Optimal bounds for a colorful Tverberg–Vrećica type problem

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Abstract

We prove the following optimal colorful Tverberg–Vrećica type transversal theorem: For prime r and for any $k+1$ colored collections of points C^ℓ in $\mathbb{R}^d$, $C^\ell = \biguplus C_i^\ell$, $|C^\ell| = (r-1)(d-k+1)+1$, $|C_i^\ell| \leq r-1$, $\ell = 0,\ldots,k$, there are partition of the collections C^ℓ into colorful sets $F_1^\ell,\ldots,F_r^\ell$ such that there is a k-plane that meets all the convex hulls $\operatorname{conv}(F_j^\ell)$, under the assumption that $r(d-k)$ is even or $k=0$.

Along the proof we obtain three results of independent interest: We present two alternative proofs for the special case $k=0$ (our optimal colored Tverberg theorem (2009)), calculate the cohomological index for joins of chessboard complexes, and establish a new Borsuk–Ulam type theorem for $(\mathbb{Z}_p)^m$-equivariant bundles that generalizes results of Volovikov (1996) and Živaljević (1999).

1 Introduction

In their 1993 paper [TV93] H. Tverberg and S. Vrećica presented a conjectured common generalization of some Tverberg type theorems, some ham sandwich type theorems and many intermediate results. See [Živ99] for a further collection of implications.

Conjecture 1.1 (Tverberg–Vrećica Conjecture)**.** *Let $0 \leq k \leq d$ and let $C^0,\ldots,C^k$ be finite point sets in $\mathbb{R}^d$ of cardinality $|C^\ell| = (r_\ell - 1)(d-k+1)+1$. Then one can partition each C^ℓ into r_ℓ sets $F_1^\ell,\ldots,F_{r_\ell}^\ell$ such that there is a k-plane P in $\mathbb{R}^d$ that intersects all the convex hulls $\operatorname{conv}(F_j^\ell)$, $0 \leq \ell \leq k$, $1 \leq j \leq r_\ell$.*

The Tverberg–Vrećica Conjecture has been verified for the following special cases:

- $k = d$ (trivial),
- $k = 0$ (Tverberg's theorem [Tve66]),
- $k = d-1$ (Tverberg & Vrećica [TV93]),
- for $k = d-2$ a weakened version was shown in [TV93] (one requires two more points for each C^ℓ),
- k and d are odd, and $r_0 = \cdots = r_k$ is an odd prime (Živaljević [Živ99]),

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- $r_0 = \cdots = r_k = 2$ (Vrećica [Vre08]), and
- $r_\ell = p^{a_\ell}$, $a_\ell \geq 0$, for some prime p, and $p(d-k)$ is even or $k = 0$ (Karasev [Kar07]).

In this paper we consider the following colorful generalization of the Tverberg–Vrećica conjecture.

Conjecture 1.2. *Let $0 \leq k \leq d$, $r_\ell \geq 2$ $(\ell = 0, \dots, k)$ and let C^ℓ $(\ell = 0, \dots, k)$ be subsets of $\mathbb{R}^d$ of cardinality $|C^\ell| = (r_\ell - 1)(d-k+1)+1$. Let the C^ℓ be colored,*

$$C^\ell \;=\; \biguplus C_i^\ell,$$

such that no color class is too large, $|C_i^\ell| \leq r_\ell - 1$. Then we can partition each C^ℓ into sets $F_1^\ell, \dots, F_{r_\ell}^\ell$ that are colorful *(in the sense that $|C_i^\ell \cap F_j^\ell| \leq 1$ for all i, j, ℓ) and find a k-plane P that intersects all the convex hulls* $\mathrm{conv}(F_j^\ell)$.

The Tverberg–Vrećica Conjecture 1.1 is the special case of the previous conjecture when all color classes are given by singletons. The main result of this paper is the following special case.

Theorem 1.3 (Main Theorem)**.** *Let r be prime and $0 \leq k \leq d$ such that $r(d-k)$ is even or $k = 0$. Let C^ℓ $(\ell = 0, \dots, k)$ be subsets of $\mathbb{R}^d$ of cardinality $|C^\ell| = (r-1)(d-k+1)+1$. Let the C^ℓ be colored,*

$$C^\ell \;=\; \biguplus C_i^\ell,$$

such that no color class is too large, $|C_i^\ell| \leq r - 1$. Then we can partition each C^ℓ into colorful sets $F_1^\ell, \dots, F_r^\ell$ and find a k-plane P that intersects all the convex hulls $\mathrm{conv}(F_j^\ell)$.

In Section 5 we will see that this theorem is quite tight in the sense that it becomes false if one single color class C_i^ℓ has r_ℓ elements and all the other ones are singletons.

Since we will prove Theorem 1.3 topologically it has a natural topological extension, Theorem 3.1.

Recently we had obtained the first case $k = 0$ using equivariant obstruction theory [BMZ09]. In Section 2 we present two alternative proofs, based on the configuration space/test map scheme from [BMZ09]. The first one is more elementary and shorter; it uses a degree argument. The second proof puts the first one into the language of cohomological index theory. For this, we calculate the cohomological index of joins of chessboard complexes. This allows for a more direct proof of the case $k = 0$, which is the first of two keys for the Main Theorem 1.3.

The second key is a new Borsuk–Ulam type theorem for equivariant bundles. We establish it in Section 4, and prove the Main Theorem in Section 5. The new Borsuk–Ulam type theorem can also be applied to obtain an alternative proof of Karasev's above-mentioned result from [Kar07]; see Section 5. Karasev has also obtained a colored version of the Tverberg–Vrećica conjecture, different from ours, even for prime powers, which can also alternatively be obtained from our new Borsuk–Ulam type theorem.

2 The topological colored Tverberg problem revisited

In [BMZ09] we have shown the following new colored version of the topological Tverberg theorem. It is the special case $k = 0$ of the Topological Main Theorem 3.1.

Theorem 2.1 ([BMZ09])**.** *Let $r \geq 2$ be prime, $d \geq 1$, and $N := (r-1)(d+1)$. Let σ_N be an N-dimensional simplex with a partition of the vertex set into "color classes" $C_0, \dots, C_m$ such that $|C_i| \leq r - 1$ for all i.*

Then for every continuous map $f : \sigma_N \to \mathbb{R}^d$ there are r disjoint rainbow faces $F_1, \dots, F_r$ of σ_N (that is, $|C_i \cap F_j| \leq 1$) such that

$$f(F_1) \cap \cdots \cap f(F_r) \neq \emptyset.$$

This implies the optimal colored Tverberg theorem (the Bárány–Larman conjecture) for the case when the number of disjoint faces is a prime minus one, even its topological extension. This conjecture being proven implies new complexity bounds in computational geometry; see the introduction of [BMZ09] for three examples.

In this section we present two new proofs of Theorem 2.1. The first one uses an elementary degree argument. The second proof puts the first one into the language of cohomological index theory, as developed by Fadell and Husseini [FH88]. Even though the second proof looks more difficult it actually allows for a more direct path, since it avoids the non-topological reduction of Lemma 2.2. This requires more index calculations, which however are valuable since they provide a first key step towards our proof of the Main Theorem 1.3 in Section 5.

