Mini-course: Spectral sequences

April 11, 2014
SpectralSequences02.tex

Abstract

These are the lecture notes for a mini-course on spectral sequences held at Max-Planck-Institute for Mathematics Bonn in April 2014. The covered topics are: their construction, examples, extra structure, and higher spectral sequences.

1 Introduction

Spectral sequences are computational devices in homological algebra. They appear essentially everywhere where homology appears: In algebra, topology, and geometry.

They have a reputation of being difficult, and the reason for this is two-fold: First the indices may look unmotivated and too technical at the first glance, and second, computations with spectral sequences are in general not at all straight-forward or even possible.

There are essentially two situations in which spectral sequences arise:

1. From a filtered chain complex: A $\mathbb{Z}$-filtration of a chain complex (C, d) is a sequence of subcomplexes F_p such that $F_q \subseteq F_p$ for all $q \leq p$.

2. From a filtered space X and a generalized (co)homology theory h: A $\mathbb{Z}$-filtration of a topological space X is a family $(X_p)_{p\in\mathbb{Z}}$ of open subspaces such that $X_q \subseteq F_p$ whenever $q \leq p$.

More generally, in 1.) the chain complexes may live over any abelian category, and in 2.) the category of spaces can be replaced by the category of spectra; then all important cases of spectral sequences known to the author are covered. But let us keep the presentation as elementary as possible.

The most common scenario for an application is the following: Suppose we want to compute the homology $H(C)$ of a chain complex C, or the generalized homology $h(C)$ of a space X. Then we filter C respectively X in a useful way, and the associated spectral sequences give in some sense a recipe to compute $H(C)$ and $h(X)$ from their "first pages", which consists of the terms $H(F_p/F_{p-1})$ and $h(X_p, X_{p-1})$, respectively.

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Remark 1.1 (Exact couples)**.** Seemingly more general spectral sequences come from Massey's exact couples, which we will not discuss in these notes, since the author does not know of any spectral sequence in practice that does not arise from one of the two situations above.

There are many standard text books that cover spectral sequences, for example McCleary [11], Spanier [12], Hatcher [8], Weibel [14], Cartan–Eilenberg [3], Gelfand–Manin [6], Bott–Tu [2], Switzer [13], and many more. A reader interested in the early history of spectral sequences should take a look at McCleary [10].

2 Construction

Notation. Let $\mathbb{Z}_n := \{(p_1, \ldots, p_n) \in \mathbb{Z}^n \mid p_1 \leq \ldots \leq p_n\}$. We put a partial order on $\mathbb{Z}_n$ as follows: Put $(p_1, \ldots, p_n) \leq (p'_1, \ldots, p'_n)$ if and only if $p_i \leq p'_i$ for all $1 \leq i \leq n$.

Moreover, whenever X and Y are subgroups of the same abelian group, we write X/Y for $(X+Y)/Y \cong X/(X \cap Y)$.

2.1 Spectral sequence of a filtered chain complex

In these notes we want to forget about the grading of chain complexes, since this will not be of any importance. An ungraded chain complex (C, d) consists simply of an abelian group C together with an endomorphism $d : C \to C$ such that $d \circ d = 0$. The homology of C is defined as $H(C) := \ker(d)/\mathrm{im}(d)$.

Every ordinary chain complex

$$\ldots \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} \ldots, \quad d_i \circ d_{i+1} = 0,$$

can be made into an ungraded chain complex by simply forgetting about the grading: Simply put $C := \bigoplus_i C_i$ and $d := \bigoplus_i d_i$, i.e. $d((c_i)_{i\in\mathbb{Z}}) = (d_{i-1}(c_{i-1}))_{i\in\mathbb{Z}}$. From now on, all chain complexes will be ungraded.

A subcomplex (C', d') of (C, d) is a subgroup $C' \subseteq C$ such that $d(C') \subseteq C'$, with the inherited differential $d' = d|_{C'}$.

A $\mathbb{Z}$-filtration of C is a family of subcomplexes $(F_p)_{p\in\mathbb{Z}}$ of C such that $F_q \subseteq F_p$ whenever $q \leq p$. The data $(C, d, (F_p)_{\mathbb{Z}})$ is called a filtered (ungraded) chain complex.

Suppose that $(C, d, (F_p)_{\mathbb{Z}})$ is a filtered chain complex. Let us augment the filtration with $F_{-\infty} := 0$ and $F_\infty := C$. For any integers $z \leq p \geq q \leq b$ define an abelian group

$$S[z, q, p, b] := \frac{F_p \cap d^{-1}(F_z)}{d(F_b) + F_q}.$$

Here, p is the *filtration degree*, q the *quotient degree*, b the *boundary degree*, and z the *cycle degree*. So far these are just a bunch of abelian groups. Some of them have a simpler form, for example

$$S[p, q, p, q] \cong F_p/F_q, \quad \text{for } q \leq p, \tag{1}$$

and

$$S[q, q, p, p] \cong H(F_p/F_q), \qquad \text{for } q \leq p, \tag{2}$$

and in particular

$$S[-\infty, -\infty, \infty, \infty] \cong H(C),$$

which is usually the goal of computation. So let's write

$$S[\infty] := S[-\infty, -\infty, \infty, \infty].$$

The collection of groups

$$S[p, p-1, p, p-1] \cong F_p/F_{p-1}, \quad p \in \mathbb{Z}, \tag{3}$$

is called the *zero page*, and the collection of groups

$$S[p-1, p-1, p, p] \cong H(F_p/F_{p-1}), \quad p \in \mathbb{Z}, \tag{4}$$

is called the *first page*, both of which are usually more easy to understand than $H(C)$.

An equivalent description of $S[z, q, p, b]$ for $(z, q, p, b) \in \mathbb{Z}_4$ is the following. For any $(z, q, p) \in \mathbb{Z}_3$ there is a short exact sequence of chain complexes,

$$0 \to F_q/F_z \to F_p/F_z \to F_p/F_q \to 0,$$

where the maps are induced by inclusion. In the associated long exact sequence (actually an exact triangle),

$$H(F_q/F_z) \longrightarrow H(F_p/F_z) \longrightarrow H(F_p/F_q) \xrightarrow{\;d_{zqp}\;} H(F_q/F_z)$$

let $d_{zqp} : H(F_p/F_q) \to H(F_q/F_z)$ denote the connecting homomorphism. Then

$$S[z, q, p, b] \cong \frac{\ker\big(d_{zqp} : H(F_p/F_q) \to H(F_q/F_z)\big)}{\operatorname{im}\big(d_{bpq} : H(F_b/F_p) \to H(F_p/F_q)\big)}, \qquad \text{for } (z, q, p, b) \in \mathbb{Z}_4. \tag{5}$$

Now, there exist two kind of relations between such groups: Differentials and extensions. Let's start with the simpler one.

2.2 Extensions

For any $(z_1, q_1, p_1, b_1) \leq (z_2, q_2, p_2, b_2) \in \mathbb{Z}_4$, there is a map induced by inclusion of chain complexes,

$$\ell : S[z_1, q_1, p_1, b_1] \to S[z_2, q_2, p_2, b_2]. \tag{6}$$

For notational simplicity we don't put indices on ℓ.

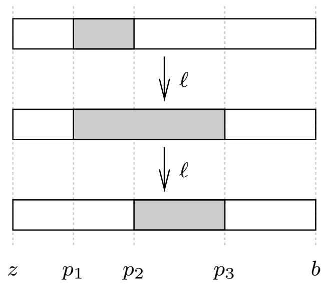

Figure 1: Schematic figure for (7). Here and below we draw $S[z,q,p,b]$ as two strips from z to p and from q to b, and we draw the overlap gray.

Lemma 2.1. *Let $z \le p_1 \le p_2 \le p_3 \le b$. Then there is a short exact sequence*

$$0 \to S[z,p_1,p_2,b] \xrightarrow{\ell} S[z,p_1,p_3,b] \xrightarrow{\ell} S[z,p_2,p_3,b] \to 0. \tag{7}$$

See Figure 1. We leave this as an easy exercise. This yields many relations between these groups. In practice, the following special case is used almost exclusively:

Lemma 2.2. *Set $G_p := S[-\infty,-\infty,p,\infty]$, for $p \in \mathbb{Z}$. They form a filtraton of $S[\infty]$,*

$$\ldots \subseteq G_{p-1} \subseteq G_p \subseteq G_{p+1} \subseteq \ldots \subseteq S[\infty].$$

Moreover the filtration quotients G_p/G_q have a very simple form

$$G_p/G_q \cong S[-\infty,q,p,\infty]. \tag{8}$$

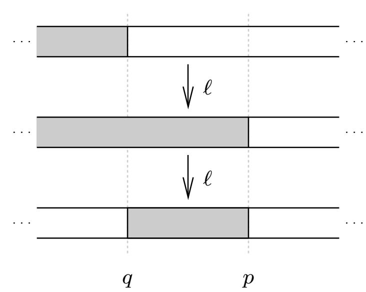

Figure 2: Schematic figure for (8), drawn as a short exact sequence.

Proof. Apply (7) with $z = p_1 = -\infty$, $b = \infty$, $p_3 = p$, and $p_2 = q$. □

Thus one can "more or less" compute $S[\infty]$ from the groups $S[-\infty,q,p,\infty]$ by iteratively computing G_p. At each step one computes G_p as an extension of $S[-\infty,p-1,p,\infty]$ by G_{p-1} (which was obtained in the previous step). There are several serious difficulties though: Without further knowledge, this extension problem is not uniquely solvable. Moreover, the filtration $(G_p)_{\mathbb{Z}}$ may not be *exhaustive* (i.e. $\bigcup_p G_p = S[\infty]$) and it may not be *bounded below* (i.e. $G_p = 0$ for some p). We will address these problems in Section 2.5 below.

