A parametrized version of the Borsuk–Ulam–Bourgin–Yang–Volovikov theorem

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Abstract

We present a parametrized version of Volovikov's powerful Borsuk–Ulam–Bourgin–Yang type theorem, based on a new Fadell-Husseini type ideal-valued index of G-bundles which makes computations easy.

As an application we provide a parametrized version of the following waist of the sphere theorem due to Gromov, Memarian, and Karasev–Volovikov: Any map f from an n-sphere to a k-manifold ($n \geq k$) has a preimage $f^{-1}(z)$ whose epsilon-neighborhoods are at least as large as the epsilon-neighborhoods of the equator S^{n-k} (if $n = k$ we further need that f has even degree).

1 Introduction

Volovikiv's theorem. Volovikov presented in [Vol92, Theorem 1] a strong Bourgin–Yang type theorem on point coincidences that also generalizes the Borsuk–Ulam theorem. The notation, in particular the index $\mathrm{ind}_G^{FH}(X)$ of a G-space, will be explained in section 3.

Theorem 1.1 (Volovikov)**.** *Let $q = p^k$ be a prime power, $G = \mathbb{Z}_p^k$ the corresponding elementary Abelian group, and let X be a free G-space of index $\mathrm{ind}_G^{FH}(X) \subseteq H^{*\geq m(p^k-1)+N}(G)$ with $N \geq 1$. Let M be a compact m-manifold that is orientable if $p > 2$. Suppose the $f^* : H^*(M) \to H^*(X)$ is zero for $i \geq 1$.*
Then the set

$$S := \{x \in X \mid |f(G \cdot x)| = 1\}$$

is non-empty and has index $\mathrm{ind}_G^{FH} S \subseteq H^{*\geq N}(G)$.

For $k = 1$, this theorem was already obtained in Volovikov [Vol80] and [Vol83]. Karasev and Volovikov [KV11] generalized theorem 1.1 further to non-orientable manifolds and to arbitrary groups $\mathbb{Z}_p^k \subseteq G \subseteq \mathrm{Syl}_p(S_q)$.

The main methodological tool for this paper is a parametrized version of Volovikov's theorem, which we state in Section 4.

Many other parametrized versions of the Borsuk–Ulam theorem are known. We refer to Jaworowski [Jaw81a], [Jaw81b] and [Jaw04], Dold [Dol88], Nakaoka [Nak84] and [Nak89], Fadell–Husseini [FH87a], [FH88] and [FH89], Živaljević–Vrećica [ŽV90], Izydorek–Jaworowski [IJ92],

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Izydorek–Rybicki [IR92], Mramor-Kosta [MK95], Volovikov [Vol96], Koikara–Mukerjee [KM96], Živaljević [Živ99], de Mattos–dos Santos [dMdS07], Crabb–Jaworowski [CJ09], Schick–Simon–Spiecż–Toruńczyk [SSST11], Blagojević–M.–Ziegler [BMZ11] and [Mat11], and Singh [Sin11].

Main application. Gromov proved in [Gro03] the following version of the Borsuk–Ulam theorem.

Theorem 1.2 (Gromov's waist of the sphere theorem)**.** *Let $f : S^n \to \mathbb{R}^k$ be a continuous map where $n \geq k \geq 0$.*
Then there exists a point $z \in \mathbb{R}^k$ such that for any $\varepsilon > 0$,

$$\mathrm{vol}_n\Big(U_\varepsilon\big(f^{-1}(z)\big)\Big) \geq \mathrm{vol}_n\Big(U_\varepsilon\big(S^{n-k}\big)\Big).$$

Here, vol_n *denotes the standard measure on S^n, $U_\varepsilon(X)$ denotes the ε-neighborhood of a set $X \subseteq S^n$ with respect to the standard metric on S^n, and S^{n-k} is the $(n-k)$-dimensional equator of S^n.*

Memarian [Mem09] gave a more detailed proof of Gromov's theorem. Karasev and Volovikov [KV11] generalized it to maps $f : S^n \to M$ of even degree from the n-sphere to arbitrary k-manifolds M, see figure 1.

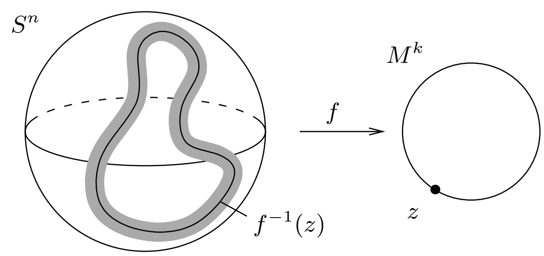

Figure 1: Example of the Gromov–Memarian–Karasev–Volovikov theorem for $n = 2$ and $M = S^1$. In this example, $f^{-1}(z)$ is not a large preimage.

The main application of this paper is a parametrized version of this Gromov–Memarian–Karasev–Volovikov theorem.

Theorem 1.3 (Parametrized Gromov–Memarian–Karasev–Volovikov waist of the sphere theorem)**.** *Let $f : B \times S^n \to E$ be a bundle map over B, where $S^n \hookrightarrow B \times S^n \to B$ is the trivial S^n bundle over B and $M \hookrightarrow E \xrightarrow{p} B$ is a fiber bundle over B whose fiber is a k-manifold M. If $n = k$ then we further assume that the fiber maps $f_b : S^n \to M$ have even degree at every base point $b \in B$.*
Then there exist a subset $Z \subseteq E$ such that for all $z \in Z$ and $\varepsilon > 0$,

$$\mathrm{vol}_n\Big(U_\varepsilon\big(f^{-1}(z)\big)\Big) \geq \mathrm{vol}_n\Big(U_\varepsilon\big(S^{n-k}\big)\Big), \tag{1}$$

and such that Z is the set of limit points of a sequence of subsets $Z_i \subseteq E$ with

$$(p_E|_Z)^* : H^*(B; \mathbb{F}_2) \to H^*(Z_i; \mathbb{F}_2)$$

being injective. Here, vol_n *is the standard measure on the fiber S^n over $p_E(z)$, $U_\varepsilon(.)$ denotes the ε-neighborhood in that fiber, and H^* denotes Čech cohomology.*

Remark 1.4. It may happen the set Z^* of all points $z \in E$ that satisfy the volume bound (1) for all $\varepsilon > 0$ has the property that $H^*(B; \mathbb{F}_2) \to H^*(Z^*; \mathbb{F}_2)$ is not injective. For example this happens when $M = S^n$, the rank $n+1$ vector bundle associated to p has non-trivial Stiefel–Whitney classes, and f is wrapping enough to make $Z^* = E$.

