Successive Spectral Sequences

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Abstract

If a chain complex is filtered over a poset I, then for every chain in I we obtain a spectral sequence. In this paper we define a spectral system that contains all these spectral sequences and relates their pages via differentials, extensions, and natural isomorphisms. We also study an analog of exact couples that provides a more general construction method for these spectral systems.

This turns out to be a good framework for unifying several spectral sequences that one would usually apply one after another. Examples are successive Leray–Serre spectral sequences, the Adams–Novikov spectral sequence following the chromatic spectral sequence, successive Grothendieck spectral sequences, and successive Eilenberg–Moore spectral sequences.

1 Introduction

Motivation. When one applies spectral sequences successively it fairly often happens that there are several ways to do this. In this paper we unify these ways in one *spectral system* (short for system of spectral sequences).

As an example, consider the following tower of two fibrations,

$$\begin{tikzcd} F \arrow[r, "i_{FE}"] & E \arrow[d, "p_{EN}"] \\ M \arrow[r, "i_{MN}"] & N \arrow[d, "p_{NB}"] \\ & B. \end{tikzcd} \tag{1}$$

Our aim is to compute $H_*(E)$ from $H_*(F)$, $H_*(M)$, and $H_*(B)$. We could apply a Leray–Serre spectral sequence to compute $H_*(N)$ from $H_*(M)$ and $H_*(B)$ as an intermediate step, and then apply another Leray–Serre spectral sequence to derive

$H_*(E)$ from $H_*(F)$ and $H_*(N)$. Alternatively we might define $P := p_{EN}^{-1}(i_{MN}(M)) = (p_{NB} \circ p_{EN})^{-1}(\mathrm{pt})$ and get another sequence of fibrations,

$$\begin{tikzcd}
F \arrow[r, "i_{FP}"] & P \arrow[r, "i_{PE}"] \arrow[d, "p_{PM}"] & E \arrow[d, "p_{EB}"] \\
& M & B.
\end{tikzcd} \tag{2}$$

Here we can apply two different but related Leray–Serre spectral sequences with intermediate step $H_*(P)$. The following diagram illustrates this 'associativity law',

$$\begin{tikzcd}
H_*(F), H_*(M), H_*(B) \arrow[r, "\mathrm{id}\times\mathrm{LS}"] \arrow[d, "\mathrm{LS}\times\mathrm{id}"] & H_*(F), H_*(N) \arrow[d, "\mathrm{LS}"] \\
H_*(P), H_*(B) \arrow[r, "\mathrm{LS}"] & H_*(E).
\end{tikzcd}$$

Under certain conditions on the fibrations we construct in Section 5 a spectral system with "second page" $H_*(B, H_*(M, H_*(F)))$ that converges to $H_*(E)$.

Outline. In Section 2 we construct a spectral system for chain complexes that are filtered over an arbitrary poset.

In Section 3 we treat the special case of chain complexes that are $\mathbb{Z}$-filtered in several ways and prove basic properties, some of which have only trivial analogs in usual spectral sequences. We show that there are several interesting non-trivial connections between the 0-page and the usual goal of computation $H_*(C)$.

Section 4 is about exact couple systems, which lead to spectral systems that are more general than the ones coming from I-filtered chain complexes, in the same way as exact couples generalize $\mathbb{Z}$-filtered chain complexes. We study basic properties such as differentials, extensions, natural isomorphisms, splitting principles, and multiplicative structures.

We give the following examples of spectral systems:

1. A basic running example in this paper is the spectral system for the generalized homology theory of a I-filtered space X. It has many desirable properties, and the next two examples are instances of it.

2. As mentioned above there is a spectral system for iterated fibrations, generalizing the Leray–Serre and Atiyah–Hirzebruch spectral sequences, see Section 5.

3. In Section 6 we construct a generalization of Grothendieck's spectral sequence in the setting of a composition of several functors. We also provide a general and natural condition for the existence of a product structure, which seems new also for the ordinary Grothendieck spectral sequence.

4. The E_2-page of the Adams–Novikov spectral sequence is the limit of the chromatic spectral sequence; we show how to put this into a single spectral system in Section 7.
5. In Section 8 we construct a spectral system for a cube of fibrations, where one would usually apply Eilenberg–Moore spectral sequences successively.

This paper starts at a much slower pace than it ends in order for it to be more accessible to a broad audience. Background on spectral sequences can be found in McCleary [McC01], Spanier [Spa66], Hatcher [Hat04], Weibel [Wei94], Cartan–Eilenberg [CE56], Gelfand–Manin [GM03], Bott–Tu [BT82], Switzer [Swi75], and many more. A detailed account on the early history of spectral sequences is given in McCleary [McC99].

Previous generalizations of spectral sequences. Several very useful extensions of spectral sequences are known: Po Hu's [Hu98, Hu99] *transfinite spectral sequences* are the closest among them to this paper. They consists of terms $E^r_{p,q}$ as in the standard setting except that p, q, r are elements in the Grothendieck group $G(\omega^\alpha)$, which in case $\alpha = n \in \mathbb{N}$ is isomorphic to $\mathbb{Z}^n$ with the lexicographic order. Thus indices $P \in G(\omega^\alpha) = \mathbb{Z}^n$ correspond to the sets $A(p_1, \ldots, p_n; \mathrm{id}_{\mathbb{Z}^n})$ in Section 3.2.1, which makes transfinite spectral sequences basically a substructure of spectral systems, using the lexicographic connection from Lemma 3.12 (that transfinite spectral sequences need to have $G(\omega^\alpha)$-graded pages as in [Hu99] is actually not necessary). On the other hand, that spectral systems have a richer structure means that not all transfinite spectral sequences will naturally generalize to spectral systems. Hu gave several examples of transfinite spectral sequences.

Deligne studied in connection with his mixed Hodge structures [Del71] spectral sequences of chain complexes with respect to several $\mathbb{Z}$ -filtrations. Compare [Del71, §1] with Section 3 of this paper.

Behrens [Beh12] constructed transfinite versions of the Atiyah–Hirzebruch, EHP, Goodwillie, and homotopy spectral sequences; see also Behrens [Beh06].

The exact couple systems of Section 4 are canonical substructures of perverse sheaves (Beĭlinson–Bernstein–Deligne [BBD82]) and more generally Fáry functors (Grinberg–MacPherson [GM99]). Thus the latter give also rise to spectral systems over the poset of open sets of the underlying space.

2 The spectral system of a generalized filtration

Throughout Sections 2, 3, and 4, we will simplify notation by omitting the grading of chain complexes and homology; compare with Remark 2.6.

Let (C, d) be a chain complex, which has several subcomplexes F_i, $i \in I$. Then the family $F := \{F_i \mid i \in I\}$ can be seen as a generalized filtration, since we do not require any inclusion relations in F. Let us give I the structure of a poset by $i \leq j$ for $i, j \in I$ if and only if $F_i \subseteq F_j$. We say that C is filtered over the poset I, or I-filtered for short.

Every chain of countable size in I gives rise to a spectral sequence that converges to $H(C)$ (under some standard assumptions). The questions we are interested in are how they are related, and whether there is a reasonable larger device that contains all those spectral sequences.

Interesting generalized filtrations arise when C is filtered over $\mathbb{Z}$ in two or more different ways, see Section 3.

Another basic example to have in mind comes from an I-filtered space X, that is, at family of subspaces $(X_i)_{i\in I}$ of X, ordered by inclusion. The singular chain complex $C_*(X)$ of X is then filtered by $F_i := \mathrm{im}(C_*(X_i) \hookrightarrow C_*(X))$, $i \in I$. Similarly, the singular cochain complex $C^*(X)$ of X is filtered by $F^i := \ker(C^*(X) \to C^*(X_i))$, $i \in I^*$, I^* being the dual poset of I (same elements, reversed relations); compare with Remarks 2.4 and 2.5.

2.1 Construction and basic properties

For two subgroups X and Y of a larger abelian group, we write

$$X/Y := X/(X \cap Y) \cong (X + Y)/Y,$$

which will simplify notation considerably.

For $p, q, z, b \in I$, we define

$$S_{bq}^{pz} := \frac{F_p \cap d^{-1}(F_z)}{d(F_b) + F_q}. \tag{3}$$

We call p as usual the *filtration degree*, z the *cycle degree*, b the *boundary degree*, and q the *quotient degree* (q has here a different meaning as in the standard notation $E^r_{p,q}$). In what follows we will only consider the S-terms for which $F_q \subseteq F_p$. If F is closed under taking intersections and sums, then we may assume that

$$F_z \subseteq F_p \supseteq F_q \subseteq F_b, \tag{4}$$

since we can intersect F_z by F_p and add F_q to F_b without changing the S-term.