But suppose we can solve these problems, how do we compute $S[-\infty, p-1, p, \infty]$? Well, for this we need differentials.

2.3 Differentials

Suppose that two quadruples (z_1, q_1, p_1, b_1) and (z_2, q_2, p_2, b_2) in $\mathbb{Z}_4$[1] satisfy

$$z_2 = p_1 \text{ and } q_2 = b_1. \tag{9}$$

Compare with the first two rows of (10).

$$\begin{array}{ccccccccccccc}
F_{z_1} & \subseteq & F_{q_1} & \subseteq & F_{p_1} & \subseteq & F_{b_1} & & & & & & \\
 & & & & \| & & \| & & & & & & \\
 & & & & F_{z_2} & \subseteq & F_{q_2} & \subseteq & F_{p_2} & \subseteq & F_{b_2} & & \\
 & & & & & & & & \| & & \| & & \\
 & & & & & & & & F_{z_3} & \subseteq & F_{q_3} & \subseteq & F_{p_3} \subseteq F_{b_3}
\end{array} \tag{10}$$

Then d induces a well-defined differential

$$d_2 : S[z_2, q_2, p_2, b_2] \longrightarrow S[z_1, q_1, p_1, b_1]. \tag{11}$$

Then it is an easy exercise (do it!) to check that

$$\ker(d_2) = S[q_1, q_2, p_2, b_2]$$

and

$$\operatorname{coker}(d_2) = S[z_1, q_1, p_1, p_2].$$

Putting both together we get:

Lemma 2.3. *Suppose that we have a sequence of such differentials,*

$$S[z_3, q_3, p_3, b_3] \xrightarrow{d_3} S[z_2, q_2, p_2, b_2] \xrightarrow{d_2} S[z_1, q_1, p_1, b_1], \tag{12}$$

such that (9) *and the corresponding inclusions for d_3 are fulfilled, namely $z_3 = p_2$ and $q_3 = b_2$, see* (10)*. Then we can compute the homology at the middle term and get*

$$\frac{\ker(d_2)}{\operatorname{im}(d_3)} = S[q_1, q_2, p_2, p_3]. \tag{13}$$

[1]Actually we don't need $(z_i, q_i, p_i, b_i) \in \mathbb{Z}_4$. For the differentials it is enough to assume $z_i \leq p_i \geq q_i \leq b_i$, however this makes notation more complicated and less didicatic, and this generality will be used only at the zero page.

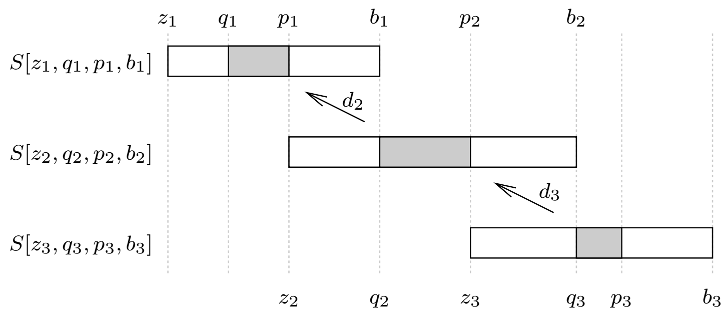

Figure 3: Schematic figure for (12) and (10).

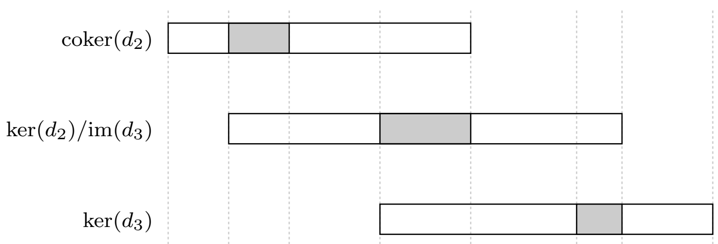

Figure 4: Same as Figure 3 but with $S[z_1, q_1, p_1, b_1]$ replaced by coker d_2, $S[z_2, q_2, p_2, b_2]$ by the homology $\ker(d_2)/\mathrm{im}(d_3)$, and $S[z_3, q_3, p_3, b_3]$ by $\ker(d_3)$.

Example 2.4. Consider $S[p, q, p, q] \cong F_p/F_q$ from (1). Then d induces a differential from this group to itself, and taking homology yields $S[q, q, p, p] \cong H(F_p/F_q)$ from (2). The differential of course agrees under the isomorphism (1) with the one induced on F_p/F_q.

In particular for $q = p - 1$, this means that taking homology at the zero page (3) yields the first page (4).

For $p = \infty$ and $q = -\infty$ this means that taking homology of C yields $H(C)$, a tautology.

Now let's generalize this example. For $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $p \in \mathbb{Z}$, define

$$E_p^r := S[p - r, p - 1, p, p - 1 + r]. \tag{14}$$

Then d induced differentials

$$\ldots \xrightarrow{d_r} E_{p+r}^r \xrightarrow{d_r} E_p^r \xrightarrow{d_r} E_{p-r}^r \xrightarrow{d_r} \ldots, \tag{15}$$

which are usually denoted as d_r, and taking homology at E_p^r yields E_p^{r+1}. In symbols,

$$E_*^{r+1} = H(E_*^r, d_r). \tag{16}$$

In particular, $E_p^0 = F_p/F_{p-1}$ constitute the zero page, and $E_p^1 = H(F_p/F_{p=1})$ the first page.

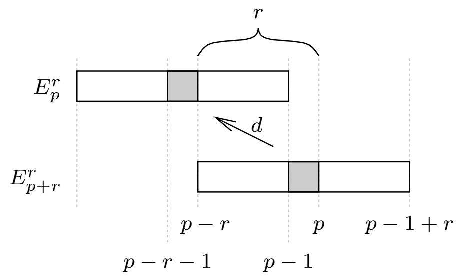

Figure 5: Schematic figure for $d : E_{p+1}^r \to E_p^r$.

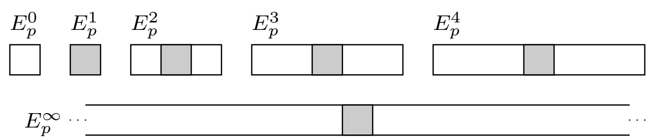

Figure 6: Schematic figure for E_p^r for $r = 0, 1, 2, 3, 4, \infty$.

A word of warning: E_p^{r+1} is clearly a *subquotient* of E_p^r, i.e. a quotient of a subgroup. Taking infinitely often homology with respect to the respective differentials yields a abelian group, say $\widetilde{E}_p^\infty$, which is of course a subquotient of E_p^r for every $r \in \mathbb{Z}_{\geq 0}$. However in general, E_p^∞ is only a subquotient of $\widetilde{E}_p^\infty$. They need not coindice. A simple example is a constant filtration, where we choose two proper subcomplexes $0 \subset F \subset F' \subset C$ and put $F_p = F$ for $p \in \mathbb{Z}< 0$ and $F_p = F'$ for $p \in \mathbb{Z}_{\geq 0}$: Here $\widetilde{E}_0^\infty = H(F'/F)$, whereas $E_0^\infty = \frac{F' \cap d^{-1}(0)}{d(C)+F}$.

The simplest sufficient criterion for equality of $\widetilde{E}_p^\infty$ and E_p^∞ is that the filtration is *bounded*, i.e. when there exists integers p_1 and p_2 such that $F_{p_1} = 0$ and $F_{p_2} = C$, since then $E_p^r = E_p^\infty$ for all $r \geq p_2 - p_1$. A more general criterion is discussed in Section 2.5.

Remark 2.5. Let us remark that some authors (not all) write E_p^∞ for what we wrote $\widetilde{E}_p^\infty$, this may cause some confusion. But in practise, for most spectral sequences both coincide.

2.4 Standard recipe

Lemmas 2.3 and 2.1 together yield the ordinary spectral sequence of the filtered chain complex $(C, d, (F_p)_\mathbb{Z})$: It relates the zero page E_*^0 and the first page E_*^1 to the usual goal of computation $H(C)$ as above: First compute E_*^r for all $r \in \mathbb{Z}_{\geq 0}$, by iterating (16). If (F_p) is bounded, E_p^r stabilizes at E_p^∞. Finally, $H(C)$ has a filtration (which is natural with respect to (F_p)) whose filtration quotients are E_p^∞.

Grading. Most often, C is a graded chain complex, i.e. a graded abelian group and d is a graded homomorphism of degree $+1$ or -1. Then the $S[z, q, p, b]$ are graded abelian groups as well, all maps induced by inclusion 6 are graded of degree 0, and all differentials (11) are graded of the same degree as d. In particular, E_p^r, $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, is a graded group as well. If d is of degree -1, i.e. C is a homology chain complex, then the standard notation is

$$E_{pq}^r \tag{17}$$

for the $(p+q)$'th graded piece of E_p^r. Note that this q has a different meaning than the q above! The q in (17) is called the *complementary degree*, and $p+q$ is called the *total degree*. In particular, the differentials (15) on E_*^r, $r \in \mathbb{Z}_{\geq 0}$, decompose as differentials

$$\ldots \xrightarrow{d_r} E_{p+r,q-r+1}^r \xrightarrow{d_r} E_{p,q}^r \xrightarrow{d_r} E_{p-r,q+r-1}^r \xrightarrow{d_r} \ldots \tag{18}$$

Similarly, the filtration of $H(C)$ from Lemma 2.1 decomposes as filtrations of the graded pieces $H_n(C)$, $n \in \mathbb{Z}$, and the associated filtration quotients for $H_n(C)$ are E_{pq}^∞ with $p+q=n$. Given $r \geq 0$, the collection of all E_{pq}^r is called the *r'th page* E^r, and it is usually displayed in a 2-dimensional coordinate system with E_{pq}^r at coordinates (p,q), as in Figure 7.