The paper is organized as follows: We briefly discuss what we mean by parametrized theorems in section 2 (this section is rather philosophical and the reader may skip it without danger). In section 3 we define the index theories for G-bundles that are used in this paper and we discuss their basic properties. The parametrized Borsuk–Ulam–Bourgin–Yang–Volovikov–Karasev theorem is stated in section 4 and it is proved in section 5. Finally, we prove the parametrized waist of the sphere theorem 1.3 in section 6.

2 Parametrized discrete geometry

Many more theorems in discete geometry apart from Gromov's waist theorem have a parametrized version. A large class of such theorems are those that can be proved via what we would call the *preimage method*: For these theorems, the solution set can be described as a preimage $f^{-1}(Z)$ for some usually equivariant map $X \to Y \supset Z$, such that an invariant such as the equivariant bordism class of $f^{-1}(Z)$ does not vanish, which implies that the solution set is nonempty. This is a specialization of the Configuration Space/Test Map scheme, see Živaljević [Živ96], [Živ98].

Some theorems from discrete geometry turn out to admit "stronger" parametrizations than others. Let's make this precise. Consider a theorem T that asserts for every valid input datum $d \in D$ the existence of a solution s in the space of candidates of solutions X. Here, D and T are assumed to be topological spaces. There may be several natural choices for D and especially for X.

Let us assume right away that X is a fiber bundle over D, $p : X \to D$, and that the solution set $S(d)$ for $d \in D$ lies in the fiber over d. If X does not naturally have such a structure, then simply replace it with the trivial bundle $\mathrm{pr}_2 : X \times D \to D$. Thus $S : D \to 2^X$ is a set-valued section of p. In discrete geometry it is often upper hemicontinuous, i.e. its graph is closed.

The strongest form of a parametrization for theorem T would be a section $s : D \to X$ such that $s(d) \in S(d)$ for all $d \in D$. This appears often when T admits a constructive existence proof. Let's call this a solution selection map.

Convex solution sets. Slightly weaker parametrizations occur when there is set-valued function $S' : D \to 2^X$ with $S'(d) \subseteq S(d)$ such that $S'(d)$ is convex, where here we require $p : X \to D$ to be a vector bundle.

The easiest example is probably Helly's theorem [Hel23] (see also Matoušek [Mat02]).

Theorem 2.1 (Helly)**.** *Any given family of convex sets $C_i, i \in I$ in $\mathbb{R}^d$ have a point in common if any $d+1$ of them do.*

For an input datum $d = (C_i)_{i \in I}$ (topologized by the Hausdorff topology with respect to some finite metric on $\mathbb{R}^d$, and the product topology) the solution set $S(d) = \bigcap_i C_i$ is already convex. However there is no solution selection map, a counter-example is depicted in figure 2.

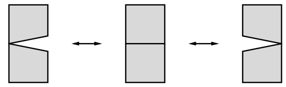

Figure 2: Example showing that there is no solution selection map for Helly's theorem.

A parametrized Helly theorem on vector bundles was proved and used by Dol'nikov [Dol87] and [Dol92] to establish the *center transversal theorem*, which is a generalization and interpolation between Banach's ham sandwich theorem and Rado's center point theorem.

Theorem 2.2 (Rado)**.** *Let μ be a probability measure on the Borel-σ-algebra of $\mathbb{R}^d$. Then there exists a point $c \in \mathbb{R}^d$, called the center point of μ, such that any halfspace H that contains C satisfies $\mu(H) \geq 1/(d+1)$.*

Since the solutions set of all centerpoints of a given measure μ is convex, we get again immediately a parametrized version. This was first observed by Živaljević–Vrećica [ŽV90] who used it to prove the center transversal theorem, independently from Dol'nikov.

Weak parametrizations. Most often the there is not even a convex set valued solution selection function. Still there might be more quantitative assertions about the solution set $\mathbb{S} := \cup_{d\in D} S(d)$, such as the *injectivity* of the map induced in some continuous cohomology $H^*(D) \to H^*(\mathbb{S})$, or the *surjectivity* of the map induced in some homology $H_*(\mathbb{S}) \to H_*(D)$.

A basic example is of course the Borsuk–Ulam theorem [Bor33].

Theorem 2.3 (Borsuk–Ulam)**.** *Any map $f : S^d \to \mathbb{R}^d$ sends some pair of antipodal points $x, -x \in S^d$ to the same point $f(x) = f(-x)$.*

The solution set represents the generator of $H_0(\mathbb{R}P^d; \mathbb{F}_2) = \mathbb{F}_2$, meaning that the generic number of solutions is odd. Jaworowski's parametrized version [Jaw81a] concerns bundle maps from an S^d-bundle to a rank d vector bundle φ over the same base space B. He proved that if all Stiefel–Whitney classes up to ω_i of φ are trivial, then $H^{d-i}(B; \mathbb{F}_2) \to H^{d-i}(\mathbb{S}; \mathbb{F}_2)$ is injective, H^* being any continuous cohomology and d being the cohomology dimension of B. For many more versions see the above references in the paragraph below remark 1.4.

Another example is Tverberg's theorem [Tve66], [Tve81].

Theorem 2.4 (Tverberg)**.** *Let $N := (r-1)(d+1)$. Any $N+1$ points in $\mathbb{R}^d$ can be partitioned into r parts whose convex hulls have a point in common.*

We could replace $\mathbb{R}^d$ by some rank d vector bundle φ over a base space D, and replace the given point set by $N+1$ sections in φ. The union $\mathbb{S}$ of the solution sets of Tverberg's theorem for every fiber of φ will be over generic points $d \in D$ only a finite point set. But one can show that if $r = p^k$ is prime power then φ induces an *injection* on Čech-cohomology $H^*(D; \mathbb{F}_p) \to H^*(\mathbb{S}; \mathbb{F}_p)$, see Živaljević [Živ99], Vrećica [Vre03], Karasev [Kar07], and Blagojević–M.–Ziegler [BMZ11]. These parametrized versions are then used to prove cases of the so-called transversal versions of Tverberg's theorem, the Tverberg–Vrećica conjecture [TV93]. More transversal versions of standard theorems in discrete geometry can be found in Karasev [Kar07], [Kar09b], and Montejano–Karasev [MK11]

3 Topological notations

Let us fix some notation that will be used throughout the paper.

All spaces are paracompact, all maps are continuous. By a bundle we simply mean a map $X \to B$, where X is called total space and B base space. A G-bundle is a G-map $X \longrightarrow_G B$ from a G-space X to a *trivial* G-space B. Base spaces will always be trivial G-spaces in this paper. In particular, G acts fiberwise on X. When we write $F \hookrightarrow E \to B$ we mean a fiber bundle. Fiber bundles will always be locally trivial. A (Serre) G-fibration is a G-bundle with the G-equivariant lifting property for G-CW-complexes (usually G-fibrations are also defined for base spaces with non-trivial G-action, but not in this paper).

