A tight colored Tverberg theorem for maps to manifolds

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Abstract

Any continuous map of an N-dimensional simplex Δ_N with colored vertices to a d-dimensional manifold M must map r points from disjoint rainbow faces of Δ_N to the same point in M; assuming that $N \geq (r-1)(d+1)$, no r vertices of Δ_N get the same color, and our proof needs that r is a prime. A face of Δ_N is called *rainbow face* if all vertices have different colors.

This result is an extension of our recent new colored Tverberg theorem, the special case of $M = \mathbb{R}^d$. It is also a generalization of Volovikov's 1996 topological Tverberg theorem for maps to manifolds, which arises when all color classes have size 1 (i.e., without color constraints); for this special case Volovikov's proofs, as well as ours, work when r is a prime power.

1 Introduction

Recently, we formulated a new version of the 1992 colored Tverberg conjecture by Bárány and Larman [1], and proved this new version in the case of primes.

Theorem 1.1 (Tight colored Tverberg theorem [3])**.** *For $d \geq 1$ and a prime $r \geq 2$, set $N := (d+1)(r-1)$, and let the $N+1$ vertices of an N-dimensional simplex Δ_N be colored such that all color classes are of size at most $r-1$.*

Then for every continuous map $f : \Delta_N \to \mathbb{R}^d$, there are r disjoint faces $F_1, \ldots, F_r$ of Δ_N such that the vertices of each face F_i have all different colors, and such that the images under f have a point in common: $f(F_1) \cap \cdots \cap f(F_r) \neq \emptyset$.

Here a *coloring* of the vertices of the simplex Δ_N is a partition of the vertex set into color classes, $C_1 \uplus \cdots \uplus C_m$. The condition $|C_i| \leq r-1$ implies that there are at least $d+2$ different color classes. In the following, a face all whose vertices have different colors, $|F_j \cap C_i| \leq i$ for all i, will be called a *rainbow face*.

Theorem 1.1 is tight in the sense that it fails for maps of a simplex of smaller dimension, or if r vertices have the same color. It implies an optimal result for the Bárány–Larman conjecture in the case where $r+1$ is a prime, and an asymptotically-optimal bound in general; see [3]. The special case where all vertices of Δ_N have different colors, $|C_i| = 1$, is the prime case of the topological Tverberg theorem of Bárány, Shlosman & Szűcs [2].

In this paper we present an extension of Theorem 1.1 that treats continuous maps $\Delta_N \to M$ from the N-simplex to an arbitrary d-dimensional manifold M in place of $\mathbb{R}^d$.

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Theorem 1.2 (Tight colored Tverberg theorem for M)**.** *For $d \geq 1$ and a prime $r \geq 2$, set $N := (d+1)(r-1)$, and let the $N+1$ vertices of an N-dimensional simplex Δ_N be colored such that all color classes are of size at most $r-1$.*

Then for every continuous map $f : \Delta_N \to M$ to a d-dimensional manifold, the simplex Δ_N has r disjoint rainbow faces whose images under f have a point in common.

Theorem 1.2 without color constraints (that is, when all color classes are of size 1, and thus all faces are rainbow faces) was previously obtained by Volovikov [9], using different methods. His proof (as well as ours in the case without color constraints) works for prime powers r; see Section 3.1.

The prime power case for the colored version, Theorem 1.2, seems however out of reach at this point, even in the case $M = \mathbb{R}^d$. Similarly, there currently does not seem to be a viable approach to the case without color constraints, even for $M = \mathbb{R}^d$, when r is not a prime power. This is the remaining open case of the topological Tverberg conjecture [2].

We expect that the conclusion of Theorem 1.2 remains valid if we only consider a continuous map $f : R \to M$, where R denotes the subcomplex of rainbow faces of Δ_N. This is trivial in the case when M is contractible, but not in general. See the discussion in Section 3.2.

