On the Square Peg Problem and its relatives II

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Abstract

The Square Peg Problem asks whether every *continuous* simple closed curve in the plane contains the four vertices of a square. In this paper we prove it for an open class of curves, which is not given by a smoothness condition, and thus we do not need to assume the curve to be injective.

Furthermore we show that every smooth planar curve whose angular convexity is at most $60°$ inscribes a rectangle with aspect ratio $\sqrt{3}$.

Finally we disprove a conjecture of Cantarella on the parity of inscribed squares for immersed curves in the plane. We derive the correct parity, also for inscribed rectangles.

1 Introduction

The Square Peg Problem was first posed by O. Toeplitz in 1911:

Conjecture 1.1 (Square Peg Problem, Toeplitz [Toe11]). *Every continuous embedding $\gamma : S^1 \to \mathbb{R}^2$ contains four points that are the vertices of a square.*

In its full generality Toeplitz' problem is still open. So far it has been solved affirmatively for curves that are "smooth enough", by various authors for varying smoothness conditions [**?**]. All of these proofs are based on the fact that smooth curves inscribe generically an odd number of squares. In a previous paper [Mat09] we proved the following so far strongest version. Its proof easily extends to curves in arbitrary metric spaces, so we state it in this more general form.

Theorem 1.2. *Let $\gamma : S^1 \to X$ be an embedded circle in a metric space (X, d). Assume that there is an $0 < \varepsilon < 2\pi$ such that γ contains no (or generically an even number of) special trapezoids of size ε. Then γ inscribes a square. That is, there exist four pairwise distinct points $P_1, \ldots, P_4 \in \gamma$ such that*

$$d(P_1, P_2) = d(P_2, P_3) = d(P_3, P_4) = d(P_4, P_1) \text{ and } d(P_1, P_3) = d(P_2, P_4).$$

Here a **special trapezoid** on a curve γ is a 4-tuple of pairwise distinct points $x_1, \ldots, x_4 \in S^1$ lying counter-clockwise on S^1 such that the points $P_i := \gamma(x_i)$ satisfy

$$d(P_1, P_2) = d(P_2, P_3) = d(P_3, P_4) > d(P_4, P_1) \text{ and } d(P_1, P_3) = d(P_2, P_4).$$

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The **size** of this special trapezoid is defined as the length of the counter-clockwise arc in S^1 from x_1 to x_4.

This paper splits into three parts.

1.) The first result is the first known *open* set of curves in the compact-open topology (equivalently, in $((\mathbb{R}^2)^{S^1}, ||.||_\infty))$ for which the Square Peg Problem holds. It does neither require the curve to be smooth nor injective; see Section 2 for the rather simple proof and variations of the statement.

Theorem 1.3. *Let A denote the annulus $\{x \in \mathbb{R}^2 \mid 1 \leq ||x|| \leq 1 + \sqrt{2}\}$. Suppose that $\gamma : S^1 \to A$ is a continuous closed curve in A that represents the generator of $\pi_1(A) = \mathbb{Z}$. Then γ inscribes a square of side length at least $\sqrt{2}$.*

Figure 1 shows an example. This Theorem does *not* contain all previous known classes of curves for which the Square Peg Problem is proved. It is the first partial result on the Square Peg Problem that bounds the size of an inscribed square from below.

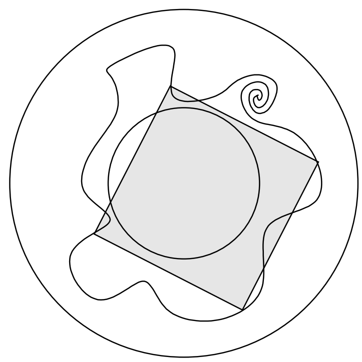

Figure 1: Example for Theorem 1.3.

2.) The second result is a first non-trivial case for the "Rectangular Peg Problem".