The configuration space/test map scheme

Suppose we are given a continuous map

$$f : \sigma_N \longrightarrow \mathbb{R}^d$$

and a coloring of the vertex set $\mathrm{vert}(\sigma_N) = [N+1] := \{1, \ldots, N+1\} = C_0 \uplus \cdots \uplus C_m$ such that $|C_i| \leq r-1$. We want to find a colored Tverberg partition $F_1, \ldots, F_r$.

As in [BMZ09] we construct a test-map F out of f. Let $f^{*r} : (\sigma_N)^{*r} \longrightarrow_{\mathbb{Z}_r} (\mathbb{R}^d)^{*r}$ be the r-fold join of f, which is equivariant with respect to the $\mathbb{Z}_r$-action that shifts the join constituents cyclically. Since we are interested in pairwise disjoint faces $F_1, \ldots, F_r$, we restrict the domain of f^{*r} to the r-fold 2-wise deleted join of σ_N, $(\sigma_N)^{*r}_{\Delta(2)} = [r]^{*(N+1)}$. (See [Mat03] for an introduction to these notions.) Since we are interested in colorful F_js, we restrict the domain further to the subcomplex

$$K := (C_0 * \ldots * C_m)^{*r}_{\Delta(2)} = [r]^{*|C_0|}_{\Delta(2)} * \cdots * [r]^{*|C_m|}_{\Delta(2)}.$$

The space $[n]^{*m}_{\Delta(2)}$ is known as the *chessboard complex* $\Delta_{n,m}$. Hence K can be written as

$$K = \Delta_{r,|C_0|} * \cdots * \Delta_{r,|C_m|}. \tag{1}$$

Thus by restricting the domain of f^{*r} to K we get a $\mathbb{Z}_r$-equivariant map

$$F'' : K \longrightarrow_{\mathbb{Z}_r} (\mathbb{R}^d)^{*r}.$$

Let $\mathbb{R}[\mathbb{Z}_r] \cong \mathbb{R}^r$ be the regular representation of $\mathbb{Z}_r$ and $W_r \subseteq \mathbb{R}^r$ the orthogonal complement of the all-one vector $\mathbb{1} = e_1 + \cdots + e_r$. We write W_r^{d+1} for $(W_r)^{\oplus(d+1)}$. The orthogonal projection $p : \mathbb{R}^r \longrightarrow_{\mathbb{Z}_r} W_r$ yields a $\mathbb{Z}_r$-equivariant map

$$\begin{array}{lll} (\mathbb{R}^d)^{*r} & \longrightarrow_{\mathbb{Z}_r} & W_r^{d+1} \\ \sum_{j=1}^r \lambda_j x_j & \longmapsto & (p(\lambda_1, \ldots, \lambda_r), p(\lambda_1 x_{1,1}, \ldots, \lambda_r x_{r,1}), \ldots, p(\lambda_1 x_{1,d}, \ldots, \lambda_r x_{r,d}). \end{array}$$

The composition of this map with F'' gives us the test-map F',

$$F' : K \longrightarrow_{\mathbb{Z}_r} W_r^{d+1}.$$

The pre-images $(F')^{-1}(0)$ of zero correspond exactly to the colored Tverberg partitions. Hence the image of F' contains 0 if and only if the map f admits a colored Tverberg partition. Suppose that 0 is not in the image, then we get a map

$$F : K \longrightarrow_{\mathbb{Z}_r} S(W_r^{d+1}) \tag{2}$$

into the representation sphere by composing F' with the radial projection map. We will derive contradictions to the existence of such an equivariant map.

The first proof of the non-existence establishes a key special case of Theorem 2.1, which implies the general result by the following reduction.

Lemma 2.2 ([BMZ09]). *It suffices to prove Theorem 2.1 for $m = d+1$ with $|C_0| = \cdots = |C_d| = r-1$ and $|C_{d+1}| = 1$.*

For the elementary proof of this lemma see [BMZ09, Reduction of Thm. 2.1 to Thm. 2.2]. This lemma is the special case $k = 0$ of Lemma 5.1, which we prove later.

Therefore it suffices to consider $K = K' * [r]$ where $K' = (\Delta_{r,r-1})^{*(d+1)}$. Let $M = F|_{K'} : K' \to S(W_r^{d+1})$ be the restriction of F to K'. The chessboard complex $\Delta_{r,r-1}$ for $r \geq 3$ is a connected orientable pseudo-manifold, hence K' is one as well. For $r = 2$, K' is the boundary of a $d+1$-dimensional cross-polytope, hence a d-sphere. The dimensions $\dim K' = N - 1 = \dim S(W_r^{d+1})$ coincide. Thus we can talk about the degree $\deg(M) \in \mathbb{Z}$. Here we are not interested in the actual sign, hence we do not need to fix orientations. Since K' is a free $\mathbb{Z}_r$-space and S^{N-1} is $(N-2)$-connected, the degree $\deg(M)$ is uniquely determined modulo r: This is because M is unique up to $\mathbb{Z}_r$-homotopy on the codimension one skeleton of K', and changing M on top-dimensional cells of K' has to be done $\mathbb{Z}_r$-equivariantly, hence it affects $\deg(M)$ by a multiple of r.

To determine $\deg(M) \mod r$, we let f be the affine map that takes the vertices in C_0 to $-\mathbb{1} = -(e_1 + \cdots + e_d)$ and the vertices in C_i $(1 \leq i \leq d)$ to e_i, where e_i is the ith standard basis vector of $\mathbb{R}^d$. The singleton C_{d+1} does not matter, we can choose it arbitrarily in $\mathbb{R}^d$. Let $P \in S(W_r^{d+1})$ be the normalization of the point $(p(1, \ldots, 1, 0), 0, \ldots, 0) \in W_r^{d+1}$. The pre-image $M^{-1}(P)$ is exactly the set of barycenters of the $(r-1)!^{d+1}$ top-dimensional faces of $K' \cap (\Delta_{r-1,r-1})^{*(d+1)}$. With $\Delta_{r-1,r-1}$ we mean the full subcomplex $[r-1]^{*(r-1)}_{\Delta(2)}$ of $\Delta_{r,r-1}$. One checks that all pre-images of P have the same pre-image orientation. This was essentially done in [BMZ09] when we calculated that $c_f(\Phi) = (r-1)!^d\zeta$. Hence

$$\deg(M) = \pm(r-1)!^{d+1} = \pm 1 \mod r. \tag{3}$$

Alternatively one can take any map $m : \Delta_{r,r-1} \longrightarrow_{\mathbb{Z}_r} S(W_r)$, show that its degree is ± 1 by a similar pre-image argument in dimension $d = 1$, and deduce that

$$\deg(M) = \deg(m^{*(d+1)}) = \deg(m)^{d+1} = \pm 1 \mod r.$$

First proof of Theorem 2.1. Since $\deg(M) \neq 0$, M is not null-homotopic. Thus M does not extend to a map with domain $K' * [1] \subseteq K$. Therefore the test-map F of (2) does not exist. □

Remark 2.3. The degree $\deg(M)$ is even uniquely determined modulo $r!$. To see this one uses the $\mathfrak{S}_r$-equivariance of M and the fact that M is given uniquely up to $\mathfrak{S}_r$-homotopy on the non-free part, which lies in the codimension one skeleton of K'. The latter can be shown with the modified test-map F_0 from [BMZ09]. This might possibly be an Ansatz for a proof of the affine version of Theorem 2.1 for non-primes r.

