Optimal bounds for the colored Tverberg problem

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Abstract

We prove a "Tverberg type" multiple intersection theorem. It strengthens the prime case of the original Tverberg theorem from 1966, as well as the topological Tverberg theorem of Bárány et al. (1980), by adding color constraints. It also provides an improved bound for the (topological) colored Tverberg problem of Bárány & Larman (1992) that is tight in the prime case and asymptotically optimal in the general case. The proof is based on relative equivariant obstruction theory.

1 Introduction

Tverberg's theorem from 1966 [17] [12, Sect. 8.3] claims that any family of $(d+1)(r-1)+1$ points in $\mathbb{R}^d$ can be partitioned into r sets whose convex hulls intersect; a look at the codimensions of intersections shows that the number $(d+1)(r-1)+1$ of points is minimal for this.

In their 1990 study of halving lines and halving planes, Bárány, Füredi & Lovász [2] observed "we need a colored version of Tverberg's theorem" and provided a first case, for three triangles in the plane. In response to this, Bárány & Larman [3] in 1992 formulated the following general problem and proved it for the planar case.

The colored Tverberg problem: *Determine the smallest number $t = t(d, r)$ such that for every collection $\mathcal{C} = C_0 \sqcup \cdots \sqcup C_d$ of points in $\mathbb{R}^d$ with $|C_i| \geq t$, there are r disjoint subcollections $F_1, \ldots, F_r$ of $\mathcal{C}$ satisfying*

$$|F_i \cap C_j| \leq 1 \quad \text{for every} \quad i \in \{1, \ldots, r\},\ j \in \{0, \ldots, d\}, \ \text{and} \ \operatorname{conv}(F_1) \cap \cdots \cap \operatorname{conv}(F_r) \neq \emptyset.$$

A family of such disjoint subcollections $F_1, \ldots, F_r$ that contain at most one point from each *color class* C_i is called a *rainbow r-partition.* (We do not require $F_1 \cup \cdots \cup F_r = \mathcal{C}$ for this.) Multiple points are allowed in these collections of points, but then the cardinalities have to account for these.

A trivial lower bound is $t(d, r) \geq r$: Collections $\mathcal{C}$ with only $(r-1)(d+1)$ points in general position do not admit an intersecting r-partition, again by codimension reasons.

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Bárány and Larman showed that the trivial lower bound is tight in the cases $t(1,r) = r$ and $t(2,r) = r$, presented a proof by Lovász for $t(d,2) = 2$, and conjectured the following equality.

The Bárány–Larman conjecture: *$t(d,r) = r$ for all $r \geq 2$ and $d \geq 1$.*

Still in 1992, Živaljević & Vrećica [18] established for r prime the upper bound $t(d,r) \leq 2r-1$. The same bound holds for prime powers according to Živaljević [23]. The bound for primes also yields bounds for arbitrary r: For example, one gets $t(d,r) \leq 4r-3$, since there is a prime p (and certainly a prime power!) between r and $2r$.

As in the case of Tverberg's classical theorem, one can consider a topological version of the colored Tverberg problem.

The topological Tverberg theorem: ([4] [13, Sect. 6.4]) *Let $r \geq 2$ be a prime power, $d \geq 1$, and $N = (d+1)(r-1)$. Then for every continuous map of an N-simplex Δ_N to $\mathbb{R}^d$ there are r disjoint faces $F_1, \dots, F_r$ of Δ_N whose images under f intersect in $\mathbb{R}^d$.*

The topological colored Tverberg problem: *Determine the smallest number $t = tt(d,r)$ such that for every simplex Δ with $(d+1)$-colored vertex set $\mathcal{C} = C_0 \sqcup \dots \sqcup C_d$, $|C_i| \geq t$, and every continous map $f : \Delta \to \mathbb{R}^d$ there are r disjoint faces $F_1, \dots, F_r$ of Δ satisfying*

$$|F_i \cap C_j| \leq 1 \text{ for every } i \in \{1, \dots, r\},\ j \in \{0, \dots, d\}, \text{ and } \ f(F_1) \cap \dots \cap f(F_r) \neq \emptyset.$$

The family of faces $F_1, \dots, F_r$ is called a *topological rainbow partition.*

The argument from [18] and [23] gives the same upper bound $tt(d,r) \leq 2r-1$ for r a prime power, and consequently the upper bound $tt(d,r) \leq 4r-3$ for arbitrary r. Notice that $t(d,r) \leq tt(d,r)$.

The topological Bárány–Larman conjecture: *$tt(d,r) = r$ for all $r \geq 2$ and $d \geq 1$.*

The Lovász proof for $t(d,2) = 2$ presented in [3] is topological and thus also valid for the topological Bárány–Larman conjecture. Therefore $tt(d,2) = 2$.

The general case of the topological Bárány–Larman conjecture would classically be approached via a study of the existence of an $\mathfrak{S}_r$-equivariant map

$$\Delta_{r,|C_0|} * \dots * \Delta_{r,|C_d|} \quad \longrightarrow_{\mathfrak{S}_r} \quad S(W_r^{\oplus(d+1)}) \ \simeq \ S^{(r-1)(d+1)-1}, \tag{1}$$

where W_r is the standard $(r-1)$-dimensional real representation of $\mathfrak{S}_r$ obtained by restricting the coordinate permutation action on $\mathbb{R}^r$ to $\{(\xi_1, \dots, \xi_r) \in \mathbb{R}^r : \xi_1 + \dots + \xi_r = 0\}$ and $\Delta_{r,n}$ denotes the $r \times n$ chessboard complex $([r])^{*n}_{\Delta(2)}$; cf. [13, Remark after Thm. 6.8.2]. However, we will establish in Proposition 4.1 that this approach fails when applied to the colored Tverberg problem directly, due to the fact that the square chessboard complexes $\Delta_{r,r}$ admit $\mathfrak{S}_r$-equivariant collapses that reduce the dimension.

In the following, we circumvent this problem by a different, particular choice of parameters, which produces chessboard complexes $\Delta_{r,r-1}$ that are closed pseudomanifolds and thus do not admit collapses.

2 Statement of the main results

Our main result is the following strengthening of (the prime case of) the topological Tverberg theorem.