Let $q = p^k$ be a prime power. In this paper we consider only symmetry groups G with $\mathbb{Z}_p^k \subseteq G \subseteq \mathrm{Syl}_p(S_q)$. Here, S_q is the symmetric group on q elements, $\mathbb{Z}_p = \mathbb{Z}/(p\mathbb{Z})$, and $\mathrm{Syl}_p(S_q) = \mathbb{Z}_p \wr \ldots \wr \mathbb{Z}_p$ is some p-Sylow subgroup of S_q. Cohomology groups $H^*(X)$ always denote Čech cohomology with $\mathbb{F}_p$-coefficients, which are constant coefficients except when we are talking about equivariant cohomology. In that case, the coefficients $\mathbb{F}_p$ are twisted by the sign of the permutation (remember that $G \subseteq S_q$).

For a G-space X, we write $X_G := EG \times_G X$, which is the total space of the fibration $X \hookrightarrow X_G \to B$ called Borel construction. The G-equivariant cohomology (equivariant bundle cohomology, or Borel cohomology) of X is $H^*_G(X) := H^*(X_G)$. In particular for a trivial G-space B we have $H^*_G(B) = H^*(B_G) = H^*(G) \otimes H^*(B)$.

Let $W_q := \{x \in \mathbb{R}^q \mid \sum x_i = 0\}$ denote the standard representation of S_q. The G-equivariant Euler class of a G-bundle $F \hookrightarrow E \to B$ is the ordinary Euler class of the bundle $F \hookrightarrow E_G \to B_G$. The Euler class $e(V)$ of a G-representation V is the equivariant Euler class of the bundle $V \hookrightarrow V \to \mathrm{pt}$, that is, the ordinary Euler class of $V \hookrightarrow V_G \to BG$.

For a vector bundle $F \hookrightarrow E \to B$ we denote the associated sphere and disk bundles by $S(F) \hookrightarrow S(E) \to B$ and $D(F) \hookrightarrow D(E) \to B$.

3.1 Index theories

In many situations one wants to disprove the existence of G-equivariant maps $X \longrightarrow_G Y \backslash Z$, or more generally that for some G-map $f : X \to Y \supset Z$ the preimage $f^{-1}(Z)$ is 'large' in some specific sense.

In our situation we are interested in G-bundle maps $f : X \to Y \supset Z$ over some trivial base space B. In this paper we will connect two different index theories, the first of which was defined and studied by Fadell and Husseini [FH87b], [FH88].

Definition 3.1. Let $f : X \to B$ be a G-bundle. The **Fadell–Husseini index** of f is defined as

$$\mathrm{ind}^{FH}_{B,G}(X) := \ker(H^*_G(B) \xrightarrow{f^*} H^*_G(X)) \subseteq H^*(G) \otimes H^*(B).$$

When B is a point, we also write

$$\mathrm{ind}^{FH}_G(X) := \mathrm{ind}^{FH}_{\mathrm{pt},G}(X) \subseteq H^*(G).$$

Lemma 3.2 (Properties of $\mathrm{ind}^{FH}_{B,G}$)**.** *Let $f : X \longrightarrow_G B$ and $g : Y \longrightarrow_G B$ be G-maps with B a trivial G-space as above. Then:*
a) If there is a G-bundle map $h : X \longrightarrow_G Y$, that is $f = g \circ h$, then

$$ind^{FH}_{B,G}(X) \supseteq \mathrm{ind}^{FH}_{B,G}(Y).$$

b) If X is n-connected then

$$\operatorname{ind}_G^{FH}(X) \subseteq H^{*\geq n+2}(G).$$

c) If $F \hookrightarrow X \to B$ is a G-fibration and F is n-connected, then

$$\operatorname{ind}_{B,G}^{FH}(X) \subseteq H^{*\geq n+2}(BG \times B).$$

d) If $f : F \times B \to B$ is the projection to the second coordinate, then

$$\operatorname{ind}_{B,G}^{FH}(F \times B) = \operatorname{ind}_G(F) \otimes H^*(B).$$

e) If $G = \mathbb{Z}_p^k$ then for any G-space X, $\operatorname{ind}_G^{FH}(X) = \emptyset$ if and only if X has a fixed point.
f) If $\operatorname{ind}_{B,G}^{FH}(X) \cap H^0(G) \otimes H^*(B) = 0$, then $f^* : H^*(B) \to H^*(X)$ is injective.

Proof. a) follows immediately from the definition and b) is the special case of c) for $F = X$ and $B = \text{pt}$. c) follows from chasing the Leray–Serre spectral sequence of $F \hookrightarrow X_G \xrightarrow{f_G} B_G$: Note that the index defining map $H_G^*(B) \xrightarrow{f^*} H_G^*(X)$ coincides with the bottom edge homomorphism. Hence the elements of the index $\operatorname{ind}_{B,G}^{FH}(X)$ are exactly the elements in the bottom row of the spectral sequence that lie in the image of some differential. Since F is n-connected, the only non-zero differentials hit the bottom row in filtration degree $n+2$ or higher.
d) follows from Künnet's theorem. For e), see tom Dieck [tD87, Prop. 3.14, p. 196]. f) follows immediately from the definition. □

3.2 A spectral sequence based index

Let $f : X \to B$ be a G-bundle. If f is not a G-fibration then we replace X by $X' := \{(x,\gamma) \mid x \in X, \gamma : I \to B, \gamma(0) = f(x)\}$ and f by the map $f' : X' \to B$ that sends (x,γ) to $\gamma(1)$. This replacement makes f into a G-fibration, it is functorial, and if f is a already a G-fibration then f and f' are G-fiber homotopy equivalent. This gives several ways to define spectral-sequence based indices of f. For example, Blagojević–Blagojević–McCleary [BBM11] defined the spectral sequence witness of a pair of G-spaces X and X' which gives a criterion for the non-existence of G-maps $X \to X'$.

The index we will be interested in in this paper is the Leray-Serre spectral sequence of the map $X'_G \to B$ given by $[e,(x,\gamma)] \mapsto f(\gamma(1))$. Here we need that B is a trivial G-space. There is a natural map from $X'_G \to B$ to $B_G \to B$, where $B_G = BG \times B$, which induces a morphism of associated spectral sequences.

Note that the spectral sequence of $B_G \to B$ collapes at $E_2^{*,*} = H^*(B) \otimes H^*(G)$. Also, any map bundle map $X \to Y$ over B gives rise to a commutative triangle of maps between the associated spectral sequences.