Conjecture 1.4 (Rectangular Peg Problem)**.** *Every C^∞ embedding $\gamma : S^1 \to \mathbb{R}^2$ contains four points that are the vertices of a rectangle with a prescribed aspect ratio $r > 0$.*

We state this conjecture for smooth curves only, since already this seems to be a hard problem. It is equivalent to stating this conjecture for piecewise linear curves. So far it is only known to hold in the case $r = 1$, that is, for inscribed squares. The proof in Griffiths' paper [Gri91] contains unfortunately an error in the calculation of intersection numbers, see [Mat08] for details. The difficulty comes from the fact that, counted with orientations, every smooth curve contains generically zero rectangles of a prescribed aspect ratio. E.g. an ellipse contains two rectangles with opposite orientations. This makes a standard topological approach, called *configuration space-test map method*, fail as we will show in Section 3.1. Further geometric arguments are needed to attack the problem.

In Section 3 we prove the following first non-square special case of the Rectangular Peg Problem.

Theorem 1.5. *Let $\gamma : S^1 \to \mathbb{R}^2$ be a C^∞ curve whose angular convexity is at most $60°$. Then γ inscribes a rectangle with aspect ratio $\sqrt{3}$.*

Here we call a smooth plane curve γ to have **angular convexity** at most α, if the signed curvature of γ restricted to any arc is at most α; see Figure 2. The proof uses a hidden symmetry that appears for $r = \sqrt{3}$, which is a geometric piece of information.

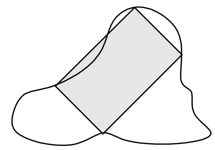

Figure 2: Example for Theorem 1.5.

3.) In Section 4 we deal with immersed planar curves and the parity of their inscribed squares. Cantarella [**?**] conjectured that this parity is an isotopy invariant and he stated a precise formula based on examples. We disprove Cantarella's conjecture and state in Theorem 4.1 how the parity can be computed from the angles at the intersection points. Theorem 4.2 gives a similar formula for the parity of inscribed rectangles of a fixed aspect ratio.

2 Squares on curves

In this section we prove Theorem 1.3 from the introduction together with the following two versions, whose proofs are very similar. See Figures 2.1 and 2.2 for examples.

Theorem 2.1. *Let S denote the area $\{x \in \mathbb{R}^2 \mid 1 \leq ||x||_\infty \leq 3\} = \{x \in \mathbb{R}^2 \mid 1 \leq |x_{1,2}| \leq 3\}$. Suppose that $\gamma : S^1 \to S$ is a continuous closed curve in S that represents the generator of $\pi_1(S) = \mathbb{Z}$. Then γ inscribes a square of side length at least $\sqrt{2}$.*

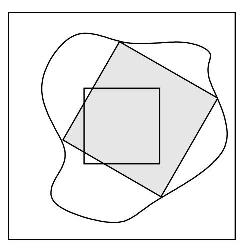

Figure 3: Example for Theorem 2.1.

Theorem 2.2. *Let Δ be an equilateral triangle in $\mathbb{R}^2$ whose center point is the origin. Let T be the closure of $((1+\sqrt{3}) \cdot \Delta)\backslash\Delta$. Suppose that $\gamma : S^1 \to T$ is a continuous closed curve in T that represents the generator of $\pi_1(T) = \mathbb{Z}$. Then γ inscribes a square of side length at least $2\sqrt{3} - 3$.*

It seems to be desireable to extend this method for much more general shapes in order to possibly prove the Square Peg Problem for all curves.

The proofs of Theorems 1.3, 2.1, and 2.2 follow from the following lemma.

Lemma 2.3. *Let A be a subset of $\mathbb{R}^2$. Let S_A be the set of 4-tuples $(P_1, \ldots, P_4) \in A^4$ that form the vertices of a possibly degenerate square in counter-clockwise order. Let C be*

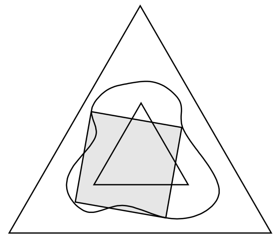

Figure 4: Example for Theorem 2.2.

a connected component of an ε-neighborhood of S_A that does not contain degenerate squares, that is, points of the form (P, P, P, P). Let $\widetilde{\gamma} : S^1 \to A$ be a generic curve that contains an odd number of squares in C. Then every continuous curve $\gamma : S^1 \to A$ that is homotopic to $\widetilde{\gamma}$ in A contains a square in C as well.

