ON THE SQUARE PEG PROBLEM AND SOME RELATIVES

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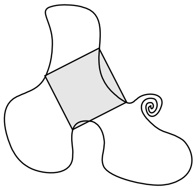
Abstract. The Square Peg Problem asks whether every *continuous* simple closed planar curve contains the four vertices of a square. This paper proves this for the largest so far known class of curves.

Furthermore we solve an analogous Triangular Peg Problem affirmatively, state topological intuition why the Rectangular Peg Problem should hold true, and give a fruitful existence lemma of edge-regular polygons on curves. Finally, we show that the problem of finding a regular octahedron on embedded spheres in $\mathbb{R}^3$ has a "topological counter-example", that is, a certain test map with boundary condition exists.

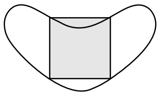
1. Introduction

The Square Peg Problem was first posed by O. Toeplitz in 1911:

Conjecture 1.1 (Square Peg Problem, Toeplitz [Toe11])**.** *Every continuous embedding $\gamma : S^1 \to \mathbb{R}^2$ contains four points that are the vertices of a square.*



The name Square Peg Problem might be a bit misleading: We do not require the square to lie inside the curve, otherwise there are easy counter-examples:



Toeplitz' problem has been solved affirmatively for various restricted classes of curves such as convex curves and curves that are "smooth enough", by various authors; the strongest version so far was due to W. Stromquist [Str89, Thm. 3] who established the Square Peg Problem for "locally monotone" curves. All known proofs are based on the fact that "generically" the number of squares on a curve is odd, which can be measured in various topological ways. See [Pak08], [VrŽi08], [CDM10], and [Mat08] for surveys. For general embedded plane curves, the problem is still open.

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We start our discussion in Section 2 with a description of a convenient parameter space for the polygons on a given curve. We then present a proof idea due to Shnirel'man [Shn44] (in a modern version, in terms of a bordism argument), which establishes the Square Peg Problem for the class of smooth curves.

Then we prove it for a new class of curves, which includes W. Stromquist's locally monotone curves. The first drawing above is an example that lies in this new class, but not in Stromquist's. See Definition 2.2 and Corollary 2.5 for an interesting special case. The proof generalises to curves in metric spaces under a suitable definition of a square; see Remark 2.6.3).

In Section 3 we ask the analogous question for equilateral triangles instead of squares and get a positive answer, even in a larger generality. Similarly we look for edge-regular polygons on curves, that is, polygons whose edges are all of the same length. In Section 4 we prove the existence of an interesting family of ε-close edge-regular polygons on smooth curves and deduce some immediate corollaries. Section 5 deals with the existence of rectangles with a given aspect ratio on smooth curves. I have no proof for this, but we will see some intuition why those rectangles should exist.

The last section, Section 6, treats higher-dimensional analogs. We ask for d-dimensional regular crosspolytopes on smoothly embedded $(d-1)$-spheres in $\mathbb{R}^d$. The Square Peg Problem for smooth curves is the case $d = 2$. The problem is open for all $d \geq 3$, but we use Koschorke's obstruction theory [Kos81] to derive that for $d = 3$, a natural topological approach for a proof fails: The strong test map in question exists.

This paper is an extended extract of [Mat08, Chap. III]. Some of the new results have been announced in [Mat09].

2. SQUARES ON CURVES

2.1. **Notations and the parameter space of polygons on curves.** For any space X, we denote by

$$\Delta_{X^n} := \{(x, \dots, x) \in X^n\}$$

the thin diagonal of X^n. For an element x of the unit circle $S^1 \cong \mathbb{R}/_{\mathbb{Z}}$ and $t \in \mathbb{R}$ we define $x + t \in S^1$ as the counter-clockwise rotation of x by the angle $2\pi t$ around 0. Let $\sigma^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1}_{\geq 0} \mid \sum t_i = 1\}$ be the standard n-simplex.

The natural **parameter space** of polygons is

$$P_n := S^1 \times \sigma^{n-1}.$$

It parametrises polygons on S^1 or on some given curve $S^1 \to \mathbb{R}^\infty$ by their vertices in the following way

$$\varphi :\ P_n \to (S^1)^n :\ (x; t_0, \dots, t_{n-1}) \mapsto (x, x + t_0, x + t_0 + t_1, \dots, x + \sum_{i=0}^{n-2} t_i).$$

The so parametrised polygons are the ones that are lying counter-clockwise on S^1. The map φ is not injective, as all $(x; 0, \dots, 0, 1, 0, \dots, 0)$ are mapped to the same point $(x, \dots, x)$; but it is injective on $P_n \backslash (S^1 \times \mathrm{vert}(\sigma^{n-1}))$, and on this set φ bijects to $(S^1)^n \backslash \Delta_{(S^1)^n}$. Let $P_n^\circ = S^1 \times (\sigma^{n-1})^\circ$ denote the interior of P_n. The map φ identifies P_n° with the set of n-tuples of pairwise distinct points in counter-clockwise order on S^1. We define the boundary as $\partial P_n^\circ := P_n \backslash P_n^\circ$.

We let $\mathbb{Z}_n := \mathbb{Z}/_{n\mathbb{Z}} =: \langle\varepsilon\rangle$ act on P_n by

$$\varepsilon \cdot (x; t_0, \ldots, t_{n-1}) = (x + t_0; t_1, \ldots, t_{n-1}, t_0).$$

This corresponds to a cyclic relabeling of the vertices of the parametrised polygon.

2.1.1. *A substitution.* The following coordinate transformation makes the $\mathbb{Z}_n$-action on P_n look nicer. We substitute $(x; t_0, \ldots, t_{n-1}) \in P_n$ by $(x^*; t_0, \ldots, t_{n-1})$, where $x^* := x + \sum_{k=1}^{n-1} \frac{n-k}{n} \cdot t_{k-1} \in S^1$. In terms of the new coordinates,

$$\varepsilon \cdot (x^*; t_0, \ldots, t_{n-1}) = (x^* + \frac{1}{n}; t_1, \ldots, t_{n-1}, t_0).$$

2.1.2. *Further notations.* When we talk about an *arc* on S^1 from a point x to y, we always mean the arc that goes counter-clockwise. For $x, y \in S^1$, we denote by $y - x$ the length of the arc from x to y, normalised with the factor $\frac{1}{2\pi}$. For an n-tuple $(x_1, \ldots, x_n) \in \varphi(P_n) \subset S^n$ we write

$$[x_1, \ldots, x_n] := (x_1; x_2 - x_1, x_3 - x_2, \ldots, x_n - x_{n-1}, 1 - \sum_{k=2}^{n}(x_k - x_{k-1})) \in P_n.$$

The function $[\ldots] : \varphi(P_n) \to (S^1)^n$ is right-inverse to φ, but not continuous.