If the inclusions

$$F_z \subseteq F_q \text{ and } F_p \subseteq F_b,$$

are fulfilled then S_{bq}^{pz} can be written as

$$\begin{aligned} S_{bq}^{pz} &= \mathrm{im}\big(H\left(F_p/F_z\right) \xrightarrow{\mathrm{incl}_*} H\left(F_b/F_q\right)\big) && (5) \\ &= \frac{\ker\big(H(F_p/F_q) \to H(F_q/F_z)\big)}{\mathrm{im}\big(H(F_b/F_p) \to H(F_p/F_q)\big)}. && (6) \end{aligned}$$

We may assume that F contains $F_{-\infty} := 0$ (the zero complex) and $F_{\infty} := C$. Then our goal of computation is $H(C) = S^{\infty,-\infty}_{\infty,-\infty}$., which we call the *limit* of S. (Note that for us the limit is part of the structure of S, it does not imply any kind of convergence in the usual sense.)

As with ordinary spectral sequences of a $\mathbb{Z}$-filtration, there are the following ways to relate different S-terms.

2.2 Extension property

If $F_z \subseteq F_{p_0} \subseteq \ldots \subseteq F_{p_n} \subseteq F_b$, then $S^{p_n,z}_{b,p_0}$ is a successive group extension of the $S^{p_i,z}_{b,p_{i-1}}$, $i = 1 \ldots n$, since the sequence

$$0 \to S^{p_i,z}_{b,p_{i-1}} \to S^{p_{i+1},z}_{b,p_{i-1}} \to S^{p_{i+1},z}_{b,p_i} \to 0$$

is exact.

2.3 Differential

Suppose that two quadruples (p_1, z_1, b_1, q_1) and (p_2, z_2, b_2, q_2) in I^4 satisfy

$$F_{z_2} \subseteq F_{p_1} \text{ and } F_{q_2} \subseteq F_{b_1}. \tag{7}$$

Then d induces a well-defined differential

$$d : S^{p_2 z_2}_{b_2 q_2} \longrightarrow S^{p_1 z_1}_{b_1 q_1},$$

which we also denote by d.[1]

Now suppose that we have a sequence of such differentials,

$$S^{p_3 z_3}_{b_3 q_3} \xrightarrow{d_3} S^{p_2 z_2}_{b_2 q_2} \xrightarrow{d_2} S^{p_1 z_1}_{b_1 q_1},$$

such that (7) and the corresponding inclusions for d_3 are fulfilled. It might help to visualize this like that:

$$\begin{array}{ccccccccccc}
F_{z_1} & F_{q_1} & \subseteq & F_{p_1} & F_{b_1} & & & & & & \\
 & & & \rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} & & & & & & \\
 & & & F_{z_2} & F_{q_2} & \subseteq & F_{p_2} & F_{b_2} & & & \\
 & & & & & & \rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} & & & \\
 & & & & & & F_{z_3} & F_{q_3} & \subseteq & F_{p_3} & F_{b_3}
\end{array} \tag{8}$$

If

$$q_2 = b_1 \text{ and } F_{z_2} \cap F_{q_1} = F_{z^*} \text{ for some } z^* \in I,^{2} \tag{9}$$

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[1] The condition (7) is not most general. In fact we only need to assume that $d(F_{p_2}) \cap F_{z_2} \subseteq F_{p_1}$ and $d(F_{q_2}) \subseteq d(F_{b_1}) + F_{q_1}$.

[2] For example if $z_2 = p_1$ we can take $z^* = q_1$.

then the kernel of d_2 is given by

$$\ker(d_2) = S^{p_2,z^*}_{b_2,q_2}.$$

If

$$z_3 = p_2 \text{ and } F_{b_2} + F_{p_3} = F_{b^*} \text{ for some } b^* \in I, \text{[3]} \tag{10}$$

then the cokernel of d_3 is given by

$$\operatorname{coker}(d_3) = S^{p_2,z_2}_{b^*,q_2}.$$

If both conditions (9) and (10) hold then we can compute the homology

$$\frac{\ker(d_2)}{\operatorname{im}(d_3)} = S^{p_2,z^*}_{b^*,q_2}. \tag{11}$$

Interesting special case: if all four vertical inclusions in Diagram (8) hold with equality then

$$\frac{\ker(d_2)}{\operatorname{im}(d_3)} = S^{p_2,q_1}_{p_3,q_2}. \tag{12}$$

Example 2.1. We call the collection of all E-terms of the form

$$S^{pp}_{qq} = F_p/F_q$$

the 0-*page*. Here, diagram (8) has the pattern

$$\begin{array}{cccccccc}
p & q & p & q & & & & \\
 & & p & q & p & q & & \\
 & & & & p & q & p & q
\end{array}$$

The induced differential $S^{pp}_{qq} \to S^{pp}_{qq} \to S^{pp}_{qq}$ coincides with the boundary map that d induces on F_p/F_q. By (12) the homology of the middle term is

$$S^{pq}_{pq} = H(F_p/F_q).$$

We call the collection of all these S-terms the 1-*page*. By (5) the 1-page determines all other relevant S-terms as long as we know the maps between them that are induced by inclusion $F_p \hookrightarrow F_b$. However in many applications, only the S-terms of the 0- and 1-page where p covers q are given.

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[3] For example if $b_2 = q_3$ we can take $b^* = p_3$.

2.4 Remarks

Remark 2.2 (Isomorphic S-terms)**.** For general index posets I it may happen that two S-terms with different indices are naturally isomorphic (natural with respect to filtration preserving maps between I-filtered chain complexes). This does occur not in the classic case $I = \mathbb{Z}$, but it does when $I = \mathbb{Z}^n$, $n \geq 2$, where it turns out the be very useful. We postpone this connection to Sections 3.1 and 4.2.

Remark 2.3 (Limits, convergence, comparison)**.** As in ordinary spectral sequences, depending on the given filtration we might need to take differentials or do extensions an infinite number of times in order to connect the 0-page to $H(C)$. The convergence and comparison theorems from Eilenberg–Moore [EM62], Boardman [Boa99], McCleary [McC01], and Weibel [Wei94] go over in the general setting without difficulties. For simplicity we will ignore this, for example by assuming that the given filtration is of *finite height*, that is, only finitely many different terms F_p appear in any chain of the filtration. If C is graded then we will require this only degree-wise.

Remark 2.4 (Surjections)**.** A generalized filtration of a chain complexes is a commutative diagram of chain complexes where all maps are injections. We could do all constructions of this section dually, starting with a diagram of surjections instead of injections. This leads essentially to the same spectral system, since one can filter C by the kernels of the given surjections from C to all the chain complexes in the diagram and take the spectral system of this filtration.

Remark 2.5 (Cohomology)**.** Most of the paper is written in terms of homology. If the reader prefers cohomology, all one has to do is dualizing the underlying poset I, which makes increasing filtrations in this paper decreasing. For a particular instance, see Example 4.5. All maps stay the same, except for morphisms between spectral systems, which are now contravariant functors.

Remark 2.6 (Grading)**.** Instead of omitting the grading of a chain complex in the notation, we can work with more general ungraded chain complexes, that is an abelian group C (or an object in some other abelian category) together with an endomorphism $d : C \to C$ with $d^2 = 0$. All constructions in this and the next two sections work equally in the graded and ungraded case. For example if C has the structure of a graded abelian group (graded over $\mathbb{Z}$ or any other abelian group), d is a graded homomorphism of a fixed degree, and the F_i are graded subgroups, then the S-terms are graded as well and the differential is graded with the same degree as d.

2.5 The spectral sequence of a $\mathbb{Z}$-filtration

Suppose that the filtration F is indexed over the integers, $F = (F_p)_{p\in\mathbb{Z}}$. Then the well-known *spectral sequence of the $\mathbb{Z}$-filtration F* is defined as

$$E^r_{p,q} := (S^{p,p-r}_{p+r-1,p-1})_{(p+q)},$$

together with differentials

$$d : E^r_{p,q} \to E^r_{p-r,q+r-1}.$$

In this grading, d is a differential of bidegree $(-r, r-1)$ in $E^r_{*,*} := \oplus_{p,q} E^r_{p,q}$. Equation (11) implies that

$$H(E^r_{*,*}) = E^{r+1}_{*,*}.$$

The E^∞-term is

$$E^\infty_{p,q} = (S^{p,-\infty}_{\infty,p-1})_{(p+q)},$$

hence the above extension property recovers (or at least relates to) $H_*(C) = S^{\infty,-\infty}_{\infty,-\infty}$ by iterated group extensions.

Remark 2.7. Note that the spectral sequence of a $\mathbb{Z}$-filtration contains only a small subset of the S^{pz}_{bq}-terms, namely those where $p - q = 1$ and $b - q = p - z = r$. This is definitely a good choice, since it connects the E^0-page to $H_*(C)$ by a sequence of differentials and extensions in a direct fashion, which contains often all necessary information. Often there is also additional structure such as products. Sometimes it might be reasonable to look at other S-terms as well, since for a $\mathbb{Z}$-filtration there are many connecting differentials and extensions.[4]

For example it can be reasonable to look at terms

$$E^{r_b,r_z}_{p,q} := (S^{p,p-r_z}_{p+r_b-1,p-1})_{(p+q)},$$

that is, the pages are now indexed over all $(r_b, r_z) \in \mathbb{Z}^2_{\geq 0}$. For $r_b = r_z =: r$ this coincides with $E^r_{p,q}$. There is a connecting differential $d : E^{r_{b1},r_{z1}}_{p_1,q_1} \to E^{r_{b2},r_{z2}}_{p_2,q_2}$ if $p_1 + q_1 = p_2 + q_2 + 1$, $r_{z1} = r_{b2}$ and $p_1 = p_2 + r_{z1}$, and we have

$$\ker d = E^{r_{b1},r_{z1}+1}_{p_1,q_1}$$

and

$$\operatorname{coker} d = E^{r_{b2}+1,r_{z2}}_{p_2,q_2}.$$

One can also mix differentials and extensions along the way (but then the S-notation seems more convenient).