Here, by a generic curve $\widetilde{\gamma}$ we mean a curve such that the corresponding test-map that measures squares in C hits the test-space smoothly and transversally.

The proof of Lemma 2.3 is a simple bordism argument.

Proof of Theorem 1.3. We may assume that γ is actually a curve in the interior of A. The other cases follow by a limit argument, for which we use that on each approximating curve we can find a square of size at least $\sqrt{2}$. Some subsequence of this sequence of squares will then converge to a non-degenerate square of the given curve.

By compactness γ is a curve in $A' := \overline{U_\varepsilon(\overline{A})}$ for some $\varepsilon > 0$. Now we can apply Lemma 2.3, where we choose $\widetilde{\gamma}$ to being an ellipse in A'. $\square$

The proofs of Theorems 2.1, and 2.2 are analogous.

3 Rectangles on curves

The Rectangular Peg Problem is a very challenging and from the author's point of view the most beautiful open problem in this area of inscribing and circumscribing problems. In Section 3.1 we show that the standard topological approach, the *configuration space/test map scheme*, fails to prove the Rectangular Peg Problem 1.4 since the test map in question exists.

Then we prove Theorem 1.5 under some technical assumptions concerning transversality; see Section 3.2. We show in Section 3.3 that these assumptions can be made. These technicalities seem not to be obvious in advance for two reasons: The natural group action on one solution manifold (namely P) is in general not free; and transversality has to be achieved for several maps simultaneously since we need to relate solution manifolds of different maps in the proof.

3.1 The test-map exists

In this section we show that a standard topological approach, called configuration space-test map method, does not work to prove the Rectangular Peg Problem 1.5.

Assume we are given a smooth simple closed planar curve $\gamma : S^1 \hookrightarrow \mathbb{R}^2$. Let $P_4 \subset (S^1)^4$ be the set of four pairwise distinct points on the circle that lie counter-clockwise on it. Then

$P_4 \cong S^1 \times (\sigma^3)^\circ$, where $(\sigma^3)^\circ$ denotes the interiour of the 3-simplex. We construct from γ a natural test map,

$$\begin{aligned} t : P_4 \quad & \longrightarrow \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}, \\ (x_1, x_2, x_3, x_4) \quad & \longmapsto (v, l, a), \end{aligned}$$

where v is the difference between the midpoints of the diagonals in the quadrilateral $(\gamma(x_1), \ldots, \gamma(x_4))$, l is the difference of the length of these diagonals, and a is the aspect ratio.

We let $\mathbb{Z}_2 = \mathbb{Z}/_{2\mathbb{Z}} = \{\bar{0}, \bar{1}\}$ act on P_4 by $\bar{1} \cdot (x_1, x_2, x_3, x_4) = (x_3, x_4, x_1, x_2)$. The map t is then $\mathbb{Z}_2$-equivariant with respect to the corresponding group action on $\mathbb{R}^4$. Since γ is smooth, there is an $\varepsilon > 0$, such that t maps no point of $B := U_{[\varepsilon]}(\partial(P_4)) \cap P_4$ to zero, where U_ε denotes the closed ε-neighborhood and ∂P the topological boundary of $P \subset (S^1)^4$. The map $t|_B : B \to \mathbb{R}^4 \backslash \{0\}$ is uniquely given up to $\mathbb{Z}_2$-homotopy. $R := t^{-1}(0, 0, r)$ is the set of rectangles of aspect ratio r whose vertices lie conter-clockwise on γ. R is generically a zero-dimensional $\mathbb{Z}_2$-manifold. Using the pre-image orientation for R, $\mathbb{Z}_2$ acts orientation preserving on R. Therefore R determines an element $[R]$ in the oriented zero-dimensional bordism group $\Omega_0(P_4/_{\mathbb{Z}_2}) \cong \mathbb{Z}$, which is the primary obstruction for extending $t|_U : U \to \mathbb{R}^4 \backslash \{0\}$ to a map $P_4 \to \mathbb{R}^4 \backslash \{0\}$. If γ is an ellipse then R consists of two orbits, since an ellipse inscribes exactly two rectangles. Computation shows that their orientation is opposite. Therefore $[R] = 0$ and the obstruction class vanishes. Since this is the only obstruction, we can find a map $t' : P_4 \to \mathbb{R}^4 \backslash \{0\}$ such that $t'|_B = t|_B$. That is there is no purely topological argument that can show the existence of a rectangle of aspect ratio r on γ, at least as long as we are not using more geometric information.