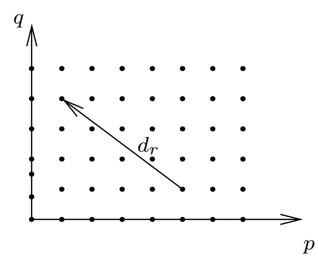

Figure 7: The coordinate system for E^r, with one of the differentials drawn for $r=4$.

If d is of degree $+1$, i.e. C is a cohomology chain complex, then people usually prefer decreasing filtrations of C, i.e.

$$C \supseteq \ldots \supseteq F^{p-1} \supseteq F^p \supseteq F^{p+1} \supseteq \ldots \supseteq 0,$$

which can be turned into the previous situation of inscreasing filtrations via $F_p := \overline{F^{-p}}$, where the bar means that we negate the grading of F^{-p}. Then as above we get abelian groups E_{pq}^r, but the standard notation now is

$$E_r^{pq} := E_{-p,-q}^r, \tag{19}$$

and the differentials go in the opposite direction,

$$\ldots \xrightarrow{d_r} E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1} \xrightarrow{d_r} \ldots$$

2.5 Standard convergence and comparison theorems

We saw that there might be some serious problems to connect E^1 to $S[\infty]$. First, E_p^∞ might be a proper subquotient of $\widetilde{E}_p^\infty$; compare with Section 2.3. Second, $S[\infty]$ might not be uniquely determinded by E_p^∞, $p \in \mathbb{Z}$, compare with Section 2.2.

Definition 2.6. Let $(F_p)_{\mathbb{Z}}$ be a filtration of a chain complex C. It is called

1. *bounded below* if $F_p = 0$ for some p,
2. *bounded* if $F_p = 0$ for some p and $F_{p'} = C$ for some p',
3. *Hausdorff* if $\bigcap_p F_p = 0$,
4. *exhaustive* if $\bigcup_p F_p = C$, and
5. *complete* if $C = \varprojlim_p C/F_p$.

Definition 2.7. A spectral sequence is said to *converge* (to $S[\infty]$) if

1. $E_p^\infty = \widetilde{E}_p^\infty$ for all p,
2. $S[\infty] = \bigcup_p G_p$ and $0 = \bigcap_p G_p$, and
3. $S[\infty] = \varprojlim_p S[\infty]/G_p$,

where as above, $G_p := S[-\infty, -\infty, p, \infty]$. If only (1) holds then the spectral sequence *weakly converges* to $S[\infty]$, and if (1) and (2) hold it *approaches/abuts* $S[\infty]$.

Theorem 2.8 (Convergence)**.** *If $(F_p)_{\mathbb{Z}}$ is a filtration of C, which is bounded below and exhaustive. Then the associated spectral sequence converges to $S[\infty]$.*

Theorem 2.9 (Comparison)**.** *Let C and C' be chain complexes, both with a complete and exhaustive filtration. Let $f : C \to C'$ be a filtration preserving map. It induces a morphism between the associated spectral sequences. If for some $r \geq r_0$, $f_* : E_p^r(C) \to E_p^r(C')$ is an isomorphism for all p, then $f_* : H(C) \to H(C')$ is an isomorphism as well.*

These are the standard theorems; there are more refined versions, see e.g. Boardman [1].

Remark 2.10. The standard definition of a homology spectral sequence (as given for example in Weibel) consists only of the data E_p^r for $r_0 \leq r < \infty$ together with the differentials d_r (18) such that $E^{r+1} = H(E^r, d)$. Personally I prefer to include all $S[z, q, p, b]$ (in particular also E^∞ and $S[\infty]$) into the data of a spectral sequence and call $S[\infty]$ the limit of the spectral sequence, though this is just a name and does not mean any convergence (I beg for forgiveness if this causes confusion).

2.6 Spectral sequence of a filtered space

Now let X be a filtered space with (increasing) filtration $(X_p)_{p\in\mathbb{Z}}$ and let h be a generalized homology theory, i.e. a functor from pairs of spaces to abelian groups that satisfies all Eilenberg–Steenrod axioms except for possibly the dimension axiom.

For any $(z,q,p) \in \mathbb{Z}_3$ there is a long exact sequence for the triple (X_p, X_q, X_z),

$$h(X_q, X_z) \longrightarrow h(X_p, X_z) \longrightarrow h(X_p, X_q), \qquad h(X_p, X_q) \xrightarrow{d_{zqp}} h(X_q, X_z)$$

and we call the connecting homomorphism d_{zqp}. Then we use the analog of (5) to define

$$S[z,q,p,b] := \frac{\ker\big(d_{zqp} : h(X_p, X_q) \to h(X_q, X_z)\big)}{\operatorname{im}\big(d_{bpq} : h(X_b, X_p) \to h(X_p, X_q)\big)}, \qquad \text{for } (z,q,p,b) \in \mathbb{Z}_4. \tag{20}$$

Note that here there is no reasonable zero page, but this causes no problem, we simply start at

$$S[q,q,p,p] \cong h(X_p, X_q), \quad q \le p,$$

in particular the first page given by all

$$S[p-1,p-1,p,p] \cong h(X_p, X_{p-1}), \quad p \in \mathbb{Z}.$$

As for filtered chain complexes we get natural maps induced by inclusion (6) and differentials (11), though the well-definedness and naturality of the differentials is a bit more technical here. Moreover, Lemmas 2.1 and 2.3 hold in the same wording for 20. Define E_p^r via the same formula (14). Then we get the same recipe as in Section 2.4 to relate the corresponding first page $E_p^1 = h(X_p, X_{p-1})$ to $S[\infty] = h(X)$.

Cohomology version. Consider the same filtered space X as above, but now consider a generalized cohomology theory h. To simplify indices, make $(X_p)_\mathbb{Z}$ into a decreasing filtration via $X^p := X_{-p}$. Similarly to above, for $(z,q,p) \in \mathbb{Z}_3$ let d^{pqz} denote the connecting homomorphism in the long exact sequence in h of the triple (X^z, X^q, X^p). Define

$$S[z,q,p,b] := \frac{\ker\big(d^{pqz} : h(X^q, X^p) \to h(X^z, X^q)\big)}{\operatorname{im}\big(d^{bpq} : h(X^p, X^b) \to h(X^q, X^p)\big)}, \qquad \text{for } (z,q,p,b) \in \mathbb{Z}_4.$$

Then similarly as above, $S[q,q,p,p] = h(X^q, X^p)$, $q \le p$, and thus $E_1^p = h(X^{p-1}, X^p)$ and $S[\infty] = h(X)$.

3 Examples of spectral sequences

3.1 Leray–Serre and Atiyah–Hirzebruch spectral sequences

Let

$$F \xrightarrow{i} E \xrightarrow{\pi} B$$

be a Serre fibration, whose base space B is a connected CW-complex (actually it would be enough to assume that B has the homotopy type of a CW-complex). Let h be a generalized homology theory. The Leray–Serre spectral sequence is a spectral sequence with

$$S[\infty] = h(E).$$

It is constructed from the following filtration of the total space E. Let $B^{(p)}$ denote the p-skeleton of B. Define $E_p := \pi^{-1}(B^{(p)})$. This is a filtration of closed subspaces of E, which creates no problem. If you want you can take instead the preimages of small neighborhoods of the skeleta $B^{(p)}$ that deformation retract to $B^{(p)}$. Then one can check that in the associated spectral sequence,

$$E^1_p = h(E_p, E_{p-1}) \cong C_p(B; h(F))$$

is the p-th graded piece of the cellular chain complex of B with local coefficients in $h(F)$. (We will not give any proofs here, but instead refer to the standard text books mentioned in the introduction.) The local coefficient system comes from the canonical action of $\pi_1(B)$ on $h(F)$. Moreover, the second page is given by

$$E^2_p = H_p(B; h(F)),$$

the p'th cellular homolgoy of B with local coefficients in $h(F)$. If h is $\mathbb{Z}$-graded then $E^2_{p,q} = H_p(B; h_q(F))$.

Analogously, if h is a generalized cohomology theory then there is an associated cohomology spectral sequence with

$$E_2^p = H^p(B; h(F)),$$

and if h is $\mathbb{Z}$-graded then $E_2^{pq} = H^p(B; h^q(F))$.

Let us remark that when h is complex K-theory, then E_r^p is naturally $\mathbb{Z}/2$-graded (similarly with any other periodic generalized (co)homology theory). If you prefer $\mathbb{Z}$-gradings, of course E_r^p can also be regarded as being $\mathbb{Z}$-graded with periodic rows: $E_r^{pq} = E_r^{p,q+2}$.

Switzer [13] calls these two spectral sequences the Leray–Serre spectral sequences. Originally this was called the Leray–Serre spectral sequence only when h is (co)homology with coefficients in some abelian group or ring. For an arbitrary generalized (co)homology theory h and when $F = \text{pt}$ and $E = B$, this was originally called the Atiyah–Hirzebruch spectral sequence. The unification seems to be due to Dold [4].

3.2 Spectral sequence of a double complex

A (homology) *double complex* D is a chain complex of chain complexes, i.e. a $\mathbb{Z}^2$-graded family of abelian groups (or objects in an arbitrary fixed abelian category) $(D_{ij})_{i,j\in\mathbb{Z}}$ together with boundary homomorphisms $d_1 : D_{ij} \to D_{i-1,j}$ and $d_2 : D_{ij} \to D_{i,j-1}$ such that $d_1 \circ d_2 = d_2 \circ d_1$. (Some other authors instead require $d_1 \circ d_2 + d_2 \circ d_1 = 0$; these two equations can be turned into each other, for example by negating d_2 at every (ij) for which i is odd.)