Smooth means C^∞ for us. An *ε-close square* is a quadrilateral whose ratios between the edges and diagonals are up to an ε-error the ones of a square. The precise definition will not matter. We will use "ε-closeness" with other polygons analogously.

2.2. **Shnirel'man's proof for the smooth Square Peg Problem.** We start with L. G. Shnirel'man's proof [Shn44], since it is in my point of view the most beautiful one. The following presentation uses transversality and a bordism argument; in Shnirel'man's days, these notions had not been formalised and baptised yet, but his argument works like this.

Proof. Suppose that γ is smooth. P_4° parametrises quadrilaterals on γ. Let $f : P_4 \to \mathbb{R}^6$ be the function that measures the four edges and the two diagonals of the quadrilaterals,

$$\begin{aligned} f : P_4 \quad &\longrightarrow \quad \mathbb{R}^4 \times \mathbb{R}^2 \\ [x_1, x_2, x_3, x_4] \quad &\longmapsto \quad (||\gamma(x_1) - \gamma(x_2)||, ||\gamma(x_2) - \gamma(x_3)||, ||\gamma(x_3) - \gamma(x_4)||, \\ &\qquad ||\gamma(x_4) - \gamma(x_1)||, ||\gamma(x_1) - \gamma(x_3)||, ||\gamma(x_2) - \gamma(x_4)||) \end{aligned} \tag{1}$$

We can compose f with the quotient map $\mathbb{R}^6 \to \mathbb{R}^6/_{\Delta_{\mathbb{R}^4} \times \Delta_{\mathbb{R}^2}} \cong \mathbb{R}^4$ and get $f : P_4 \to \mathbb{R}^4$. The test-map f' measures squares, since $Q := (f')^{-1}(0) \backslash \Delta_{(S^1)^4} = (f')^{-1}(0) \cap P_4^\circ$ is the set of all squares that lie counter-clockwise on γ. f' is $\mathbb{Z}_4$-equivariant with respect to the natural $\mathbb{Z}_4$-actions. We can deform $\widetilde{f}$ relative to a small neighborhood of ∂P_4° equivariantly by a small ε-homotopy to make 0 a regular value of f'. So Q becomes a zero-dimensional $\mathbb{Z}_4$-manifold (note that Q lies in P_4°, which is free) of ε-close squares. If we deform the curve smoothly to another curve (e.g. the ellipse), which can also happen in $\mathbb{R}^4$ to construct such a homotopy easily, then Q changes by a $\mathbb{Z}_4$-bordism. This bordism stays away from the boundary of P_4°, if the homotopy is chosen smoothly, since then no curve inscribes ε-close squares which have arbitrarily small edges (the angles get too close to π). Hence Q represents a unique class $[Q]$ in the zero-dimensional unoriented bordism group $\mathcal{N}_0(P_4^\circ/_{\mathbb{Z}_4}) \cong H^0(P_4^\circ/_{\mathbb{Z}_4}; \mathbb{Z}_2) \cong \mathbb{Z}_2$. If γ is an ellipse then 0 is a regular value of f' and Q consists of one point. Hence $[Q]$ is the generator of $\mathbb{Z}_2$, so Q is non-empty for any smooth curve γ. Taking a convergent subsequence of ε-close squares finishes the proof. $\square$

If γ is only continuous one might try to approximate it with smooth curves and then take a convergent subsequence of the squares that we get on them. The problem is to guarantee that this subsequence does not converge to a square that degenerates to a point. Natural candidates for which this works are continuous curves with bounded total curvature without cusps, see Cantarella, Denne & McCleary [CDM10]. So far, nobody managed to do this for all continuous curves.

Shnirel'man's proof can be refined to get a slightly stronger result.

Corollary 2.1 (of the proof)**.** *We may assume that γ goes counter-clockwise around its interior. Then one can find and order four vertices of a square on γ, such that they lie counter-clockwise on γ and also label the square counter-clockwise.*

Proof. This can be achieved by restricting Q in the above proof to the set of squares $[x_1, x_2, x_3, x_4] \in P_4$ that are labeled by $(\gamma(x_1), \dots, \gamma(x_4))$ in counter-clockwise order. Along a path in the bordism this cannot change (here we take a bordism that is induced by a deformation of the curve in the plane). If γ is an ellipse then it is clear that the restricted Q is equal to Q, so it represents the generator in $\mathcal{N}_0(P_4^\circ/_{\mathbb{Z}_4})$. $\square$

2.3. **New cases of the Square Peg Problem.**

First of all we will establish the main theorem of this section, which gives a larger class of curves for which inscribed squares exist. Then we deduce two handy corollaries that are more directly applicable.

Let $\gamma : S^1 \to \mathbb{R}^2$ be a simple closed curve (that is, injective and continuous). We need some preparation. Let $f : P_4 \to \mathbb{R}^6$ be the corresponding test map that measure the four edges and two diagonals, which was defined in equation (1) in Section 2.2. For $y_1, y_4 \in S^1$, $y_1 \neq y_4$, let

$$P_4(y_1, y_4) := \{[y_1, x_2, x_3, y_4] \in P_4^\circ\}$$

the set of all quadrilaterals counter-clockwise on S^1 where the first and last vertex are given. For a path $y : S^1 \to (S^1)^2 \backslash \Delta_{(S^1)^2}$, $y(t) = (y_1(t), y_2(t))$, we define

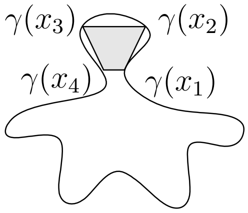
$$P_4(y) := \bigcup_{t \in S^1} P_4(y(t)) = \{[y_1(t), x_2, x_3, y_4(t)] \in P_4^\circ \mid t \in S^1\}.$$

Definition 2.2. We call a quadrilateral on γ given by $[x_1, x_2, x_3, x_4]$ **special** if

$$f([x_1, x_2, x_3, x_4]) = (a, a, a, b, e, e) \text{ with } a \geq b, \text{ for some reals } a, b, e.$$

The **size** of a special quadrilateral $[x_1, x_2, x_3, x_4]$ is the normalised arc length $x_4 - x_1$.