The smooth Square Peg Problem can be solved using this configuration space-test map scheme, since squares are more symmetric. Here the group of symmetry is $\mathbb{Z}_4$ and on an ellipse we find only one $\mathbb{Z}_4$ orbit of squares. In the case $r = \sqrt{3}$ there is a "hidden" symmetry that we will use in the next section to prove Theorem 1.5.

3.2 Inscribed Rectangles with aspect ratio $\sqrt{3}$

Now we prove Theorem 1.5 but leave all technical details concerning transversality to the subsequent Section 3.3. Suppose we are given a smooth curve $\gamma : S^1 \hookrightarrow \mathbb{R}^2$.

We define a map

$$\begin{aligned} f : (S^1)^4 \quad & \longrightarrow_G \quad \mathbb{R}^2 \times S^1 \\ (x_1, x_2, y_1, y_2) \quad & \longmapsto \quad (v, \alpha), \end{aligned}$$

where v is again the difference between the midpoints of the diagonals in the quadrilateral $(\gamma(x_1), \gamma(y_1), \gamma(x_2), \gamma(y_2))$ and α is the mod-180° angle between these diagonals (we measure angles always in counter-clockwise sense). If one diagonal is degenerate to a point we take the tangent of γ at this point to define α.

The map f is $G := \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\bar{0}_x, \bar{1}_x\} \times \{\bar{0}_y, \bar{1}_y\}$-invariant, where G acts on $(S^1)^4$ by $\bar{1}_x \cdot (x_1, x_2, y_1, y_2) = (x_2, x_1, y_1, y_2)$ and $\bar{1}_y \cdot (x_1, x_2, y_1, y_2) = (x_1, x_2, y_2, y_1)$.

Let $P := f^{-1}(0, 60°)$ be the set of parallelograms on γ having a 60°-angle modulo 180° between their diagonals. We call them 60°-parallelograms. We may assume that P is a union of connected 1-dimensional submanifolds K_i of $(S^1)^4$,

$$P = K_1 \cup \ldots \cup K_n, \; K_i \cong S^1,$$

where the union is disjoint execpt that points $(x_1, x_2, y_1, y_2) \in P$ of the form $x_1 = x_2$ or $y_1 = y_2$ occur exactly twice (for all technicalities, see Section 3.3). This is because P might contain parallelograms where one diagonal is degenerate to a point. These are exactly the points of P on that G does not act freely. However G acts freely on the disjoint union $K_1 \uplus \ldots \uplus K_n$. We denote $(S^1)^4/_G = M^2$ where $M := (S^1)^2/_{\mathbb{Z}_2}$ is the Möbius strip. The first factor M parametrises x_1 and x_2 without their order and the second M parametrises y_1 and y_2. Let $L_1 \uplus \ldots \uplus L_m \subset M^2$ be the quotient manifold $(\bigcup K_i)/_G$, which has corners at the points where it touches ∂M^2. Then L represents an element in the 1-dimensional unoriented bordism group $N_1(M^2) \cong N_1((S^1)^2) \cong (\mathbb{Z}_2)^2$, since all embedded circles γ are isotopic in the plane and G-homotopies of f change $K_1 \uplus \ldots \uplus K_n$ by a G-bordism.

If γ is the unit circle then we see that P is the disjoint union of four circles that all get identified by G. Their quotient L is one circle that represents $(\bar{1}, \bar{1}) \in N_1(M^2) \cong (\mathbb{Z}_2)^2$, where $\bar{1} \in \mathbb{Z}_2$ is the generator.