In this paper it will be enough to consider the E_∞-page.

Definition 3.3. Let $f : X \to B$ be a G-bundle. We define the **E_∞-index** of $X \to B$ as

$$\operatorname{ind}_{G,B}^{\infty}(X) := \ker\left(E_\infty^{*,*}(B_G \to B) \to E_\infty^{*,*}(X'_G \to B)\right) \subseteq H^*(G) \otimes H^*(B).$$

By the multiplicativity of the Leray–Serre spectral sequence $\operatorname{ind}_{G,B}^{\infty}(X)$ is a bi-homogeneous ideal in $H^*(G) \otimes H^*(B)$.

Lemma 3.4 (Properties of $\mathrm{ind}^{\infty}_{B,G}$)**.** *Let $X \to B$ be a G-bundle.*
a) Let $X \to Y$ be a map of G-bundles over B. Then

$$\mathrm{ind}^{\infty}_{B,G}(X) \supseteq \mathrm{ind}^{\infty}_{B,G}(Y).$$

b) If $F \times B \to B$ is the projection to the second coordinate then

$$\mathrm{ind}^{\infty}_{B,G}(F \times B) = \mathrm{ind}^{FH}_{\mathrm{pt},G}(F) \otimes H^*(B) = \mathrm{ind}^{FH}_{B,G}(F \times B).$$

c) If $\mathrm{ind}^{\infty}_{B,G}(X) \subseteq H^{*\geq 1}(G) \otimes H^*(B)$, then $f^* : H^*(B) \to H^*(X)$ is injective. If moreover B is a compact manifold, then the map in singular homology $f_* : H_*(U_\varepsilon(X)) \to H_*(B)$ is surjective for any $\varepsilon > 0$.

Proof. a) is a trivial chase in the diagram of the index defining spectral sequences. b) is trivial. c) follows from the definition and the edge-homomorphism. □

Comparing ind^{FH} **and** ind^{∞}**.** Although both indices ind^{FH} and ind^{∞} are similarly defined, there are substantial differences: First of all, ind^{∞} is a $\mathbb{Z}^2$-graded ideal of $H^*(G) \otimes H^*(B)$, whereas ind^{FH} is only a $\mathbb{Z}$-graded ideal (with respect to the total grading).

Definition 3.5. We define the **leading term** $\mathrm{lt}(\alpha)$ of a homogeneous element $\alpha \in H^*(G) \otimes H^*(B)$ as the first non-zero α_i, where $\alpha = \alpha_0 + \alpha_1 + \ldots$ with $\alpha_i \in H^*(G) \otimes H^i(B)$. This extends degree-wise (with respect to the total degree) to a maps of sets $\mathrm{lt} : H^*(G) \otimes H^*(B) \to H^*(G) \otimes H^*(B)$.

Lemma 3.6. *Any G-bundle $X \to B$ satisfies $\mathrm{lt}\big(\mathrm{ind}^{FH}_{B,G}(X)\big) \subseteq \mathrm{ind}^{\infty}_{B,G}(X)$.*

However, the non-leading bihomogeneous parts of $\alpha \in \mathrm{ind}^{FH}_{B,G}(X)$ may not be in $\mathrm{ind}^{\infty}_{B,G}(X)$. Also bihomogeneous elements $\beta \in \mathrm{ind}^{\infty}_{B,G}(X)$ may not lie in $\mathrm{ind}^{FH}_{B,G}(X)$, because $\beta \in \mathrm{ind}^{\infty}_{B,G}(X)$ means that its image is zero in some filtration quotient of $H^*_G(X)$.

Example 3.7. As an example, let $p : X \to B$ be the associated circle bundle of the tangent bundle of $\mathbb{R}P^2$. Suppose that $G = \mathbb{Z}_2$ acts antipodally on each fiber of X. Then, $H^*(G) = \mathbb{F}_2[t]$ and $H^*(B) = \mathbb{F}_2[u]/(u^3)$. The associated fibration of Borel constructions, $X_{\mathbb{Z}_2} \to BG \times B$, is a circle bundle with (mod 2) Euler class $e = u^2 + t^2$. This is the generator for $\mathrm{ind}^{FH}_{B,G}(X)$. We have $\mathrm{lt}(e) = t^2 \in \mathrm{ind}^{\infty}_{B,G}(X)$, and in fact t^2 is the generator of $\mathrm{ind}^{\infty}_{B,G}(X)$. Hence in this example, the other bihomogeneous part u^2 of $e \in \mathrm{ind}^{FH}_{B,G}(X)$ does not lie in $\mathrm{ind}^{\infty}_{B,G}(X)$, and the bihomogeneous element $t^2 \in \mathrm{ind}^{\infty}_{B,G}(X)$ does not lie in $\mathrm{ind}^{FH}_{B,G}(X)$.

3.2.1 More general versions

The following versions and generalizations of ind^{∞} may be useful for different problems, but we won't need them in this paper.

We can define to any G-bundle $f : X \to B$ the **spectral sequence valued index** $\mathrm{ind}^{SS}_{G,B}(X)$ as the Leray–Serre spectral sequence of $X'_G \to B$ together with the morphism of spectral sequences from the spectral sequence of $B_G \to B$. In abstract terms this index is a functor from the category of G-bundles over B to the category whose objects are morphisms from one spectral sequence to the spectral sequence of $B_G \to B$.

Two other useful indices can be defined in a similar way using the fibrations $X \hookrightarrow X_G \to BG$ and $F \hookrightarrow X'_G \to B_G$, where F is the homotopy fiber of $X \to B$. If B is a point then both of them coincide. The latter contains all information of the Fadell–Husseini index, since $\operatorname{ind}_{G,B}(X)$ is the set of all elements in the 0-row of the spectral sequence of $X'_G \to B_G$ that are in the image of some differential.

In some sense these three spectral sequence valued indices can be unified using a "higher spectral sequence" that will be constructed in [Mat12]. The corresponding sequence of two fibrations is $X_G \to B_G \to B$.

4 Parametrized Volovikov theorem

The main methodological tool in this paper is the following parametrized Volovikov theorem. It relates the ∞-index of the configuration space X to the Fadell–Husseini index of the solution set S.