Theorem 2.1. *Let $r \geq 2$ be prime, $d \geq 1$, and $N := (r-1)(d+1)$. Let Δ_N be an N-dimensional simplex with a partition of the vertex set into parts ("color classes")*

$$\mathcal{C} \quad = \quad C_0 \sqcup \dots \sqcup C_m,$$

with $|C_i| \leq r-1$ for all i.

Then for every continous map $f : \Delta_N \to \mathbb{R}^d$, there are r disjoint "rainbow" faces $F_1, \dots, F_r$ of Δ_N whose images under f intersect, that is,

$$|F_i \cap C_j| \leq 1 \text{ for every } i \in \{1, \dots, r\},\ j \in \{0, \dots, m\}, \text{ and } \ f(F_1) \cap \dots \cap f(F_r) \neq \emptyset.$$

The requirement $|C_i| \le r-1$ forces that there are at least $d+2$ non-empty color classes. Theorem 2.1 is tight in the sense that there would exist counter-examples f if $|C_0| = r$ and $|C_1| = \ldots = |C_m|$.

Our first step will be to reduce Theorem 2.1 to the following special case.

Theorem 2.2. *Let $r \ge 2$ be prime, $d \ge 1$, and $N := (r-1)(d+1)$. Let Δ_N be an N-dimensional simplex with a partition of the vertex set into $d+2$ parts*

$$\mathcal{C} \quad = \quad C_0 \sqcup \cdots \sqcup C_d \sqcup C_{d+1},$$

with $|C_i| = r-1$ for $i \le d$ and $|C_{d+1}| = 1$.

Then for every continous map $f : \Delta_N \to \mathbb{R}^d$, there are r disjoint faces $F_1, \ldots, F_r$ of Δ_N satisfying

$$|F_i \cap C_j| \le 1 \text{ for every } i \in \{1, \ldots, r\},\ j \in \{0, \ldots, d+1\}, \text{ and } \ f(F_1) \cap \cdots \cap f(F_r) \ne \emptyset.$$

Reduction of Theorem 2.1 to Theorem 2.2. Suppose we are given such a map f and a coloring $C_1 \sqcup \cdots \sqcup C_m$ of the vertex set of Δ_N. Let $N' := (r-1)m$ and $C_{m+1} := \emptyset$. We enlarge the color classes C_i by $N' - N = (r-1)(m-(d+1))$ new vertices and obtain color classes $C'_1, \ldots, C'_{m+1}$, such that $C_i \subseteq C'_i$ for all i, and $|C'_1| = \cdots = |C'_m| = r-1$ and $|C'_{m+1}| = 1$. We construct out of f a new map $f' : \Delta_{N'} \to \mathbb{R}^{d'}$, where $d' := m-1$, as follows: We regard $\mathbb{R}^d$ as the subspace of $\mathbb{R}^{d'}$ where the last $d'-d$ coordinates are zero. So we let f' be the same as f on the N-dimensional front face of $\Delta_{N'}$. We assemble the further $N'-N$ vertices into $d'-d$ groups $V_1, \ldots, V_{d'-d}$ of $r-1$ vertices each. The vertices in V_i shall be mapped to e_{d+i}, the $(d+i)$st standard basis vector of $\mathbb{R}^{d'}$. We extend this map linearly to all of $\Delta_{N'}$ and we obtain f'. We apply Theorem 2.2 to f' and the coloring $C'_1, \ldots, C'_{m+1}$ and obtain disjoint faces $F'_1, \ldots, F'_r$ of $\Delta_{n'}$. Let $F_i := F'_i \cap \Delta_N$ be the intersection of F'_i with the N-dimensional front face of $\Delta_{N'}$. By construction of f', the intersection $f'(F'_1) \cap \cdots \cap f'(F'_r)$ lies in R^d. Therefore, already $F_1, \ldots, F_r$ is a colorful Tverberg partition for f', and hence it is for f: We have $f(F_1) \cap \cdots \cap f(F_r) = \emptyset$. □

Such a reduction previously appears in Sarkaria's proof for the prime power Tverberg theorem [16, (2.7.3)]; see also Longueville's exposition [10, Prop. 2.5].

Remark 2.3. Soon after completion of the first version of the preprint for this paper we noticed (see [7, Sect. 2]) that Theorem 2.2 also has a simpler proof, using degrees rather than equivariant obstruction theory; a very similar proof was provided by Vrećica and Živaljević [19]. We provide it in [7] as a special case of a Vrećica–Tverberg type transversal theorem, accompanied by much more complete cohomological index calculations, which also yield a second new proof that establishes Theorem 2.1 directly, without a reduction to Theorem 2.2.

The simpler proof, however, does not imply that the equivariant map proposed by the natural configuration space/test map scheme of Theorem 4.2 *does* exists if r divides $(r-1)!^d$. This we prove at the end of the current paper.

Either of our Theorems 2.1 and 2.2 immediately implies the topological Tverberg theorem for the case when r is a prime, as it holds for an *arbitrary* partition of the vertex set into color classes of the specified sizes. Thus it is a "constrained" Tverberg theorem as discussed recently by Hell [8].

It remains to be explored how the constraints can be used to derive lower bounds for the number of Tverberg partitions; compare Vućić & Živaljević [20] [13, Sect. 6.3].

More importantly, however, Theorem 2.2 implies the topological Bárány–Larman conjecture for the case when $r+1$ is a prime, as follows.

Corollary 2.4. *If $r+1$ is prime, then $t(d,r) = tt(d,r) = r$.*

Proof. We prove that if $r \ge 3$ is prime, then $tt(d, r-1) \le r-1$. For this, let Δ_{N-1} be a simplex with vertex set $\mathcal{C} = C_0 \sqcup \cdots \sqcup C_d$, $|C_i| = r-1$, and let $f : \Delta_{N-1} \to \mathbb{R}^d$ be continuous. Extend this to a map $\Delta_N \to \mathbb{R}^d$, where Δ_N has an extra vertex v_N, and set $C_{d+1} := \{v_N\}$. Then Theorem 2.1 can be applied, and yields a topological colored Tverberg partition into r parts. Ignore the part that contains v_N. □

Using estimates on prime numbers one can derive from this tight bounds for the colored Tverberg problem also in the general case. The classical Bertrand's postulate ("For every r there is a prime p with $r+1 \leq p < 2r$") can be used here, but there are also much stronger estimates available, such as the existence of a prime p between r and $r + r^{6/11+\varepsilon}$ for arbitrary $\varepsilon > 0$ if r is large enough according to Lou & Yao [11].