3 Several $\mathbb{Z}$-filtrations

Suppose our given chain complex C is $\mathbb{Z}$-filtered in n different ways,

$$\ldots \subseteq F^{(i)}_{k-1} \subseteq F^{(i)}_k \subseteq F^{(i)}_{k+1} \subseteq \ldots, \quad 1 \leq i \leq n. \tag{13}$$

In this section we construct a spectral system that incorporates all the spectral sequences of the n filtrations.

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[4] A particular example, where this might be useful, are spectral sequence valued indices of G-bundles, see [Mat13].

We denote $\overline{\mathbb{Z}} := \mathbb{Z} \cup \{-\infty, \infty\}$, which is a lattice ordered by $\leq$. $\overline{\mathbb{Z}}^n$ is the product poset, ordered by $P = (p_1, \ldots, p_n) \geq Q = (q_1, \ldots, q_n)$ if and only $p_i \geq q_i$ for all i. Let $D(\overline{\mathbb{Z}}^n)$ denote the set of downsets $p \subseteq \overline{\mathbb{Z}}^n$, that is, all subsets p that satisfy: if $Q \leq P$ and $P \in p$ then $Q \in p$. $D(\overline{\mathbb{Z}}^n)$ will be the underlying poset for our spectral system.

We extend the n $\mathbb{Z}$-filtrations (13) by setting $F^{(i)}_{-\infty} := \bigcap_{k\in\mathbb{Z}} F^{(i)}_k$ and $F^{(i)}_{\infty} := C$.[5] For any $P \in \overline{\mathbb{Z}}^n$ we define

$$F_P := F^{(1)}_{p_1} \cap \ldots \cap F^{(n)}_{p_n}.$$

For any downset $p \in D(\overline{\mathbb{Z}}^n)$ we define

$$F_p := \sum_{P\in p} F_P \subseteq C, \tag{14}$$

where the sum denotes taking interior sums of subcomplexes of C. Thus, $(F_p)_{p\in D(\overline{\mathbb{Z}}^n)}$ is a $D(\overline{\mathbb{Z}}^n)$-filtration of C.

Definition 3.1. Let I be a distributive lattice with meet denoted $\cap$ and join denoted $\cup$. We call an I-filtration $(G_p)_{p\in I}$ *distributive* if $G_{a\cap b} = G_a \cap G_b$ and $G_{a\cup b} = G_a + G_b$ for all $a, b \in I$.

Example 3.2. For distributivity of the above filtration $(F_p)_{p\in D(\overline{\mathbb{Z}}^n)}$ only $F_{a\cap b} = F_a \cap F_b$ needs to be checked. Dually we could have defined a $D(\overline{\mathbb{Z}}^n)$-filtration $F^p := \bigcap_{P\notin p-\mathbb{1}} F^{(1)}_{p_1} + \ldots + F^{(n)}_{p_n}$, which in turn is distributive if and only if $F^{a\cup b} = F^a + F^b$ for all a, b. If one of (F_p) and (F^p) is distributive and the underlying $\mathbb{Z}$-filtrations (13) are finite then $(F_p) = (F^p)$. (This finiteness assumption can be omitted if the filtrations are completely distributive.)

Example 3.3. If C is $\mathbb{Z}^n$-graded and (13) are the n canonical filtrations of C then (F_p) is clearly distributive.

Remark 3.4. We could have started this section right away from an arbitrary $D(\overline{\mathbb{Z}}^n)$-filtration (F_p). All statements below will only require it to be distributive. (In practice, $\overline{\mathbb{Z}}^n$-filtrations usually arise as the common refinement of n $\mathbb{Z}$-filtrations.)

3.1 Naturally isomorphic S-terms

Let us regard $\overline{\mathbb{Z}}^n$ as the vertices of a graph G, where two elements $P, Q \in \overline{\mathbb{Z}}^n$ are connected by an edge if and only if $P \leq Q$ or $P \geq Q$. In this sense, we call a subset $A \subseteq \overline{\mathbb{Z}}^n$ *connected* if and only if the induced subgraph $G[A]$ is connected.

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[5] The reader should not wonder why we did not set $F^{(i)}_{-\infty} := 0$ or alternatively $F^{(i)}_{\infty} := \sum_{k\in\mathbb{Z}} F^{(i)}_k$. The reason for this definition is that it makes (14) very natural. E.g. for $n = 1$, we have $F_\emptyset = 0$, $F_{\{-\infty\}} = \bigcap_{k\in\mathbb{Z}} F^{(1)}_k$, $F_{\mathbb{Z}\cup\{-\infty\}} = \sum_{k\in\mathbb{Z}} F^{(1)}_k$, and $F_{\overline{\mathbb{Z}}} = C$.

Definition 3.5. To any downsets $z \leq q \leq p \leq b$ in $D(\overline{\mathbb{Z}}^n)$, let

$$Z(z, q, p, b) \subseteq \overline{\mathbb{Z}}^n$$

denote the union of all connected components of $p \backslash z$ that intersect $p \backslash q$, and let

$$B(z, q, p, b) \subseteq \overline{\mathbb{Z}}^n$$

denote the union of all connected components of $b \backslash q$ that intersect $p \backslash q$.

We call S^{pz}_{bq} with $z \leq q \leq p \leq b$ a *reduced S-term* if $p \backslash z = Z(z, q, p, b)$ and $b \backslash q = B(z, q, p, b)$.

The following lemma shows that any S^{pz}_{bq} with $z \leq q \leq p \leq b$ has an isomorphic reduced S-term $S^{p\widetilde{z}}_{\widetilde{b}q}$ (canonical with respect to maps between $\overline{\mathbb{Z}}^n$-filtered chain complexes).

Lemma 3.6 (Reducing S^{pz}_{bq}). *Let $z, q, p, b \in D(\overline{\mathbb{Z}}^n)$ be downsets with $z \leq q \leq p \leq b$. Let $Z := Z(z, q, p, b)$ and $B := B(z, q, p, b)$. Define $\widetilde{z} := p \backslash Z$ and $\widetilde{b} := q \cup B$. Then the inclusions $z \hookrightarrow \widetilde{z}$ and $\widetilde{b} \hookrightarrow b$ induce a commutative diagram of four natural isomorphisms of S-terms,*

$$\begin{tikzcd}
S^{pz}_{\widetilde{b}q} \arrow[r, "\cong"] \arrow[d, "\cong"'] & S^{p\widetilde{z}}_{\widetilde{b}q} \arrow[d, "\cong"] \\
S^{pz}_{bq} \arrow[r, "\cong"] & S^{p\widetilde{z}}_{bq},
\end{tikzcd} \tag{15}$$

and $S^{p\widetilde{z}}_{\widetilde{b}q}$ is reduced.

Later we will prove an similar statement for exact couple systems, Lemma 4.16, which has a more conceptual proof, however it does not imply the full Lemma 3.6.

Proof. First we prove that the left map in (15) is an isomorphism. Surjectivity is trivial, since on both sides elements are represented by the same group $F_p \cap d^{-1}(F_z)$. In order to prove injectivity, let $x \in F_p \cap d^{-1}(F_z)$ represent an element that goes to zero in S^{pz}_{bq}, that is, $x \in F_p \cap d^{-1}(F_z) \cap (F_q + d(F_b))$. Note that $F_b = F_{\widetilde{b}} + F_C$, where C is the downset of $b \backslash \widetilde{b}$. Thus x is of the form $x_q + d(x_{\widetilde{b}}) + d(x_C)$ where $x_q \in F_q$, $x_{\widetilde{b}} \in F_{\widetilde{b}}$, and $x_C \in F_C$. From $q \subseteq p$ and $x \in F_p$ it follows that $d(x_{\widetilde{b}}) + d(x_C) \in F_p$. The key point is that $b \backslash \widetilde{b}$ and B are disconnected, which implies that $d(x_C) \in F_q$. Thus $x \in F_q + d(F_{\widetilde{b}})$. This means that x represents zero in $S^{pz}_{\widetilde{b}q}$. Analogously, the right map is an isomorphism.