Theorem 4.1 (Parametrized Volovikov theorem)**.** *Let*

- $q = p^k$,
- *G be a subgroup of S_q such that $\mathbb{Z}_p^k \subseteq G \subseteq \mathrm{Syl}_p(S_q)$,*
- *B be a path-connected trivial G-space,*
- *$p_X : E_X \to B$ be a G-bundle,*
- *Y be a paracompact space and let G act on Y^q by permuting the coordinates,*
- *$Y \hookrightarrow E_Y \to B$ be a fiber bundle,*
- *M be a connected (paracompact) smooth m-manifold,*
- *$M \hookrightarrow E_M \to B$ be a fiber bundle.*

Let $i : E_X \longrightarrow_G E_Y^{\oplus q}$ be a G-bundle map over B,

$$\begin{tikzcd} E_X \arrow[rr, "i"] \arrow[dr, "p_X"'] & & E_Y^{\oplus q} \arrow[dl, "p_Y^{\oplus q}"] \\ & B & \end{tikzcd}.$$

Let F be a fiber bundle map,

$$\begin{tikzcd} E_Y \arrow[rr, "F"] \arrow[dr, "p_Y"'] & & E_M \arrow[dl, "p_M"] \\ & B & \end{tikzcd}.$$

Assume that over some (and hence any) base point $b \in B$, the map $F_b := F|_{p_Y^{-1}(b)} = F|_Y$ induces the zero map in positive cohomology $H^{*\geq 1}(M, \mathbb{F}_p) \to H^{*\geq 1}(Y, \mathbb{F}_p)$.

Then the "solution set"

$$S := \{x \in E_X \mid i(x) = (y_1, \ldots, y_q) \text{ satisfies } F(y_1) = \ldots = F(y_q)\}$$

has the following index bound: If $\alpha \in \mathrm{ind}^{FH}_{B,G}(S)$ *is a homogeneous element then*

$$\mathrm{lt}(\alpha)e(W_q)^m \in \mathrm{ind}^{\infty}_{B,G}(E_X).$$

In particular if $\mathrm{ind}^{\infty}_{B,G}(E_X) \cap \big(e(W_q)^n \otimes H^*(B)\big) = \emptyset$ *then*

$$H^*(B, \mathbb{F}_p) \xrightarrow{(p_M|_Z)^*} H^*(Z, \mathbb{F}_p)$$

is injective, where

$$Z := F^q(i(S)) \cong \{z \in E_M \mid z = F(y_1) = \ldots = F(y_q) \\ \text{for some } x \in E_X, (y_1, \ldots, y_q) = i(x)\}.$$

Remark 4.2. In applications of theorem 4.1, E_X is usually the configuration space (the space of solution candidates), which is parametrized over B, and M is also naturally given by our description of the solution set as a preimage. But what about $Y \hookrightarrow E_Y \to B$? The assumption of the theorem that $F^{\oplus q} \circ i$ has to factor over E_Y is important for rather technical reasons. Two cases for the choice of E_Y usually appear:

1. $i : E_X \longrightarrow_G E_Y^{\oplus q}$ is an inclusion of G-spaces.
2. If $G = \mathbb{Z}_p^k$ and $E_X \to B$ is a fiber bundle, then one can simply choose $E_Y := E_X$ and let $i : E_X \to E_Y^{\oplus q}$ be defined as $x \mapsto (g^{-1}x)_{g \in G}$.

Remark 4.3. The set $Z \subseteq E_M$ is in general much more complicated than the image of a section of $p_M : E_M \to B$ (p_M may not even admit a section).

Remark 4.4 (Desirable extensions)**.** It would be useful to have a version of theorem 4.1 that relates E_X and S using the *same* index theory, such that one can apply the theorem iteratively.

If the "parametrized Nakaoka lemma" $H^*_G(E_M^{\oplus q}) \cong H^*(G; H^*(E_M^{\oplus q})$ is true (and if this isomorphism is natural in p_M) then we would have the following relation: There exists $e' \in H_G^{m(q-1)}(B)$ with $\mathrm{lt}(e') = e(W_q)^m \otimes 1$ and $e' \cdot \mathrm{ind}^{FH}_{B,G}(S) \subseteq \mathrm{ind}^{FH}_{B,G}(E_X)$.

5 Proof of the parametrized Volovikov theorem

In large parts we follow the proof of Volovikov [Vol92] (see §5 and in particular the proof of lemma 3) and Karasev–Volovikov [KV11].

We denote the q-fold Withney sum of E_M by $M^q \hookrightarrow E_M^{\oplus q} \to B$. Let Δ_{M^q} denote the thin diagonal $\{(m, \ldots, m) \in M^q\}$ of M^q. Similarly, let $\Delta_{M^q} \hookrightarrow \Delta_{E_M^{\oplus q}} \to B$ be the thin diagonal subbundle of $E_M^{\oplus q}$.

$G \subseteq S_q$ acts on M^q and $E_M^{\oplus q}$ by permuting coordinates. Their fixed-point sets are $\Delta_{M^q} \cong M$ and $\Delta_{E_M^{\oplus q}} \cong E_M$, respectively.

Some closed tubular neighborhood $N\big(\Delta_{E_M^{\oplus q}}\big) \subset E_M^{\oplus q}$ can be regarded as a disc bundle $D^{m(q-1)} \hookrightarrow N\big(\Delta_{E_M^{\oplus q}}\big) \to \Delta_{E_M^{\oplus q}} \cong E_m$ of some G-vector bundle $W_q^{\oplus m} \hookrightarrow W \xrightarrow{\varphi} \Delta_{E_M^{\oplus q}}$, the normal bundle of $\Delta_{E_M^{\oplus q}}$ in $E_M^{\oplus q}$.

Let τ be the rank m vector bundle over $\Delta_{E_M^{\oplus q}}$ whose fiber at a point $e \in \Delta_{E_M^{\oplus q}}$ is the tangent space $T_e M_b$, where $M_b \cong M$ is the fiber of $\Delta_{E_M^{\oplus q}} \to B$ that contains e. Then $\varphi \oplus \tau = \tau^{\oplus q}$, and φ is stably equivalent to $\tau^{\oplus(q-1)}$. For $p > 2$, $q-1$ is even. Hence φ is $\mathbb{F}_p$-orientable. Furthermore all non-zero elements of G have an odd order if $p > 2$, thus the G-action preserves the $\mathbb{F}_p$-orientation on φ.

Therefore W has a G-equivariant mod-p Thom class $\tau_{E_M,G} \in H_G^{m(q-1)}(D(W), S(W))$, which is the ordinary mod-p Thom class of the bundle $W_q^m \hookrightarrow W_G \to \big(\Delta_{E_M^{\oplus q}}\big)_G$, where $\big(\Delta_{E_M^{\oplus q}}\big)_G = \Delta_{E_M^{\oplus q}} \times BG$.

By excision we regard $\tau_{E_M,G}$ as an element in $H_G^{m(q-1)}\big(E_M^{\oplus q}, E_M^{\oplus q} \backslash \Delta_{E_M^{\oplus q}}\big)$. By the Thom isomorphism this group is isomorphic to $H_G^0\big(\Delta_{E_M^{\oplus q}}\big) = \mathbb{F}_p$.