Corollary 2.5. *$r \leq t(d,r) \leq tt(d,r) \leq 2r-2$ for all $d \geq 1$ and $r \geq 2$.*
$r \leq t(d,r) \leq tt(d,r) \leq (1+o(1))\, r$ for $d \geq 1$ and $r \to \infty$.

Proof. The first, explicit estimate is obtained from Bertrand's postulate: For any given r there is a prime p with $r+1 \leq p < 2r$. We use $|C_i| \geq 2r-2 \geq p-1$ to derive the existence of a colored Tverberg $(p-1)$-partition, which in particular yields an r-partition since $p-1 \geq r$.

The second, asymptotic estimate uses the Lou & Yao bound instead. □

Remark 2.6. The colored Tverberg problem as originally posed by Bárány & Larman [3] in 1992 was different from the version we have given above (following Bárány, Füredi & Lovász [2] and Vrećica & Živaljević [18]): Bárány and Larman had asked for an upper bound $N(d,r)$ on the cardinality of the union $|\mathcal{C}|$ that together with $|C_i| \geq r$ would force the existence of a rainbow r-partition. This original formulation has two major disadvantages: One is that the Vrećica–Živaljević result does not apply to it. A second one is that it does not lend itself to estimates for the general case in terms of the prime case.

However, our Corollary 2.4 also solves the original version for the case when $r+1$ is a prime.

The colored Tverberg problem originally arose as a tool to obtain complexity bounds in computational geometry. As a consequence, our new bounds can be applied to improve these bounds, as follows. Note that in some of these results $t(d,d+1)^d$ appears in the exponent, so even slightly improved estimates on $t(d,d+1)$ have considerable effect. For surveys see [1], [12, Sect. 9.2], and [22, Sect. 11.4.2].

Let $S \subseteq \mathbb{R}^d$ be a set in general position of size n, that is, such that no $d+1$ points of S are on a hyperplane. Let $h_d(n)$ denote the number of hyperplanes that bisect the set S and are spanned by the elements of the set S. According to Bárány [1, p. 239],

$$h_d(n) = O(n^{d-\varepsilon_d}) \qquad \text{with} \qquad \varepsilon_d = t(d,d+1)^{-(d+1)}.$$

Thus we obtain the following bound and equality.

Corollary 2.7. *If $d+2$ is a prime then*

$$h_d(n) = O(n^{d-\varepsilon_d}) \qquad \textit{with} \qquad \varepsilon_d = (d+1)^{-(d+1)}.$$

For general d, we obtain e.g. $\varepsilon_d \geq (d+1)^{-(d+1)-O(\log d)}$.

Let $\mathcal{C} \subseteq \mathbb{R}^d$ be a finite set. A *$\mathcal{C}$-simplex* is the convex hull of some collection of $d+1$ points of $\mathcal{C}$. The second selection lemma [12, Thm. 9.2.1] claims that for an n-point set $\mathcal{C} \subseteq \mathbb{R}^d$ and the family $\mathcal{F}$ of $\alpha\binom{n}{d+1}$ $\mathcal{C}$-simplices with $\alpha \in (0,1]$ there exists a point contained in at least $c \cdot \alpha^{s_d}\binom{n}{d+1}$ $\mathcal{C}$-simplices of $\mathcal{F}$. Here $c = c(d) > 0$ and s_d are constants. For dimensions $d > 2$, the presently known proof gives that $s_d \approx t(d,d+1)^{d+1}$. Again, Corollary 2.5 yields the following, much better bounds for the constant s_d.

Corollary 2.8. *If $d+2 > 4$ is a prime then the second selection lemma holds for $s_d = (d+1)^{d+1}$, and in general e.g. for $s_d = (2d+2)^{d+1}$.*

Let $X \subset \mathbb{R}^d$ be an n element set. A *k-facet* of the set X is an oriented $(d-1)$-simplex $\mathrm{conv}\{x_1,\ldots,x_d\}$ spanned by elements of X such that there are exactly k points of X on its strictly positive side. When $n-d$ is even $\frac{n-d}{2}$-facets of the set X are called *halving facets*. From [12, Thm. 11.3.3] we have a new, better estimate for the number of halving facets.

Corollary 2.9. *For $d > 2$ and $n-d$ even, the number of halving facets of an n-set $X \subset \mathbb{R}^d$ is $O(n^{d-\frac{1}{(2d)^d}})$.*

3 The Configuration Space/Test Map scheme

According to the "deleted joins" version the general "Configuration Space/Test Map" (CS/TM) scheme for multiple intersection problems, as pioneered by Sarkaria, Vrećica & Živaljević, and others, formalized by Živaljević, and exposited beautifully by Matoušek [13, Chap. 6], we proceed as follows.

Assume that we want to prove the existence of a rainbow r-partition for arbitrary colored point sets $\mathcal{C} = C_0 \sqcup C_1 \sqcup \cdots \sqcup C_k$ in $\mathbb{R}^d$ with $|C_i| = t_i$. So we have to show that there is no (affine) map

$$f : C_0 * C_1 * \cdots * C_k \longrightarrow \mathbb{R}^d,$$

for which no r images of disjoint simplices from the simplicial complex (join of discrete sets) $C_0 * C_1 * \cdots * C_k$ intersect in $\mathbb{R}^d$. (Compare Živaljević [22, Sect. 11.4.2].)