For the bottom map in (15): Injectivity is immediate, since $z \subseteq \widetilde{z}$. In order to prove surjectivity, let $x \in F_p \cap d^{-1}(F_{\widetilde{z}})$ represent an element in $S^{p\widetilde{z}}_{bq}$. Note that $F_p = F_z + F_Y + F_{Z'}$, where Y is the downset of $\widetilde{z} \backslash z$ and Z' is the downset of Z. Thus x is of the form $x_z + x_Y + x_{Z'}$. Since $z \leq q$ and $d(x_z) \in F_z$, x and $x_Y + x_{Z'}$ represent the same element in $S^{p\widetilde{z}}_{bq}$. The key is again that $\widetilde{z} \backslash z$ and Z are disconnected. Thus, since $d(x_Y) + d(x_{Z'}) \in F_{\widetilde{z}}$, we have $d(x_{Z'}) \in F_z$. Since also $x_Y \in F_q$, x and $x_{Z'}$ represent the same element in $S_{bq}^{p\widetilde{z}}$. Since $x_{Z'} \in F_p \cap d^{-1}(F_z)$, $x_{Z'}$ already represents an element in S_{bq}^{pz}. This shows surjectivity. Analogously, the top map is an isomorphism.

By definition, $S_{\widetilde{b}q}^{p\widetilde{z}}$ is reduced. □

Remark 3.7 (Other canonical forms for S-terms)**.** Given $z \leq q \leq p \leq b$, the lemma replaces z and b by the largest $\widetilde{z} \leq q$ and smallest $\widetilde{b} \geq p$ such that $S_{\widetilde{b}q}^{p\widetilde{z}}$ is isomorphic to S_{bq}^{pz}. Instead we could take also the *smallest* $\widetilde{z} \leq z$ and the *largest* $\widetilde{b} \geq b$, obtaining a different normal form, which might be useful in some specific cases.

Lemma 3.8 (Natural isomorphisms)**.** *Suppose (F_p) is a distributive $D(\overline{\mathbb{Z}}^n)$-filtration of C. Let $z, q, p, b \in D(\overline{\mathbb{Z}}^n)$ be downsets with $z \leq q \leq p \leq b$. Then $Z := Z(z, q, p, b)$ and $B := B(z, q, p, b)$ determine S_{bq}^{pz} up to natural isomorphism, that is, all $z \leq q \leq p \leq b$ with the same Z and B give naturally isomorphic S-terms.*

Proof. This can be proved similarly as the previous lemma. More elegantly, we can first reduce to the case where $z = p \backslash Z$, $z' = p' \backslash Z$, $b = q \cup B$, and $b' = q' \cup B$ using the previous lemma. Then the assertion follows from Lemma 4.16 (the associated exact couple system is excisive). □

Thus when we are only interested in the isomorphism class we may write S_{bq}^{pz} simply as S_B^Z. When differentials are involved we usually prefer the notation S_{bq}^{pz}.

Remark 3.9 (Splitting S-terms)**.** Suppose $z \leq q \leq p \leq b$, and let $p \backslash q = X_1 \dot{\cup} X_2$. Let Z_1 be the union of connected components of $p \backslash z$ that intersect X_1. Let B_1 be the union of connected components of $b \backslash q$ that intersect X_2. Analogously, define Z_2 and B_2. Assume further that $Z_1 \cap Z_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$. Then there is a natural isomorphism

$$S_{bq}^{pz} \cong S_{B_1}^{Z_1} \oplus S_{B_2}^{Z_2}.$$

Remark 3.10. Lemma 3.8 also holds more generally under the weaker condition $b \geq q \leq p \geq z$, which includes terms S_{qq}^{pp} of the 0-page, in the following way: S_{bq}^{pz} is determined up to natural isomorphism by Z, B and $p \backslash q$. Note that $p \backslash q = Z \cap B$ does not need to hold anymore.

3.2 Enough differentials

Consider a distributive $D(\overline{\mathbb{Z}}^n)$-filtration (F_p) of C. A priori it is not be clear what is the best way to connect the S-terms $S_{qq}^{pp} = F_p/F_q$, $p \succ q$, from the 1-page to the goal of computation $S_{\infty,-\infty}^{\infty,-\infty} = H_*(C)$ by a collection of kernels, cokernels, extensions, limits, and natural isomorphisms. The answer depends on the particular situation and there are several choices.

3.2.1 Lexicographic connections

Based on the well-known spectral sequence of a $\mathbb{Z}$-filtration, the most apparent way to connect the 1-page to $H(C)$ is the following.

The basic downset we are studying in this section is the following. For $0 < k \leq n$ and $p_1, \ldots, p_k \in \overline{\mathbb{Z}}$ we define

$$A(p_1, \ldots, p_k) := \{X \in \overline{\mathbb{Z}}^n \mid (x_1, \ldots, x_k) \leq_{\text{lex}} (p_1, \ldots, p_k)\},$$

where $\leq_{\text{lex}}$ denotes the lexicographic order.

More generally we will consider linear transforms of $A(p_1, \ldots, p_k)$. Let $\varphi \in \mathrm{GL}(n, \mathbb{Z})$ be a unimodular matrix with only non-negative integer entries.[6] Usual choices are $\varphi = \mathrm{id}_{\mathbb{Z}^n}$, which means no shearing, and the matrix with all ones on and above the diagonal, which will be used for the definition of the 2-page in Section 3.2.2. Let $\varphi_{1\ldots k}(X)$ denote the first k components of $\varphi(X)$.

For $0 < k \leq n$, and $P = (p_1, \ldots, p_k) \in \mathbb{Z}^k$ we define

$$\begin{aligned} A(p_1, \ldots, p_k; \varphi) &:= \{X \in \overline{\mathbb{Z}}^n \mid \varphi_{1\ldots k}(X) \leq_{\text{lex}} (p_1, \ldots, p_k)\} \\ &= \{X \in \overline{\mathbb{Z}}^n \mid \varphi(X) \leq_{\text{lex}} (p_1, \ldots, p_k, \infty^{n-k})\}, \end{aligned}$$

which is a downset of $\overline{\mathbb{Z}}^n$ since $\varphi \in (\mathbb{Z}_{\geq 0})^{n \times n}$. Note that for $R \in \mathbb{Z}^n$,

$$A(p_1 + r_1, \ldots, p_k + r_k; \varphi) = A(p_1, \ldots, p_k; \varphi) + \varphi^{-1}(R).$$

Let's fix an offset $Q = (q_1, \ldots, q_n) \in \mathbb{Z}^n$ with $Q \geq_{\text{lex}} 0$. Usual choices are $Q = 0$, which means no offset, and $Q = \mathbb{1} := (1, \ldots, 1)$, which will be used in Section 3.2.2. Let $e_1, \ldots, e_n$ denote the standard basis vectors of $\mathbb{Z}^n$. We define

$$\begin{aligned} p(p_1, \ldots, p_k; r, \varphi, Q) &:= A(p_1, \ldots, p_k; \varphi), \\ q(p_1, \ldots, p_k; r, \varphi, Q) &:= A(p_1, \ldots, p_k - 1; \varphi) \\ &= p(p_1, \ldots, p_k; r, \varphi, Q) - \varphi^{-1}(e_k), \\ z(p_1, \ldots, p_k; r, \varphi, Q) &:= A(p_1 - q_1, \ldots, p_k - q_k - r; \varphi) \\ &= p(p_1, \ldots, p_k; r, \varphi, Q) - \varphi^{-1}(Q + re_k), \\ b(p_1, \ldots, p_k; r, \varphi, Q) &:= A(p_1 + q_1, \ldots, p_k + q_k + r - 1; \varphi) \\ &= q(p_1, \ldots, p_k; r, \varphi, Q) + \varphi^{-1}(Q + re_k). \end{aligned}$$

We extend the definitions for $k = 0$ and $r = 1$ by $p(-; 1, \varphi, Q) = b(-; 1, \varphi, Q) = C$ and $q(-; 1, \varphi, Q) = z(-; 1, \varphi, Q) = 0$, where "$-$" denotes the empty sequence. We write the associated S-terms as $S^{pz}_{bq}(p_1, \ldots, p_k; r, \varphi, Q)$. Only the first k components of Q matter for this S-term.

If one likes to think of usual spectral sequences as a book, then one can think of these S-term as words on page r of chapter k. There are n chapters, which are counted downwards, every chapter starts on page 1, with a page index shift given by Q.

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[6] Actually we only need that in every column of φ the top non-zero entry is positive. However this yields no generalization, since $A(P_{1,\ldots,k}; \varphi) = A(T_{1,\ldots k}(P); T\varphi)$ for any lower-triangular matrix T with ones on the diagonal.