Matoušek, Wagner, and Tancer [MTW10] found a point configuration for the non-prime case $r = 4$ where the degree is 0. In their example however, the desired colored Tverberg partition does exist nevertheless.

Index computations

Let H^* denote Čech cohomology with $\mathbb{Z}_r$-coefficients, where r is prime. The equivariant cohomology of a G-space X is defined as

$$H^*_G(X) \ := \ H^*(EG \times_G X),$$

where EG is a contractible free G-CW complex and $EG \times_G X := (EG \times X)/G$. The classifying space of G is $BG := EG/G$. If $p : X \to B$ is furthermore a projection to a trivial G-space B, we denote the *cohomological index* of X over B, also called the *Fadell–Husseini index* [FH88], by

$$\mathrm{Ind}^B_G(X) \ := \ \ker\left(H^*_G(B) \xrightarrow{p^*} H^*_G(X)\right) \ \subseteq \ H^*_G(B) \cong H^*(BG) \otimes H^*(B).$$

If $B = \mathrm{pt}$ is a point then one also writes $H^*_G(\mathrm{pt}) = H^*(G)$ and $\mathrm{Ind}_G(X) := \mathrm{Ind}^{\mathrm{pt}}_G(X)$.

The cohomological index has the four properties

- Monotonicity: If there is a bundle map $X \longrightarrow_G Y$ then

$$\mathrm{Ind}^B_G(X) \supseteq \mathrm{Ind}^B_G(Y). \tag{4}$$

- Additivity: If $(X_1 \cup X_2, X_1, X_2)$ is excisive, then

$$\mathrm{Ind}^B_G(X_1) \cdot \mathrm{Ind}^B_G(X_2) \subseteq \mathrm{Ind}^B_G(X_1 \cup X_2).$$

- Joins:

$$\mathrm{Ind}^B_G(X) \cdot \mathrm{Ind}^B_G(Y) \subseteq \mathrm{Ind}^B_G(X * Y).$$

- Subbundles: If there is a is a bundle map $f : X \longrightarrow_G Y$ and a closed subbundle $Z \subseteq Y$ then

$$\mathrm{Ind}_G^B(f^{-1}(Z)) \cdot \mathrm{Ind}_G^B(Y) \subseteq \mathrm{Ind}_G^B(X). \tag{5}$$

The first two properties imply the other two. The last one uses furthermore the continuity of Čech cohomology H^*. For more information about this index theory see [FH87] and [FH88].

If r is odd then the cohomology of $\mathbb{Z}_r$ as a $\mathbb{Z}_r$-algebra is

$$H^*(\mathbb{Z}_r) = H^*(B\mathbb{Z}_r) \cong \mathbb{Z}_r[x, y]/(y^2),$$

where $\deg(x) = 2$ and $\deg(y) = 1$. If r is even, then $r = 2$ and $H^*(\mathbb{Z}_2) \cong \mathbb{Z}_2[t]$, $\deg t = 1$.

Theorem 2.4. *Let r be a prime. Let K be an n-dimensional connected free $\mathbb{Z}_r$-CW complex and let S be an n-dimensional $(n-1)$-connected free $\mathbb{Z}_r$-CW complex. If there is a $\mathbb{Z}_r$-map $M : K \longrightarrow_{\mathbb{Z}_r} S$ that induces an isomorphism on H^n, then*

$$\mathrm{Ind}_{\mathbb{Z}_r}^{\mathrm{pt}}(K) = H^{*\geq n+1}(B\mathbb{Z}_r).$$

Proof. The $\mathbb{Z}_r$-equivariant map $M : K \longrightarrow_{\mathbb{Z}_r} S$ induces a map of fibrations,

$$\begin{tikzcd}
K \arrow[r, "M"] \arrow[d] & S \arrow[d] \\
E\mathbb{Z}_r \times_{\mathbb{Z}_r} K \arrow[r, "(\mathrm{id}, M)"] \arrow[d] & E\mathbb{Z}_r \times_{\mathbb{Z}_r} K \arrow[d] \\
B\mathbb{Z}_r \arrow[r, "\mathrm{id}"] & B\mathbb{Z}_r.
\end{tikzcd}$$

Consequently, M induces a morphism $E_*^{*,*}(M)$ between associated Leray–Serre spectral sequences $E_*^{*,*}(K)$ and $E_*^{*,*}(S)$, see Figure 1. It has the property that $E_2^{*,0}(M) = id_{H^*(B\mathbb{Z}_r)}$. For background on Leray–Serre spectral sequences see [McC01, Chapters 5 and 6]. Moreover, the nth rows $E_2^{*,n}(K) = H^*(\mathbb{Z}_r; H^n(K))$ and $E_2^{*,n}(S) = H^*(\mathbb{Z}_r; H^n(S))$ at the E_2-pages are identified via $E_2^{*,n}(M)$.

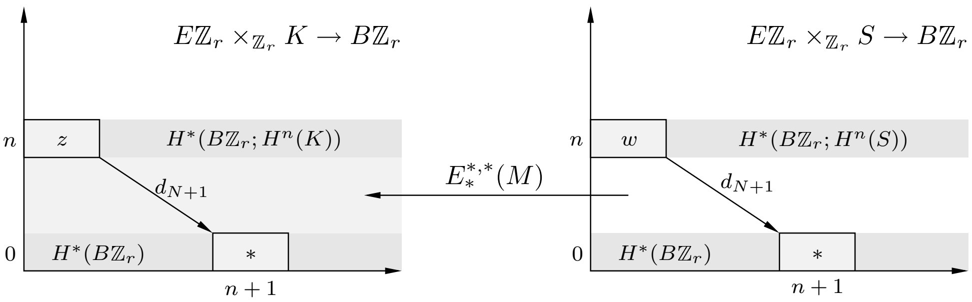

Figure 1: The morphism M^* between the spectral sequences $E_*^{*,*}(S)$ and $E_*^{*,*}(K)$.