The "deleted joins" configuration space/test map scheme now suggests to take a r-fold deleted join of this map f, where one has to take an r-fold 2-wise deleted join in the domain and an r-fold r-wise deleted join in the range; cf. [13, Chap. 6.3]. Thus we arrive at an equivariant map

$$f^{*r}_{\Delta(2)} : \quad \Delta_{r,|C_0|} * \Delta_{r,|C_1|} * \cdots * \Delta_{r,|C_k|} \quad \longrightarrow_{\mathfrak{S}_r} \quad (\mathbb{R}^d)^{*r}_{\Delta} \;\subset\; \mathbb{R}^{r\times(d+1)} \backslash T \;\simeq\; S(W_r^{\oplus(d+1)}). \tag{2}$$

Here

- the simplicial complex $X := \Delta_{r,|C_0|} * \Delta_{r,|C_1|} * \cdots * \Delta_{r,|C_k|}$ on the left hand side is a join of $k+1$ chessboard complexes, where $\Delta_{r,|C_i|} = (C_i)^{*r}_{\Delta(2)}$ is the chessboard complex on r rows and $|C_i|$ columns, on which $\mathfrak{S}_r$ acts by permuting the r rows.
  This is a simplicial complex on $r(|C_0|+|C_1|+\cdots+|C_k|)$ vertices, of dimension $|C_0|+|C_1|+\cdots+|C_k|-1$ if $|C_i| \le r$, and of dimension $\max\{|C_0|, r\} + \max\{|C_1|, r\} + \cdots + \max\{|C_k|, r\} - 1$ in general.
  Points in X can be represented in the form $\lambda_1 x_1 + \cdots + \lambda_r x_r$, where x_i is a point in (a simplex of) the i-th copy of the complex $C_0 * C_1 * \cdots * C_k$, and the $\lambda_i \ge 0$, $\sum_i \lambda_i = 1$, denote a convex combination.
- $(\mathbb{R}^d)^{*r}_{\Delta}$ is a deleted join, which is most easily represented as a subset of the space of all real $r \times (d+1)$-matrices for which not all rows are equal, and where $\mathfrak{S}_r$ acts by permuting the rows. To factor out the diagonal T, which is the $(d+1)$-dimensional subspace of all matrices for which all rows are equal, we subtract the average of all rows from each row, which maps this equivariantly to $W_r^{\oplus(d+1)} \backslash \{0\}$, the space of all real $r \times (d+1)$-matrices with column sums equal to zero but for which not all rows are zero, and where $\mathfrak{S}_r$ still acts by permuting the rows. This in turn is homotopy equivalent to the sphere $S(W_r^{\oplus(d+1)}) = (S^{r-2})^{*(d+1)} = S^{(r-1)(d+1)-1} = S^{N-1}$, where $\pi \in \mathfrak{S}_r$ reverses the orientation exactly if $(\operatorname{sgn} \pi)^{d+1}$ is negative.
- The action of $\mathfrak{S}_r$ is non-free exactly on the subcomplex $A := (\Delta_{r,|C_0|} * \ldots * \Delta_{r,|C_m|})^{\emptyset,\emptyset} \subset X$ given by all the points $\lambda_1 x_1 + \cdots + \lambda_r x_r$ such that $\lambda_i = \lambda_j = 0$ for two distinct row indices $i < j$. These lie in simplices that have no vertices in the rows i and j, so the transposition π_{ij} fixes these simplices pointwise.
- The map $f^{*r}_{\Delta(2)} : X \to \mathbb{R}^{r\times(d+1)}$ suggested by the "deleted joins" scheme takes the point $\lambda_1 x_1 + \cdots + \lambda_r x_r$ and maps it to the $r \times (d+1)$-matrix in $\mathbb{R}^{r\times(d+1)}$ whose k-th row is $(\lambda_k, \lambda_k f(x_k))$. For an arbitrary map f, the image of A under $f^{*r}_{\Delta(2)}$ does not intersect the diagonal T: If $\lambda_i = \lambda_j = 0$, then not all rows $(\lambda_k, \lambda_k f(x_k))$ can be equal, since $\sum_k \lambda_k = 1$.
  However, for the following we replace $f^{*r}_{\Delta(2)}$ by the map $F_0 : X \to \mathbb{R}^{r\times(d+1)}$ that maps $\lambda_1 x_1 + \cdots + \lambda_r x_r$, to the $r \times (d+1)$-matrix whose k-th row is $(\lambda_k, (\Pi_{\ell=1}^r \lambda_\ell) f(x_k))$. The two maps $f^{*r}_{\Delta(2)}$ and F_0 are homotopic as maps $A \to \mathbb{R}^{r\times(d+1)} \setminus \{T\}$ by a linear homotopy, so the resulting extension problems are equivalent by [15, Prop. 3.15(ii)]. The advantage of the map F_0 is that its restriction to A is independent of f.

Thus we have established the following.

Proposition 3.1 (CS/TM scheme for the generalized topological colored Tverberg problem)**.** *If for some parameters $(d, r, k; t_0, \ldots, t_k)$ the $\mathfrak{S}_r$-equivariant extension (2) of the map $F : A \to \mathbb{R}^{r\times(d+1)} \backslash T$ does not exist, then the colored Tverberg r-partition exists for all continuous $f : C_0 * C_1 * \cdots * C_k \to \mathbb{R}^d$ with $|C_i| \ge t_i$.*

Vrećica & Živaljević achieve this for $(d, r, d; 2r-1, \ldots, 2r-1)$ and prime r by applying a Borsuk–Ulam type theorem to the action of the subgroup $\mathbb{Z}_r \subset \mathfrak{S}_r$, which acts freely on the join of chessboard complexes if r is a prime. However, they loose a factor of 2 from the fact that the chessboard-complexes $\Delta_{r,t}$ of dimension $r-1$ are homologically $(r-2)$-connected only if $t \geq 2r-1$; compare [5], [21], and [14].

Our Theorem 2.2 claims this for $(d, r, d+1; r-1, \ldots, r-1, 1)$. To prove it, we will use relative equivariant obstruction theory, as presented by tom Dieck in [15, Sect. II.3].

4 Proof of Theorem 2.2

First we establish that the scheme of Proposition 3.1 fails when applied to the colored Tverberg problem directly.

Proposition 4.1. *For all $r \geq 2$ and $d \geq 1$, with $N = (r-1)(d+1)$, an equivariant $\mathfrak{S}_r$-equivariant map*

$$F : (\Delta_{r,r})^{*(d+1)} \quad \longrightarrow_{\mathfrak{S}_r} \quad W_r^{\oplus(d+1)} \setminus \{0\} \;\simeq\; S^{N-1}$$

exists.