The *total complex* of D is the chain complex $(\mathrm{Tot}(D), d)$ where

$$\mathrm{Tot}(D) := \bigoplus_{i,j} C_{ij}$$

and

$$d(c_{ij}) := d_1(c_{ij}) + (-1)^i d_2(c_{ij}), \quad c_{ij} \in C_{ij}.$$

$\mathrm{Tot}(D)$ is $\mathbb{Z}$-graded, the n'th graded piece being $\bigoplus_{i+j=n} C_{ij}$.

In many siutations one wants to compute $H_*(\mathrm{Tot}(D))$. There are two canonical $\mathbb{Z}$-filrations of $\mathrm{Tot}(D)$, leading to two spectral sequences.

Horizontal filtration. Let $F_p^{(1)} := \bigoplus_{i,j:i\leq p} C_{ij} \subseteq \mathrm{Tot}(D)$. Then one easily checks that in the associated spectral sequence,

$$E^0_{pq} = C_{pq}, \quad E^1_{pq} = H_q(C_{p,*}, d_2), \quad E^2_{pq} = H_p(H_q(C_{*,*}, d_2), d_1),$$

and of course $S[\infty] = H_*(\mathrm{Tot}(D))$.

Vertical filtration. Let $F_p^{(2)} := \bigoplus_{i,j:j\leq p} C_{ij} \subseteq \mathrm{Tot}(D)$. Then one checks that in the associated spectral sequence,

$$E^0_{pq} = C_{qp}, \quad E^1_{pq} = H_q(C_{*,p}, d_1), \quad E^2_{pq} = H_p(H_q(C_{*,*}, d_1), d_2),$$

and again $S[\infty] = H_*(\mathrm{Tot}(D))$.

In practice, one often plays these two spectral sequences out against each other. in particular if often happens that for one of these two filtrations, E^2_{pq} is concentrated in one row or in one column, such that the second page already agrees with $H_*(\mathrm{Tot}(D))$. An example for this is the Grothendieck spectral sequence from the next section.

Cohomology version. If D is a cohomology double complex, i.e. d_i is of bidegree $(+1, 0)$ and d_2 is of digree $(0, +1)$, then one gets two analogous spectral sequences: Simply negate both coordinates to reduce to the homological setup, and negate the coordinates back using the convention (19). Then the two cohomology spectral sequences satisfy

$$E_0^{pq} = C^{pq}, \quad E_1^{pq} = H^q(C^{p,*}, d_2), \quad E_2^{pq} = H^p(H^q(C^{*,*}, d_2), d_1),$$

respectively

$$E_0^{pq} = C^{qp}, \quad E_1^{pq} = H^q(C^{*,p}, d_1), \quad E_2^{pq} = H^p(H^q(C^{*,*}, d_1), d_2).$$

Both satisfy $S[\infty] = H^*(\mathrm{Tot}(D))$.

3.3 Grothendieck spectral sequence

Let $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{A}_2$ be abelian categories such that $\mathcal{A}_0$ and $\mathcal{A}_1$ have enough projectives. Then Grothendieck's spectral sequence [7] computes the left derived functors of a composition of two right-exact functors

$$\mathcal{A}_0 \xrightarrow{F_1} \mathcal{A}_1 \xrightarrow{F_2} \mathcal{A}_2 \tag{21}$$

from the left derived functors of F_1 and F_2, assuming that F_1 sends projective objects to F_2-acyclic objects. More precisely, for any object $A \in \mathcal{A}_0$, the second page is given by

$$E^2_{pq} = L_p F_2 \circ L_q F_1(A)$$

and it converges to

$$L_{p+q}(F_2 \circ F_1)(A).$$

CE-resolutions. The construction of Grothendieck's spectral sequence is based on the Cartan–Eilenberg resolution (or CE-resolutions for short) for chain complexes, see Cartan–Eilenberg [3]. The original construction of CE-resolutions is somewhat technical (it uses the horse shoe lemma, see Cartan–Eilenberg [3] or Weibel [14]), and a more conceptual construction (using relative homological algebra, see Eilenberg–Moore [5]) needs machinery that we don't want to assume here. But the intuition is as follows.

Call a chain complex P_* *CE-projective*, if it is a locally finite sum of complexes of the form

$$\ldots \to 0 \to 0 \to P \to 0 \to 0 \to \ldots$$

or of the form

$$\ldots \to 0 \to 0 \to P \xrightarrow{\mathrm{id}_P} P \to 0 \to 0 \to \ldots$$

where P is a projective object. Furthermore, call a map of chain complexes $f : C_* \to D_*$ *CE-surjective*, if $f_i : C_i \to D_i$ and $f_i|_{Z_i(C_*)} : Z_i(C_*) \to Z_i(D_*)$ are surjective for all i, where $Z_i(C_*) := \ker(d : C_i \to C_{i-1})$. Using this, one can construct CE-resolutions as usual: In order to resolve a chain complex C_*, take a CE-surjective map $g : P_* \to C_*$ with P_* being CE-projective (it exists!), and iterate with the kernel of g. For our purposes the fundamental property is that for any CE-resolution $P_{**} \to C_*$, taking homology with respect to the horizontal differential (imagine C_* as a chain complex on the x-axis) yields projective resolutions $H_k(P_{**}) \to H_k(C_*)$ of the $H_k(C_*)$.

Construction of Grothendieck's spectral sequence. Suppose we are in the above setup, and consider $A \in \mathcal{A}_0$. Let $P_* \to A$ be a projective resolution of A, and consider P_* as a chain complex lying on the non-negative x-axis. Apply F_1 to P_* and take a Cartan–Eilenberg resolution $Q_{**} \to F_1(P_*)$, which we consider as a double complex in the first quadrant of $\mathbb{Z}^2$, the resolution degree corresponding to the y-coordinate. Define $D_{**} := F_2(Q_{**})$. There are two spectral sequences associated to the double complex D_{**}. The one coming from the vertical filtration has

$$E^1_{pq} = H_q(D_{*,p}, d_1) = F_2(H_q(Q_{*,p}, d_1))$$

by the special form of CE-projective chain complexes. Now $H_q(Q_{*,p}, d_1)$ is a projective resolution of $H_q(F_1(P_*))$. Thus

$$E^2_{pq} = L_p F_2(H_q(F_1(P_*))) = L_p F_2 \circ L_q F_1(A).$$

We call this the Grothendieck spectral sequence.

In order to determine what it converges to, consider the spectral sequence coming from the horizontal filtration. It has

$$E^1_{pq} = H_q(D_{p,*}, d_2) = H_q(F_2(Q_{p,*}), d_2).$$

Now we assumed that F_1 sends projective objects from $\mathcal{A}_0$ to F_2-acyclic ones. This implies that $E^1_{pq} = 0$ for all $q \geq 1$ and

$$E^1_{p0} = H_0(F_2(Q_{p,*}), d_2) = F_2(F_1(P_p)).$$

Thus also E^2 is concentrated in the row $q = 0$ with

$$E^2_{p0} = L_p(F_2 \circ F_1)(A),$$

which therefore coincides with $H_p(\mathrm{Tot}(D_{**}))$, the sum of which is also $S[\infty]$ for Grothendieck's spectral sequence.

The construction itself was quite short, but more work is needed to show that it is well-defined from the second page on, and that the spectral sequence is natural with respect to morphisms in $\mathcal{A}_0$.

Cohomology version. Analogously there is a cohomological version of Grothendieck's spectral sequence in the situation of three abelian categories $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$ such that A_0 and A_1 have enough injectives. If then we have a sequence of left-exact functors (21) such that F_1 sends injective objects of $\mathcal{A}_0$ to F_2-acyclic objects of $\mathcal{A}_1$, then for any $A \in \mathcal{A}_0$ there is a cohomological Grothendieck spectral sequence with

$$E_2^{pq} = R^p F_2 \circ R^q F_1(A)$$

and $S[-\infty, -\infty, \infty, \infty] = R^*(F_2 \circ F_1)(A)$.

3.4 Bockstein spectral sequence

Let $C_*(X)$ denote the cellular chain complex of a CW-complex X of finite type (i.e. only finitely many cells in each dimension). The following will also work immediately for any other free chain complex of finite rank in each degree.

Fix a prime ℓ, and consider $\mathbb{Z}[1/\ell] = \{a\ell^n \mid a \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}\} \subset \mathbb{Q}$. Consider the chain complex $C = C_*(X) \otimes \mathbb{Z}[1/\ell]$. We filter it via

$$0 \subseteq \ldots \subseteq F_p \subseteq F_{p+1} \subseteq \ldots \subseteq C \tag{22}$$

with $F_p := C_*(X) \otimes \ell^{-p}\mathbb{Z} = C_*(X; \ell^{-p}\mathbb{Z})$. The associated spectral sequence is called the *Bockstein spectral sequence*. If $[\ell]$ denotes the multiplication-by-ℓ map then $[\ell] : F_p \to F_{p-1}$ is clearly an isomorphism for any $p \in \mathbb{Z}$. Thus the Bockstein spectral sequence has periodic columns, i.e. $E^r_p \cong E^r_{p+1}$, more precisely,

$$E^r_{pq} \cong E^r_{p+1,q-1}, \qquad \text{for all } r, p, q. \tag{23}$$

This periodicity is quite special and does not happen very often. In particular, all F_p are isomorphic to $F_0 = C_*(X)$.