At the E_∞-pages both spectral sequences have to satisfy $E_\infty^{p,q} = 0$ whenever the total degree $p + q \geq n+1$. This is because K is free $\mathbb{Z}_r$-space, hence $H^*_{\mathbb{Z}_r}(K) \cong H^*(K/\mathbb{Z}_r)$, which is zero in degrees $* \geq n+1$. The same holds for S. Therefore, the elements $E_*^{*\geq n+1,0}(S) = H^{*\geq n+1}(\mathbb{Z}_r)$ in the bottom row of the spectral sequence $E_*^{*,*}(S)$ must be hit by some differential. These differentials can come only from the nth row at the E_{n+1}-page (this argument even gives us the $H^*(\mathbb{Z}_r)$-module structure of the nth row). Hence there is a non-zero transgressive element $w \in E_2^{0,n}(S) = H^0(\mathbb{Z}_r; H^n(S)) = H^n(S)^{\mathbb{Z}_r}$, that is, $d_{n+1}(w) \neq 0$. Let $z := E_r^{0,n}(M)(w) \in E_2^{0,n}(K) = H^n(K)^{\mathbb{Z}_r}$. Then $d_i(z) = d_i(E_r^{0,n}(M)(w)) = E_r^{i,n-i+1}(M)(d_i(w))$, which is zero for $i \leq n$. Therefore z survives at least until E_{n+1}. Analogously, the whole nth row survives until E_{n+1}. We know that all elements in $E_{n+1}^{*\geq 1,n}(K)$ have to die eventually, so they do it exactly on page E_{n+1}. Thus these elements are exactly the elements whose differentials make the part $E_*^{*\geq n+2,0}$ of the bottom row vanish.

We claim that no non-zero differential can arrive at the bottom row on an earlier page of $E_*^{*,*}(K)$. Assume that $d_i(\alpha) = x^a y^b \in E_i^{*,0}$ for some α and $i \leq n$. This would imply that $d_i(x^k\alpha) = x^{a+k}y^b$ for all $k > 0$. But we already know that the elements in $E_*^{*\geq n+2,0}(K)$ survive until page E_{n+1}, which gives the desired contradiction.

Therefore at $E_\infty(K)$, the non-zero part of the bottom row is $H^{*\leq n}(\mathbb{Z}_r)$. The index defining map $H^*_{\mathbb{Z}_r}(\mathrm{pt}) \to H^*_{\mathbb{Z}_r}(K)$ is the edge homomorphism, which is the composition

$$H^*_{\mathbb{Z}_r}(\mathrm{pt}) \xrightarrow{\cong} E_2^{*,0}(K) \twoheadrightarrow E_\infty^{*,0} \hookrightarrow H^*_{\mathbb{Z}_r}(K).$$

Therefore the index of K is everything in the bottom row that got hit by a differential, that is,

$$\mathrm{Ind}^{\mathrm{pt}}_{\mathbb{Z}_r}(K) = H^{*\geq n+1}(B\mathbb{Z}_r).$$ □

We apply this theorem to the above maps $M : K' \to S(W_r^{d+1})$ and $(M * \mathrm{id}) : K' * [r] \to S(W_r^{d+1}) * [r]$.

Corollary 2.5. *The $\mathbb{Z}_r$-index of $K' = (\Delta_{r,r-1})^{*(d+1)}$ is*

$$\mathrm{Ind}^{\mathrm{pt}}_{\mathbb{Z}_r}(K') = H^{*\geq N}(B\mathbb{Z}_r)$$

and the $\mathbb{Z}_r$-index of $K' * [r]$ is

$$\mathrm{Ind}^{\mathrm{pt}}_{\mathbb{Z}_r}(K' * [r]) = H^{*\geq N+1}(B\mathbb{Z}_r).$$

Using the first part of this corollary we can compute the index for more general joins of chessboard complexes.

Corollary 2.6. *Let $0 \leq c_0, \dots, c_m \leq r-1$ and let $s := \sum c_i$. Let $K := \Delta_{r,c_0} * \dots * \Delta_{r,c_m}$. Then*

$$\mathrm{Ind}^{\mathrm{pt}}_{\mathbb{Z}_r}(K) = H^{*\geq s}(B\mathbb{Z}_r).$$

Proof. Let $L := \Delta_{r,r-1-c_0} * \dots * \Delta_{r,r-1-c_m}$ and $K' := (\Delta_{r,r-1})^{*(m+1)}$. Then $\dim K = s-1$ and $\dim K' = (r-1)(m+1)-1$. We calculate $\dim K' + 1 = (\dim K + 1) + (\dim L + 1)$. There is an inclusion $K' \longrightarrow_{\mathbb{Z}_r} K * L$. This implies

$$\mathrm{Ind}_{\mathbb{Z}_r}(K') \supseteq \mathrm{Ind}_{\mathbb{Z}_r}(K * L) \supseteq \mathrm{Ind}_{\mathbb{Z}_r}(K) \cdot \mathrm{Ind}_{\mathbb{Z}_r}(L). \tag{6}$$

Since K is a free $\mathbb{Z}_r$-space, $H^*_{\mathbb{Z}_r}(K) = H^*(K/\mathbb{Z}_r)$, hence

$$\mathrm{Ind}_{\mathbb{Z}_r}(K) \supseteq H^{*\geq \dim K+1}(B\mathbb{Z}_r), \tag{7}$$

and analogously

$$\mathrm{Ind}_{\mathbb{Z}_r}(L) \supseteq H^{*\geq \dim L+1}(B\mathbb{Z}_r). \tag{8}$$

The dimension $a := \dim K'$ is odd if r is odd. Using Corollary 2.5, we find that $\mathrm{Ind}_{\mathbb{Z}_r}(K') = H^{\geq a+1}(B\mathbb{Z}_r) = \langle x^{\frac{a+1}{2}} \rangle$ if r is odd, and $\mathrm{Ind}_{\mathbb{Z}_r}(K') = \langle t^{a+1} \rangle$ if $r = 2$. Together with equation (6), the inclusions (7) and (8) have to hold with equality. □

It is interesting that the last argument of the proof would fail for odd r if $a+1$ was odd, due to the relation $y^2 = 0$ in $H^*(\mathbb{Z}_r)$.

Now we plug in the configuration space K from (1) and obtain the second proof of Theorem 2.1.

Second proof of Theorem 2.1. According to the monotonicity of the index, see (4), the existence of the test-map $F : K \longrightarrow_{\mathbb{Z}_r} S(W_r^{d+1})$ of (2) would imply that

$$\mathrm{Ind}^{\mathrm{pt}}_{\mathbb{Z}_r}(K) \supseteq \mathrm{Ind}^{\mathrm{pt}}_{\mathbb{Z}_r}(S(W_r^{d+1})).$$

This is a contradiction since $\mathrm{Ind}^{\mathrm{pt}}_{\mathbb{Z}_r}(K) = H^{*\geq N+1}(B\mathbb{Z}_r)$ and $\mathrm{Ind}^{\mathrm{pt}}_{\mathbb{Z}_r}(S(W_r^{d+1})) = H^{*\geq N}(B\mathbb{Z}_r)$, as $S(W_r^{d+1})$ is an $(N-1)$-dimensional free $\mathbb{Z}_r$-sphere. □

On the surface this proof seems to be a more difficult reformulation of the first proof. However, its view point is essential for the transversal generalization, since we do not rely on the geometric tools of the Reduction Lemma 2.2 anymore, and such a reduction lemma does not seem to exist for the Tverberg–Vrećica type transversal theorem. Thus we need to use the more general configuration space $\Delta_{r,|C_0|} * \dots * \Delta_{r,|C_m|}$ of (1) instead of $(\Delta_{r,r-1})^{*(d+1)} * [r]$.