Let S denote the set of all special quadrilaterals in P_4. The following figure shows a special quadrilateral of small size on γ.



Theorem 2.3. *Suppose there is a path* $y : S^1 \to (S^1)^2 \backslash \Delta_{(S^1)^2} \cong P_2^\circ$*,* $y(t) = (y_1(t), y_4(t))$*, that represents a generator in* $\pi_1((S^1)^2 \backslash \Delta_{(S^1)^2}) \cong \pi_1(S^1) \cong \mathbb{Z}$*. If* γ *does not inscribe a square then the mod-*2 *intersection number of* $P_4(y)$ *and* S *is* 1*.*

The mod-2 intersection number will be described in the proof. The proof is based on equivariant obstruction theory, which was first used in connection to the Square Peg Problem by Vrećica and Živaljević [VrŽi08]. The second part of our proof will be very close to what they did. One can of course use different topological methods, but their way is quite straight-forward and beautiful. Another point of view will be sketched in the remarks 2.6.

Proof. $P_4(y)$ can be parametrised by $g : S^1 \times \sigma^2 \to P_4(y)$, where S^1 parametrises y and σ^2 the three arc lengths between the points $y_1(t)$, x_2, x_3 and $y_4(t)$. The map g is injective if and only if y is.

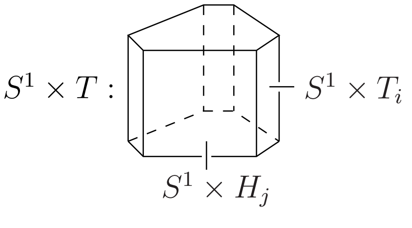
The mod-2 intersection number in the theorem is defined as the mod-2 intersection number of $f(g(S^1 \times \sigma^2))$ and $V := \{(a, a, a, b, e, e) \in \mathbb{R}^6 \mid a \geq b\}$ in $\mathbb{R}^6$. This is only well-defined if $f(g(S^1 \times \partial\sigma^2)) \cap V = \emptyset$ and $\mathrm{im}(f \circ g) \cap \partial V = \emptyset$. The former is trivially fulfilled, the latter if and only if no quadrilateral on γ given by $P_4(y)$ is a square (this is interesting if one deformes y; compare with Remark 2.6.1.). The map $f \circ g$ could now be deformed by a homotopy rel $S^1 \times \partial\sigma^2$, such that at no time it intersects the boundary of V, and such that it becomes transversal to V. The intersection number then counts the pre-images of V under $f \circ g$ modulo 2.

Suppose that γ does not inscribe a square, but the described mod-2 intersection number is zero. We want to derive a contradiction.

For some $\varepsilon \in (0, \frac{1}{2})$ (later we might choose $\varepsilon = \frac{1}{3}$), let $T = T^\varepsilon \subset \sigma^3$ be a polytope obtained from a tetrahedron by cutting an open vertex figure of size ε from the vertices (we delete all points $(t_0, \dots, t_3) \in \sigma^3$ that have an entry $> 1 - \varepsilon$). The four vertices of σ^3 are given by the standard basis vectors $e_0, \dots, e_3$ of $\mathbb{R}^3$. The four corresponding triangular facets of T are denoted by $T_0, \dots, T_3$, and their opposite hexagonal facets by $H_0, \dots, H_3$.

$S^1 \times T_3 \subset P_4$ parametrises the 4-tuples $(x_1, \dots, x_4) \in (S^1)^4$ with $x_4 - x_1 = \varepsilon$.

Here is a sketch of T in one dimension smaller where we draw $S^1 \times T \subset P_4$ as a cylinder whose the bottom and top face are identified:



We will construct for some small $\delta > 0$ an $\mathbb{Z}_4$-equivariant map

$$h : S^1 \times T^\varepsilon \longrightarrow_{\mathbb{Z}_4} S^1 \times T^\delta$$

that satisfies the following conditions:

(1) h maps $S^1 \times H_i$ to $S^1 \times H_i$, $0 \leq i \leq 3$,
(2) h is prescribed on $S^1 \times T_3^\varepsilon \subset P_4$ as

$$h(t; t_0, t_1, t_2, t_3 = 1 - \varepsilon) := (y_1(t); \lambda_t t_0, \lambda_t t_1, \lambda_t t_2, y_1(t) - y_4(t)),$$

where $\lambda_t > 0$ is chosen uniquely such that the last four entries sum up to one, that is, we want $h(t; \_, \_, \_, 1-\varepsilon) \in P_4(y_1(t), y_2(t))$.

The second condition prescribes h on all $S^1 \times T_i$, $i = 0, \dots, 3$, since h is $\mathbb{Z}_4$-equivariant.

Now we construct h. If $y = (y_1, y_4)$ is given by $(\mathrm{id}_{S^1}, \mathrm{id}_{S^1} + \varepsilon)$, then we can choose $\delta = \varepsilon$ and $h = \mathrm{id}_{S^1 \times T^\varepsilon}$. Otherwise there is a homotopy $Y_s : S^1 \to (S^1)^2 \backslash \Delta_{(S^1)^2}$, $s \in [0,1]$, from y to the previous one. For each time $s \in [0,1]$ we can now ask how to find an h_s as above for Y_s. If we only require condition (2) then this is a homotopy extension problem. Since $(S^1 \times T^\varepsilon, S^1 \times (T_0 \cup \ldots \cup T_3))$ is a pair of free $\mathbb{Z}_4$-CW-complexes, we can solve this. The standard proof for this gives a solution that automatically satisfies condition (1) at each time, so especially for y. Therefore h exists.