Consider the diagram of restrictions

$$\begin{tikzcd}
\tau_{E_M.G} \in H_G^{m(q-1)}\big(E_M^{\oplus q}, E_M^{\oplus q}\backslash \Delta_{E_M^{\oplus q}}\big) \arrow[r, "\cong"] \arrow[d] & H_G^{m(q-1)}\big(D(W), S(W)\big) \arrow[d] \\
\gamma_{E_M,G} \in H_G^{m(q-1)}\big(E_M^{\oplus q}\big) \arrow[r] \arrow[d] & H_G^{m(q-1)}\big(\Delta_{E_M^{\oplus q}}\big) \arrow[d] \\
\gamma_{M,G} \in H_G^{m(q-1)}(M^q) \arrow[r] \arrow[dr] & H_G^{m(q-1)}(\Delta_{M^q}) \arrow[d] \\
 & e(W_q^{\oplus m}) \in H_G^{m(q-1)}(\mathrm{pt}^q)
\end{tikzcd}$$

Let $\gamma_{E_M,G}$ denote image of $\tau_{E_M,G}$ in $H_G^{m(q-1)}\big(E_M^{\oplus q}\big)$. By commutativity of the top square, $\gamma_{E_M,G}$ maps to the Euler class $e(\varphi) \in H_G^{m(q-1)}\big(\Delta_{E_M^{\oplus q}}\big)$ of φ. Thus, when further restricting to $H_G^{m(q-1)}(\mathrm{pt}^q)$, pt^q being some point in $\Delta_{E_M^{\oplus q}}$, $\gamma_{E_M,G}$ maps to $e(W_q)^m$.

Remark 5.1. In case B is a manifold, $\tau_{E_M,G}$ can be constructed as the Poincaré dual of $E_rG \times_G \Delta_{E_M^{\oplus q}}$ in $E_rG \times_G E_M^{\oplus q}$, where E_rG is an r-connected free G-manifold and $r \geq m(q-1)$, using the canonical isomorphism

$$H^{m(q-1)}\big(E_rG \times_G \big(E_M^{\oplus q}, E_M^{\oplus q}\backslash \Delta_{E_M^{\oplus q}}\big)\big) \xrightarrow{\cong} H_G^{m(q-1)}\big(E_M^{\oplus q}, E_M^{\oplus q}\backslash \Delta_{E_M^{\oplus q}}\big).$$

See Volovikov [Vol92] and Karasev–Volovikov [KV11] for details in the case when $B = \mathrm{pt}$.

Let $\gamma_{E_X} := (F^q \circ i)^*(\gamma_{E_M,G}) \in H_G^{m(q-1)}(E_X)$.

Claim 5.2. *The restiction of γ_{E_X} to $E_X \backslash S$ is zero in $H_G^{m(q-1)}(E_X \backslash S)$.*

Proof. From the long exact sequence of the pair $\big(E_M^{\oplus q}, E_M^{\oplus q}\backslash \Delta_{E_M^{\oplus q}}\big)$ we see that $\gamma_{E_M,G}$ restricts to zero in $H_G^{m(q-1)}\big(E_M^{\oplus q}\backslash \Delta_{E_M^{\oplus q}}\big)$. Since $F^q \circ i$ sends the pair (E_X, S) to $\big(E_M^{\oplus q}, \Delta_{E_M^{\oplus q}}\big)$, the claim follows. $\square$

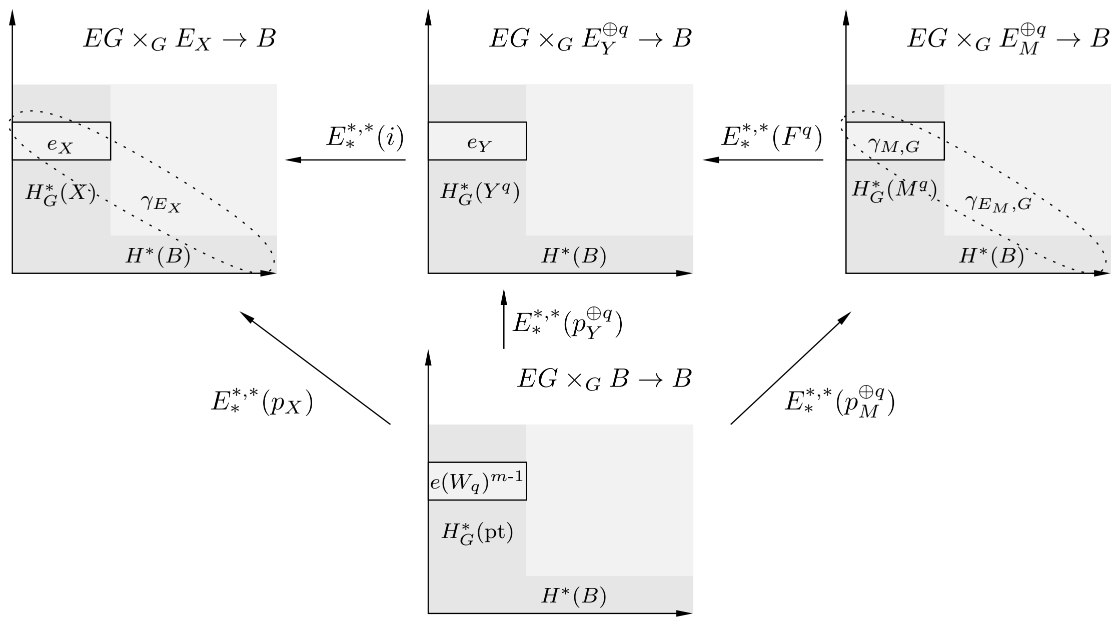

Figure 3: Seconds pages of the spectral sequences of $(E_X)_G$, $(E_Y^{\oplus q})_G$, $(E_M^{\oplus q})_G$, and B_G over B.

The constructions of $\tau_{E_M,G}$ and $\gamma_{E_M,G}$ are natural with respect to taking subgroups of G and restrictions of B. When restricting E_M to a fiber $M_b \cong M$ over some base point $b \in B$, $\gamma_{E_M,G}$ restricts to an element $\gamma_{M,G} \in H_G^{m(q-1)}(M^q)$.

Now consider the diagram of spectral sequences in figure 3. Here, X is the homotopy fiber of $p_X : E_X \to B$, and $e_X \in H^*_G(X)$ and $e_Y \in H^*_G(Y^{\oplus q})$ are the natural images of $e(W_q)^{m-1}$.