3 The configuration space/test map scheme

The proof of our Main Theorem 1.3 is based on a configuration space/test map scheme for vector bundles. Such a proof scheme was already used in [Dol87], [Dol92], [Živ99], [Vre08] and [Kar07]. Our progress in this paper stems from the topological index calculations of Section 2 and from the Borsuk–Ulam type Theorem 4.1 in Section 4.

The proof gives actually the following more general topological version. The Main Theorem is the special case when all maps f_ℓ are affine.

Theorem 3.1 (Topological Main Theorem). *Let r be prime and $0 \le k \le d$ such that $r(d-k)$ is even or $k = 0$. Let C^ℓ $(\ell = 0, \dots, k)$ be sets of cardinality $|C^\ell| = (r-1)(d-k+1)+1$, which we identify with the vertex sets of simplices $\sigma_{|C^\ell|-1}$. We color them*

$$C^\ell \;=\; \biguplus C_i^\ell,$$

such that no color class is too large, $|C_i^\ell| \le r-1$. Let

$$f_\ell : \sigma_{|C^\ell|-1} \to \mathbb{R}^d$$

be continuous maps. Then we can find r disjoint rainbow faces $F_1^\ell, \dots, F_r^\ell$ in each simplex $\sigma_{|C^\ell|-1}$ (that is, $|F_j^\ell \cap C_i^\ell| \le 1$) and a k-plane $P \subseteq \mathbb{R}^d$ that intersects all the sets $f_\ell(F_j^\ell)$.

The proof scheme for our situation works as follows. Suppose we are given C^ℓs, f_ℓs and r as in the assertion of Theorem 3.1 together with the colorings

$$C^\ell \;=\; \biguplus_{i=0}^{m_\ell} C_i^\ell.$$

A collection of rainbow faces F_j^ℓ of the simplices $\sigma_{|C^\ell|-1}$ admits a common k-plane P that intersects all images $f(F_j^\ell)$ if and only if one can project these images orthogonally to a $(d-k)$-dimensional subspace of $\mathbb{R}^d$ (namely the orthogonal complement of P) such that the convex hulls of the projected F_j^ℓs have a point in common (this point is the image of P under the projection).

Calculations turn out to be easier if we look first at the set of colored Tverberg points of all projections of one single fixed C^ℓ and then show that the corresponding sets for all C^ℓs have to intersect.

Fix an $\ell \in \{0, 1, \dots, k\}$. Let $B := G_{d,d-k}$ be the Grassmannian manifold of all $(d-k)$-dimensional subspaces of $\mathbb{R}^d$ and $\gamma \to B$ the tautological bundle over B. For definitions and context, see Chapter 5 of [MS74]. Let ε denote the trivial line bundle over B. Let $B \times W_r$ be the trivial bundle over B with fiber W_r, which was defined in Section 2. Let $E := (B \times W_r) \oplus \gamma^{\oplus r}$. The group $G := \mathbb{Z}_r$ acts on $[r]$ by left translations and on E by fiberwise shifting the coordinates cyclically. E is a G-bundle over the trivial G-space B whose fixed-point subbundle $\Delta := E^G = (B \times W_r)^G \oplus (\gamma^{\oplus r})^G \cong \gamma$ is the thin diagonal bundle.

The space

$$K \;:=\; \Delta_{r,|C_0^\ell|} * \dots * \Delta_{r,|C_{m_\ell}^\ell|} = (C_0^\ell * \ldots * C_{m_\ell}^\ell)^{*r}_{\Delta(2)} \subseteq (\sigma_{|C^\ell|-1})^{*r} \tag{9}$$

will again be the configuration space. For each $b \in B$, we can compose the map f_ℓ with the orthogonal projection to the $(d-k)$-space given by b, which can be identified with the fiber over b in γ. This is gives function

$$B \times \sigma_{|C^\ell|-1} \to \gamma,$$

which is bundle map over B, $B \times \sigma_{|C^\ell|-1}$ being the trivial bundle over B. Doing the analogous construction as in Section 2, we get a $\mathbb{Z}_r$-equivariant bundle map

$$B \times K \xrightarrow{M} E,$$

where the r join coefficients in $K = (C_0^\ell * \ldots * C_{m_\ell}^\ell)^{*r}_{\Delta(2)}$ are mapped into the r trivial summands ε of E. Define $T^\ell := \mathrm{im}(M) \cap \Delta$, which is the set of colored Tverberg points of the respective projected sets

$\mathrm{im}(F_\ell)$. Each point of $T^\ell \subseteq \Delta$, which lies in the fiber over say $b \in B$, lies in the intersection of the images $\mathrm{im}(F_j^\ell)$ of r disjoint rainbow faces projected to b.

Hence we need to show that

$$T^0 \cap \cdots \cap T^k \neq \emptyset. \tag{10}$$

We will apply our results on the index of the configuration space K, derived in Corollary 2.6, and tools from Section 4 to show that this is indeed the case. The proof of Theorem 3.1 continues in Section 5.

4 A new Borsuk–Ulam type theorem

In this section we prove the following Borsuk–Ulam type theorem. It is the second topological main step towards the proof of Theorem 1.3. This theorem will be applied in combination with the subsequent intersection Lemma 4.3.

Theorem 4.1 (Borsuk–Ulam type)**.** *Let*

- *p be a prime,*
- *$G = (\mathbb{Z}_p)^m$ an elementary abelian group,*
- *K a G-CW-complex with index $\mathrm{Ind}_G^{\mathrm{pt}}(K) \subseteq H^{*\geq n+1}(BG;\mathbb{Z}_p)$,*
- *B a connected, trivial G-space,*
- *$E \xrightarrow{\phi} B$ a G-vector bundle (all fibers carry the same G-representation),*
- *$\Delta := E^G \to B$ the fixed-point subbundle of $E \to B$,*
- *$C \to B$ its G-invariant orthogonal complement subbundle ($E = C \oplus \Delta$),*
- *F be the fiber of the sphere bundle $S(C) \to B$.*

Suppose that

- *$n = \mathrm{rank}(C)$,*
- *$\pi_1(B)$ acts trivially on $H^*(F;\mathbb{Z}_p)$ (that is, $C \to B$ is orientable if $p \neq 2$), and*
- *we are given a G-bundle map M,*

$$\begin{tikzcd} B \times K \arrow[rr, "M"] \arrow[dr, "pr_1"'] & & E = C \oplus \Delta \arrow[dl, "\phi"] \\ & B & \end{tikzcd}.$$

Then for $S := M^{-1}(\Delta)$ and $T := M(S) = \mathrm{im}(M) \cap \Delta$ the maps induced by projection

$$H^*(B;\mathbb{Z}_p) \xrightarrow{(pr_1|_S)^*} H^*_G(S;\mathbb{Z}_p) \qquad \text{and} \qquad H^*(B;\mathbb{Z}_p) \xrightarrow{(\phi|_T)^*} H^*(T;\mathbb{Z}_p)$$

are injective.