P does not contain parallelograms that have an edge that is degenerate to a point. Hence the x_1 and x_2-coordinates will always differ from the y_1 and y_2-coordinates at any point $(x_1, x_2, y_1, y_2) \in P$. Therefore the circles L_i can only represent the elements $(\bar{0}, \bar{0})$ and $(\bar{1}, \bar{1})$ of $N_1(M^2) \cong (\mathbb{Z}_2)^2$.

Now we come to the "hidden symmetry", that is, the geometric piece of information that is the key in this proof. Let $W := \{(\alpha, \beta, \gamma) \in (S^1)^3 \mid \alpha + \beta + \gamma = 0^\circ \mod 180^\circ\}$. We define a map

$$\begin{aligned} F : (S^1)^6 \quad &\longrightarrow \quad (\mathbb{R}^2)^3 \times W \\ (x_1, x_2, y_1, y_2, z_1, z_2) \quad &\longmapsto \quad (m_x, m_y, m_z, \alpha_x, \alpha_y, \alpha_z), \end{aligned}$$

where m_x is the mid-point of the segment $(\gamma(x_1), \gamma(x_2))$, α_x is the mod-180°-angle to some fixed axis in $\mathbb{R}^2$, and analogously for the the other coordinates. F is equivariant with respect to the natural actions of the wreath product $K := (\mathbb{Z}_2)^3 \rtimes \mathbb{Z}_3$. Let

$$\widetilde{S} := F^{-1}(\Delta_{(\mathbb{R}^2)^3} \times \{(60^\circ, 60^\circ, 60^\circ)\}).$$

We may assume that $\widetilde{S}$ is a 0-dimensional free K-manifold. We define $S := \widetilde{S}/_K$ to be the set of *stars*. Every star $s \in S$ contains three 60°-parallelograms on γ, namely P_{xy}, P_{yz} and P_{zx}. Modulo G they lie in some components L_i, L_j and L_k (they are not necessarily pairwise distinct). We say that this star s relates L_i, L_j and L_k. Saying this is unique up to cylic permutation of L_i, L_j and L_k. So we can draw a directed graph D whose nodes are the components of L, and we draw for each star a directed triangle $L_i \to L_j \to L_k \to L_i$.

Assume that γ does not contain a rectangle of aspect ratio $\sqrt{3}$. These are exactly the rectangles whose diagonals cross in a 60°-angle. Then all 60°-parallelograms on γ are *skinny* or *fat* in the sense that the x-diagonal is longer or shorter than the y-diagonal. By continuity this does not change along the components of L. Hence we can call the L_i's fat or skinny.

Recall that $[L_i] \in N_1(M^2)$ is $(\bar{0}, \bar{0})$ or $(\bar{1}, \bar{1})$. Correspondingly, we say that the *winding number* $w(L_i)$ of L_i is $\bar{0}$ (even) or $\bar{1}$ (odd), respectively.

Let $x, y : M^2 \to M$ be the projections to the first and to the second factor, respectively. An arc $L_i \to L_j$ in the graph D corresponds to an intersection of $y(L_i)$ and $x(L_j)$. The number of such intersections is

$$\sharp(y(L_i) \cap x(L_j)) = w(L_i) \cdot w(L_j) \mod 2. \tag{1}$$

We will derive a contradiction by double counting the number of stars $\sharp S$.

By (1), components of L with even winding number will have no influence on what follows. Let s be the number of skinny components of L with odd winding number, and let f be the number of fat components of L with odd winding number.