Lemma 3.11 (Big steps)**.** *With the above definitions,*

1. $S^{pz}_{bq}(p_1,\dots,p_n;0,\varphi,0) = F_{A(p_1,\dots,p_n;\varphi)}/F_{A(p_1,\dots,p_n-1;\varphi)}$.
2. $S^{pz}_{bq}(p_1,\dots,p_n;1,\varphi,0) = H_*(F_{A(p_1,\dots,p_n;\varphi)}/F_{A(p_1,\dots,p_n-1;\varphi)})$.
3. *The differential* $d : C \to C$ *induces differentials*
$$S^{pz}_{bq}(p_1,\dots,p_k;r,\varphi,Q) \to S^{pz}_{bq}(p_1-q_1,\dots,p_k-q_k-r;r,\varphi,Q).$$
Taking homology at $S^{pz}_{bq}(p_1,\dots,p_k;r,\varphi,Q)$ *with respect to this differential yields* $S^{pz}_{bq}(p_1,\dots,p_k;r+1,\varphi,Q)$.
4. *If the filtration is finite,* $S^{pz}_{bq}(p_1,\dots,p_k;r,\varphi,Q)$ *equals* $S^{pz}_{bq}(p_1,\dots,p_k;\infty,\varphi,Q)$ *for* r *large enough.*
5. *The terms* $S^{pz}_{bq}(p_1,\dots,p_k;\infty,\varphi,Q)$ *are the filtration quotients of a* $\mathbb{Z}$*-filtration of* $S^{pz}_{bq}(p_1,\dots,p_{k-1};1,\varphi,Q)$*. More precisely, there is a canonical* $\mathbb{Z}$*-filtration* $(F_i)_{i\in\mathbb{Z}}$
$$0 \subseteq \ldots \subseteq F_i \subseteq F_{i+1} \subseteq \ldots \subseteq S^{pz}_{bq}(p_1,\dots,p_{k-1};1,\varphi,Q)$$
such that $S^{pz}_{bq}(p_1,\dots,p_k;\infty,\varphi,Q) \cong F_{p_k}/F_{p_k-1}$.
6. $S^{pz}_{bq}(-;1,\varphi,Q) = H(C)$.

The lemma says that starting with $S^{pz}_{bq}(p_1,\dots,p_n;0,\varphi,Q)$ or $S^{pz}_{bq}(p_1,\dots,p_n;1,\varphi,Q)$ (which are part of the 0-page and 1-page, respectively, in case the offset Q is zero), we can repeat steps 3.), 4.), and 5.) for $k = n, n-1, \dots, 1$ until we arrive at 6.). Permuting the coordinates of $\overline{\mathbb{Z}}^n$ gives different connections. See Figure 1, where the sets $p\backslash z$ and $b\backslash q$ are depicted by the ruled areas ▨ and ▧, respectively. They overlap in $p\backslash q$, which is thus ruled in both ways.

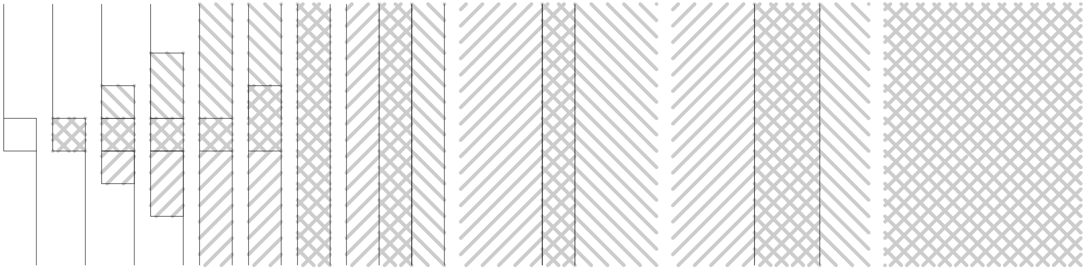

Figure 1: Some S-terms in the big step connection for $n = 2$, $\varphi = \mathrm{id}_{\mathbb{Z}^2}$, and $Q = 0$. The first five S-terms are for $k = 2$ and $r = 0, 1, 2, 3$, and ∞. The sixth S-term indicates the extension process. The following three S-terms are for $k = 1$ and $r = 1, 2$, and ∞. The last two S-terms indicate the second extension process, leading to $H(C)$.

Step 3.) in Lemma 3.11 increases B by the set of boxes

$$b(p_1,\dots,p_k;r+1,\varphi,Q)\backslash b(p_1,\dots,p_k;r,\varphi,Q) =$$
$$\{X \in \overline{\mathbb{Z}}^n \mid \varphi_{1\ldots k}(X) = (p_1+q_1,\dots,p_k+q_k+r)\},$$

which gets bigger and bigger as k decreases from n to 1, and similarly with Z. Alternatively, we could do small steps only, adding only a box at a time to B and Z.

In order to recycle the above notation, in the next lemma we think of Q being the page index, and we always let $r = 0$.

Lemma 3.12 (Small steps)**.** *Suppose that the filtration is finite. Let* $P = (p_1, \dots, p_n) \in \mathbb{Z}^n$ *and* $Q = (q_1, \dots, q_n) \in \mathbb{Z}^n$*.*

1. $S^{pz}_{bq}(P; 0, \varphi, 0) = F_{A(P;\varphi)}/F_{A(P-e_n;\varphi)}$.
2. $S^{pz}_{bq}(P; 0, \varphi, e_n) = H_*(F_{A(P;\varphi)}/F_{A(P-e_n;\varphi)})$.
3. *The differential* $d : C \to C$ *induces differentials*
$$S^{pz}_{bq}(P; 0, \varphi, Q) \to S^{pz}_{bq}(P - Q; 0, \varphi, Q).$$
Taking homology at $S^{pz}_{bq}(P; 0, \varphi, Q)$ *with respect to this differential yields the page* $S^{pz}_{bq}(P; 0, \varphi, Q + e_n)$*.*
4. *If the filtration is finite and* $q_k, \dots, q_n$*,* $k \geq 2$*, are large enough then* $S^{pz}_{bq}(P; 0, \varphi, Q)$ *is isomorphic to* $S^{pz}_{bq}(P; 0, \varphi, Q^*)$ *with* $Q^* = (q_1, \dots, q_{k-1}, q_k + 1, -q_{k+1}, \dots, -q_n)$*.*
5. *If the filtration is finite and* $q_1, \dots, q_n$ *are large enough then* $S^{pz}_{bq}(P; 0, \varphi, Q)$ *is isomorphic to* $S^{pz}_{bq}(P; 0, \varphi, \infty^n)$*.*
6. *The terms* $S^{pz}_{bq}(P; 0, \varphi, \infty^n)$ *are filtration quotients of* $H(C)$*. More precisely, there is a canonical filtration* $(G_P)_{P \in \mathbb{Z}^n}$*,* $G_P \subseteq H(C)$*, with* $G_P \subseteq G_{P'}$ *if* $P \leq_{\text{lex}} P'$*, such that* $S^{pz}_{bq}(P; 0, \varphi, \infty^n) \cong G_P/G_{P-e_n}$*.*

The lemma says that starting with 1.) or 2.) we can repeat steps 3.) and 4.) until we reach 5.), which can be used to compute 6.). Note that in steps 3.) and 4.), Q increases in the lexicographic order. Hence for finite filtrations this is indeed a finite connection from the 1-page to $H_*(C)$. See Figure 2.

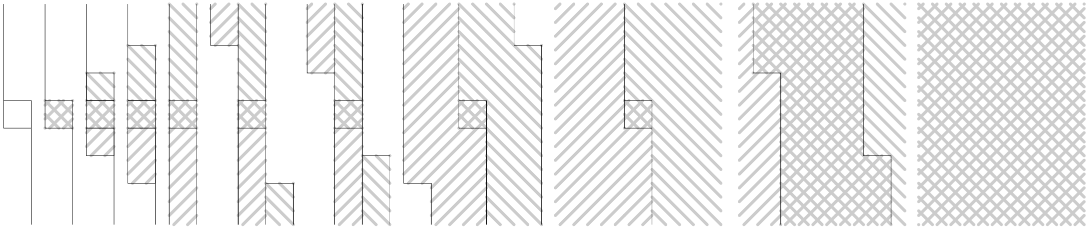

Figure 2: Some S-terms in the small step connection for $n = 2$, $\varphi = \mathrm{id}_{\mathbb{Z}^2}$. The first nine S-terms are for $Q = (0, 0)$, $(0, 1)$, $(0, 2)$, $(0, 3)$, $(0, \infty)$, $(1, -3)$, $(1, -2)$, $(2, 2)$, and (∞, ∞).

Permuting the coordinates of $\overline{\mathbb{Z}}^n$ gives different connections. We can even use different permutations for different P, until we reached $Q = \infty^n$. At that point 4.) we reached naturally isomorphic S-terms by Lemma 3.8, hence we can proceed with 5.)

Moreover we can merge the ideas of Lemmas 3.11 and 3.12 and get many more connections based on the lexicographic ordering.

3.2.2 Secondary connection

The following connection between the 1-page and $H(C)$ will be in particular useful for successive spectral sequences for which the limit of one spectral sequence is the *second* page of the next one.

We start with connecting the 1-page to the '2-page', for which we actually need to take n times homology.

Let $\varphi_k : \mathbb{Z}^n \to \mathbb{Z}^n$, $0 \leq k \leq n$, be the automorphism given by

$$\varphi_k(X) = (x_{k+1}, x_{k+2}, \ldots, x_n, \sum_{i=1}^{k} x_i, \sum_{i=2}^{k} x_i, \ldots, x_k). \tag{16}$$

For $P \in \overline{\mathbb{Z}}^n$ and $1 \leq k \leq n$, we define

$$T_P^k := A(\varphi_k(P); \varphi_k) = \{X \in \overline{\mathbb{Z}}^n \mid \varphi_k(X) \leq_{\text{lex}} \varphi_k(P)\}.$$

In other words, $A(0^n; \text{id}) = \varphi_k(T_P^k - P)$. Explicitly,

$$T_P^k = \big\{X \in \overline{\mathbb{Z}}^n \mid \big((x_{k+1}, \ldots, x_n) = (p_{k+1}, \ldots, p_n) \text{ and } \sum_{i=1}^{k} x_i + \delta_{X_{1\ldots k} <_{\text{lex}} P_{1\ldots k}} \leq \sum_{i=1}^{k} p_i\big) \\ \text{or } (x_{k+1}, \ldots, x_n) <_{\text{lex}} (p_{k+1}, \ldots, p_n)\big\}.$$

Note that

$$T_P^k \backslash \{P\} = T_{P+e_{k-1}-e_k}^k.$$

Here we set $e_0 := 0$, $e_{-1} := -e_n$, such that we don't need to treat the cases $k = 0$, $k = 1$, and $k \geq 2$ separately.