Remark 4.2. The theorem generalizes

- a lemma of Volovikov [Vol96], which is the special case when $B = \mathrm{pt}$ and K is $(n-1)$-$\mathbb{Z}_p$-acyclic,
- and Theorem 4.2 of Živaljević [Živ99], from whose proof one can extract the special case when $m = 1$ and K is $(n-1)$-$\mathbb{Z}_p$-acyclic. and in particular
- the Borsuk–Ulam theorem, which is the special case when $G = \mathbb{Z}_2$, $B = \mathrm{pt}$, $K = S^n$, $E = \mathbb{R}^n$, E and K with antipodal action.

Proof of Theorem 4.1. We use Čech cohomology with $\mathbb{Z}_p$-coefficients.

(1.) Let $b \in B$ be the point over which F is the fiber in the sphere bundle $S(C) \to B$. We denote by $E_*^{*,*}(F)$ and $E_*^{*,*}(S(C))$ the Leray–Serre spectral sequences associated to the fibrations

$$F \hookrightarrow EG \times_G F \to BG \times b \tag{11}$$

and

$$F \hookrightarrow EG \times_G S(C) \to BG \times B, \tag{12}$$

respectively, see Figure 2. For details on the Leray–Serre spectral sequence, see [McC01, Chapters 5 and 6].

The E_2-page $E_2^{*,*}(F)$ has only two non-zero rows, the 0-row and the $(n-1)$-row. The local coefficients in $E_2^{p,q}(F) = H^p(BG, H^q(F))$ are given by the $\pi_1(BG)$-module structure on $H^q(F)$. Since $G = \pi_1(BG)$ is an elementary abelian group and F is a sphere, the $H^*(BG)$-module structure on $H^q(F)$ is trivial, for the G action on $H^q(F)$ is induced by homeomorphisms $F \to F$, and the degree of this homeomorphism has to be 1 if p is odd. Therefore

$$E_2^{p,q}(F) = H^p(BG, H^q(F)) = H^p(BG) \otimes H^q(F).$$

The differentials are $H^*(BG)$-homomorphism, and

$$E_n^{*,n-1}(F) = E_2^{*,n-1}(F) = H^*(BG) \otimes H^{n-1}(F)$$

is a $H^*(BG)$-module generated by

$$1 \in E_n^{0,n-1}(F) = H^0(BG) \otimes H^{n-1}(F),$$

where 1 is regarded as the generator of $H^{n-1}(F)$. Hence there is a non-vanishing differential in $E_*^{*,*}(F)$ if and only if the differential $d_n : E^{0,n-1}(F) \to E^{n,0}(F)$ is non-zero. Since F is fixed-point free, the edge homomorphism $H^*(BG) \to H_G^*(F)$ is not injective [Die87, Prop. 3.14, p. 196]. Thus there must be a non-vanishing differential. Therefore there is a non-zero element $\alpha = d_n(1) \in \mathrm{Ind}_G^{\mathrm{pt}}(F)$ of degree n.

(2.) Now the inclusion $F \hookrightarrow S(C)$ gives a bundle map from (11) to (12),

$$\begin{array}{ccc} EG \times_G F & \longrightarrow & EG \times_G S(C) \\ \downarrow & & \downarrow \\ BG \times b & \longrightarrow & BG \times B \end{array} \tag{13}$$

which induces a morphism of associated Leray–Serre spectral sequences $E_*^{*,*}(S(C)) \to E_*^{*,*}(F)$, see Figure 2.

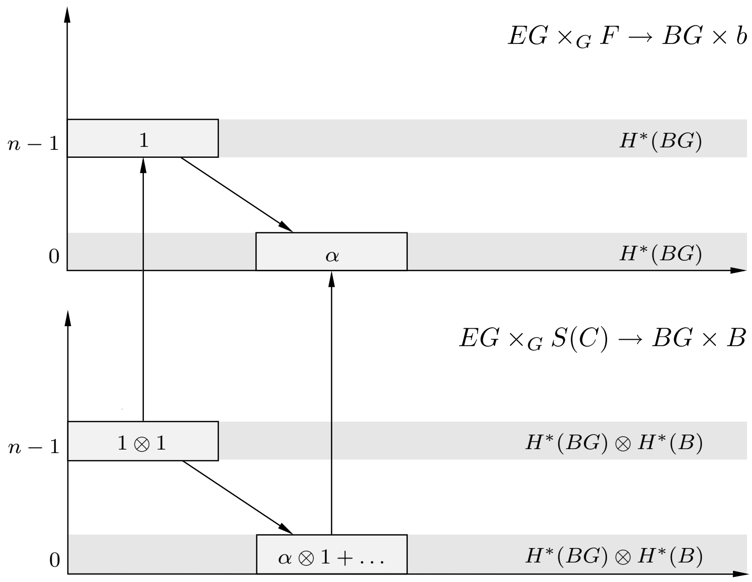

Figure 2: The morphism of spectral sequences induced by the bundle map (13).

The E_2-page of $E_*^{*,*}(S(C))$ is $E_2^{p,q}(S(C)) = H^p(BG \times B, H^q(F))$, where the local coefficients are given by the $\pi_1(BG \times B)$-module structure on $H^q(F)$. Since $H^*(F)$ is a trivial $G \times \pi_1(B)$-module, the 0- and $(n-1)$-rows of this spectral sequence are given by

$$E_2^{p,q}(S) = H^p(BG \times B; H^q(F)) = H^p(BG \times B) = \bigoplus_{i=0}^{p} H^i(BG) \otimes H^{p-i}(B), \text{ for } q \in \{0, n-1\}.$$

The morphism of the spectral sequences $E_*^{*,*}(S(C)) \to E_*^{*,*}(F)$ on the 0-row and on the $(n-1)$-row of the E_2-page,

$$\bigoplus_{i=0}^{p} H^i(BG) \otimes H^{p-i}(B) \to H^p(BG),$$

is zero on $\bigoplus_{i=1}^{p} H^i(BG) \otimes H^{p-i}(B)$. On $H^p(BG) \otimes H^0(B) = H^p(BG)$ it is just the identity. The differential of the generator $1 \otimes 1 \in H^0(BG) \otimes H^0(B)$ of $E_n^{0,n-1}(S(C))$ hits an element $\gamma \in E_n^{n-1,0}(S(C))$ of the bottom row $\bigoplus_{i=0}^{n} H^i(BG) \otimes H^{n-i}(B)$. Since the differentials commute with morphisms of spectral sequences, γ is an element in $\mathrm{Ind}_G^B(S(C)) \subseteq H^*(BG) \otimes H^*(B)$ that restricts to α under the map $\bigoplus_{i=0}^{n} H^i(BG) \otimes H^{n-i}(B) \to H^n(BG)$, hence $\gamma \neq 0$. Since α and γ are of degree n, γ has the form

$$\gamma = \alpha \otimes 1 + \sum_i \delta_i \otimes \varepsilon_i, \tag{14}$$

for some δ_i and ε_i with $\deg \delta_i + \deg \varepsilon_i = n$ and $\deg \delta_i \leq n-1$.