2 Proof

We prove Theorem 1.2 in two steps:

- First, a geometric reduction lemma implies that it suffice to consider only manifolds M that are of the form $M = N \times I^g$. Here I stands for the interval $[0, 1]$ and N is another manifold. This is done in Section 2.1.
- In the second step, we prove Theorem 1.2 for maps $\Delta_N \to N \times I^g$ via the configuration space/test map scheme and Fadell–Husseini index theory, see Sections 2.2 and 2.4.

In the second step we rely on the computation of the Fadell–Husseini index of joins of chessboard complexes that we obtained in [4].

2.1 A geometric reduction lemma

The topological calculations in the next section will require that M has trivial cohomology in high dimensions. More precisely we need

$$(r-1)\dim(M) > r \cdot \mathrm{cohdim}(M), \tag{1}$$

where $\mathrm{cohdim}(M)$ is the cohomology dimension of M. We may assume that the inequality (1) holds by the following reduction lemma.

Lemma 2.1. *Theorem 1.2 for parameters (d, r, M, f) can be derived from the case with parameters $(d', r', M', f') = (d+1, r, M \times I, f')$, where the continuous map f' is defined in the proof.*

Proof. Suppose we have to prove the theorem for the parameters (d, r, M, f). Let $d' = d+1$, $r' = r$, and $M' = M \times I$. Then $N' := (d'+1)(r-1) = N + r - 1$. Let $v_0, \ldots, v_N, v_{N+1}, \ldots, v_{N'}$ denote the vertices of $\Delta_{N'}$. We regard Δ_N as the front face of $\Delta_{N'}$ with vertices $v_0, \ldots, v_N$. We give the new vertices $v_{N+1}, \ldots, v_{N'}$ a new color. Define a new map $f' : \Delta_{N'} \to M'$ by

$$\lambda_0 v_0 + \ldots + \lambda_{N'} v_{N'} \longmapsto (f(\lambda_0 v_0 + \ldots + \lambda_{N-1} v_{N-1} + (\lambda_N + \ldots + \lambda_{N'}) v_n), \lambda_{N+1} + \ldots + \lambda_{N'}) .$$

Suppose we can show Theorem 1.2 for the parameters (d', r', M', f'). That is, we found a Tverberg partition $F_1', \ldots, F_r'$ for these parameters. Put $F_i := F_i' \cap \Delta_N$. Since f' maps the front face Δ_N to $M \times \{0\}$ and since $\Delta_{N'}$ has only $r-1 < r$ vertices more than Δ_N, already the F_i will intersect in $M \times \{0\}$. Hence the r faces $F_1, \ldots, F_r$ form a solution for the original parameters (d, r, M, f). This reduction is sketched in Figure 1. $\square$

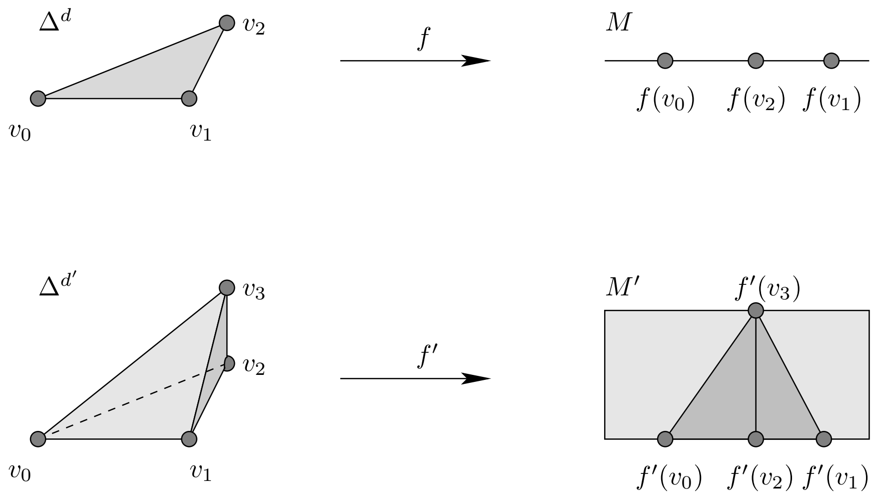

Figure 1: Exemplary reduction in the case $d = 1$, $r = 2$, $N = 2$.