Proof. For any facet of the $(r-1)$-dimensional chessboard complex $\Delta_{r,r}$ there is a collapse which removes the facet together with its subfacet obtained by deleting the vertex in the r-th column. Performing these collapses simultaneously, we see that $\Delta_{r,r}$ collapses $\mathfrak{S}_r$-equivariantly to an $(r-2)$-dimensional subcomplexes of $\Delta_{r,r}$, and thus $(\Delta_{r,r})^{*(d+1)}$ equivariantly retracts to a complex whose dimension is only $(d+1)(r-1)-1 = N-1$.

Thus there is no obstruction to the construction of such an equivariant map: Any generic map $f : \mathcal{C} \to \mathbb{R}^d$ induces such an equivariant map on the $(N-2)$-skeleton, and since the action of $\mathfrak{S}_r$ is free on the open $(N-1)$-simplices, there is no obstruction for the equivariant extension of the map to $W_r^{\oplus(d+1)} \setminus \{0\} \simeq S^{N-1}$. ☐

We now specialize the general scheme of Proposition 3.1 to the situation of Theorem 2.2. Thus we have to show the following.

Proposition 4.2. *Let $r \geq 2$ and $d \geq 1$ be integers, and $N = (r-1)(d+1)$.*
An $\mathfrak{S}_r$-equivariant map

$$F : (\Delta_{r,r-1})^{*d} * \Delta_{r,r-1} * [r] \quad \longrightarrow_{\mathfrak{S}_r} \quad W_r^{\oplus(d+1)} \setminus \{0\}$$

that extends the equivariant map $F_0|_A$ which on the non-free subcomplex of the domain,

$$A \;=\; ((\Delta_{r,r-1})^{*d} * \Delta_{r,r-1} * [r])^{\emptyset,\emptyset},$$

maps $\lambda_1 x_1 + \cdots + \lambda_r x_r$ with $\lambda_i = \lambda_j = 0$, $i < j$ to the $r \times (d+1)$-matrix with i-th row $(\lambda_i, 0)$, exists if and only if

$$r \mid (r-1)!^d.$$

The vertex set of $X = (\Delta_{r,r-1})^{*d} * \Delta_{r,r-1} * [r]$ may be represented by a rectangular array of size $r \times ((r-1)(d+1)+1)$, which carries the $d+1$ chessboard complexes $\Delta_{r,r-1}$ lined up from left to right, and in the last column has the chessboard complex $\Delta_{r,1} = [r]$, which is just a discrete set. (See Figure 1)

The join of chessboard complexes $(\Delta_{r,r-1})^{*d} * \Delta_{r,r-1} * [r]$ has dimension $(r-1)(d+1) = N$, while the target sphere has dimension $N-1$. On both of them, $\mathfrak{S}_r$ acts by permuting the rows.

While the chessboard complexes $\Delta_{r,r}$ collapse equivariantly to lower-dimensional complexes, the chessboard complexes $\Delta_{r,r-1}$ are closed oriented pseudomanifolds of dimension $r-2$ and thus don't collapse; for example, $\Delta_{3,2}$ is a circle and $\Delta_{4,3}$ is a torus. We will read the maximal simplices of such a complex from left to right, which yields the orientation cycle in a special form with few signs that will be very convenient.

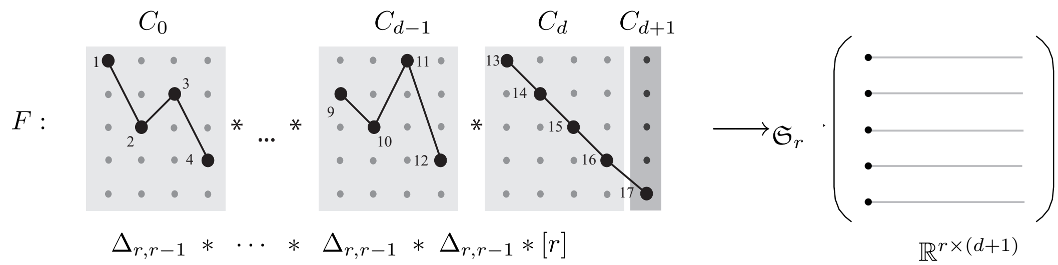

Figure 1: The vertex set, and one facet in Φ of the combinatorial configuration space for $r = 5$.

Lemma 4.3. (cf. [5] [14], [9, p. 145]) *For $r > 2$, the chessboard complex $\Delta_{r,r-1}$ is a connected, orientable pseudomanifold of dimension $r-2$. Therefore*

$$H_{r-2}(\Delta_{r,r-1};\mathbb{Z}) = \mathbb{Z}$$

and an orientation cycle is

$$z_{r,r-1} \;=\; \sum_{\pi\in\mathfrak{S}_r} (\operatorname{sgn}\pi)\langle(\pi(1),1),\ldots,(\pi(r-1),r-1)\rangle. \tag{3}$$

$\mathfrak{S}_r$ acts on $\Delta_{r,r-1}$ by permuting the rows; this affects the orientation according to $\pi\cdot z_{r,r-1} = (\operatorname{sgn}\pi)z_{r,r-1}$.

Here we use the usual notation $\langle w_0,\ldots,\widehat{w_i},\ldots,w_k\rangle$ for an oriented simplex with ordered vertex set $(w_0,\ldots,w_k)$ from which the vertex w_i is omitted.

Proof of Proposition 4.2. For $r = 2$, since $2 \nmid 1$, this says that there is no equivariant map $S^N \to S^{N-1}$, where both spheres are equipped with the antipodal action: This is the Borsuk–Ulam theorem (and the Lovász proof). Thus we may now assume that $r \geq 3$.

Let $X := (\Delta_{r,r-1})^{*(d+1)} * [r]$ be our combinatorial configuration space, $A \subset X$ the non-free subset, and $F_0 : A \to_{\mathfrak{S}_r} S(W_r^{\oplus(d+1)})$ the prescribed map that we are to extend $\mathfrak{S}_r$-equivariantly to X.