(3.) Formula (5) of Section 2 yields

$$\mathrm{Ind}_G^B(S) \cdot \mathrm{Ind}_G^B(S(C)) \subseteq \mathrm{Ind}_G^B(B \times K) = \mathrm{Ind}_G^{\mathrm{pt}}(K) \otimes H^*(B).$$

We know that $\mathrm{Ind}_G^{\mathrm{pt}}(K) \subseteq H^{*\geq n+1}(BG)$, and in (14) we got an element $\gamma \in \mathrm{Ind}_G^B(S(C))$. We claim that $\mathrm{Ind}_G^B(S) \subseteq H^*(BG) \otimes H^*(B)$ does not contain any non-zero element of the form $1 \otimes \beta$, $\beta \in H^*(B)$. Indeed, if $1 \otimes \beta \in \mathrm{Ind}_G^B(S) \backslash \{0\}$, then

$$(1 \otimes \beta) \cdot \gamma = \alpha \otimes \beta + \sum_i \delta_i \otimes (\beta \cdot \varepsilon_i) \in \mathrm{Ind}_G^{\mathrm{pt}}(K) \otimes H^*(B).$$

Since $\deg(\delta_i) < \deg(\alpha) = n$, this implies that $\alpha \in \mathrm{Ind}_G^{\mathrm{pt}}(K)$. This contradicts $\mathrm{Ind}_G^{\mathrm{pt}}(K) \subseteq H^{*\geq n+1}(BG)$.

Hence the following composition is injective,

$$H^*(B) \xrightarrow{1\otimes \mathrm{id}} H^*(BG) \otimes H^*(B) \to H_G^*(S),$$

where both maps are induced by projection. The following diagram is induced by the obvious maps

$$\begin{tikzcd}
H_G^*(S) & H_G^*(T) = H^*(BG) \otimes H^*(T) \arrow[l] & H^*(T) \arrow[l] \\
& H^*(B) \arrow[ul, "\text{inj.}"] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

which shows that $H^*(B) \to H^*(T)$ has to be injective as well. $\square$

We will use Theorem 4.1 together with the following intersection lemma.

Lemma 4.3 (Intersection lemma). *Let p be prime and $\Delta \xrightarrow{pr} B$ be a vector bundle over an $\mathbb{Z}_p$-orientable compact manifold B, whose mod-p Euler class $e := e(\Delta) \in H^*(B;\mathbb{Z}_p)$ satisfies $e^k \neq 0$. Let $T_0, \ldots, T_k \subseteq \Delta$ be closed sets such that $H^{\dim B}(B;\mathbb{Z}_p) \xrightarrow{(pr|_{T_i})^*} H^{\dim B}(T_i;\mathbb{Z}_p)$ is injective for all i. Then*

$$T_0 \cap \cdots \cap T_k \neq \emptyset.$$

A proof for the case where p is prime can be extracted from [Živ99]. For the convenience of the reader and at the request of the referee we repeat the argument.

Proof of Lemma 4.3. In the case $k = 0$ we need to show that $T_0 \neq \emptyset$. This is true since $(pr|_{T_0})^*$ is injective. So let us assume that $k \geq 1$.

Let $D(\Delta)$ and $S(\Delta)$ be the disc and sphere bundles of Δ. As a bundle, Δ is $\mathbb{Z}_p$-orientable since $e \neq 0$, and B is a $\mathbb{Z}_p$-orientable manifold. Hence $D(\Delta)$ is a $\mathbb{Z}_p$-orientable manifold as well. We may assume that the T_is lie in the interior of $D(\Delta)$ by rescaling the fibers. Let $b := \dim(B)$ and $r := \mathrm{rank}(\Delta)$. Let

$\tau \in H^r(D(\Delta), S(\Delta))$ be the Thom class. We have $e = i^*(\tau)$, where $i : (B, \emptyset) \to (D(\Delta), S(\Delta))$ is the inclusion. Hence $\tau^{k+1} = e^k\tau \neq 0$, since $e^k \neq 0$ and multiplication by τ is the Thom isomorphism.

Suppose that $T_0 \cap \cdots \cap T_k = \emptyset$. Then there are open neighborhoods V_i of T_i such that $V_0 \cap \cdots \cap V_k = \emptyset$. We will derive a contradiction to that. Since the Čech-cohomology of T_i is the limit of the ordinary cohomology over open neighborhoods in E, we can make the neighborhoods V_i smaller such that the maps $H^b(B; \mathbb{Z}_p) \xrightarrow{(pr|_{V_i})^*} H^b(V_i; \mathbb{Z}_p)$ are injective for all i. Therefore the universal coefficient theorem for cohomology implies that $H_b(V_i; \mathbb{Z}_p) \to H_b(B; \mathbb{Z}_p)$ is surjective, because we use field coefficients.

In the following commutative diagram, the vertical arrows are Poincaré–Alexander–Lefschetz duality and the bottom map j_i is induced by inclusion.

$$\begin{tikzcd}
H_b(V_i) \arrow[r] \arrow[d, "\cong"'] & H_b(B) \arrow[r, "\cong"] & H_b(D(\Delta)) \arrow[d, "\cong"] \\
H^r(D(\Delta), D(\Delta)\backslash V_i) \arrow[rr, "j_i"] & & H^r(D(\Delta), S(\Delta))
\end{tikzcd}$$

Since τ is the Poincaré dual of the orientation class $[B] \in H^b(B) = H^b(D(\Delta))$ and the top map is surjective, we find an element $\alpha_i \in H^r(D(\Delta), D(\Delta)\backslash V_i)$ such that $j_i(\alpha_i) = \tau$. Hence

$$\tau^{k+1} = j_0(\alpha_0) \cdot \ldots \cdot j_k(\alpha_k) = j(\alpha_0 \cdot \ldots \cdot \alpha_k),$$

where j is the map induced by inclusion,

$$j : H^{r(k+1)}(D(\Delta), D(\Delta)\backslash(V_0 \cap \ldots \cap V_k)) \to H^{r(k+1)}(D(\Delta), S(\Delta)).$$

Since the image of the map j contains $\tau^{k+1} \neq 0$, we find that $V_0 \cap \ldots \cap V_k \neq 0$. $\square$

Remark 4.4. The lemma can be extended to all positive integers p. We need to change the argument only at the point where we need $H_b(V_i; \mathbb{Z}_p) \to H_b(V_i; \mathbb{Z}_p)$ to be surjective. This can be proven for $\mathbb{Z}_p$-coefficients by a (non-standard) universal coefficient theorem that computes homology from cohomology, using $\mathbb{Z}_p$ as the base ring and the fact that $\mathbb{Z}_p$ is an injective $\mathbb{Z}_p$-module. For details see [Mat11].

5 Proof of the colored Tverberg–Vrećica type theorem

Now we have all the topological tools to prove the Main Theorem. We continue from where we stopped at in Section 3. We need to prove (10), that is,

$$T_0 \cap \ldots \cap T_k \neq \emptyset.$$

Continued proof of Theorem 3.1. First we assume that $p = 2$ or that d and k are odd. The remaining case, when p is odd and d and k are even, will be a consequence of an elementary reduction lemma at the end of the proof.