If the reduction lemma is applied $g = 1 + \lfloor \frac{d}{r-1} \rfloor$ times, the problem is reduced from the arbitrary parameters (d, r, M, f) to parameters (d'', r'', M'', f'') where $M'' = M \times I^g$. Thus M'' has vanishing cohomology in its g top dimensions. Therefore $(r-1)\dim(M'') > r \cdot \mathrm{cohdim}(M'')$.

Having this reduction in mind, in what follows we may simply assume that the manifold M already satisfies inequality (1).

2.2 The configuration space/test map scheme

Suppose we are given a continuous map

$$f : \Delta_N \longrightarrow M,$$

and a coloring of the vertex set $\mathrm{vert}(\Delta_N) = [N+1] = C_0 \uplus \cdots \uplus C_m$ such that the color classes C_i are of size $|C_i| \leq r - 1$. We want to find a colored Tverberg partition, that is, pairwise disjoint rainbow faces $F_1, \ldots, F_r$ of Δ_N, $|F_j \cap C_i| \leq 1$, whose images under f intersect.

The test map F is constructed using f in the following way. Let $f^{*r} : (\Delta_N)^{*r} \longrightarrow_{\mathbb{Z}_r} M^{*r}$ be the r-fold join of f. Since we are interested in pairwise disjoint faces $F_1, \ldots, F_r$, we restrict the domain of f^{*r} to the simplicial r-fold 2-wise deleted join of Δ_N, $(\Delta_N)^{*r}_{\Delta(2)} = [r]^{*(N+1)}$. This is the subcomplex of $(\Delta_N)^{*r}$ consisting of all joins $F_1 * \ldots * F_r$ of pairwise disjoint faces. (See [8] for an introduction to these notions.) Since we are interested in colored F_js, we restrict the domain further to the subcomplex

$$(C_0 * \cdots * C_m)^{*r}_{\Delta(2)} = [r]^{*|C_0|}_{\Delta(2)} * \cdots * [r]^{*|C_m|}_{\Delta(2)}.$$

This is the subcomplex of $(\Delta_N)^{*r}$ consisting of all joins $F_1 * \ldots * F_r$ of pairwise disjoint rainbow faces. The space $[n]^{*m}_{\Delta(2)}$ is known as the *chessboard complex* $\Delta_{n,m}$ [8]. We write

$$K := (\Delta_{r,|C_0|}) * \cdots * (\Delta_{r,|C_m|}). \tag{2}$$

Hence we get a map

$$F' : K \longrightarrow_{\mathbb{Z}_r} M^{*r}.$$

Let $T_{M^{*r}} := \{\sum_{i=1}^{r} \frac{1}{r} \cdot x \ : \ x \in M\}$ be the thin diagonal of M^{*r}. Its complement $M^{*r} \backslash T_{M^{*r}}$ is called the topological r-fold r-wise deleted join of M and it is denoted by $M^{*r}_{\Delta(r)}$.

The preimages $(F')^{-1}(T_{M^{*r}})$ of the thin diagonal correspond exactly to the colored Tverberg partitions. Hence the image of F' intersects the diagonal if and only if f admits a colored Tverberg partition.

Suppose that there f admits no colored Tverberg partition, then we get a *test map*

$$F : K \longrightarrow_{\mathbb{Z}_r} M^{*r}_{\Delta(r)}. \tag{3}$$

We will derive a contradiction to the existence of such an equivariant map using the Fadell–Husseini index theory.

2.3 The Fadell–Husseini index

Let in the following H^* denote singular or Čech cohomology with $\mathbb{F}_r$-coefficients, where r is prime, and G a finite group.