Since $\dim(X) = N$ and $\dim S(W_r^{\oplus(d+1)}) = N-1$ with $\operatorname{conn} S(W_r^{\oplus(r+1)}) = N-2$, by [15, Sect. II.3] the existence of an $\mathfrak{S}_r$-equivariant extension $(\Delta_{r,r-1})^{*(d+1)} * [r] \to S(W_r^{\oplus(d+1)})$ is equivalent to the vanishing of the primary obstruction

$$\mathfrak{o} \;\in\; H^N_{\mathfrak{S}_r}\big(X, A; \Pi_{N-1}(S(W_r^{\oplus(d+1)}))\big).$$

The Hurewicz isomorphism gives an isomorphism of the coefficient $\mathfrak{S}_r$-module with a homology group,

$$\Pi_{N-1}(S(W_r^{\oplus(r+1)})) \;\cong\; H_{N-1}(S(W_r^{\oplus(r+1)});\mathbb{Z}) \quad =: \quad \mathcal{Z}.$$

As an abelian group this module $\mathcal{Z} = \langle\zeta\rangle$ is isomorphic to $\mathbb{Z}$. The action of the permutation $\pi \in \mathfrak{S}_r$ on the module $\mathcal{Z}$ is given by

$$\pi\cdot\zeta \;=\; (\operatorname{sign}\pi)^{d+1}\zeta.$$

Computing the obstruction cocycle. We will now compute an obstruction cocycle $\mathfrak{c}_f$ in the cochain group $C^N_{\mathfrak{S}_r}\big(X, A; \mathcal{Z}\big)$, and then show that for prime r the cocycle is not a coboundary, that is, it does not vanish when passing to $\mathfrak{o} = [\mathfrak{c}_f]$ in the cohomology group $H^N_{\mathfrak{S}_r}\big(X, A; \mathcal{Z}\big)$.

For this, we use a specific general position map $f : X \to \mathbb{R}^d$, which induces a map $F : X \to \mathbb{R}^{r\times(d+1)}$; the value of the obstruction cocycle $\mathfrak{c}_f$ on an oriented maximal simplex σ of X is then given by the signed intersection number of $F(\sigma)$ with the test space, the diagonal T. (Compare [15] and [6].)

Let $e_1,\ldots,e_d$ be the standard basis vectors of $\mathbb{R}^d$, set $e_0 := 0 \in \mathbb{R}^d$, and denote by $v_0,\ldots,v_N$ the set of vertices of the N-simplex Δ_N in the given order, that is, such that $C_i = \{v_{i(r-1)},\ldots,v_{(i+1)(r-1)-1}\}$ for $i \leq d$ and $C_{d+1} = \{v_{(d+1)(r-1)}\}$. Let $f : \|\Delta_N\| \to \mathbb{R}^d$ be the linear map defined on the vertices by

$$\begin{cases} v_i \;\overset{f}{\longmapsto}\; e_{\lfloor i/(r-1)\rfloor} & \text{for } 0 \leq i \leq N-1,\\ v_N \;\overset{f}{\longmapsto}\; \frac{1}{d+1}\sum_{i=0}^{d} e_i, & \end{cases}$$

that is, such that the vertices in C_i are mapped to the vertex e_i of the standard d-simplex for $i \leq d$, while $v_N \in C_{d+1}$ is mapped to the center of this simplex.

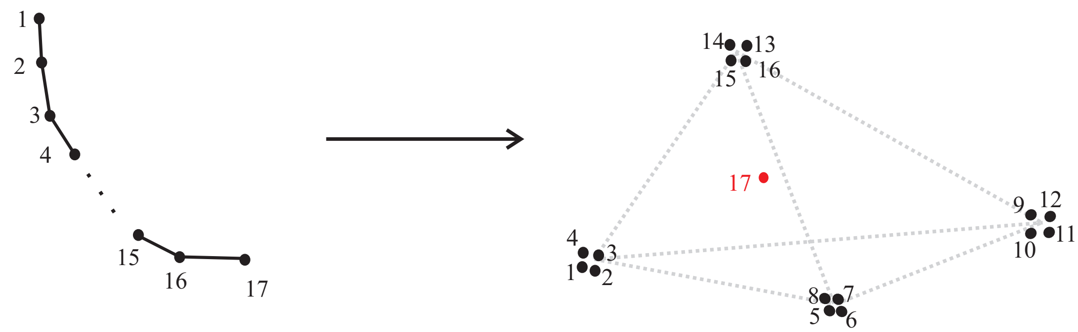

Figure 2: The map $f : \|\Delta^{16}\| \to \mathbb{R}^3$ in the case $d = 3$ and $r = 5$

This induces a linear map $f : C_0 * \cdots * C_{d+1} \to \mathbb{R}^d$ and thus an equivariant map $F : X \to \mathbb{R}^{r\times(d+1)}$, taking $\lambda_1 x_1 + \cdots + \lambda_r x_r$ to the $r \times (d+1)$-matrix whose k-th row is $(\lambda_k, (\Pi_{\ell=1}^r \lambda_\ell)x_k)$, which extends the prescribed map $F_0 : A \to \mathbb{R}^{r\times(d+1)} \backslash T$. The intersection points of the image of F with the diagonal T correspond to the colored Tverberg r-partitions of the configuration $\mathcal{C} = C_0 \sqcup \cdots \sqcup C_{d+1}$ in $\mathbb{R}^d$. Since $\lambda_1 = \cdots = \lambda_r = \frac{1}{r}$ at all these intersection points, we find that F is in general position with respect to T.

The only Tverberg r-partitions of the point configuration $\mathcal{C}$ (even ignoring colors) are given by $r-1$ d-simplices with its vertices at $e_0, e_1, \ldots, e_d$, together with one singleton point (0-simplex) at the center. Clearly there are $(r-1)!^d$ such partitions.

We take representatives for the $\mathfrak{S}_r$-orbits of maximal simplices of X such that from the last $\Delta_{r,r-1}$ factor, the vertices $(1,1), \ldots, (r-1, r-1)$ are taken.

On the simplices of X we use the orientation that is induced by ordering all vertices left-to-right on the array of Figure 1. This orientation is $\mathfrak{S}_r$-invariant, as permutation of the rows does not affect the left-to-right ordering.