The configuration space K is of dimension $(r-1)(d-k)$. The ranks of E and Δ are $r(d-k+1)-1$ and $d-k$. We claim that the orthogonal complement bundle C of Δ in E is $\mathbb{Z}_r$-orientable. Since all vector bundles are $\mathbb{Z}_2$-orientable, we only need to deal with the case where r is odd. Then $r-1$ is even and C is stably isomorphic to γ^{r-1}, which is an even power of a bundle, hence orientable. Therefore we can apply the Borsuk–Ulam type Theorem 4.1 and get that $H^*(B) \to H^*(T_i)$ is injective. To apply the Intersection Lemma 4.3, we need that $e^k \neq 0$ for the mod-r Euler class $e \in H^{d-k}(B)$ of $\Delta \cong \gamma$.

If $r = 2$ then e is the top Stiefel–Whitney class w_{d-k}, whose k-th power is the mod-2 fundamental class of B (see e.g. [Hil80, Lemma 1.2]), which proves the theorem in this case. Now we come to the case where r is odd. If $\operatorname{rank}(\gamma) = d-k$ is odd then the mod-r Euler class is zero and our method yields no conclusion. If $d-k$ is even then we may assume that d and k are odd, otherwise we prove the theorem for $d' = d+1$ and $k' = k+1$ first and use the reduction lemma 5.1 below afterwards. Then $e^k \neq 0$ was proved in Proposition 4.9 of [Živ99], based on [FH88]. In fact, he even shows it for the tautological bundle over the *oriented* Grassmannian. Since this bundle is the pullback of γ we are done by naturality of the Euler class. Now the Intersection Lemma 4.3 gives that $T_0 \cap \cdots \cap T_k \neq \emptyset$. Hence by the preliminary work of Section 3 we are done.

Finally we prove the elementary Reduction Lemma 5.1 that also implies the case when p is odd and d and k are even.

Lemma 5.1 (Reduction Lemma). *If Conjecture 1.2 holds for parameters $(d, k, r_0, \ldots, r_k)$ then so it does for $(d', k', r_0, \ldots, r_{k-1})$ with $d' := d-1$ and $k' := \max(k-1, 0)$.*

Proof. We will prove only the case $k \geq 1$, since the case $k = 0$ is exactly the reduction that is used in the proof of Lemma 2.2 [BMZ09].

Assume that Conjecture 1.2 is true for parameters $(d, k, r_0, \ldots, r_k)$ and suppose we are given colored sets $C^0, \ldots, C^{k-1} \subseteq \mathbb{R}^{d-1}$ where we have to partition C^ℓ into r_ℓ pieces such that some $(k-1)$-dimensional plane meets the convex hulls of all pieces. To do this, view $\mathbb{R}^{d-1}$ as the hyperplane in $\mathbb{R}^d$ where the last coordinate is zero, and define $C^k \subset \mathbb{R}^d$ to be a set of $(r_k - 1)(d - k + 1) + 1$ points all of which lie close to $(0, \ldots, 0, 1)$. We color C^k in an arbitrary way. For example, we may give each point a different color. Since Conjecture 1.2 is true for $(d, k, r_0, \ldots, r_k)$, we can partition the sets C^ℓ appropriately and find a k-plane P meeting all of the convex hulls of the pieces. Since P goes through the convex hull of C^k, it cannot be fully contained in $\mathbb{R}^{d-1}$. Therefore $P \cap \mathbb{R}^{d-1}$ is a $(k-1)$-plane intersecting the convex hulls of the pieces of the sets $C^0, \ldots, C^{k-1}$. □

This finishes the proof of the Main Theorem. □

Our proof of the Main Theorem does not extend to prime powers $r_\ell = p^{a_\ell}$ over the same prime p. The basic reason is that the degree of M vanishes modulo r if and only if r divides $(r-1)!^d$ (see (3)). Therefore this proof can only work if r is a prime or if $r = 4$ and $d = 1$. For $k = 0$, even using the full symmetry group $\mathfrak{S}_r$ does not help since an $\mathfrak{S}_r$-equivariant test-map exists if and only if r divides $(r-1)!^d$; see [BMZ09]. To see this one needs to take a closer look at the obstruction; the degree proof from Section 2 does not yield this.

A new proof for Karasev's result [Kar07]

We can extend the above proof to arbitrary powers of a fixed prime p if all color classes are singletons, or in other words, if we omit all the color constraints. In this case, the configuration space K of (9) becomes

$$K = [r_\ell]^{*(N+1)} = (\sigma_N)^{*r_\ell}_{\Delta(2)},$$

which is the r_ℓ-wise 2-fold deleted join of an N-simplex. It follows from the connectivity relation $\operatorname{conn}(A * B) \geq \operatorname{conn}(A) + \operatorname{conn}(B) + 2$ for CW-complexes that K is $(N-1)$-connected. As symmetry group we take instead of $\mathbb{Z}_r$ a subgroup $G \cong (\mathbb{Z}_p)^{m_\ell}$ of S_{r_ℓ} that acts fixed-point freely on $[r_\ell]$. By the connectivity of K, $\operatorname{Ind}^{\mathrm{pt}}_G(K) \subseteq H^{*\geq N+1}(BG)$, as we can directly deduce from the Leray–Serre spectral sequence of $K \hookrightarrow EG \times_G K \to BG$. We obtain Karasev's result.

Theorem 5.2 ([Kar07]). *The Tverberg–Vrećica Conjecture 1.1 holds in the special case $r_\ell = p^{a_\ell}$, where p is a prime such that $p(d-k)$ is even or $k = 0$.*

Tightness of the Main Theorem 1.3

Observation 5.3. For any $0 \leq k \leq d-1$, $0 \leq \ell^* \leq k$, $r_{\ell^*} \geq 2$, we can choose point sets $C^\ell \subset \mathbb{R}^d$ of size $|C^\ell| = (r_\ell - 1)(d - k + 1) + 1$ and make all the color classes singletons except for one single color class $C_0^{\ell^*}$ of size r_{ℓ^*} such that there are no colorful partitions of the C^ℓs into r_ℓ pieces each that admit a common k-dimensional transversal.

Proof. Let V^ℓ, $0 \leq \ell \leq k$, be pairwise parallel $(d-k)$-dimensional affine subspaces of $\mathbb{R}^d$ such that their projections to an orthogonal k-space are the $k+1$ vertices of a k-simplex. On each V^ℓ we place a standard point configuration C^ℓ: Take a $(d-k)$-simplex σ^ℓ in V^ℓ, let C^ℓ have $r_\ell - 1$ points on each vertex of σ^ℓ and put the last vertex of C^ℓ into the center c^ℓ of σ^ℓ.

The sets C^ℓ admit only one Tverberg point, namely c^ℓ. Hence a potential common k-dimensional transversal P must intersect all c^ℓ. Since the V^ℓ have been chosen generically enough, P is uniquely determined and $P \cap V^\ell = \{c^\ell\}$.

Now we color the points of an arbitrary C^{ℓ^*} at an arbitrary vertex of σ^{ℓ^*} red, together with a further point at another vertex. Even if all other color classes in C^{ℓ^*} are singletons there will be no colored Tverberg partition of C^{ℓ^*}. Together with $P \cap V^{\ell^*} = \{c^{\ell^*}\}$ this proves the observation. □

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