Hence we get a test map

$$t := pr \circ f \circ h : \ S^1 \times T \xrightarrow{f \circ h}_{\mathbb{Z}_4} \mathbb{R}^6 \backslash (\Delta_{\mathbb{R}^4} \times \Delta_{\mathbb{R}^2}) \xrightarrow[\simeq_{\mathbb{Z}_4}]{pr} \mathbb{R}^4 \backslash \{0\}$$

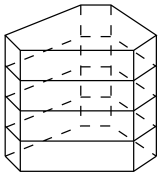
which is avoiding $0 \in \mathbb{R}^4$, since we assumed that γ inscribes no square.

The range $\mathbb{R}^4 \backslash \{0\}$ of t is a product of the standard $\mathbb{Z}_4$-representation $W_4 := \mathbb{R}^4/_{\Delta_{\mathbb{R}^4}}$ and $U := \mathbb{R}^2/_{\Delta_{\mathbb{R}^2}}$ ($\varepsilon \cdot u = -u$, $u \in U$), with 0 deleted. The corresponding components of t are t_W and t_U. The images $f_0, \dots, f_3 \in W_4$ of the four standard basis vectors $e_0, \dots, e_3$ of $\mathbb{R}^4$ span a tetrahedron which defines a fan with apex in 0 and with four facets, which we label by $F_0, \dots, F_3$, such that $-f_i \in F_i$. $V \subset \mathbb{R}^6$ projects under pr in $\mathbb{R}^4 = W_4 \times U$ to $V' := \mathbb{R}_{\leq 0} \cdot (-f_3) \times \{0\}$.

We have enough information to disprove the existence of t using an obstruction argument. Assume that only the restriction of t to $\partial(S^1 \times T) = S^1 \times \partial T$ is given, we look whether we can extend it.

We are allowed to deform t by an arbitrary $\mathbb{Z}_4$-homotopy. First of all we make t transversal to V' on $S^1 \times T_3$ relative to its boundary (and extend this deformation $\mathbb{Z}_4$-equivariantly). Let $t^{-1}(V') \cap (S^1 \times T_3) = \{p_1, \dots, p_{2k}\}$.

From now on we write $S^1 \times T \subset P_n$ in the coordinates that were introduces in Section 2.1.1. We see that it has a simple $\mathbb{Z}_4$-CW-complex structure with only one four-dimensional $\mathbb{Z}_4$-cell orbit:



One three-cell e shall be $* \times T$, $* \in S^1$. We may assume that $t(\partial(e) \cap T_3) \cap V' = \emptyset$ and analogously for the other T_i, since there are only finitely many points $* \in S^1$ which are forbidden in this way (namely the S^1-coordinates of the p_i and their $\mathbb{Z}_4$-translates).

Note that $t_W(S^1 \times H_i) \subset F_i \backslash \{0\}$. This is because on such points the t_i-coordinate is zero, hence the corresponding edge of the parametrised quadrilateral is zero and thus minimal among all edges. Therefore we can $\mathbb{Z}_4$-deform t_U on a sufficiently small neighborhood of $S^1 \times (H_0 \cup \ldots \cup H_3)$ such that t_U becomes zero on $S^1 \times (H_0 \cup \ldots \cup H_3)$ and such that during no time of this deformation change the new intersections of $t(S^1 \times T_3)$ and V'.

By the *degree* of a map $S^{n-1} \to \mathbb{R}^n \backslash \{0\}$ we mean the degree of the normalised map to S^{n-1}, or the scaling factor of the induced map on homology $H_{n-1}(\_)$.

Since $t(\partial(e) \cap T_3) \cap V = \emptyset$, we can also deform t on a small neighborhood of $\partial(e) \cap T_3$ such that $t_W|_{\partial(e)\cap T_3}$ lies in $F_0 \cup F_1 \cup F_2$ and such that $t_U|_{\partial(e)\cap T_3}$ is zero, without changing the intersections of $t(S^1 \times T_3)$ and V. Suppose we have extended t on e such that t_U is positive on the interior of e. Then t_U is negative on the interior of $\varepsilon \cdot e$ ($\mathbb{Z}_4 = \langle \varepsilon \rangle$). Let E be the 4-cell of $S^1 \times T$ that has e and $\varepsilon \cdot e$ as boundary faces. The degree of t_W on ∂e is one.

Recall $t^{-1}(V) \cap (S^1 \times T_3) = \{p_1, \ldots, p_{2k}\}$. If $2k = 0$, then one could also deform t on $\partial E \cap \partial(S^1 \times T)$ as we did with t on ∂e. In this case, $t|_{\partial E}$ is homotopic to the suspension of $t_W|_{\partial e}$, hence it was of degree 1. However for every $p_i \in \partial E$ the degree changes by one. This also happens at the other facets $\partial E \cap (S^1 \times T_i)$ of ∂E with the $\mathbb{Z}_4$-translates of $\{p_1, \ldots, p_{2k}\}$. In total there are $2k$ such points, hence the degree of $t|_{\partial E}$ is odd. If $t|_e$ was chosen differently, the degree of $t|_E$ would change twice ± the same number, once for e and once for $\varepsilon \cdot e$. Hence one cannot extend t to E, contradiction. □

Corollary 2.4. *Suppose there is a path* $y : S^1 \to (S^1)^2 \backslash \Delta_{(S^1)^2} = P_2^\circ$, $y(t) = (y_1(t), y_4(t))$, *that represents a generator in* $\pi_1((S^1)^2 \backslash \Delta_{(S^1)^2}) \cong \pi_1(S^1) \cong \mathbb{Z}$. *If* $P_4(y) \cap S = \emptyset$, *then* γ *circumscribes a square.*

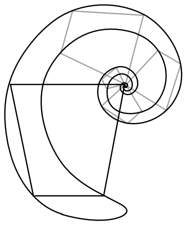
Proof. The mod-2 intersection number of Theorem 2.3 is here trivially zero. □

Corollary 2.5. *Suppose there is an* $\varepsilon \in (0, 1)$, *such that* γ *inscribes no (or generically an even number of) special quadrilateral of size* ε. *Then* γ *circumscribes a square.*

Proof. Use Theorem 2.3 with $y_1 := \mathrm{id}_{S^1}$ and $y_4 := \mathrm{id}_{S^1} + \varepsilon$. □

Remarks 2.6. 1.) An alternative view point is to look at S as a 1-dimensional manifold, after one made f transversal to V by a small ε-homotopy, at first on $P_4(y)$ and then on P_4. Here a technical trick is to choose ε not as a constant but as a function on P_4° that becomes arbitrarily small at the boundary, such that all technicalities work out. What Theorem 2.3 measures is the following.