The *equivariant cohomology* of a G-space X is defined as

$$H^*_G(X) \ := \ H^*(EG \times_G X),$$

where EG is a contractible free G-CW complex and $EG \times_G X := (EG \times X)/G$. The classifying space of the group G is $BG := EG/G$.

If X is a G-space, we denote the *cohomological index* of X, also called the *Fadell–Husseini index* [6, 7], by

$$\mathrm{Ind}_G(X) \ := \ \ker \left(H^*_G(\mathrm{pt}) \xrightarrow{p^*} H^*_G(X) \right) \ \subseteq \ H^*_G(\mathrm{pt}),$$

the kernel of the map in cohomology induced by the projection from X to a point.
The cohomological index is monotone in the sense that if there is a G-map $X \longrightarrow_G Y$ then

$$\mathrm{Ind}_G(X) \supseteq \mathrm{Ind}_G(Y). \tag{4}$$

If r is odd then the cohomology of $\mathbb{Z}_r$ with $\mathbb{F}_r$-coefficients as an $\mathbb{F}_r$-algebra is

$$H^*(\mathbb{Z}_r) = H^*(B\mathbb{Z}_r) \cong \mathbb{F}_r[x, y]/(y^2),$$

where $\deg(x) = 2$ and $\deg(y) = 1$. If r is even, then $r = 2$ and $H^*(\mathbb{Z}_r) \cong \mathbb{F}_2[t]$, $\deg t = 1$.

The index of the complex K was computed in [4, Corollary 2.6]:

Theorem 2.2. $\mathrm{Ind}_{\mathbb{Z}_r}(K) = H^{*\geq N+1}(B\mathbb{Z}_r)$.

Therefore in the proof of Theorem 1.2 it remains to show that $\mathrm{Ind}_{\mathbb{Z}_r}(M^{*r}_{\Delta(r)})$ contains a non-zero element in dimension less or equal to N. Indeed, the monotonicity of the index (4) implies thereby the non-existence of a test map (3), which in turn implies the existence of a colored Tverberg partition.

2.4 The index of the deleted join of the manifold

We have inclusions

$$T_{M^{*r}} \hookrightarrow \left\{ \sum \lambda_i x_i \in M^{*r} \ : \ \lambda_i > 0, \sum \lambda_i = 1, x_i \in M \right\} = M \times \Delta_{r-1} \hookrightarrow M^{*r}.$$

Since M is a smooth $\mathbb{Z}_r$-invariant manifold there exists a $\mathbb{Z}_r$-equivariant tubular neighborhood of $T_{M^{*r}}$ in M^{*r}, see [5]. Its closure can be described as the disk bundle $D(\xi)$ of an equivariant vector bundle ξ over M. We denote its sphere bundle by $S(\xi)$. The fiber F of ξ is as a $\mathbb{Z}_r$-representation the $(d+1)$-fold sum of W_r, where $W_r = \{x \in \mathbb{R}[\mathbb{Z}_r] \ : \ x_1 + \ldots + x_r = 0\}$ is the so-called augmentation ideal of $\mathbb{R}[\mathbb{Z}_r]$.

The representation sphere $S(F)$ is of dimension $N-1$. It is a free $\mathbb{Z}_r$-space, hence its index is

$$\mathrm{Ind}_{\mathbb{Z}_r}(S(F)) = H^{*\geq N}(B\mathbb{Z}_r). \tag{5}$$

This can be directly deduced from the Leray–Serre spectral sequence associated to the Borel construction $E\mathbb{Z}_r \times_{\mathbb{Z}_r} S(F) \to B\mathbb{Z}_r$, noting that the images of the differentials give precisely the index of $S(F)$, which can be seen from the edge-homomorphism.

The Leray–Serre spectral sequence associated to the fibration $S(\xi) \to M$ collapses at E_2, since $N = (r-1)(d+1) \geq d+1$ and hence there is no differential between non-zero entries. Thus the map $i^* : H^{N-1}(S(\xi)) \to H^{N-1}(S(F))$ induced by inclusion is surjective.