We know that $[L] = \sum_i [L_i] = (\bar{1}, \bar{1})$, thus

$$s + f = 1 \mod 2.$$

Note that no star relates three skinny or three fat 60°-parallelograms with each other. Hence every star gives exactly one arc from a skinny to a fat component of L. Modulo 2 and using (1), there are congruent $s \cdot f = 0 \mod 2$ of these arcs. Therefore,

$$\sharp S = 0 \mod 2.$$

On the other hand, every star relates three components, two of which are skinny or two of which are fat. So every star gives exactly one arc between two skinny components or between two fat components. Using (1), the number of arcs between skinny components modulo two is

$$s^2 = s \mod 2,$$

and the number of arcs between fat components modulo two is

$$f^2 = f \mod 2.$$

Together this gives,

$$\sharp S = s + f = 1 \mod 2.$$

This is a contradiction, which finishes the proof of Theorem 1.5. □

3.3 Technical Details

In the previous section we assumed that the set of inscribed 60°-parallelograms P is a 1-dimensional manifold in the 4-manifold M^2. Also the set of stars should be finite. At the same time, when two parallelograms p_1 and p_2 have a common diagonal $y(p_1) = x(p_2)$ they form a star. Thus there should be another parallelogram p_3 such that $x(p_1) = y(p_3)$ and $y(p_2) = x(p_3)$. These triple intersection points come from the geometry, but they are in some sense *not generic*. That is, we need to be careful on how to make the test-maps f and F simultaneously transversal in oder to keep the geometric propery of a star and without violating the equivariance. We solve this issue by perturbing the following two maps.

Let

$$m : (S^1)^2 \to \mathbb{R}^2$$

be the map that sends $(x_1, x_2) \in (S^1)^2$ to the mid-point $\frac{\gamma(x_1)+\gamma(x_2)}{2}$. Let

$$\alpha : (S^1)^2 \to S^1$$

be the map that sends $(x_1, x_2) \in (S^1)^2$ to the mod-180° angle of the line through $\gamma(x_1)$ and $\gamma(x_2)$ and some fixed line in the plane. The maps f and F can written in terms of m and α,

$$f(x_1, x_2, y_1, y_2) = (m(y_1, y_2) - m(x_1, x_2), \alpha(y_1, y_2) - \alpha(x_1, x_2))$$

and similarly F.

Let $\varphi_i : S^1 \to [0,1]$, $i = 1 \dots k$, be a partition of unity of S^1 subordinate to a covering of S^1 with small ε-balls. We will perturb the maps m and α with two sets of parameters $S_m := ([-\varepsilon, +\varepsilon]^2)^{\binom{k+1}{2}}$ and $S_\alpha := [-\varepsilon, +\varepsilon]^{\binom{k+1}{2}}$ as follows:

$$\begin{aligned} m' : & S_m \times (S^1)^2 && \longrightarrow \mathbb{R}^2 \\ & (s_m, x_1, x_2) && \longmapsto m(x_1, x_2) + \sum_{i \leq j} (\varphi_i(x_1)\varphi_j(x_2) + \varphi_i(x_2)\varphi_j(x_1)) \cdot (s_m)_{i,j}, \end{aligned}$$

and

$$\begin{aligned} \alpha' : & S_\alpha \times (S^1)^2 && \longrightarrow S^1 \\ & (s_\alpha, x_1, x_2) && \longmapsto \alpha(x_1, x_2) + \sum_{i \leq j} (\varphi_i(x_1)\varphi_j(x_2) + \varphi_i(x_2)\varphi_j(x_1)) \cdot (s_\alpha)_{i,j}, \end{aligned}$$

This defined analogous functions $f' : S_m \times S_\alpha \times (S^1)^4 \longrightarrow_G \mathbb{R}^2 \times S^1$ and $F' : S_m \times S_\alpha \times (S^1)^6 \longrightarrow_K (\mathbb{R}^2)^3 \times W$. Because of the additional parameter space f' and F' are transversal to the respective test-spaces $\{(0, 60°)\}$ and $\Delta_{(\mathbb{R}^2)^3} \times \{60°, 60°, 60°\}$. By the transversality theorem [**?**, p. 68], for almost all choices $s := (s_m, s_\alpha)$ (up to a zero set), the perturbations $f'_s := f'(s, \_)$ and $F'_s := F'(s, \_)$ are transversal to the test-spaces as well. Similarly one can show that for almost all s, $y(K_i)$ intersects $x(K_j)$ transversally for all i, j.