Since $C_*(X)$ is free, there is a short exact sequence,

$$0 \to F_{p-1} \hookrightarrow F_p \to C_*(X; \mathbb{F}_\ell) \to 0, \tag{24}$$

where we identified $(\ell^{-p}\mathbb{Z})/(\ell^{-p+1}\mathbb{Z})$ with $\mathbb{F}_\ell = \mathbb{Z}/\ell\mathbb{Z}$. From this we can read off

$$E^0_{0,q} = C_q(X; \mathbb{F}_\ell), \quad E^1_{0,q} = H_q(X; \mathbb{F}_\ell), \qquad \text{for } q \in \mathbb{Z}.$$

The short exact sequence of chain complexes (24) induces an exact triangle in homology,

$$H_*(X) \xrightarrow{[\ell]_*} H_*(X) \xrightarrow{\text{mod}} H_*(X; \mathbb{F}_\ell). \tag{25}$$
$$H_*(X; \mathbb{F}_\ell) \xrightarrow{\partial} H_*(X)$$

The composition $\beta := \text{mod} \circ \partial : H_*(X; \mathbb{F}_\ell) \to H_*(X; \mathbb{F}_\ell)$, which is of degree -1, is called *Bockstein homomorphism*, and one can show that the differential d_1 on E^1 coincides with the composition of $\beta : E^1_{0,q} \to E^1_{0,q-1}$ with the isomorphism $E^1_{0,q-1} \cong E^1_{-1,q}$ from (23). That's what the spectral sequence is named after.

The original construction of the Bockstein spectral sequence by Browder (and perhaps Moore) came from regarding (25) as a particular $\mathbb{Z}$-graded exact couple (which we did not explain here), which yields a "spectral sequence" with just one column per page, and also no extension process was considered, since $E^\infty_{0,*}$ was already considered as the limit. In the above construction,

$$S[\infty] = H_*(X; \mathbb{Z}[1/\ell]) = H_*(X) \otimes \mathbb{Z}[1/\ell]. \tag{26}$$

Let $Z_*(X) = \ker(d : C_*(X) \to C_*(X))$ and $B_*(X) = \operatorname{im}(d : C_*(X) \to C_*(X))$ denote the cycles and boundaries of $C_*(X)$. Then by definition E^∞ is given by

$$E^\infty_{0,*} = \frac{F_0 \cap d^{-1}(0)}{d(C) + F_{-1}} = \frac{Z_*(X)}{B_*(X)[1/\ell] + \ell C_*(X)} = (H_*(X)/\text{torsion}) \otimes \mathbb{F}_\ell.$$

The last equality uses that $C_*(X)$ is of finite type. It is also special here that the E^∞-page has a simple description. This is one reason why in this spectral sequence E^∞ is more interesting than than $S[-\infty, -\infty, \infty, \infty]$. Another one is that E^∞ sees no torsion of $H_*(X)$, whereas (26) artificially sees the non-ℓ-torsion, since $\mathbb{Z}/q^k \otimes \mathbb{Z}[1/\ell] = \mathbb{Z}/q^k$ for primes $q \neq \ell$.

Remark 3.1. The filtratoin (22) is not complete, i.e. C does not equal the inverse limit $\widehat{C} := \varprojlim C/F_p$ (unless $X = \emptyset$). Thus from this point of view it makes sense to replace C by $\widehat{C} = C_*(X) \otimes \mathbb{Q}_\ell$, and F_p by

$$\widehat{F}_p = \varprojlim_q F_p/F_q = C_*(X) \otimes \{a \in \mathbb{Q}_\ell \mid \nu_\ell(a) \geq -p\},$$

where $\mathbb{Q}_\ell$ are the ℓ-adic numbers and ν_ℓ is the ℓ-adic valuation, i.e. $\nu(\ell^n x) := n$ whenever $x \in \mathbb{Z}_\ell^*$. Both spectral sequences have naturally isomorphic E^r-pages for all $0 \leq r \leq \infty$.

4 Extra structure

4.1 Naturality

Let $(C, d, (F_p)_\mathbb{Z})$ and $(C', d', (F'_p)_\mathbb{Z})$ be two filtered chain complexes. A chain map $f : (C, d) \to (C', d')$ is called *filtration preserving* if $f(F_p) \subseteq F'_p$ for all p.

Any such f induces canonical homomorphisms

$$f_* : S[z, q, p, b] \to S'[z, q, p, b], \quad (z, q, p, b) \in \mathbb{Z}_4.$$

Moreover they all maps induced by inclusion ℓ (6) and with all differentials (11).

The same holds for filtration preserving maps between spaces $X \to X'$ and their associated spectral sequences.

This should not be underestimated.

Example 4.1 (Edge-homomorphisms in Leray–Serre spectral sequence)**.** Consider the Leray–Serre spectral sequence of the fibration

$$F \hookrightarrow E \xrightarrow{\pi} B, \tag{27}$$

over a connected CW-complex B and for a generalized homology theory h.

There is a canonical map of fibrations from

$$F \hookrightarrow F \to \mathrm{pt} \tag{28}$$

to (27), and another one from (27) to

$$\mathrm{pt} \hookrightarrow B \to B. \tag{29}$$

For clarity, here they are in one diagram:

$$\begin{array}{ccccc}
F & \longrightarrow & F & \longrightarrow & \mathrm{pt} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
F & \longrightarrow & E & \longrightarrow & B \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathrm{pt} & \longrightarrow & B & \longrightarrow & B
\end{array}$$

Both of them induce maps between the associated Leray–Serre spectral sequences. This is very useful, since the spectral sequence for (28) is very easy to understand: Its E^2_p-term it is $H_p(\mathrm{pt}; h(F))$, which is $h(F)$ for $p = 0$, and zero otherwise. I.e. this spectral sequence collapses at E^2. Diagram chasing now yields that the canonical map $h(F) \to h(E)$ factors as

$$h(F) \xrightarrow{\cong} E^2_0 \twoheadrightarrow E^\infty_0 \hookrightarrow h(E).$$

This composition is one of the two so-called *edge-homomorphisms* in the Leray–Serre spectral sequence.

Similarly, the spectral sequence for (29) has E^2-page $H_p(B; h(\mathrm{pt}))$. In case h is homology with coefficients in some group G, then this simplifies to $H_p(B; G)$ and the spectral sequence collapses again. Diagram chasing yields, that the canonical map $H_p(E; G) \to H_p(B; G)$ factors as

$$H_p(E; G) \twoheadrightarrow E^\infty_{p,0} \hookrightarrow E^2_{p,0} \xrightarrow{\cong} H_p(B, G).$$

This is the other *edge-homomorphism*.

There are analogous edge-homomorphisms and for generalized cohomology theories h in the associated cohomology spectral sequence E^{pq}_r, namely $h(E) \to h(F)$ factors as

$$h(E) \twoheadrightarrow E^0_\infty \hookrightarrow E^0_2 \xrightarrow{\cong} h(F),$$

and if furthermore $h = H^*(\_; G)$ then $H^p(B; G) \to H^p(E; G)$ factors as

$$H^p(B; G) \xrightarrow{\cong} E^{p,0}_2 \twoheadrightarrow E^{p,0}_\infty \hookrightarrow H^p(E; G).$$

The generalizations to non-connected B are left to the reader.

4.2 Multiplicative structure

Often cohomological spectral sequences E_r^p have a *multiplicative structure*, which means the following. All E_r^p for $r \geq r_0$ and $S[\infty]$ are modules over some ring R, and there are R-bilinear maps

$$\cdot_r : E_r^{p_1} \times E_r^{p_2} \to E_r^{p_1+p_2}$$

for all $r \geq r_0$ (usually r_0 is 0, 1, or 2), which are graded if E_r^p is. They are required to satisfy a *Leibnitz rule* (usually the sign also depends on the grading of E_r^p),

$$d_r(x \cdot_r y) = d_r(x) \cdot_r y \pm x \cdot_r d_r(y), \quad x \in E_r^{p_1}, y \in E_r^{p_1},$$

and a compatibility condition: For any $r > r' \geq r_0$, $\cdot_r$ is induced from $\cdot_{r'}$. In other words, if $x' \in E_{r'}^{p_1}$ is a lift of $x \in E_r^{p_1}$ and $y' \in E_{r'}^{p_2}$ a lift of $y \in E_r^{p_2}$, then $x' \cdot_{r'} y' \in E_{r'}^{p_1+p_2}$ is required to be a lift of $x \cdot_r y \in E_r^{p_1+p_2}$.

Moreover, one requires a product on $S[\infty]$, i.e. an R-bilinear map

$$\cdot : S[\infty] \times S[\infty] \to S[\infty]$$

which induces the product d_∞ on E_∞^*.

Usually, if a spectral sequence has a useful multiplicative structure then $\cdot_{r_0}$ and $\cdot$ have a meaningful description.

Example 4.2 (Multiplicative structure for Leray–Serre spectral sequence)**.** Suppose that h is a multiplicative generalized cohomology theory. Then the Leray–Serre spectral sequence for $F \hookrightarrow E \to B$ has a multiplicative structure from $r_0 = 2$ on. Moreover, at E^2, the product conincides (perhaps up to signs, depending on the convention) with the cup product on $H^*(B; h(F))$, and at $S[\infty]$ it coincides with the product on $h(E)$.

4.3 Examples

An excellent selection of exercises can be found in Hatcher [8]. They show how much can be squeezed out of a spectral sequence from various kinds of partial information.

5 Higher spectral sequences

Consider a chain complex (C, d) that is filtered in n different compatible[2] ways over the integers, or alternatively, a space X that is filtered in n different ways over the integers together with a generalized homology theory h_*. Then there is an associated "higher spectral sequence" with $S[\infty]$ being $H_*(C)$ respectively $h_*(X)$.

They allow more flexibility than ordinary spectral sequences. For example, starting from the first page, there are always not only one but n differentials to choose from (until we reach the first extension step, see below).