$P_4(y)$ can be seen as a "membrane", which separates P_4 into two components if y is injective. If γ circumscribes no square then there is an *odd* number of paths in S that pass through $P_4(y)$ and approach the boundary at $S^1 \times e_3$, e_3 being the one vertex of σ^3. These paths might look very chaotic close to the boundary. On the other side of the membrane $P_4(y)$, this odd number of paths cannot all end in each other. One of them has to end somewhere else. It might end suddenly in P_4°, which means that it found a square, or it might end somewhere else at ∂P_4°. My hope was that the latter is not possible, but it is:

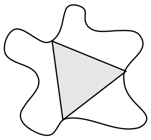


The drawn path of special quadrilaterals starts in the middle of the spiral at $S^1 \times e_3$ with a quadrilateral that is degenerate to a point, and it stops when x_1 and x_4 moved together again, $x_4 - x_1 = 1$.

2.) The corollaries are sometimes good for proving the existence of a square, if the curve is piecewise C^1 but has cusps (points in which the tangent vector changes the direction). This however works not in a large generality as the previous example shows.

3.) The whole Section 2.3 deals with the curve *intrinsically*, since the only datum of γ we used is the distances between points on γ. If we define a square in a metric space (X, d) to be a 4-tuple $(x_0, \ldots, x_3) \in X^4$ such that $d(x_0, x_1) = d(x_1, x_2) = d(x_2, x_3) = d(x_3, x_0)$ and $d(x_0, x_2) = d(x_1, x_3)$, then the whole section also works for curves $\gamma : S^1 \to X$. More generally, X does not need to fulfill the triangle inequality. In other words, we do not need an embedded curve but a distance defining function $d : S^1 \times S^1 \to \mathbb{R}$ that is continuous, positive definite, and symmetric.

3. Equilateral Triangles on Curves



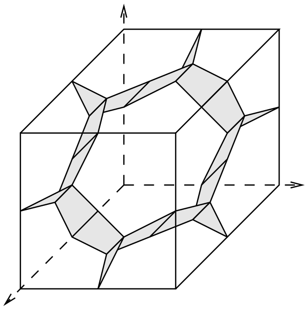
For our first result, suppose we are given a symmetric distance function d on the circle. This might occur if we embed S^1 into a metric space and pull back the metric.

Theorem 3.1 ("Triangular Peg Problem")**.** *Let $d : S^1 \times S^1 \longrightarrow \mathbb{R}$ be a continuous function satisfying $d(x, y) = d(y, x)$. Then there are three points $x, y, z \in S^1$, not all of them equal, forming an equilateral triangle, that is $d(x, y) = d(y, z) = d(z, x)$.*

Proof. We use the configuration-space test-map scheme. Suppose there is a curve admitting no such triangle. This induces us an S_3-equivariant map

$$(S^1)^3 \backslash \Delta_{(S^1)^3} \xrightarrow{S_3} \mathbb{R}^3 \backslash \Delta_{\mathbb{R}^3} \simeq S^1.$$

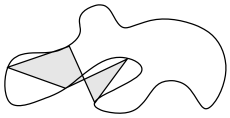
The configuration space $(S^1)^3 \backslash \Delta_{(S^1)^3}$ deformation retracts equivariantly to the following figure.



It is a nice exercise in equivariant obstruction theory to show that such a map cannot exist. For more details see [Mat08, Chap. III.3] $\square$

If the distance function comes from a planar continuous embedding $\gamma : S^1 \to \mathbb{R}^2$ then M. J. Nielsen [Nie92] has proven much more. Then there are even infinitely many triangles inscribed in the γ which are similar to a given triangle T, and if one fixes a vertex of smallest angle in T then the set of the corresponding vertices on γ is dense in γ. In the next section we show even a bit more if γ is smooth. For example the latter holds true for any angle and the set of corresponding vertices is all of γ.

4. Polygons on Curves



This section is very similar to independently obtained results of V. V. Makeev [Mak05] and J. Cantarella, E. Denne and J. McCleary [CDM10].

So far we asked about triangles and quadrilaterals that we can find up to similarity on curves. There are several possibilities to generalise this problem to other polygons, and the most natural one seems to consider fixed edge ratios. Suppose we are given an non-degenerate planar n-gon. If we take the quotient of the first $n-1$ edges by the last one, we get $n-1$ *edge ratios* $\rho_1, \ldots, \rho_{n-1} \in \mathbb{R}_{>0}$. They are characterised by the property that any number of $\rho_1, \ldots, \rho_{n-1}, 1$ is smaller than the sum of the others.

Let $\gamma : S^1 \to \mathbb{R}^\infty$ be a given *smooth* curve. We could also let γ map into any Riemannian manifold, which would not make a difference by Nash's embedding theorem. We proceed as in Shnirel'man's proof of Section 2.2. n-gons that are lying counter-clockwise on γ are parametrised by P_n. One can measure their edges by a test map $P_n \to \mathbb{R}^n$, make this by an ε-homotopy relative to ∂P_n transversal to the vector subspace spanned by the edge length vector of the given polygon, and find the solution set S of all n-gons in P_n° with the given edge ratios as a pre-image, which then defines a unique element $[S] \in \Omega_1(P_n)$ in the one-dimensional oriented bordism group of P_n. The projection onto the first factor induces a homotopy equivalence $P_n \simeq S^1$, hence $[S] \in \mathbb{Z}$. For γ a circle we deduce that $[S] = \pm 1$. Hence $S \neq 0$. This can be interpreted in terms of winding numbers (by the *winding number* of a component I mean its bordism class in $\Omega_1(P_n) \cong \mathbb{Z}$, where fixing this isomorphism fixes orientation issues).