The Mayer–Vietoris sequence associated to the triple $(D(\xi), M^{*r}_{\Delta(r)}, M^{*r})$ contains the subsequence

$$H^{N-1}(M^{*r}_{\Delta(r)}) \oplus H^{N-1}(D(\xi)) \xrightarrow{j^*+k^*} H^{N-1}(S(\xi)) \xrightarrow{\delta} H^N(M^{*r}).$$

We see that $H^N(M^{*r})$ is zero. This follows from the formula

$$\widetilde{H}^{*+(r-1)}(M^{*r}) \cong \left(\widetilde{H}^*(M)\right)^{\otimes r},$$

as long as $N-(r-1) > re$, where e is the cohomological dimension of M. This inequality is equivalent to $d > \frac{r}{r-1}e$, which can be assumed by applying the reduction from Section 2.1 at least $\lfloor 1 + \frac{e}{r-1} \rfloor$ times. Hence we can assume that $H^N(M^{*r}) = 0$.

Furthermore inequality (1) implies $N - 1 \geq d > \operatorname{cohdim}(M)$. Hence the term $H^{N-1}(D(\xi)) = H^{N-1}(M)$ of the sequence is zero as well.

Thus the map $j^* : H^{N-1}(M^{*r}_{\Delta(r)}) \to H^{N-1}(S(\xi))$ is surjective. Therefore the composition $(j \circ i)^* : H^{N-1}(M^{*r}_{\Delta(r)}) \to H^{N-1}(S(F))$ is surjective as well. We apply the Borel construction functor $E\mathbb{Z}_r \times_{\mathbb{Z}_r} (\_) \to B\mathbb{Z}_r$ to this map and apply Leray–Serre spectral sequences, see Figure 2.

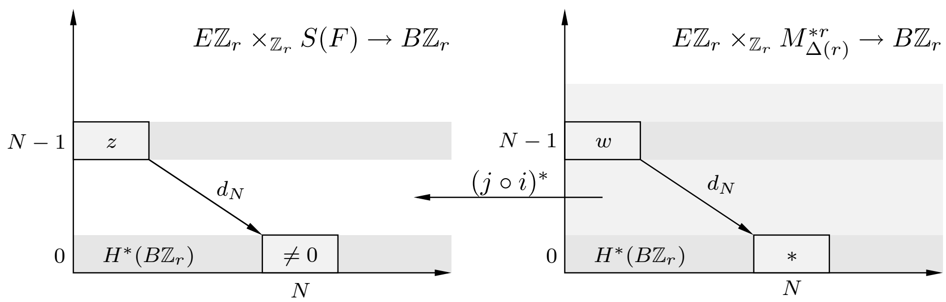

Figure 2: We associate to the map $S(F) \xrightarrow{j\circ i} M^{*r}_{\Delta(r)}$ the Borel constructions and spectral sequences to deduce that $M^{*r}_{\Delta(r)}$ contains a non-zero element in dimension N.

At the E_2-pages, the generator z of $H^{N-1}(S(F))$ has a preimage w since $(j \circ i)^*$ is surjective. At the E_N-pages $(j \circ i)^*(d_N(w)) = d_N(z)$, which is non-zero by (5). Hence $d_N(w) \neq 0$, which is an element in the kernel of the edge-homomorphism $H^*(B\mathbb{Z}_r) \to H^*_{\mathbb{Z}_r}(M^*_{\Delta(r)})$.

Therefore, the index of $M^{*r}_{\Delta(r)}$ contains a non-zero element in dimension N. This completes the proof of Theorem 1.2. □

3 Remarks

3.1 The case without color constraints

Suppose we color the vertices of Δ_N in Theorem 1.2 with pairwise distinct colors. Then all faces of Δ_N are rainbow faces, hence the condition of being a rainbow face is empty. This case was already treated by Volovikov, in a slightly stronger version.