[2]Compatibility means that the associated exact couple system (31) is excisive. This occurs for example if C is of the form $\bigoplus_{P \in \mathbb{Z}^n} C_P$ with the n canonical $\mathbb{Z}$-filtrations.

More precisely, we construct for any admissible word $\omega \in L_a^*$ (see Definition 5.11) over the alphabet

$$L := \{1, \ldots, n, 1^\infty, \ldots, n^\infty, \mathrm{x}\}.$$

a so-called ω-page, which is a collection of abelian groups $S(P;\omega)$. Here P ranges over a quotient $\mathbb{Z}^n/V_\omega \cong \mathbb{Z}^{n-k}$, where k is the number of letters x in ω. In the alphabet L, a letter $j \in [n]$ stands for taking homology with respect to the j'th differential, j^∞ denotes the same but infinitely often, and x stands for a group extension process.

In ordinary spectral sequences we have $n = 1$, and for $\omega = 1^{r-1}$ the ω-page consists of the columns in E^r_{**}, which are indexed over $P \in \mathbb{Z}$. The letter 1 stands the relation between some E^r_{**} and E^{r+1}_{**}, 1^∞ stands for the relation between some E^r_{**} and E^∞_{**}, and x for the relation between E^∞_{**} and $S[\infty]$, e.g. $S[\infty] = H(C)$ if the spectral sequence comes from a $\mathbb{Z}$-filtered chain complex C.

In Section 2.3, we define certain vectors $r^j_\omega, \delta^j_\omega \in \mathbb{Z}^n$, where r^i_ω will be the direction of the i'th differential at the ω-page, and δ^i_ω is the change of direction for the i'th differential that occurs when taking homology with respect to it. In ordinary spectral sequences, for $\omega = 1^{r-1}$ we have $r^1_\omega = r$, and $\delta^1_\omega = 1$.

Theorem 5.1 (Main theorem)**.** *Let $\omega \in L_a^*$ and $j \in [n]$ such that $\omega * j$ is admissible. Then the following holds.*

a) There are natural differentials

$$\ldots \longrightarrow S(P + r^j_\omega; \omega) \longrightarrow S(P;\omega) \longrightarrow S(P - r^j_\omega;\omega) \to \ldots. \tag{30}$$

Taking homology at $S(P;\omega)$ yields $S(P;\omega * j)$.

b) $S(P;\omega * j^\infty)$ is a natural subquotient of $S(P;\omega * j^k)$ for all $k \geq 0$.

c) There exists a natural $\mathbb{Z}$-filtration $(F_i)_{i\in\mathbb{Z}}$ of $S(P;\omega * j^\infty\mathrm{x})$,

$$0 \subseteq \ldots \subseteq F_i \subseteq F_{i+1} \subseteq \ldots \subseteq S(P;\omega * j^\infty \mathrm{x}),$$

such that $S(P + i \cdot \delta^j_\omega; \omega * j^\infty) \cong F_i/F_{i-1}$, for all $i \in \mathbb{Z}$.

In the rest of this section we explain this theorem, and how it is useful.

5.1 Preliminaries and natural isomorphisms

Let $n \geq 1$ and $[n] := \{1, \ldots, n\}$. Let $e_1, \ldots, e_n$ be the standard basis vectors in $\mathbb{Z}^n$, and $\mathbb{1} := (1, \ldots, 1)^t \in \mathbb{Z}^n$. $\mathbb{Z}^n$ is a poset via $(x_1, \ldots, x_n) \leq (x'_1, \ldots, x'_n)$ if and only if $x_i \leq x'_i$ for all i.

Throughout this section, let $I := D(\mathbb{Z}^n)$ denote the lattice of downsets of $\mathbb{Z}^n$. I has minimum $-\infty := \emptyset$ and maximum $\infty := \mathbb{Z}^n$. We write $I_k := \{(p_1, \ldots, p_k) \in I^k \mid p_1 \geq \ldots \geq p_k\}$, which is again a poset via $(p_1, \ldots, p_k) \leq (p'_1, \ldots, p'_k)$ if and only if $p_i \leq p'_i$ for all i.

As above, for us a chain complex is an abelian group C together with an endomorphism $d : C \to C$ with $d \circ d = 0$, and its homology is $H(C, d) := \ker(d)/\mathrm{im}(d)$; the grading is not of importance for us. An I-filtration of C is a family of subchain complexes $(F_p)_{p \in I}$ such that $F_q \subseteq F_p$ whenever $q \leq p$.

Similarly, if X is a topological space then an I-filtration of X is a family of open subspaces $(X_p)_{p \in I}$ such that $X_q \subseteq X_p$ whenever $q \leq p$.

Whenever we have an I-filtered chain complex (C, d), or an I-filtered space X together with a generalized homology theory h_*, we can associate a so-called exact couple system via

$$K_q^p := H(F_p/F_q) \tag{31}$$

or

$$K_q^p := h_*(X_p, X_q), \tag{32}$$

respectively, which is defined as follows.

Definition 5.2 (Exact couple system)**.** And *exact couple system over I* is a collection of abelian groups $(K_q^p)_{(p,q) \in I_2}$ together with homomorphisms $\ell^{p,q}_{p',q'} : K^p_q \to K^{p'}_{q'}$ for any $(p, q) \leq (p', q')$ and homomorphisms $k_{p,q} : K^p_q \to K^q_{-\infty}$ for any $(p, q) \in I_2$, such that the following properties are satisfied:

1. $\ell^{p',q'}_{p'',q''} \circ \ell^{p,q}_{p',q'} = \ell^{p,q}_{p'',q''}$.
2. The triangles

$$\begin{array}{ccc} K^q_{-\infty} & \xrightarrow{\ \ell^{q,-\infty}_{p,-\infty}\ } & K^p_{-\infty} \\ & {\scriptstyle k_{pq}} \nwarrow \quad \swarrow {\scriptstyle \ell^{p,-\infty}_{p,q}} & \\ & K^p_q & \end{array}$$

   are exact.

3. The diagrams

$$\begin{array}{ccc} K^p_q & \xrightarrow{\ k_{pq}\ } & K^q_{-\infty} \\ {\scriptstyle \ell^{pq}_{p'q'}} \downarrow & & \downarrow {\scriptstyle \ell^{q,-\infty}_{q',-\infty}} \\ K^{p'}_{q'} & \xrightarrow[\ k_{p'q'}\ ]{} & K^{q'}_{-\infty} \end{array}$$

   commute.

Let K be an exact couple system over I. There is a natural differential $d_{pqz} : K^p_q \to K^q_z$ for any $(p, q, z) \in I_3$ defined by $d_{pqz} := \ell^{q,-\infty}_{q,z} \circ k_{pq}$. With this we define an associated *spectral system over I* via

$$S[z, q, p, b] := \frac{\ker(d_{pqz} : K^p_q \to K^q_z)}{\mathrm{im}(d_{bpq} : K^b_p \to K^p_q)}, \quad (b, p, q, z) \in I_4. \tag{33}$$

At a first glance this is just a collection of abelian groups, one for each element in I_4, however there are many connections between them:

First note that the usual goal of computation, $K_{-\infty}^{\infty}$, appears as

$$S[\infty] := S[-\infty, -\infty, \infty, \infty].$$

Sometimes I call it the *limit* of this spectral system, which this is just a name; it does not imply any convergence or comparison property. Moreover, terms of the form $S[q, q, p, p] = K_q^p$ are usually easy to describe when q covers p, that is, when $|q \setminus p| = 1$.

The following facts are proved in [9]. For any $(b, p, q, z) \leq (b', p', q', z')$ in I_4, $\ell_{p'q'}^{pq}$ induces maps

$$S[z, q, p, b] \to S[z', q', p', b'],$$

which we call maps induced by inclusion. When there is no confusion, we abbreviate all of them as ℓ.

Lemma 5.3 (Extensions). *For any $z \leq p_1 \leq p_2 \leq p_3 \leq b$ in I, we have a short exact sequence of maps induced by inclusion,*

$$0 \to S[z, p_1, p_2, b] \to S[z, p_2, p_3, b] \to S[z, p_2, p_3, b] \to 0. \tag{34}$$

Lemma 5.4 (Differentials). *For any $(b, p, q, z), (b', p', q', z') \in I_4$ with $z \leq p'$ and $q \leq b'$ there are natural differentials*

$$d : S[z, q, p, b] \to S[z', q', p', b'], \tag{35}$$

which commute with ℓ, that is, $\ell \circ d = d \circ \ell$.

Lemma 5.5 (Kernels and cokernels). *For any $(b, p, q, z), (b', p', q', z') \in I_4$ with $z = p'$ and $q = b'$ we have*

$$\ker\left(d : S[z, q, p, b] \to S[z', q', p', b']\right) = S[q', q, p, b]$$

and

$$\operatorname{coker}\left(d : S[z, q, p, b] \to S[z', q', p', b']\right) = S[z', q', p', p]$$

Lemma 5.6 (∞-page as filtration quotients). *$K_{-\infty}^{\infty}$ can be I-filtered by*

$$G_p := \operatorname{im}(\ell : K_{-\infty}^{p} \to K_{-\infty}^{\infty}) \cong S[-\infty, -\infty, p, \infty], \quad p \in I.$$

Furthermore the S-terms on the ∞-page are filtration quotients

$$S[-\infty, q, p, \infty] \cong G_p / G_q.$$

Lemma 5.7 (∞-page as quotient kernels). *$K_{-\infty}^{\infty}$ has quotients*

$$Q_p := \frac{K_{-\infty}^{\infty}}{\ker(\ell : K_{-\infty}^{\infty} \to K_p^{\infty})} \cong S[-\infty, p, \infty, \infty], \quad p \in I.$$

Furthermore the S-terms on the ∞-page are quotient kernels

$$S[-\infty, q, p, \infty] \cong \ker(Q_q \to Q_p).$$

Definition 5.8 (Excision)**.** An exact couple system K over I is called *excisive* if for all $a, b \in I$,

$$K^a_{a\cap b} \xrightarrow{\ell} K^{a\cup b}_b$$

is an isomorphism.