Given $P = (p_1, \ldots, p_n) \in \overline{\mathbb{Z}}^n$ and $0 \leq k \leq n$, we define

$$\begin{aligned}
p(P;k) &:= T_P^k \\
q(P;k) &:= p(P;k) \backslash \{P\} = T_{P+e_{k-1}-e_k}^k \\
z(P;k) &:= p(P;k) - e_k = T_{P-e_k}^k \\
b(P;k) &:= q(P;k) + e_k = T_{P+e_{k-1}}^k \\
z^*(P;k) &:= z(P;k) \backslash \{P - e_k\} = T_{P+e_{k-1}-2e_k}^k \\
b^*(P;k) &:= b(P;k) \cup \{P + e_k\} = T_{P+e_k}^k
\end{aligned}$$

We denote the associated S-terms as $S_{bq}^{pz}(P;k)$ and $S_{b^*q}^{pz^*}(P;k)$.

Example 3.13. For $k = 0$ we have $\varphi_0 = \mathrm{id}_{\mathbb{Z}^n}$ and the above S-terms are

$$S^{pz}_{bq}(P;0) = F_{A(P)}/F_{A(P)\setminus\{P\}}, \tag{17}$$

and

$$S^{pz^*}_{b^*q}(P;0) = H(F_{A(P)}/F_{A(P)\setminus\{P\}}). \tag{18}$$

For $k = n$, φ_n is the matrix with ones on and above the diagonal and zeros otherwise, and we have

$$S^{pz}_{bq}(P;n) = S^{pz}_{bq}(\varphi_n(P); 0, \varphi_n, \mathbb{1}), \tag{19}$$

and

$$S^{pz^*}_{b^*q}(P;n) = S^{pz}_{bq}(\varphi_n(P); 1, \varphi_n, \mathbb{1}). \tag{20}$$

We call (20) the 2-*page*.

Lemma 3.14. *There are differentials "in direction $-e_k$":*

$$d : S^{pz}_{bq}(P;k) \longrightarrow S^{pz}_{bq}(P - e_k;k).$$

Taking homology at $S^{pz}_{bq}(P;k)$ yields $S^{pz^*}_{b^*q}(P;k)$.

Proof. We only need to check that $q(P) = b(P - e_k)$, $z(P) = p(P - e_k)$, $z^*(P) = q(P - e_k)$, and $b^*(P) = p(P + e_k)$. □

The inverse of φ_k is

$$\varphi_k^{-1}(y) = (y_{k+1} - y_{k+2}, y_{k+2} - y_{k+3}, \ldots, y_{n-1} - y_n, y_n, y_1, y_2, \ldots, y_{n-k}).$$

Since all entries in φ_k are non-negative, φ_k preserves the order of $\overline{\mathbb{Z}}^n$, however its inverse doesn't. This means that we can take kernels and cokernels as we did with $A(p_1, \ldots, p_k; \mathrm{id})$ in Lemma 3.11, but we might expect more natural isomorphisms. And indeed that's what we will exploit:

Let's denote

$$\begin{aligned} Z(P;k) &:= Z(z(P;k), q(P;k), p(P;k), b(P;k)), \\ Z^*(P;k) &:= Z(z^*(P;k), q(P;k), p(P;k), b^*(P;k)), \end{aligned}$$

and similarly $B(P;k)$ and $B^*(P;k)$.

Let $V_k \subset \{-1,0,1\}^k \times \{0\}^{n-k}$ be the set of all $\{-1,0,1\}$-vectors of length n with the following properties: The last $n-k$ entries are zero, the last non-zero entry is $+1$, and $+1$ and -1 appear alternating. In other words, $V_0 := \{0\} \subset \mathbb{Z}^n$ and

$$V_{k+1} := V_k \mathbin{\dot\cup} (e_{k+1} - V_k) = \{0\} \mathbin{\dot\cup} (e_1 - V_0) \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} (e_{k+1} - V_k).$$

Put $V_{-1} := \emptyset$.

Lemma 3.15. *For* $0 \leq k \leq n$,

$$\begin{aligned} Z(P;k) &= P - V_{k-1}, \qquad & B(P;k) &= P + V_{k-1}, \\ Z^*(P;k) &= P - V_k, & B^*(P;k) &= P + V_k. \end{aligned}$$

Thus for $0 \leq k \leq n-1$ *we have a natural isomorphism*

$$S^{pz^*}_{b^*q}(P;k) \cong S^{pz}_{bq}(P;k+1).$$

Proof. It suffices to prove the equalities for $B(0;k)$ and $B^*(0;k)$. The special cases $B(0;0)$, $B(0;1)$, and $B^*(0;0)$ should be checked separately, since they involve particularly defined terms e_{-1}, e_0, and V_{-1}.

Let's write $b := b(0;k)$, $b^* := b^*(0;k)$, and $q := q(0;k)$. $B(0;k)$ is the component of $b \backslash q$ containing 0, where

$$\begin{aligned} b\backslash q &= \{X \mid \varphi_k(0) \leq_{\text{lex}} \varphi_k(X) \leq_{\text{lex}} \varphi_k(e_{k-1})\} \\ &= \{X \mid 0 \leq_{\text{lex}} \varphi_k(X) \leq_{\text{lex}} e_{n-k+1} + \ldots + e_{n-1}\}. \end{aligned}$$

All such X satisfy $x_{k+1} = \ldots = x_n = 0$. Clearly, V_{k-1} is connected and $0 \in V_{k-1} \subseteq b\backslash q$. If $V_{k-1} \neq b\backslash q$, then there exists an $X \in (b\backslash q)\backslash V_{k-1}$ that is adjacent to V_{k-1}. Then for exactly one $1 \leq j \leq k$, either $x_j = \pm 2$ or ($x_j = \pm 1$ and the next non-zero entry of X equals x_j). For this j, $\phi_k(X)_{n-k+j} = \sum_{i=j}^{k} x_i$ is less than 0 or larger than 1, thus $X \notin b\backslash q$, which is the desired contradiction.

A very similar argument works for $B^*(0;k)$ using

$$\begin{aligned} b^*\backslash q &= \{X \mid \varphi_k(0) \leq_{\text{lex}} \varphi_k(X) \leq_{\text{lex}} \varphi_k(e_k)\} \\ &= \{X \mid 0 \leq_{\text{lex}} \varphi_k(X) \leq_{\text{lex}} e_{n-k+1} + \ldots + e_n\}. \end{aligned}$$

□

Using the lemmas alternately, we can connect the 0-page (17) and the 1-page (18) to the 2-page (20). From there we can proceed with Lemma 3.11 (or Lemma 3.12 if small steps are preferred) to connect $S^{pz^*}_{b^*q}(P;n)$ to $H_*(C)$. See Figure 3. In this big step lexicographic connection, on page $r = 1, \ldots$ in chapter $k = n, \ldots, 1$, the differential has direction $-\varphi_n^{-1}(re_k + Q + \langle e_{k+1}, \ldots, e_n\rangle)$. One particular vector in this $(n-k)$-dimensional affine set is $re_{k-1} - (r+1)e_k = -\varphi_n^{-1}(re_k + (1^k, 0^{n-k}))$.

3.2.3 Generalized secondary connection

In the secondary connection from the previous section, one starts from $S^{pz^*}_{b^*q}(P;0) \cong S^{pz}_{bq}(P;1)$, takes homology in direction $-e_1$, which yields $S^{pz^*}_{b^*q}(P;1) \cong S^{pz}_{bq}(P;2)$, then one takes homology in direction $-e_2$, and so on, until one arrives at the second page $S^{pz^*}_{b^*q}(P;n)$. In other words, we apply homology and natural isomorphisms alternatingly.

