} \otimes S_{nm}^{(M)} B \Big\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),\gamma(H,E))} \\
&\overset{(2)}{\eqsim} \frac{1}{2^{M/2}} \Big\| \sum_{n\in F} \sum_{m=0}^{2^M-1} \gamma_{nm} S_{nm}^{(M)} B \Big\|_{L^2(\Omega;\gamma(H,E))} \\
&\overset{(3)}{\lesssim} \frac{1}{2^{M/2}} \Big\| \sum_{n\in F} \sum_{m=0}^{2^M-1} \gamma_{nm} (t_{nm}^{(M)})^{1/2} R(t_{nm}^{(M)}, A) B \Big\|_{L^2(\Omega;\gamma(H,E))} \\
&\overset{(4)}{\eqsim} \frac{1}{2^{M/2}} \Big\| \sum_{n\in F} \sum_{m=0}^{2^M-1} \gamma_{nm} 2^{n/2} R(2^n, A) B \Big\|_{L^2(\Omega;\gamma(H,E))} \\
&\overset{(5)}{=} \Big\| \sum_{n\in F} \sum_{m=0}^{2^M-1} \mathbf{1}_{I_{nm}^{(M)}} \otimes 2^{n/2} R(2^n, A) B \Big\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),\gamma(H,E))} \\
&= \Big\| \sum_{n\in F} \mathbf{1}_{I_n} \otimes 2^{n/2} R(2^n, A) B \Big\|_{\gamma(L^2(\mathbb{R}_+,\frac{dt}{t}),\gamma(H,E))} \\
&\overset{(6)}{\eqsim} \Big\| \sum_{n\in F} \gamma_n 2^{n/2} R(2^n, A) B \Big\|_{L^2(\Omega;\gamma(H,E))}
\end{aligned}$$

with implicit constants independent of F and M. In this computation, (1) follows from property (α); (2), (5), (6) from the identity (2.1) along with the fact that the dyadic interval $I_{nm}^{(M)}$ has dt/t-measure $\eqsim 2^{-M}$; Estimate (3) follows from the γ-boundedness of the operators $U_{nm}^{(M)}$; and (4) from Lemma 3.5 applied to the points $s_n{=}2^n$ and $t_{nm}^{(M)}$ in $I_n = [2^n, 2^{n+1})$.

By the γ-Fatou lemma (see (2.2)), the above estimate implies (c). ☐

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