The exact couple system (32) is automatically excisive by the excision axiom of h_*. Note however that (31) is in general not excisive, though in many applications it is, for example when $C = \bigoplus_{P\in\mathbb{Z}^n} C_p$ (as abelian group) and $(F_p)_I$ is the canonical I-filtration given by $F_p = \bigoplus_{P\in p} C_p$.

Let us think of $J := \mathbb{Z}^n$ as an undirected graph, whose vertices are the elements of J, and $x, y \in J$ are adjacent if they are related, i.e. $x \geq y$ or $x \leq y$ (coordinate-wise). For $(b, p, q, z) \in I_4$, let $Z(z, q, p, b) \subseteq J$ denote the union of all connected components of $p \setminus z$ that intersect $p \setminus q$, and let $B(z, q, p, b) \subseteq I$ denote the union of all connected components of $b \setminus q$ that intersect $p \setminus q$.

Lemma 5.9 (Natural isomorphisms)**.** *In an excisive exact couple system K over $I = D(J)$, $S[z, q, p, b]$ is uniquely determined up to natural isomorphism by $Z := Z(z, q, p, b)$ and $B := B(z, q, p, b)$.*

We also write S^Z_B for $S[z, q, p, b]$, which is only defined up to natural isomorphisms. A word of warning: This B-Z-description of $S[z, q, p, b]$ looks quite appealing. However it may be combinatorially non-trivial to check whether some given B and Z come from some (b, p, q, z), and if so there might be several good choices. Moreover, it can be quite challenging to see whether there is a differential from $S^{Z_1}_{B_1}$ to $S^{Z_2}_{B_2}$ and what the resulting kernels and cokernels are in this case.

5.2 The construction

Throughout this section let us fix an excisive exact couple system K over $I = D(\mathbb{Z}^n)$.

Define an alphabet L,

$$L := \{1, \dots, n, 1^\infty, \dots, n^\infty, \mathrm{x}\}.$$

Remark 5.10 (Some intuition)**.** Here, a letter $j \in [n]$ stands for taking homology with respect to the j'th differential, j^∞ denotes the same but infinitely often, and x stands for a group extension process. In ordinary spectral sequences, $n = 1$, and the letter 1 stands the connection between some E^r_{**} and E^{r+1}_{**}, 1^∞ stands for the connection between some E^r_{**} and E^∞_{**}, and x for the connection between E^∞_{**} and the "limit" of the spectral sequence, e.g. $H(C)$ if the spectral sequence comes from a $\mathbb{Z}$-filtration of a chain complex C.

Let L^* denote the monoid of words of finite length with letters in L. Denote the empty word by ε, the concatenation of two words ω and ω' by $\omega * \omega'$, $\omega^n := \omega * \dots * \omega$ (n times), and the length of ω by $|\omega|$. L^*_a becomes a poset via $\tau \leq \omega$ if and only if τ is a prefix of ω, that is, a subword that starts from the beginning ($\tau = \varepsilon$ and $\tau = \omega$ are allowed).

Definition 5.11 (Admissible words)**.** Call a finite word $\omega \in L^*$ *admissible* if the following holds:

1. if j^∞ appears, the subsequent subword of ω contains neither j nor j^∞,
2. the only letter allowed directly after j^∞ is x,
3. any x occurring in ω comes directly after some j^∞.

If furthermore ω contains subwords $j^\infty\mathrm{x}$ for all $j \in [n]$ then ω is called *final*.

An exemplary final word for $n = 3$ is $123122^\infty\mathrm{x}133313^\infty\mathrm{x}111^\infty\mathrm{x}$ and any prefix of a final word is admissible. Let L_a^* denote the set of all admissible words in L^*. Define $X(\omega) \subseteq [n]$ as the set of $j \in [n]$ such that $j^\infty\mathrm{x}$ is a subword of ω, and $Y(\omega) := [n] \setminus X(\omega)$. $X(\omega)$ is so to speak the set of indices along which the extension process has been already made, and $Y(\omega)$ is the set of indices along which we still have differentials.

For $\omega \in L_a^*$, $i, j \in [n]$, we inductively define $r_\omega^i, \delta_\omega^i \in \mathbb{Z}^n$ and $B_\omega, Z_\omega \subset \mathbb{Z}^n$ as follows. Put $r_\varepsilon^i := e_i$, $r_{\omega * j^\infty}^i := r_\omega^i$, $r_{\omega * \mathrm{x}}^i := r_\omega^i$, and

$$r_{\omega * j}^i := \begin{cases} r_\omega^i & \text{if } i \neq j, \\ r_\omega^i + \delta_\omega^i & \text{if } i = j, \end{cases}$$

where $\delta_\varepsilon^i := e_i$, $\delta_{\omega * j^\infty}^i := \delta_\omega^i$, $\delta_{\omega * \mathrm{x}}^i := \delta_\omega^i$, and

$$\delta_{\omega * j}^i := \begin{cases} \delta_\omega^i & \text{if } i \in X(\omega) \cup \{j\}, \\ \delta_\omega^i - \delta_\omega^j & \text{if } i \in Y(\omega) \setminus \{j\}. \end{cases}$$

For $\omega \in [n]^*$, $\delta_\omega^i = \mathbb{1} - \sum_{k \in [n] \setminus i} r_\omega^k$.

Remark 5.12 (Some intuition 2)**.** r_ω^i will be the negated direction of the i'th differential at the ω-page, and δ_ω^i is the negated change of direction for the i'th differential that occurs when taking homology with respect to it. In ordinary spectral sequences, $n = 1$, and for $\omega = 1^{r-1}$ the ω-page consists of the columns in E_{**}^r, with $r_\omega^1 = r$, and $\delta_\omega^1 = 1$.

Further put $B_\varepsilon := \{0\}$,

$$B_{\omega * j} := B_\omega + \{0, \delta_\omega^j\}, \tag{36}$$

$$B_{\omega * j^\infty} := B_\omega + \mathbb{Z}_{\geq 0} \cdot \delta_\omega^j, \tag{37}$$

$$B_{\omega * j^\infty \mathrm{x}} := B_\omega + \mathbb{Z} \cdot \delta_\omega^j. \tag{38}$$

Here plus denotes a Minkowski sum. Thus for $\omega \in [n]^*$, B_ω can be regarded as a discrete zonotope, that is, an affine image of the vertices of an $|\omega|$-dimensional cube. See Figures 8. Define $Z_\omega := -B_\omega$, and for $P \in \mathbb{Z}^n$,

$$S(P; \omega) := S_{P + B_\omega}^{P + Z_\omega}. \tag{39}$$

Below we show that this is indeed a well-defined S-term (only up to natural isomorphism of course) by constructing $(b, p, q, z) \in I_4$ such that S_{bq}^{pz} represents $S(P; \omega)$.

Define lattices $V_\omega \subseteq \mathbb{Z}^n$ for $\omega \in L_a^*$ inductively as follows. Put $V_\varepsilon := \{0\}$, $V_{\omega * j} := V_\omega$, $V_{\omega * j^\infty} := V_\omega$, and

$$V_{\omega * j^\infty \mathrm{x}} := V_\omega + \mathbb{Z} \cdot \delta_\omega^j.$$

Alternatively, $V_\omega = B_\omega \cap Z_\omega$. For $P, P' \in \mathbb{Z}^n$ with $P - P' \in V_\omega$, $S(P; \omega) = S(P'; \omega)$. Thus we may also think of $S(P; \omega)$ as being parametrized over $P \in \mathbb{Z}^n / V_\omega$.

Definition 5.13 (ω-page)**.** Let $\omega \in L_a^*$. We call the collection of all $S(P; \omega)$, $P \in \mathbb{Z}^n / V_\omega$, the *$\omega$-page*.

For $\omega = \varepsilon$ this was called the *first page* in [9], for $\omega = 123 \ldots n$ the *second page*, and for $\omega = 1^{q_1} \ldots n^{q_n}$ a *generalized second page*, or the Q-page, where $Q = (q_1, \ldots, q_n) \in \mathbb{Z}_{\geq 0}^n$.

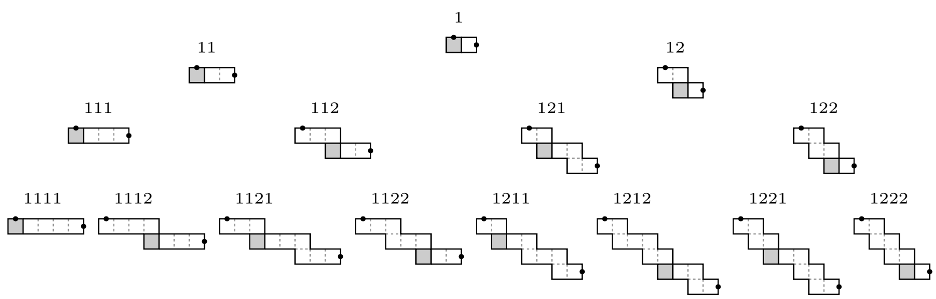

Figure 8: All B_ω with $|\omega| \leq 4$, $\omega_1 = 1$, and $n = 2$. For each B_ω, the origin is marked with a solid square, and the two points $r_\omega^i - e_i/2$ are marked with a black dot.