Lemma 4.1. *S is a disjoint union of circles that wind around $P_n \simeq S^1$ and the winding numbers add with orientation up to ± 1.* □

If all edges ratios are one, so all edges are equal, then the test map is $\mathbb{Z}_n$-equivariant. P_n is free, hence the ε-homotopy can also be equivariant, so S is a $\mathbb{Z}_n$-manifold. The generator of $\mathbb{Z}_n$ preserves the orientation of P_n if and only if it preserves the orientation of $\mathbb{R}^n$. The test-space $\Delta_{\mathbb{R}^n}$ is the fixed point set of $\mathbb{R}^n$, so $\mathbb{Z}_n$ acts on it orientation preserving. Hence $\mathbb{Z}_n$ acts on S orientation preserving, which we use in the following lemma.

Lemma 4.2. *Let $\gamma : S^1 \longrightarrow \mathbb{R}^\infty$ be a smoothly embedded curve and let n be a prime power ≥ 3. Then there is a closed one-parameter family $S^1 \longrightarrow P_n$ of polygons such that*

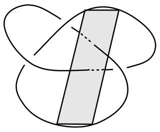
(1) *each of the polygons are ε-close edge-regular, that is, the edge ratios lie in* $[1-\varepsilon, 1+\varepsilon]$*, and*
(2) *this one-parameter family (that is, its image) is* $\mathbb{Z}_n$*-invariant.*

Proof. $\mathbb{Z}_n$ acts by permutation on the set of components of S. The lemma just claims that there is a fixed point. If there was no fixed point, all orbits had a cardinality divisible by p, where $n = p^k$. All components in one orbit have the same winding number, since $\mathbb{Z}_n$ acts on S orientation preserving and the induced action of $\mathbb{Z}_n$ on $\Omega_1(P_n)$ is trivial. Thus the sum of all winding numbers would be divisible by p, but it is ± 1, contradiction. □

Lemma 4.2 has some simple applications.

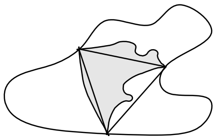
Another proof of the smooth Square Peg Problem. For a given curve γ, choose a $\mathbb{Z}_4$-invariant one-parameter family $S^1 \to P_n$ of ε-close rhombi. Then go along this family from one of these rhombi to its translate by the generator of $\mathbb{Z}_4$. What happens is that the short diagonal becomes the long diagonal, hence in the middle there was a square. Letting ε go to zero and taking a convergent subsequence of the ε-close squares finishes the proof. □

Corollary 4.3 (A Conjecture of Hadwiger, [Mak05, Thm. 4], [VrŽi08, Thm. 11])**.** *Each knot, that is, a smoothly embedded circle in* $\mathbb{R}^3$*, contains four points spanning a planar rhombus.*



Proof. As the proof before, but we look at the angle between the triangles of a triangulation of the rhombus instead of looking at a diagonal. Somewhere in the middle it has to be the straight angle (The angle has to be prevented from becoming zero, which can be done by a compactness argument). □

Corollary 4.4 (Blagojević–M., [Mat08, Thm. III.6.1])**.** *Let d_1 and d_2 be two symmetric distance functions on S^1, where d_1 is given by a smooth embedding of S^1 into a Riemannian manifold. Then there are three pairwise distinct points on S^1 forming an equilateral triangle with respect to d_1 and an isosceles triangle with respect to d_2.* □

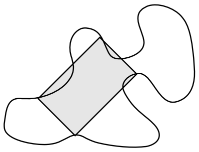


Lemma 4.2 has the following generalisation to non-prime powers n, which will be useful in the next section.

Lemma 4.5. *Let $\gamma : S^1 \longrightarrow \mathbb{R}^\infty$ be a smoothly embedded curve and let p^r be a prime power dividing $n \geq 3$. We think of $\mathbb{Z}_{p^r}$ as the subgroup of $\mathbb{Z}_n$ with p^r elements. Let P be a polygon whose edge lenths are $\mathbb{Z}_{p^r}$-invariant. Then there is a closed one-parameter family $S^1 \longrightarrow P_n$ of polygons such that*

(1) *each of the polygons have up to a factor and an ε-error the same edge lengths as P, and*
(2) *this one-parameter family is $\mathbb{Z}_{p^r}$-invariant.* □

5. Rectangles on Curves



H. B. Griffiths [Gri91] proved that every smooth planar embedded circle circumscribes a rectangle with arbitrary aspect ratio. However there are unfortunately some errors in his computation concerning orientations (see [Mat08, Chap. III.7] for details), which seem to invalidate the proof. Hence the problem is open:

Conjecture 5.1 ("Rectangular Peg Problem")**.** *For all reals $r > 0$, every smooth embedding $\gamma : S^1 \to \mathbb{R}^2$ contains four points spanning a rectangle of aspect ratio r.*

Since there is not as much symmetry in the Rectangular Peg Problem as in the Square Peg Problem, the symmetry group being $\mathbb{Z}_2$ instead of $\mathbb{Z}_4$, the number of rectangles of a fixed aspect ratio on curves will be generically even. Hence purely topological arguments will not work. But they give some intuition, here are two approaches. Assuming that Conjecture 5.1 admitted a counter-example (γ, r), both lemmas derive conclusions that seem to be unintuitive, but more geometric ideas are needed to yield a contradiction.