4 Squares on immersed curves

Toeplitz' conjecture is about studying inscribed squares on simple closed curves in the plane. There are plenty ways to generalize this problem. One possible way is to omit the requirement that γ has to be injective. Then there are several kinds of degenerate squares, which we have to deal with in that case. To keep things simple, we will only consider transversal intersections, and we will not count degenerate squares.

In this section we will prove a simple mod-2 formula for the number of squares that are inscribed in an immersed circle (or union of circles).

Let γ be a "generic" immersion of a finite union of circles in the plane.

There is a chequerboard coloring of the complement of γ, see Figure 5. That is, we color each component of $\mathbb{R}^2 \backslash \gamma$ black or white such that adjacent components get different colors. We may assume that the unbounded component is white. Let $b(\gamma)$ be the number of black components. We call a self-intersection of γ a crossing. We say that a crossing is **fat** if the black angles at this crossing are larger than 90°. The fat crossings in Figure 5 are marked by a black dot. Let $f(\gamma)$ be the number of fat crossings.

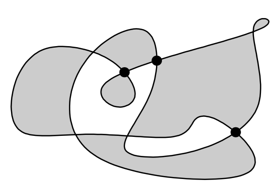

Figure 5: Chequerboard coloring associated to γ. Dots mark the fat crossings.

Theorem 4.1. *Suppose that γ is a generic immersion of finitely many circles in the plane. Then the number of non-degenerate squares inscribed in γ is congruent to $b(\gamma) + f(\gamma)$ modulo 2.*

Proof. By genericity of the curve, no inscribed square will have a vertex at a crossing. We smoothen the crossings of γ such that all white components become one big component. The number of inscribed squares increases by $f(\gamma)$ under this operation, see Figure 6. The new curve consists of $b(\gamma)$ separated simple closed curves. We can deform them by an ambient isotopy such that they become $b(\gamma)$ small ellipses and such that there is no inscribed square that touches more than one component. Therefore the resulting union of ellipes inscribes exactly $b(\gamma)$ squares. Using a bordism argument, the parity of the number of inscribes squares did not change during the isotopy. Since every ellipse inscribes exactly one square, this finishes the proof. $\square$

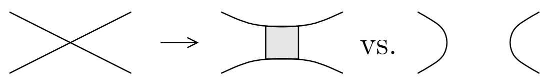

Figure 6: When we smoothen a crossing then a new square appears if and only if we opened the smaller angle.

Rectangles on immersed curves

The analog theorem for rectangles of prescribes aspect ratio $0 < r < 1$ that are inscribed in immersed circles is slightly different. Let γ be again a generic immersion of a finite union of circles in the plane, and consider again the chequerboard coloring from above. Let $0 < \alpha(r) < \pi/2$ be the angle at the intersection of the two diagonals of a rectangle with aspect ratio r. We call a self-intersection of γ α**-orthogonal**, if the angle at this crossing lies in the open interval $(\alpha, \pi - \alpha)$. Let $o(\gamma, r)$ denote the number of $\alpha(r)$-orthogonal crossings.

Theorem 4.2. *Let $0 < r < 1$. Suppose that γ is a generic immersion of finitely many circles in the plane. Then the number of non-degenerate rectangles with aspect ratio r inscribed in γ is congruent to $b(\gamma) + o(\gamma, r)$ modulo 2.*

Proof. The proof is very similar to the one of Theorem 4.1. Again, in a small neighborhood of a generic crossing we have no inscribed rectangle with aspect ratio r. When we smoothen the crossing, 0, 1, or 2 new rectangles will appear, depending on whether the angle β that we smoothen satisfies $\beta < \alpha$, $\alpha < \beta < \pi - \alpha$, or $\pi - \alpha < \beta$; compare with Figure 7.

The rest is analogous to the previous proof. $\square$

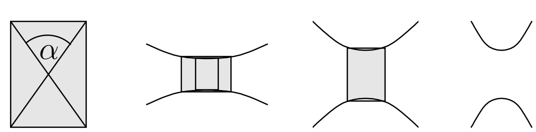

Figure 7: Smoothening a crossing changes the number of inscribed rectangles modulo 2 if and only if the crossing is α-orthogonal.

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