More generally, for arbitrary $Q = (q_1, \ldots, q_n) \geq 0$ we can start with $S^{pz}_{bq}(P;1)$, take q_1 times homology in direction $-e_1$, find a natural isomorphic page for which we

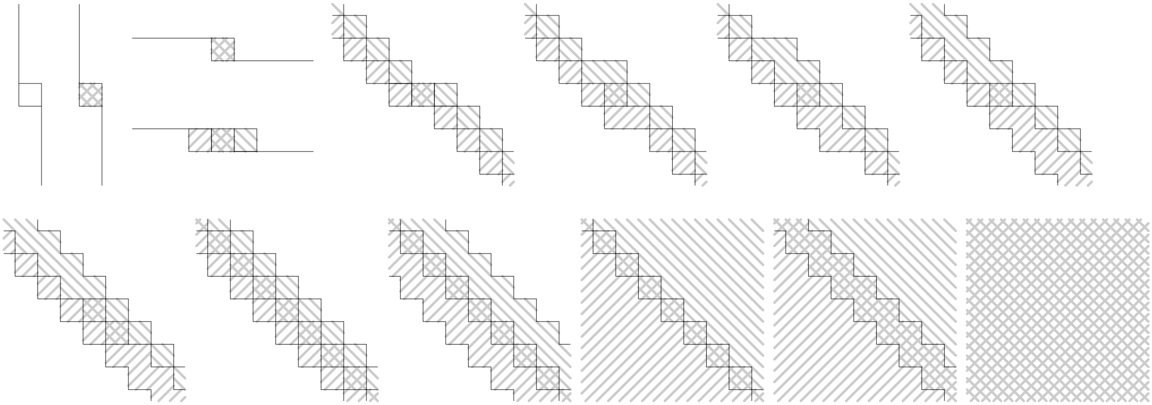

Figure 3: The six S-terms in the secondary connection for $n = 2$, followed by some S-terms in the big step lexicographic connection with offset $Q = (1, 1)$ and shearing matrix φ_2. The sixth S-term on the top is the 2-page.

can apply q_2 times homology in a direction close to $-e_2$, and so on. The secondary connection is the special case $Q = \mathbb{1}$, which might be more useful than all other cases together. Here is how it works.

Let $\varphi_k^Q \in \mathrm{GL}(n, \mathbb{Z})$, $0 \leq k \leq n$, $Q \in \mathbb{Z}^n_{\geq 0}$, be given by

$$\varphi_k^Q(X) := (x_{k+1}, \ldots, x_n, x_1 + q_1 \sum_{i=2}^{k} x_i, x_2 + q_2 \sum_{i=3}^{k} x_i, \ldots, x_k).$$

We will recycle the notation from Section 3.2.1 for the S-terms. Let $Q[n-k] := (0^{n-k}, q_1, \ldots, q_k)$.

Lemma 3.16. *For $1 \leq k \leq n$, $P \in \mathbb{Z}^n$, there is a natural isomorphism of S-terms*

$$S^{pz}_{bq}(\varphi^Q_{k-1}(P); 1, \varphi^Q_{k-1}, Q[n-k+1]) \cong S^{pz}_{bq}(\varphi^Q_k(P); 1 - q_k, \varphi^Q_k, Q[n-k]).$$

Proof (sketch). Define $a_0^Q := 0$ and $a_k^Q := e_k - \sum_{i=1}^{k-1} q_i a_i^Q \in \mathbb{Z}^n$. Explicitly,

$$a_k^Q = \sum_{i=1}^{k} (-1)^{k-i} e_i \prod_{j=i}^{k-1} (q_j - \delta_{j>i}).$$

One can show that

$$(\varphi_k^Q)^{-1} = \left(\begin{array}{c|c} 0^{k\times(n-k)} & \\ \hline \mathrm{id}_{\mathbb{Z}^{n-k}} & \end{array} \; a_1^Q \cdots a_k^Q \right).$$

Further, let $V_0^Q := \{0\}$ and $V_k^Q := \{0, \ldots, q_k\} \cdot a_k^Q + V_{k-1}^Q \subseteq \mathbb{Z}^n$. Thus V_k^Q is an 'discrete affine cube',

$$V_k^Q = (\varphi_k^Q)^{-1}(\{0\}^{n-k} \times \{0, \ldots, q_1\} \times \ldots \times \{0, \ldots, q_k\}). \tag{21}$$

For example for $Q = \mathbb{1}$ we obtain $\varphi_k^Q = \varphi_k$, $a_k^Q = e_k - e_{k-1}$, and $V_k^Q = V_k$. By induction one proves that $V_{k-1}^Q = (e_k - a_k^Q) - V_{k-1}^Q$. In particular this implies that V_k^Q is connected.

For $S_{bq}^{pz}(0; 1, \varphi_k^Q, Q[n-k])$, we have

$$b \backslash q = \{X \mid 0 \leq_{\text{lex}} \varphi_k^Q(X) \leq_{\text{lex}} Q[n-k]\}.$$

From (21) it follows $0 \in V_k^Q \subseteq b\backslash q$. Let $X \in (b\backslash q)$ be adjacent to some $Y = X \pm e_j \in V_k^Q$. Then $j \leq k$ since otherwise $\varphi_k^Q(e_j) = e_{j-k}$. Suppose $Y = X + e_j$. (The case $Y = X - e_j$ is analogous.) Then $\varphi_k^Q(Y) = \varphi_k^Q(X) + (0^{n-k}, q_1, \ldots, q_{j-1}, 1, 0^k - j)$. Since $X, Y \in b\backslash q$, we get $\varphi_k^Q(X)_{n-k+i} = q_i$ for $1 \leq i < j$ and $\leq q_i$ for $j \leq i \leq k$. Hence, X is already in V_k^Q. Therefore, V_k^Q is the connected component of $b\backslash q$ that contains 0.

We get the same statement for $S_{bq}^{pz}(0; 1 - q_{k+1}, \varphi_{k+1}^Q, Q[n-k-1])$ using a similar argument. Hence both S-terms are naturally isomorphic. □

Applying q_k times homology to

$$S_{bq}^{pz}(\varphi_k^Q(P); 1 - q_k, \varphi_k^Q, Q[n-k])$$

as in Lemma 3.11.3.) yields

$$S_{bq}^{pz}(\varphi_k^Q(P); 1, \varphi_k^Q, Q[n-k]).$$

Therefore we can apply Lemma 3.11.3.) and Lemma 3.16 alternatingly in order to connect the 0-page (17) and the 1-page (18)) to

$$S_{bq}^{pz}(\varphi_n^Q(P); 0, \varphi_n^Q, Q), \tag{22}$$

which we call the 2_Q-page. Again, using the lexicographic connections we can connect (22) to $H(C)$.

3.2.4 Further connections

There are many more ways to connect the 1-page to $H_*(C)$, which might be useful sometimes. Moreover if we know from the given C that certain terms on the 0-page vanish, more S-terms can be identified, which might give even more useful connections.

Lemma 3.17. *Let $z \leq q \leq p \leq b$ be downsets in $D(\overline{\mathbb{Z}}^n)$ such that $b\backslash z$ is finite. Then there is a connection from the 1-page to S_{bq}^{pz} using kernels and cokernels only.*

Proof. The proof is by induction on $|b\backslash z|$. If $z = q$ and $p = b$ then $S_{bq}^{pz} = H_*(F_p/F_q)$ is by definition a term on the 1-page. If $p \neq b$, then for any coordinate-wise maximal element X in $b\backslash p$ we have

$$S_{bq}^{pz} = \operatorname{coker}\left(S_{b^*, b\backslash X}^{b,p} \longrightarrow S_{b\backslash X, q}^{p,z}\right),$$

$b^* \geq b$ being arbitrary, for example $b^* = b$. If $z \neq q$, then for any coordinate-wise minimal element X in $q \backslash z$ we have

$$S_{bq}^{pz} = \ker\left(S_{b,q}^{p,z\cup X} \longrightarrow S_{q,z}^{z\cup X,z^*}\right),$$

$z^* \leq z$ being arbitrary, for example $z^* = z$. □

This lemma uses kernels and cokernels. In case one only wants to allow natural isomorphisms and taking homology as in the standard spectral sequence, the following happens: We start with S_{bq}^{pz}-terms from the 1-page with $|p\backslash q| = 1$. Then there are n possible differentials that enlarge Z and B by one box. We conjecture that this can be iterated, and at every step we have exactly n possible differentials that enlarge Z and B by at least one box when taking homology. However these operations seem to be not commutative at all, which makes them probably only useful in the above discussed special cases of secondary connections.

Note further that any S-term from the 1-page can be obtained by extensions (and limits in the case of non-finite filtrations) from 1-page S-terms S_{qq}^{pp} with $|p\backslash q| = 1$. The latter terms are naturally isomorphic to terms S_{qq}^{pp} with $p = A(p_1, \ldots, p_n)$ and $q = p\backslash\{(p_1, \ldots, p_n)\}$.

Lemma 3.18. *Let $z \leq q < p \leq b$ be downsets in $D(\overline{\mathbb{Z}}^n)$ such that $p\backslash q$ is finite. Then S_{bq}^{pz} is an iterated extension by terms $S_{bq^*}^{p^*z}$ of the form $q \leq q^* < p^* \leq p$ with $|p^*\backslash q^*| = 1$.*

Proof. This is immediate from the extension property. □

3.3 Example: n independent subcomplexes

Consider n subcomplexes $F_1, \ldots, F_n$ of a given chain complex C. The goal is to compute $H(C)$. For any $1 \leq i \leq n$ we get a $\mathbb{Z}$-filtration $(F_k^{(i)})_{k\in\mathbb{Z}}$ of C with $F_0^{(i)} := F_i$, and $F_{-k}^{(i)} := 0$ and $F_k^{(i)} := C$ for $k \geq 1$. We further assume that the induced $D(\overline{\mathbb{Z}}^n)$-filtration distributive. It induces a spectral system over $D(\overline{\mathbb{Z}}^n)$.