Lemma 5.2. *Suppose there was a counter-example (γ, r). Then for all $\varepsilon > 0$, there is a $\mathbb{Z}_2$-invariant one-parameter family $S^1 \to P_4$ of ε-close parallelograms with aspect ratio in $[r-\varepsilon, r+\varepsilon]$ and with an odd winding number, such that during the whole one-parameter family one of the diagonals stays larger than the other one.*

Proof. We would like to use Lemma 4.5 with $n = 4$, $p^r = 2$, and P a rectangle with aspect ratio r. However the solution set S of quadrilaterals on γ that have the desired edge ratios is too large. There exist *skew* quadrilaterals on γ having the same edge ratios as P, which we do not want in our solution set S since they are not parallelograms. To solve this problem, we can simply ignore them and argue that all arguments still go through. This turns out to be quite technical, but there is an easier proof:

We define another test map,

$$g : P_4 \longrightarrow \mathbb{R}^2 \times \mathbb{R}$$

that maps $[x_1, x_2, x_3, x_4]$ to

$$\big((\gamma(x_1) + \gamma(x_3)) - (\gamma(x_2) + \gamma(x_4)),$$
$$(||\gamma(x_1) - \gamma(x_2)|| + ||\gamma(x_3) - \gamma(x_4)||) - r \cdot (||\gamma(x_2) - \gamma(x_3)|| + ||\gamma(x_4) - \gamma(x_1)||)\big).$$

The pre-image $(g|_{P_4^\circ})^{-1}(0)$ is exactly the set of parallelograms on γ of aspect ratio r. Using a bordism argument, the proof works now exactly as the one of Lemma 4.5. $\square$

Remark 5.3. In Lemma 5.2, instead of looking at the set of parallelograms with aspect ratio r, we might look as well on the set of parallelograms whose diagonals intersect in an angle α, where α is the intersection angle of the diagonals in a rectangle of aspect ratio r. This gives an analogous lemma, which might be easier to deal with geometrically.

Now we come to the second lemma, which gives similarly an intuition why the Rectangular Peg Problem should hold true.

Lemma 5.4. *Suppose there was a counter-example* (γ, r)*. Then for all* $\varepsilon > 0$*, there is a* $\mathbb{Z}_4$*-invariant one-parameter family* $S^1 \to P_4$ *of* ε*-close rectangles.*

Proof. Let $f : P_4^\circ \longrightarrow_{\mathbb{Z}_4} \mathbb{R}^4 \times \mathbb{R}^2$ be the restricted map (1) from Section 2.2, measuring the edges and diagonals.

First of all we make f $\mathbb{Z}_4$-equivariantly transversal to $\Delta_{\mathbb{R}^4} \times \Delta_{\mathbb{R}^2}$ by a small δ-homotopy, and let $Q := f^{-1}(\Delta_{\mathbb{R}^4} \times \Delta_{\mathbb{R}^2})$ be the set of all squares (up to an δ-error, where δ is a function that decreases sufficiently fast near the boundary of P_4°). Then we make f $\mathbb{Z}_4$-equivariantly transversal to the $\mathbb{Z}_4$-invariant subspace $V := \{(a, b, a, b, e, e) \in \mathbb{R}^4 \times \mathbb{R}^2\}$ by a small δ-homotopy which leaves Q fixed, and let $R := f^{-1}(V)$ be the set of all rectangles on γ (up to an δ-error). If δ was chosen small enough, R consists only of ε-close rectangles.

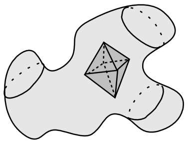
Let R_Q be the set of all components of R that contain a square. We may assume that all these components are circles, otherwise a component would come arbitrary close to the boundary of P_4, so there would be an ε-close rectangle on it with aspect ratio r. If we could do this for all ε, then a limit argument would give us a proper rectangle of aspect ratio r. So if need be, we choose a smaller ε for which this does not happen.

R is a one-dimensional $\mathbb{Z}_4$-manifold, so $\mathbb{Z}_4$ acts on R_Q as well. We decompose $R_Q = R_1 \uplus R_2 \uplus R_4$, where R_1 is the set of components with isotropy group $\langle 0 \rangle$, R_2 with isotropy group $\mathbb{Z}_2 = \langle \varepsilon^2 \rangle \subset \mathbb{Z}_4$ and R_4 with $\mathbb{Z}_4$. Now we only need to count the number of squares on each R_i.

- $\sharp Q = 4 \mod 8$, since modulo $\mathbb{Z}_4$ it is odd (see Section 2.2).
- Every component $C \in \mathbb{R}_Q$ contains an even number of squares, since while passing a square the rectangle changes from fat to skinny or vice versa (this follows from the bijectivity of the differential $\mathrm{d}f$ at points in Q).
- 4 divides $\sharp R_1$, and every component in R_1 contains two squares. So the number of squares on components of R_1 is divisible by 8.
- 2 divides $\sharp R_2$, and if a component in R_2 contains a square S, then it contains also $\varepsilon^2 \cdot S$. When it goes through a square and changes from fat to skinny, then so it does at $\varepsilon^2 \cdot S$. Hence it has to go through $4k$ squares, $k \geq 1$. Thus the number of squares on components of R_2 is divisible by 8.
- If a component C of R_4 goes through a square S and changes from fat to skinny, then it also goes through $\varepsilon \cdot S$ and changes from skinny to fat. That is, in between it had to go through an even number of squares, all of which of course belong to a different $\mathbb{Z}_4$-orbit. Hence the number of square-orbits on C is odd, $\sharp(Q \cap C) = 4 \mod 8$.

Putting this modulo 8 together, we get $\sharp R_4 = 1 \mod 2$, which is even a bit stronger than what is stated in the lemma. $\square$

6. Crosspolytopes on Spheres



H. Guggenheimer [Gug65] proved that any smoothly embedded sphere $S^{d-1} \to \mathbb{R}^d$ contains the vertices of a regular d-dimensional crosspolytope. However there is unfortunately an error in his main lemma (see [Mat08, Chap. III.9] for details), which seems to invalidate the proof. Hence the problem is open:

Conjecture 6.1 ("Crosspolytopal Peg Problem")**.** *Every smooth embedding* $\gamma : S^{d-1} \to \mathbb{R}^d$ *contains the vertices of a regular* d*-dimensional crosspolytope.*

Recently, R. N. Karasev [Kar09] has shown this conjecture to hold true for boundaries of non-angular (e.g. smooth) convex bodies, if d is an odd prime power.