Remark 5.14 (Relation between jj^∞ and j^∞)**.** Suppose $w \in L_a^*$ contains j^∞, and let w' be the same word except that j^∞ is replaced by $j^k j^\infty$ for some $k \geq 1$. Then in general, $r_\omega^i \neq r_{\omega'}^i$ and $\delta_\omega^i \neq \delta_{\omega'}^i$, but they always agree modulo $V_\omega = V_{\omega'}$. Also $B_\omega = B_{\omega'}$ and hence $S(P; \omega) = S(P; \omega')$. Moreover one can check that the differentials in the main theorem 5.1 below are the same for ω and ω'. Thus in order to speak about the ω-page it is enough to know the image of ω in the quotient semigroup $L^*/(jj^\infty \sim j^\infty)$.

Now all definitions for Theorem 5.1 are given. But how is it useful: (a) gives a connection between the first page and arbitrary ω-pages for $\omega \in [n]^*$.

Then we can proceed with (b) and take homology infinitely often in one direction. Note that as with usual spectral sequences, $S(P; \omega * j^\infty)$ may indeed be a proper subquotient of the limit of the $S(P; \omega * j^k)$, compare with Section 2.5.

Then proceed with (c), which connects to $S(P; \omega * j^\infty \mathrm{x})$. Again as with usual spectral sequences, the filtration (F_i) may be neither Hausdorff nor exhaustive, and even if they are, $S(P; \omega * j^\infty \mathrm{x})$ may not be complete with respect to (F_i). As usual these two problems in (b) and (c) can be serious, but they are the standard ones in spectral sequences.

Arriving at $S(P; \omega * j^\infty \mathrm{x})$ we can start again at (a) until ω is final.

Remark 5.15 (Multiplicative structure)**.** As usual, under certain assumptions on K there will be a multiplicative structure. The simplest instance is when K comes via (31) from an I-filtered differential algebra C, whose filtration $(F_p)_I$ satisfies $F_p \cdot F_q \subseteq F_{p+q}$, where $p+q$ denotes the Minkowski sum. Then for any $\omega \in L_a^*$ there is a natural product $S(P;\omega) \otimes S(Q;\omega) \to S(P+Q;\omega)$, which satisfies a Leibniz rule with respect to the differentials (a). Furthermore they are compatible with respect to (b) and (c) in the usual way, and for final ω it coincides with the product on $H(C)$. For details and a more general criterion see [9, 4.4].

5.3 The 2-dimensional case

The probably most frequent case (apart from the classical one, $n=1$) is $n=2$. A few more things can be said about this case:

Every final $\omega \in L_a^*$ is of the form

$$\omega = \tau * j_1^\infty \mathrm{x} j_2^k j_2^\infty \mathrm{x}, \tag{40}$$

for some $\tau \in [2]^*$, $\{j_1, j_2\} = [2]$, and $k \geq 0$. Any such ω gives a recipe to connect the first page to $S[\infty]$. This recipe is therefore already determined by τ and j_1.

Note that for all prefixes $\tau \leq \omega' \leq \omega$, $M_\tau = M_{\omega'} = M_\omega$. Let's define

$$N_\omega := (e_{j_2})^t \cdot M_\omega,$$

which is the "normal vector" along which the downsets b, p, q, z grow respectively shrink during $j_2^k j_2^\infty \mathrm{x}$. Clearly $N_\omega \geq 0$ and it is primitive (i.e. its entries are coprime), and $N = (e_i)^t$ can happen only if $i = j_2$. Also, N_ω is invariant under the relation $jj^\infty \sim j^\infty$, compare with Remark 5.14.

Observation 5.16. *Modulo $jj^\infty \sim j^\infty$, ω is uniquely determined by N_ω and j_1. Conversely, for any primitive $N^t \in \mathbb{Z}^2_{\geq 0}$ and $j_1 \in [2]$ with $N^t \neq e_{j_1}$ there is a final $\omega \in L_a^*$ of the form* (40) *such that $N = N_\omega$.*

Thus the connection determined by ω can be equivalently described by N_ω and j_1.

Proof. In fact there is a simple algorithm that determines all possible τ from N (respectively N_ω) and $j_2 = 3 - j_1$. If $j_2 = 1$, choose $(N')^t \in \mathbb{Z}^2_{\geq 0}$ such that $M := \left(\begin{smallmatrix} N \\ N' \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z}) \cap \mathbb{Z}^{2\times 2}_{\geq 0}$. If $j_2 = 2$, choose $(N')^t \in \mathbb{Z}^2_{\geq 0}$ such that $M := \left(\begin{smallmatrix} N' \\ N \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z}) \cap \mathbb{Z}^{2\times 2}_{\geq 0}$. In any case, N' is well-defined up to adding an integral multiple of N. If N'_0 is the smallest choice, then all others are of the form $N'_k := N'_0 + kN$, $k \in \mathbb{Z}_{\geq 0}$. Now one can repetitively take one of the two rows of M and subtract it from the other one such that all entries stay non-negative until one arrives at $\mathrm{id}_{\mathbb{Z}^2}$, and there is a unique way to do that. Let $q_i \in [2]$ denote the index of the column from which the other column was subtracted during the i'th round. And say there were ℓ rounds. Then $M = M_\omega$ for $\omega = \tau j_1^\infty \mathrm{x} j_2^\infty \mathrm{x}$ and $\tau := q_\ell * \ldots * q_1$. The choice of N' correspond to how often j_1 appears at the end of τ, namely k times if $N' = N'_k$. $\square$

The algorithm has similarities to the extended Euclidean algorithm applied to the first column of M.

Example 5.17. Consider an excisive exact couple system K over $I(\mathbb{Z}^2)$. Suppose we want to determine both ω for which $N_\omega = (3,5)$. For $j_1 = 1$, the algorithm runs as follows:

$$\left(\begin{smallmatrix}2&3\\3&5\end{smallmatrix}\right) \xrightarrow{2} \left(\begin{smallmatrix}2&3\\1&2\end{smallmatrix}\right) \xrightarrow{1} \left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right) \xrightarrow{2} \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \xrightarrow{1} \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),$$

Thus $\omega = 12121^\infty\mathrm{x}2^\infty\mathrm{x}$ does it. Similarly, for $j_1 = 2$ one gets $\omega = 12112^\infty\mathrm{x}1^\infty\mathrm{x}$. See Figure 9.

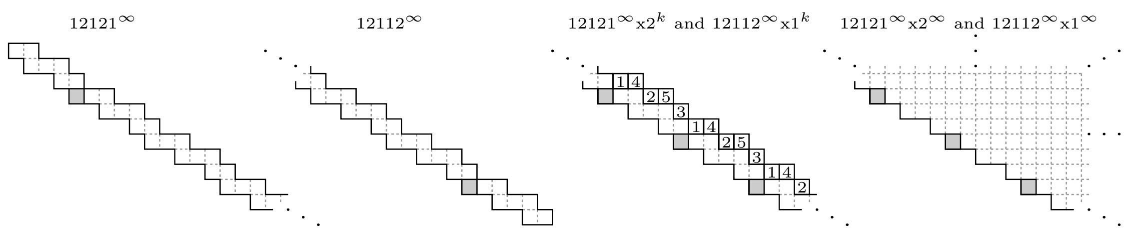

Figure 9: B_{12121^∞}, B_{12112^∞}, and $B_{12121^\infty\mathrm{x}2^k} = B_{12112^\infty\mathrm{x}1^k}$ for $0 \le k \le 5$ and for $k = \infty$. In the third figure, the squares with number i belong to $B_{12121^\infty\mathrm{x}2^k}$ if and only if $i \le k$. The solid squares depict V_σ.

Fun fact 5.18. The golden ratio can be arbitrarily well approximated by the slope of N_ω using $\tau = (12)^k$, since for this τ, $M_\tau = \left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right)^k = \left(\begin{smallmatrix}f_{2k-1}&f_{2k}\\f_{2k}&f_{2k+1}\end{smallmatrix}\right)$, where f_k are the Fibonacci numbers. However irrational slopes are not particularly useful, since one cannot connect the obtained page naturally to $S[\infty]$ (at least without further assumptions on K and without going backwards).

5.4 Example: Higher Leray–Serre spectral sequence

Consider a vertical tower of Serre fibrations,

$$\begin{array}{ccc}
F_0 & \hookrightarrow & E_0 \\
 & & \downarrow \\
\vdots & & \vdots \\
 & & \downarrow \\
F_{n-1} & \hookrightarrow & E_{n-1} \\
 & & \downarrow \\
 & & E_n
\end{array}$$

such that all E_i have the homotopy type of a CW-complex. Extend this tower trivially with $F_n := E_n$ and $E_{n+1} = \mathrm{pt}$. Fix a generalized homology theory h. Then there is a naturally associated higher spectral sequence over $I = D(\mathbb{Z}^n)$ with "second page"

$$S(P; 123...n) = H_{p_n}(F_n; H_{p_{n-1}}(F_{n-1}; \dots H_{p_1}(F_1; h(F_0)))))$$

and

$$S[\infty] = h(E).$$

5.5 Example: Higher Grothendieck spectral sequence

Consider a sequence of n right-exact functors

$$\mathcal{A}_0 \xrightarrow{F_1} \mathcal{A}_1 \xrightarrow{F_2} \dots \xrightarrow{F_n} \mathcal{A}_n$$

between abelian categories such that $\mathcal{A}_0, \dots, \mathcal{A}_{n-1}$ have enough projetives.

For any $A \in \mathcal{A}_0$, there is a naturally associated higher spectral sequence over $D(\mathbb{Z}^n)$ with "second page"

$$S(P; 123...n) = (L_{p_n} F_n) \circ \dots \circ (L_{p_1} F_1)(A).$$

If moreover for all $1 \leq i \leq n-1$, F_i sends projective objects of $\mathcal{A}_{i-1}$ to $(F_n \circ \dots \circ F_{i+1})$-acyclic objects, then

$$S[\infty] = L_*(F_n \circ \dots \circ F_1)(A).$$

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