Theorem 3.1 (Volovikov [9])**.** *Let $d \geq 1$, let $r = p^k$ be a prime power, $N := (d+1)(r-1)$, and $f : \partial\Delta_N \to M$ be a continuous map from the boundary N-simplex to a d-dimensional topological manifold. If $p = 2$ then we further assume that the degree of f is zero modulo 2. Then Δ_N has r disjoint rainbow faces whose images under f intersect.*

The proof in this paper works also for prime powers $r = p^k$ in the case without color constraints, since then

- the configuration space is the join $[r]^{*(N+1)}$, which is $(N-1)$-connected and $(\mathbb{Z}_p)^k$-free, hence its index is $H^{*\geq N+1}(B((\mathbb{Z}_p)^k))$, and
- the group $(\mathbb{Z}_p)^k$ acts fixed point freely on the sphere $S(F)$ and $\mathrm{Ind}_{(\mathbb{Z}_p)^k}(S(F))$ consequently contains an element of degree N, particularly $d_N(z)$ in the notation of Section 2.4.

3.2 Reduction to the subcomplex of rainbow faces

One could ask whether Δ_N in Theorem 1.2 can be replaced by the subcomplex R that consists of all rainbow faces. The methods of this paper seem to establish this only if we assume that sufficiently many colors are used. (The assumptions of Theorem 1.2 imply that the $N+1$ vertices of Δ_N are colored with at least $\lceil \frac{N+1}{r-1} \rceil = d+2$ colors.)

Corollary 3.2. *Let $d \geq 1$, $r \geq 2$ prime, and $N := (d+1)(r-1)$. Let the vertices of Δ_N be colored with at least $d+3+\lfloor \frac{d}{r-1} \rfloor = d+2+g$ colors such that all color classes C_i are of size $|C_i| \leq r-1$. Let R be the subcomplex of Δ_N consisting of all rainbow faces. Let $f : R \to M$ be a continuous map from R to a d-dimensional manifold M. Then R has r disjoint faces whose images under f intersect.*

The proof of Corollary 3.2 is analogous to that of Theorem 1.2. The main change occurs in the reduction to the case where the manifold is $M' = M \times I^g$, see Section 2.1. Here one needs to be a bit more careful. Instead of letting f send the $r-1$ new vertices of $\Delta_{N'}$ to points above $f(v_N)$ and giving them a new color, we send them above the images of possibly different vertices and color them with the same color as the vertex below. This has to be done in such a way that all new color classes are still of size less than r. This is possible since the number of used colors is at least $d+2+g$.

References

[1] I. Bárány, D. G. Larman. *A colored version of Tverberg's theorem*, J. London Math. Soc., 45, 1992, 314-320.

[2] I. Bárány, S. B. Shlosman, A. Szűcs. *On a topological generalization of a theorem of Tverberg*, J. London Math. Soc., 23, 1981, 158-164.

[3] P. V. M. Blagojević, B. Matschke, G. M. Ziegler. *Optimal bounds for the colored Tverberg problem*, arxiv:0910.4987v2, 2009, 11 pages.

[4] P. V. M. Blagojević, B. Matschke, G. M. Ziegler. *Optimal bounds for a colorful Tverberg–Vrećica type problem*, arxiv:0911.2692v2, 2009, 12 pages.

[5] G. E. Bredon. *Introduction to Compact Transformation Groups*, New York, Academic Press, 1972.

[6] E. Fadell, S. Husseini. *Relative cohomological index theories*, Advances in Math. 64, 1987, 1-31.

[7] E. Fadell, S. Husseini. *An ideal-valued cohomological index theory with applications to Borsuk–Ulam and Bourgin–Yang theorems*, Ergodic Theory and Dynamical Systems 8*, 1988, 73-85.

[8] J. Matoušek. *Using the Borsuk–Ulam Theorem. Lectures on Topological Methods in Combinatorics and Geometry*, Universitext, Springer 2003.

[9] A. Yu. Volovikov. *On a topological generalization of the Tverberg theorem*, Mathematical Notes 59, No. 3, 1996, 324-326.