The obstruction cocycle evaluated on subcomplexes of $(\Delta_{r,r-1})^{*d} * \Delta_{r,r-1} * [r]$. Let us consider the following chains of dimensions N resp. $N-1$ (illustrated in Figure 3), where $z_{r,r-1}$ denotes the orientation cycle for the chessboard complex $\Delta_{r,r-1}$, as given by Lemma 4.3:

$$\begin{aligned}
\Phi &= (z_{r,r-1})^{*d} * \langle (1,1), \ldots, \;\ldots\; , \ldots, (r-1,r-1), (r,r) \rangle, \\
\Omega_j &= (z_{r,r-1})^{*d} * \langle (1,1), \ldots, \;\ldots\; , \ldots, (r-1,r-1), (j,r) \rangle \qquad (1 \leq j < r), \\
\Theta_i &= (z_{r,r-1})^{*d} * \langle (1,1), \ldots, \widehat{(i,i)}, \ldots, (r-1,r-1), (r,r) \rangle \qquad (1 \leq i \leq r), \\
\Theta_{i,j} &= (z_{r,r-1})^{*d} * \langle (1,1), \ldots, \widehat{(i,i)}, \ldots, (r-1,r-1), (j,r) \rangle \qquad (1 \leq i \leq r,\ 1 \leq j < r).
\end{aligned}$$

Explicitly the signs in these chains are as follows. If σ denotes the facet $\langle (1,1), \ldots, (r-1,r-1) \rangle$ of $\Delta_{r,r-1}$, such that $\pi\sigma = \langle (\pi(1),1), \ldots, (\pi(r-1), r-1) \rangle$, then Φ is given by

$$\Phi = \sum_{\pi_1,\ldots,\pi_d \in \mathfrak{S}_r} (\operatorname{sgn}\pi_1) \cdots (\operatorname{sgn}\pi_d)\, \pi_1\sigma * \cdots * \pi_d\sigma * \langle (1,1), \ldots, (r-1,r-1), (r,r) \rangle$$

and similarly for Ω_j, Θ_i, and $\Theta_{i,j}$

The evaluation of $\mathfrak{c}_f$ on Φ picks out the facets that correspond to colored Tverberg partitions: Since the last part of the partition must be the singleton vertex v_N, we find that the last rows of the chessboard complex $Delta_{r,r-1}$ factors are not used. We may define the orientation on $S(W_r^{\oplus(d+1)})$ such that

$$\mathfrak{c}_f(\sigma * \cdots * \sigma * \langle (1,1), \ldots, (r-1,r-1), (r,r) \rangle) = +\zeta.$$

Then we get

$$\mathfrak{c}_f\big(\pi_1\sigma * \cdots * \pi_d\sigma * \langle (1,1), \ldots, (r-1,r-1), (r,r) \rangle\big) = \begin{cases} (\operatorname{sgn}\pi_1) \cdots (\operatorname{sgn}\pi_d)\, \zeta & \text{if } \pi_1(r) = \cdots = \pi_d(r) = r, \\ 0 & \text{otherwise.} \end{cases}$$

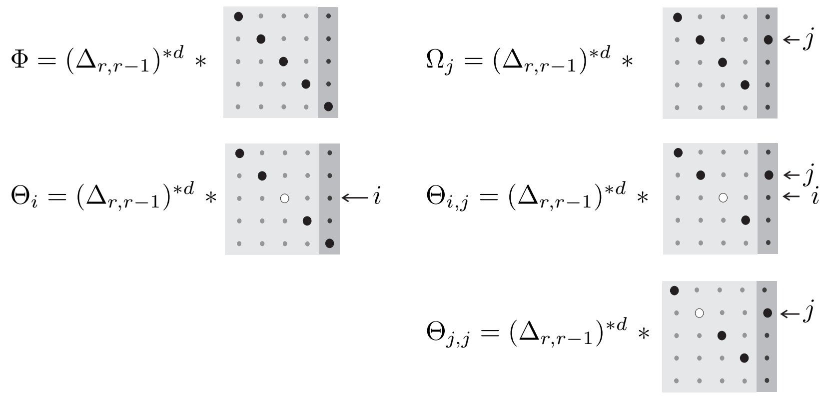

Figure 3: Schemes for the combinatorics of the chains Φ, Ω_j, Θ_i, and $\Theta_{i,j}$.

The sign $(\operatorname{sgn}\pi_1)\cdots(\operatorname{sgn}\pi_d)$ comes from the fact that F maps $\sigma * \cdots * \sigma * \langle(1,1),\ldots,(r-1,r-1),(r,r)\rangle$ and $\pi_1\sigma * \cdots * \pi_d\sigma * \langle(1,1),\ldots,(r-1,r-1),(r,r)\rangle$ to the same simplex in $W_r^{\oplus(d+1)}$, however with a different order of the vertices.

Thus,

$$\mathfrak{c}_f(\Phi) \quad = \quad (r-1)!^d\,\zeta.$$

Moreover, for any Tverberg r-partition in our configuration the last point v_N has to be a singleton, while the facets of Ω_j correspond to r-partitions where the j-th face pairs v_N with a point in C_d. Thus the cochains Ω_j do not capture any Tverberg partitions, and we get

$$\mathfrak{c}_f(\Omega_j) \quad = \quad 0 \qquad \text{for } 1 \le j < r.$$

Is the cocycle $\mathfrak{c}_f$ a coboundary? Let us assume that $\mathfrak{c}_f$ is a coboundary. Then there is an equivariant cochain $\mathfrak{h} \in C^{N-1}_{\mathfrak{S}_r}\big(X, A; \mathcal{Z}\big)$ such that $\mathfrak{c}_f = \delta\mathfrak{h}$, where δ is the coboundary operator.

In order to simplify the notation, from now on we drop the join factor $(\Delta_{r,r-1})^{*d}$ from the notation of the subcomplexes Φ, Θ_i and Ω_i. Note that the join with this complex accounts for a global sign of $(-1)^{d(r-1)}$ in the boundary/coboundary operators, since in our vertex ordering the complex $(\Delta_{r,r-1})^{*d}$, whose facets have $d(r-1)$ vertices, comes first.

Thus we have

$$\partial\Phi \quad = \quad (-1)^{d(r-1)} \sum_{i=1}^{r} (-1)^{i-1}\Theta_i$$

and similarly for $1 \le j < r$,

$$\partial\Omega_j \quad = \quad (-1)^{d(r-1)} \Big(\sum_{i=1}^{r-1} (-1)^{i-1}\Theta_{i,j} \;+\; (-1)^{r-1}\Theta_r \Big).$$

Claim 1. For $1 \le i, j < r$, $i \ne j$ we have $\mathfrak{h}(\Theta_{i,j}) = 0$.