The topological counter-example. The conjecture in general is probably very difficult and a solution would involve deeper geometric reasoning, since there is the following "topological counter-example" for $d = 3$. Suppose we are given a smooth embedding $\Gamma : S^2 \to \mathbb{R}^3$. Let $G \cong (\mathbb{Z}_2)^3 \rtimes S_3$ be the symmetry group of the regular octahedron and $G_{or} \subset G$ be the subgroup of orientation preserving symmetries. G acts on $(S^2)^6$ by permuting the coordinates in the same way as it permutes the vertices of the regular octahedron. Let G act on $\mathbb{R}^{12}$ by permuting the coordinates in the same way as it permutes the edges of the regular octahedron. The subrepresentation $(\Delta_{\mathbb{R}^{12}})^{\perp} \subset \mathbb{R}^{12}$ is denoted by Y. Let $\Delta^{fat}_{(S^2)^6}$ be the space of all 6-tuples in $(S^2)^6$ that contain at least two equal elements, that is, the fat diagonal. Let B be a small ε-neighborhood of $\Delta^{fat}_{(S^2)^6}$, where ε depends only on an isotopy of Γ to some nice embedding, that we will describe later. Then the complement $X := (S^2)^6 \backslash B$ is a free compact G-manifold with boundary and

$$X \simeq_G \{(x_1, \ldots, x_6) \in (S^2)^6 \mid x_i \text{ are pairwise distinct}\} = (S^2)^6 \backslash \Delta^{fat}_{(S^2)^6}.$$

Then Γ gives us a test map

$$t : X \longrightarrow_G Y,$$

which measures the edges of the parametrised octahedra modulo $\mathbb{1} = (1, \ldots, 1)$. Since ε was chosen small, $t|_{\partial X}$ is mapping uniquely up G-homotopy to $Y \backslash \{0\}$, if we change Γ by an isotopy. The solution set S of regular octahedra on Γ is $S := t^{-1}(0)$. The subset $S_{or} \subset S$ of positively oriented octahedra is a part of the pre-image $t^{-1}(0)$, and t induces an isomorphism of G_{or}-vector bundles over S_{or},

$$TS_{or} \oplus (i_{S_{or}})^*(X \times Y) \cong (i_{S_{or}})^*(TX),$$

where $i_{S_{or}}$ denotes the inclusion $S_{or} \hookrightarrow X$. Thus S_{or} gives us together with this normal data an element $[S_{or}]$ in the equivariant normal bordism group (see U. Koschorke [Kos81, Chap. 2])

$$\Omega_1^{G_{or}}(X, X \times Y - TX) = \Omega_1(X/_{G_{or}}, X \times_{G_{or}} Y - T(X/_{G_{or}})),$$

which is well-defined, since isotopies of Γ change S only by a normal bordism that stays away from the ∂X of ε was chosen small enough, and components of octahedra of different orientation are always separated from each other. In Koschorke's notation, $[S_{or}]$ is the obstruction

$$\widetilde{\omega}_1(\underset{\sim}{\mathbb{R}}, X \times_{G_{or}} Y, (\mathrm{id}_{\partial X}, t|_{\partial X})/_{G_{or}}),$$

where $\underset{\sim}{\mathbb{R}}$ is the trivial line bundle.

Theorem 6.2. *The above defined* $[S_{or}]$ *is zero. Hence*

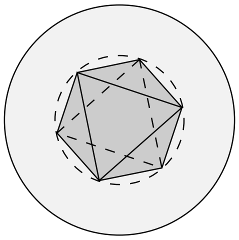
$$[S] \in \Omega_1^G(X, X \times Y - TX)$$

is zero as well.

In particular, the test map t *can be deformed* G*-equivariantly relative to* ∂X *to a map* t'*, such that* $0 \notin t'(X)$*.*

The map t' is what I call a topological counter-example.

Sketch of Proof. To construct a nice representative for $[S_{or}]$ we take the standard 2-sphere and scale it down linearly along the z-axis of $\mathbb{R}^3$. This is our Γ and we let t and S be the corresponding test map and solution set, respectively. S is a disjoint union of $16 = \frac{1}{3} \cdot \sharp G$ circles. One octahedron on the scaled sphere looks as follows (one looks along the z-axis):



If we rotate it around the z-axis then we get up to symmetry all octahedra on Γ. The G-bundles $X \times Y$ and TX are G-orientable, therefore the relevant part of Koschorke's exact sequence [Kos81, Thm. 9.3] becomes

$$\begin{aligned} H_2(X/_{G_{or}};\mathbb{Z}) \to \mathbb{Z}_2 \to \Omega_1(X/_{G_{or}}, &X \times_{G_{or}} Y - T(X/_{G_{or}})) \\ &\to H_1(X/_{G_{or}};\mathbb{Z}) \to 0. \end{aligned}$$

It is not difficult to see that the image of $[S_{or}]$ in $H_1(X/_{G_{or}};\mathbb{Z}) = H_1(G_{or};\mathbb{Z})$ is zero. This is because the 120 degree rotation of a regular octahedron around the line connecting the midpoints of two opposite triangles is an element of the commutator of G_{or}. It requires much more visualisation to see that $[S_{or}]$ is in fact the image of the generator of $\mathbb{Z}_2$. The hard part is to show that $\mathbb{Z}_2$ unfortunately lies in the image of $H_2(X/_{G_{or}};\mathbb{Z})$, which I could manage to do only with a very long program. It finds that $H_2(X/_{G_{or}};\mathbb{Z}) \cong \mathbb{Z}_4 \times (\mathbb{Z}_2)^3$, where one can choose the generators such that the first three map to zero and the last one to the generator of $\mathbb{Z}_2$.

The G_{or}-null-bordism of S_{or} can be extended to a G-null-bordism of S. By Theorem 3.1 of U. Koschorke [Kos81], we can extend the section as stated. □

Remarks to the algorithm. An economical S_6-CW-complex structure on $(S^2)^6$ is based on an S_6-cell decomposition of $\mathbb{R}^2$ of V. A. Vassiliev [Vas94], which has few high dimensional cells. $\Delta^{fat}_{(S^2)^6}$ is a subcomplex, so one can compute $H_2(X/_{G_{or}}) \cong H^{10}((S^2)^6/_{G_{or}}, (\Delta^{fat}_{(S^2)^6})/_{G_{or}})$. The Smith normal form is used to compute this cellular cohomology and the LLL-algorithm to choose nice generators. The image in $\mathbb{Z}_2$ is determined by computing second Stiefel-Whitney classes, which I implemented as obstruction classes.

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