Claim 5.3. *$E_2^{0,m(q-1)}(F^q)$ sends $\gamma_{M,G}$ to e_Y.*

Proof. By Nakaoka's lemma [Nak61], we have an isomorphism

$$H^*_G(M^q) = \mathrm{Tot}(H^*(G; H^*(M)^{\otimes q})),$$

which is natural in G and M, where Tot denotes the total complex of a bigraded complex. As an $\mathbb{F}_p[G]$-algebra, $H^*(M)^{\otimes q}$ decomposes as $\mathcal{A} + \mathcal{B}$, where $\mathcal{A} = H^0(M)^{\otimes q} = \mathbb{F}_p$ and $\mathcal{B}$ is generated by all homogeneous elements in $H^*(M)^{\otimes q}$ of positive total degree. Hence $\gamma_{M,G}$ decomposes as $\gamma_{M,G} = \gamma_a + \gamma_b$, where $\gamma_a \in H^*(G; \mathcal{A}) = H^*(G)$ and $\gamma_b \in H^*(G; \mathcal{B})$.

The composition

$$\begin{tikzcd}
H_G^{m(q-1)}(\mathrm{pt}^q) \arrow[r] \arrow[d, equal] & H_G^{m(q-1)}(M^q) \arrow[r] & H_G^{m(q-1)}(\mathrm{pt}^q) \arrow[d, equal] \\
e(W_q)^m \in H^{m(q-1)}(G; \mathcal{A}) & \gamma_a \in H^{m(q-1)}(G; \mathcal{A}) \arrow[u, hook] & H^{m(q-1)}(G; \mathcal{A})
\end{tikzcd}$$

induced by the projection on the left and by an inclusion on the right is the identity. Both maps are also individually isomorphisms on the $H^{m(q-1)}(G; \mathcal{A})$-part since the Nakaoka lemma is natural in M. Since the second map sends $\gamma_{M,G}$ to $e(W_q)^m$ we deduce that $\gamma_a = e(W_q)^m$ and the first map sends $e(W_q)^m$ to γ_a. Thus $E_2^{0,m(q-1)}(F^q)$ sends γ_a to e_Y.

$F : E_Y \to E_M$ restricts over some base point $b \in B$ to $F_b : Y^q \to M^q$, which by assumption induces zero in positive cohomology. Therefore $E_2^{0,m(q-1)}(F^q)$ will send γ_b to zero, by naturality of Nakaoka's lemma. □

From the claim follows that $E_2^{0,m(q-1)}(F^q \circ i)$ sends $\gamma_{M,G}$ to e_X. Since $(F^q \circ i)^*(\gamma_{E_M,G}) = \gamma_{E_X}$ and $\gamma_{M,G}$ is the restriction of $\gamma_{E_M,G}$ to the first column of the right spectral sequence, e_X must be the restriction of γ_{E_X} on the left spectral sequence. In other words, e_X is the leading term of γ_{E_X} in the left spectral sequence.

Now suppose we are given a homogeneous element $\alpha \in \mathrm{ind}_{B,G}^{FH}(S)$, that is, α maps to zero in $H_G^*(S)$. By claim 5.2 if follows that

$$(p_X)^*(\alpha) \cup \gamma_{E_X} = 0.$$

Thus on the E_∞-page we have that

$$E_\infty^{0,m(q-1)}(p_X)\big(\mathrm{lt}(\alpha) \cup e(W_q)^m\big) = 0.$$

This proves the general index bound for S.

For the last part of the theorem, assume that $\mathrm{ind}_{B,G}^{\infty}(E_X) \subseteq H^{*\geq m(p^k-1)+1}(G) \otimes H^*(B)$. Then the index bound yields that $\mathrm{ind}_{B,G}^{FH}(S)$ cannot contain elements in $H^0(G) \otimes H^*(B)$ except for 0. Therefore lemma 3.4 implies that $H^*(B) = H^0(G) \otimes H^*(B) \to H_G^*(S)$ is injective. The commutative diagram of natural maps

$$\begin{tikzcd} H_G^*(S) & H_G^*(Z) = H^*(G) \otimes H^*(Z) \arrow[l] & H^*(Z) \arrow[l] \\ & H^*(B) \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

implies that $H^*(B) \to H^*(Z)$ is injective as well. This finishes the proof of theorem 4.1. □

6 Sketch of proof of the parametrized waist of sphere theorem

In the case $n = k$, the parametrized waist of the sphere theorem 1.3 follows easily from a parametrized Borsuk–Ulam theorem for manifolds: Theorem 4.1 implies for the given bundle map $B \times S^n \to E$ and the antipodal $\mathbb{Z}_2$-action on the fibers of $B \times S^n$ that the set Z of all elements $z \in E_M$ whose preimage $f^{-1}(z)$ contains a pair of antipodal points has the property that $H^*(B; \mathbb{F}_2) \to H^*(Z; \mathbb{F}_2)$ is injective.

Thus we may assume $n > k$. Gromov's proof of 1.2 splits into a topological and an analytic part. The topological part is the following mass partition theorem.

Let $\mathrm{Conv}(S^n)$ denote the set of all closed convex subsets of $C \subset S^n$ with $C \neq S^n$. Let $\mathrm{Conv}^*(S^n)$ be its subset of sets with positive volume. The Hausdorff metic makes $\mathrm{Conv}(S^n)$ into a metric space. A map $c : \mathrm{Conv}^*(S^n) \to S^n$ is called a *center map*. A *partition of S^n into q convex sets* is a family of subsets $C_1, \ldots, C_q \in \mathrm{Conv}(S^n)$ with pairwise disjoint interior such that $S^n = \bigcup_i C_i$.

Theorem 6.1 (A mass partition theorem)**.** *Let $g : S^n \to M^k$ be map from the n-sphere to a k-manifold, $n > k$, let $c : \mathrm{Conv}^*(S) \to S^n$ be a center map. Then for any $q = 2^\ell$ there exists a partition of S^n into q convex sets $C_1, \dots, C_q$ with*

$$g(c(C_1)) = \ldots = g(c(C_q))$$

and

$$\mathrm{vol}(C_1) = \ldots = \mathrm{vol}(C_q).$$

Moreover the set C_i can be required to lie in the ε-neighborhood of some k-dimensional equator $E_i \subset S^n$ in case $q \geq q_0(\varepsilon)$.

The analytic part of the proof is based on involved isoperimetric inequalities that make theorem 6.1 with $\varepsilon \to 0$ imply theorem 1.2, see Gromov [Gro03], Memarian [Mem09].