Proof. We consider the effect of the transposition π_{ir}. The simplex $\langle(1,1),\ldots,\widehat{(i,i)},\ldots,(r-1,r-1),(j,r)\rangle$ has no vertex in the i-th and in the r-th row, so it is fixed by π_{ir}. The d chessboard complexes in $\Theta_{i,j}$ are invariant but change orientation under the action of π_{ir}, so the effect on the chain $\Theta_{i,j}$ is $\pi_{ir} \cdot \Theta_{i,j} = (-1)^d\Theta_{i,j}$ and hence

$$\mathfrak{h}(\pi_{ir} \cdot \Theta_{i,j}) \;=\; \mathfrak{h}((-1)^d\Theta_{i,j}) \;=\; (-1)^d\mathfrak{h}(\Theta_{i,j}).$$

On the other hand $\mathfrak{h}$ is equivariant, so

$$\mathfrak{h}(\pi_{ir}\cdot\Theta_{i,j}) \;=\; \pi_{ir}\cdot\mathfrak{h}(\Theta_{i,j}) \;=\; (-1)^{d+1}\mathfrak{h}(\Theta_{i,j})$$

since $\mathfrak{S}_r$ acts on $\mathcal{Z}$ by multiplication with $(\operatorname{sgn}\pi)^{d+1}$.

Comparing the two evaluations of $\mathfrak{h}(\pi_{ir}\cdot\Theta_{i,j})$ yields $(-1)^d\mathfrak{h}(\Theta_{i,j}) = (-1)^{d+1}\mathfrak{h}(\Theta_{i,j})$. □

Claim 2. For $1 \le j < r$ we have $\mathfrak{h}(\Theta_{j,j}) = -\mathfrak{h}(\Theta_j)$.

Proof. The interchange of the j-th row with the r-th moves $\Theta_{j,j}$ to Θ_j, where we have to account for d orientation changes for the chessboard join factors.

Thus $\pi_{jr}\Theta_{j,j} = (-1)^d\Theta_j$, which yields

$$(-1)^d\mathfrak{h}(\Theta_j) \;=\; \mathfrak{h}((-1)^d\Theta_j) \;=\; \mathfrak{h}(\pi_{jr}\Theta_{j,j}) \;=\; \pi_{jr}\cdot\mathfrak{h}(\Theta_{j,j}) \;=\; (-1)^{d+1}\mathfrak{h}(\Theta_{j,j}).$$

□

We now use the two claims to evaluate $\mathfrak{h}(\partial\Omega_j)$. Thus we obtain

$$0 \;=\; \mathfrak{c}_f(\Omega_j) \;=\; \delta\mathfrak{h}(\Omega_j) \;=\; \mathfrak{h}(\partial\Omega_j) \;=\; (-1)^{d(r-1)}\big((-1)^{j-1}\mathfrak{h}(\Theta_{j,j}) \;+\; (-1)^{r-1}\mathfrak{h}(\Theta_r)\big)$$

and hence

$$(-1)^j\mathfrak{h}(\Theta_j) \;=\; (-1)^r\mathfrak{h}(\Theta_r).$$

The final blow now comes from our earlier evaluation of the cochain $\mathfrak{c}_f$ on Φ:

$$\begin{aligned}
(r-1)!^d\cdot\zeta \;=\; \mathfrak{c}_f(\Phi) \;=\; \delta\mathfrak{h}(\Phi) \;=\; \mathfrak{h}(\partial\Phi) \;&=\; \mathfrak{h}((-1)^{d(r-1)}\sum_{j=1}^{r}(-1)^{j-1}\Theta_j)\\
&=\; -(-1)^{d(r-1)}\sum_{j=1}^{r}(-1)^j\mathfrak{h}(\Theta_j)\\
&=\; -(-1)^{d(r-1)}\sum_{j=1}^{r}(-1)^r\mathfrak{h}(\Theta_r)\\
&=\; (-1)^{(d+1)(r-1)}r\,\mathfrak{h}(\Theta_r).
\end{aligned}$$

Thus, the integer coefficient of $\mathfrak{h}(\Theta_r)$ should be equal to $\frac{(r-1)!^d}{r}\zeta$, up to a sign. Consequently, when $r \nmid (r-1)!^d$, the cocycle $\mathfrak{c}_f$ is not a coboundary, i.e. the cohomology class $\mathfrak{o} = [\mathfrak{c}_f]$ does not vanish and so there is no $\mathfrak{S}_r$-equivariant extension $X \to S(W_r^{\oplus(d+1)})$ of $F_0|_A$.

On the other hand, when $r \mid (r-1)!^d$ we can define

$$\begin{array}{lcll}
\mathfrak{h}(\Theta_j) & := & +(-1)^{(d+1)(r-1)+j+r}\cdot\frac{(r-1)!^d}{r}\cdot\zeta, & \text{for } 1 \le j \le r,\\
\mathfrak{h}(\Theta_{j,j}) & := & -(-1)^{(d+1)(r-1)+j+r}\cdot\frac{(r-1)!^d}{r}\cdot\zeta, & \text{for } 1 \le j < r,\\
\mathfrak{h}(\Theta_{i,j}) & := & 0, & \text{for } i \ne j,\ 1 \le i \le r,\ 1 \le j < r.
\end{array}$$

Here we actually do obstruction theory with respect to the filtration $(\Delta_{r,r-1})^{*d} * (\Delta_{r,r-1} * [r])^{(n)}$ of X, where $(\Delta_{r,r-1} * [r])^{(n)}$ denotes the n-skeleton of $\Delta_{r,r-1} * [r]$. The obstruction cocycle actually lies in

$$C^{r-1}_{\mathfrak{S}_r}(\Delta_{r,r-1} * [r]; \mathcal{Z}\otimes H_{(r-1)d-1}((\Delta_{r,r-1})^{*d};\mathbb{Z})),$$

and it is the coboundary of $\mathfrak{h}$. Since $\mathfrak{h}$ is only non-zero on the "cells" Θ_j and $\Theta_{j,j}$, which are only invariant under $\mathrm{id} \in \mathfrak{S}_r$, we can solve the extension problem equivariantly.

Hence for $r \mid (r-1)!^d$ an $\mathfrak{S}_r$-equivariant extension $X \to S(W_r^{\oplus(d+1)})$ exists. □

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