Every point $x \in S^n$ determines its polar hyperplane, which bisects S^n into two convex pieces. Two more points on the sphere, one for each of the two pieces, will yield a convex partition of S^n into four pieces. Iterating this, we obtain a map

$$p : X := (S^n)^{q-1} \to \mathrm{Conv}(S^n)^q.$$

Let T be the complete binary tree of height $\ell - 1$. The interior nodes of T naturally correspond to the $q-1$ sphere factors of X, and the q leaves correspond to the convex sets in the partition. Let them be labelled by $N_1, \dots, N_{q-1}$, where N_1 shall denote the root. Let the leaves of T be labelled by $L_1, \dots, L_q$. Thus the symmetry group of T, the 2-Sylow subgroup $G := \mathbb{Z}_2 \wr \ldots \wr \mathbb{Z}_2$ of the symmetric group S_q, acts on $(S^n)^{q-1}$ (with antipodal action on an S^n-factor whenever its children are exchanged, such that the partition $p(x)$ for $x \in X$ stays the same up to permutation of the indices) and on $\mathrm{Conv}(S^n)^q$ (as it acts on the leaves). This makes p into a G-equivariant map.

We would like to define a test-map

$$t : (S^n)^{q-1} \longrightarrow_{S_n} (M \times \mathbb{R})^q$$

whose k'th coordinate at $x = (x_1, \dots, x_{q-1})$ is given by

$$(f(c(p_k(x))), \mathrm{vol}(p_k(x))), \tag{2}$$

such that the preimage of $\Delta := \Delta_{(M \times \mathbb{R})^q}$ corresponds to the partitions of S^n into q convex sets of equal volume and equal g-images of their center points. However c is not continuous at some of the convex sets with zero volume. Thus we replace c in (2) by a slightly deformed map c': First, let γ_C be the shortest geodesic on S^n between $\gamma_C(0) = \pm x_1$ and $\gamma_C(1/2q) = c(C)$, where the sign in front of the vector x_1 (in the sphere corresponding to the root of T) depends on whether the leaf of T corresponding to the convex set C is on the left or on the right side of the root. If $\mathrm{vol}(C) = 0$ then γ_C might not be defined except for its end point $\gamma_C(0)$. We then define

$$c'(C) := \begin{cases} c(C) & \text{if } \mathrm{vol}(C) \geq 1/2q, \\ \gamma_C(\mathrm{vol}(C)) & \text{if } \mathrm{vol}(C) \leq 1/2q. \end{cases}$$

The so defined $t : x \mapsto \big(f(c'(p_k(x))), \mathrm{vol}(p_k(x))\big)_k$ is indeed continuous and $t^{-1}(\Delta)$ is the set of convex equipartitions of S^n such that g maps all centers of the convex parts to the same point in M.

The test-map t factors as

$$X \xrightarrow{i} Y^q \xrightarrow{(f\times \mathrm{id})^q} (M\times\mathbb{R})^q,$$

where $Y := S^n \times \mathbb{R}$.

Lemma 6.2 (An index bound for $(S^n)^{q-1}$)**.** *For $G = \mathbb{Z}_2 \wr \ldots \wr \mathbb{Z}_2 \subseteq S_q$ and $\mathbb{F}_2$-coefficients,*

$$e(W_q)^n \notin \mathrm{ind}_G^{FH}((S^n)^{q-1}).$$

Proof. Consider the map $m : (S^n)^{q-1} \to W_q^{\oplus n}$ given by

$$x \mapsto \Big(\sum_{N_i \in P_k} \pm \mathrm{pr}_{S^n\to\mathbb{R}^n}(x_i)\Big)_{k=1\ldots q},$$

where $\mathrm{pr}_{S^n\to\mathbb{R}^n} : S^n \to \mathbb{R}^n$ is the standard projection to the first n coordinates; for every leaf L_k, P_k is the set of interior nodes in T that lie on the shortest path from the root N_1 to L_k, $\ell(i)$ is the height of node i in the tree (i.e. the distance to N_1), and the sign at $N_i \in P_k$ depends on whether the path P_k continues at the right or the left subtree at node N_i.

We have that the sum of all q $\mathbb{R}^n$-coordinates of this test-map is zero, since the sum for P_k cancels with the sum for the reflected P_k. Furthermore, m is G-equivariant, and $m^{-1}(0) = \{(0,\ldots,0,\pm1)\}^{q-1}$ is the set of $(q-1)$-tuples x such that every x_i is the north or the south pole of S^n. These are regular points of m, and modulo G this is exactly one preimage. ☐

Remark 6.3 (Odd prime powers)**.** There is an analogous lemma for odd prime powers $q = p^\ell$ if n is odd: Here, $G = \mathbb{Z}_p \wr \ldots \wr \mathbb{Z}_p \subseteq S_q$, $\mathbb{Z}_p$ acts on $S^n = S^1 * \ldots * S^1$ diagonally, and we use $\mathbb{F}_p$-coefficients. The proof is the same.

Remark 6.4 (An index bound for configuration spaces)**.** Let $F_q(\mathbb{R}^{n+1})$ denotes the configuration space of q pairwise distinct points of $\mathbb{R}^{n+1}$. Hung [Hun90, §1] (see also Karasev–Volovikov [KV11, 5.2]) constructed an embedding $(S^n)^{q-1} \hookrightarrow_G F_q(\mathbb{R}^{n+1})$ as follows: The first element $x_1 \in S^n$ determines a pair of antipodal points on $\mathbb{R}^{n+1}$. The next two elements $x_2, x_3 \in S^n$ are used to split these two antipodal points into four points on $\mathbb{R}^{n+1}$. And so on. Using this embedding, lemma 6.2 provides a simple proof for

$$e(W_q)^n \notin \mathrm{ind}_G^{FH}(F_q(\mathbb{R}^{n+1})). \tag{3}$$

For an application of this index bound on convex partitions see Blagojević–Ziegler [BZ12]. More general index calculations for configuration spaces can be found in Karasev [Kar09a] and Blagojević–Lück–Ziegler [BLZ12].

Since we need only the non-vanishing of $e(W_q)^k$, we may restrict the configuration space $(S^n)^{q-1}$ to some G-invariant subspace $(S^k)^{q-1}$. Here, G-invariance means that we can choose the k-dimensional equators $S^k \subseteq S^n$ independently as long as they agree on each height (with respect to T). Choosing these equators well-distributed enough will assure the ε-neighborhood condition in theorem 6.1.

Using Volovikov's theorem 1.1 finishes the proof of theorem 6.1.

Remark 6.5. Karasev and Volovikov [KV11] observed that when we remove the condition that the C_i have to be ε-close to some k-dimensional equators of S^n, then the mass partition theorem 6.1 holds also for odd prime powers: For this they used weighted Voronoi decompositions.

A parametrized version of theorem 6.1 follows analogously using theorem 4.1. This in turn implies the parametrized waist of the sphere theorem 1.3 using the same analytic part as in Gromov [Gro03]. □

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