Example 3.19. C might arise as the singular chain complex of a space X, and the F_i as the chain complexes of n open subspaces X_i. The induced filtration is in general not excisive, but this problem can be ignored, because the associated exact couple system is excisive (compare with Example 4.20, which also deals with the case where the goal of computation is $h_*(X)$ for some generalized homology theory h_*).

Of course it makes sense to look only at the subposet I that is given by all downsets of elements in $\{0,1\}^n$ and their unions. I is isomorphic to the poset of abstract simplicial complexes on $\{1, \ldots, n\}$, ordered by inclusion. Here we need to distinguish between the empty complex $\{\emptyset\}$, which corresponds to the downset of 0^n in $D(\overline{\mathbb{Z}}^n)$, and the void complex $\emptyset$, which corresponds to $-\infty$.

Let's look only at the secondary connection (see Section 3.2.2). The only potentially non-zero S-terms $S_{bq}^{pz}(P;0)$ are those where $P \in \{0,1\}^n$, and they are given by

$$S_{bq}^{pz}(P;0) = F_P / \sum_{Q<P} F_Q \cong (\bigcap_{Q>P} F^Q)/F^P,$$

where $F_P = \bigcap_{i:p_i=0} F_i$, $F^P = \sum_{i:p_i=1} F_i$, Q runs in $\{0,1\}^n$, and empty intersections mean C.

Taking homology of each of these 2^n chain complexes yields $S_{b^*q}^{pz^*}(P;0) \cong S_{bq}^{pz}(P;1)$, $P \in \{0,1\}^n$. Then the secondary connection proceeds with an iteration over k from $k = 1, \ldots, n$: At step k, we have 2^{n-1} chain complexes

$$0 \to S_{bq}^{pz}(P+e_k;k) \xrightarrow{d} S_{bq}^{pz}(P;k) \to 0, \quad (P \in \{0,1\}^n, p_k = 0).$$

Taking homology yields $S_{b^*q}^{pz^*}(P+e_k;k) \cong S_{bq}^{pz}(P+e_k;k+1)$ and $S_{b^*q}^{pz^*}(P;k) \cong S_{bq}^{pz}(P;k+1)$, respectively. After step $k = n$, we arrived at the second page $S_{b^*q}^{pz^*}(P;n) = S_{bq}^{pz}(\varphi_n(P);1,\varphi_n,\mathbb{1})$.

We can proceed with the lexicographic connection in big steps ((19) and Lemma 3.11): Iterating over k from $k = n, \ldots, 1$, at step k we have to take homology with respect to the differentials in Lemma 3.11.3) only for the pages $r = 1, \ldots, n-k$, and then we group the pages and unify these groups using extensions (Lemma 3.11.5).

After step $k = 1$, we arrive at one remaining S-term, which is isomorphic to $H(C)$.

This spectral system should be compared to the generalized Mayer–Vietoris sequence, which gives rise to the Leray–Mayer–Vietoris spectral sequence. This spectral sequence is obtained by restricting the above filtration poset I to the skeletons of the simplex on vertex set $\{1, \ldots, n\}$. It converges to $H(\sum F_i)$ or to $H(C)$, depending on the convention whether or not one includes the full simplex in the filtration.

4 Exact couple systems

Massey [Mas52, Mas53] constructed the framework of exact couples, which give more generally rise to spectral sequences than $\mathbb{Z}$-filtrated chain complexes. So we may ask whether our generalized spectral sequences can also start directly from the 1-page, without being constructed from the 0-page.

In this section we construct basic data and axioms similar to exact couples that induce a spectral system as above that starts from the 1-page, that is, it contains only terms S_{bq}^{pz} with $z \leq q \leq p \leq b$.

We call a poset I bounded if it has a minimum and a maximum, which we denote by $-\infty$ and ∞. We regard I as a category whose objects are the elements and whose morphisms $q \to p$ are the relations $q \leq p$. For $n \geq 2$, let I_n denote the poset of n-tuples $(p_1, \ldots, p_n) \in I^n$ with $p_1 \geq p_2 \geq \ldots \geq p_n$, ordered by $(p_1, \ldots, p_n) \geq (p'_1, \ldots, p'_n)$ if and only if $p_i \geq p'_i$ for all i. As with I, I_n is also a category.

Definition 4.1. An *exact couple system* over a bounded poset I consists of the following data:

1. A functor $E : I_2 \to Ab$ from I_2 to the category of abelian groups. We write $E_q^p := E(p,q)$, $D_p := E(p,-\infty)$; and $i_{qp} : D_q \to D_p$, $j_{pq} : D_p \to E_q^p$, and $\ell_{p'q'}^{pq} : E_q^p \to E_{q'}^{p'}$ for the maps $E((q,-\infty) \leq (p,-\infty))$, $E((p,-\infty) \leq (p,q))$, and $E((p,q) \leq (p',q'))$, respectively.
2. Maps $k_{pq} : E_q^p \to D_q$ for all $(p,q) \in I_2$.

We require that it satisfies the following axioms:

1. The triangles

$$\begin{tikzcd} D_q \arrow[rr, "i_{qp}"] & & D_p \arrow[dl, "j_{pq}"] \\ & E_q^p \arrow[ul, "k_{pq}"] & \end{tikzcd}$$

   are exact.

2. The diagrams

$$\begin{tikzcd} E_q^p \arrow[r, "k_{pq}"] \arrow[d, "\ell_{p'q'}^{pq}"'] & D_q \arrow[d, "i_{qq'}"] \\ E_{q'}^{p'} \arrow[r, "k_{p'q'}"'] & D_{q'} \end{tikzcd}$$

   commute.

Remark 4.2. Equivalently, we could define an exact couple system over I as a functor $\widetilde{E} : \widetilde{I}_2 \to Ab$, where $\widetilde{I}_2$ is the same category as I_2 except that we add morphisms $\widetilde{k}_{pq} : (p,q) \to (q,-\infty)$ for all $(p,q) \in I_2$ (and all resulting compositions) that commute with the morphisms given by the relations, that is, we require $((p,-\infty) \leq (p',-\infty)) \circ \widetilde{k}_{pq} = \widetilde{k}_{p'q'} \circ ((p,q) \leq (p',q'))$. Data 2 and Axiom 2 are then automatically given and we only need to require Axiom 1.

In this wording it is also clear what maps between two exact couple systems $E : \widetilde{I}_2 \to Ab$ and $E' : \widetilde{I}_2' \to Ab$ should be, namely an order preserving map $f : I \to I'$ together with an natural transformation from E to $E' \circ (f \times f)|_{I_2}$. In what follows, we call constructions natural if they commute with maps between exact couple systems.

More generally we can take any abelian category in place of Ab.

Remark 4.3 ('Cohomological' definition)**.** It will follow from Lemma 4.8 below that we could equivalently define an exact couple system as a functor $E : I_2 \to Ab$ with E_q^p

and $\ell^{pq}_{p'q'}$ as before, $D^p := E(\infty, p)$, together with boundary maps $k^{pq} : D^p \to E^p_q$, such that the triangles

$$\begin{tikzcd}
D^q \arrow[rr, "\ell^{\infty q}_{\infty p}"] & & D^p \arrow[dl, "k^{pq}"] \\
& E^p_q \arrow[ul, "\ell^{pq}_{\infty q}"] &
\end{tikzcd}$$

are exact and $\ell^{pq}_{p'q'} \circ k^{pq} = k^{p'q'} \circ \ell^{\infty p}_{\infty p'}$ for all $(p, q) \leq (p', q')$.

Example 4.4 (Exact couple system of an I-filtered chain complex)**.** The spectral system of a I-filtered chain complex C contains an exact couple system over I with $E^p_q := H(F_p/F_q)$, $D_p := H(F_p)$, i and j are induced by inclusion, and k is the connecting homomorphism.

Example 4.5 (Exact couple system of an I-filtered space)**.** Let X be a topological space that is filtered by a bounded poset I. That is, we are given a family of closed subspaces $(X_i)_{i \in I}$ of X with inclusions $X_q \hookrightarrow X_p$ whenever $q \leq p$. We may assume that $X_{-\infty} = \emptyset$ and $X_\infty = X$.

Then for any generalized homology theory h_*, $E^p_q := h_*(X_p, X_q)$ is an exact couple system over I. Here, $D_p = h_*(X_p)$, i and j are induced by inclusions, and k_{pq} is the connecting homomorphism in the long exact sequence of the pair (X_p, X_q).

Analogously, for any generalized cohomology theory h^*, $E^p_q := h^*(X_q, X_p)$ is an exact couple system over I^*, where I^* denotes the dual poset of I^*, that is, $p \leq_I q$ if and only if $q \leq_{I^*} p$. Since $\pm\infty$ denotes minimum and maximum, $\infty_I = -\infty_{I^*}$ and $-\infty_I = \infty_{I^*}$. Thus, $X_{-\infty} = X$, $X_\infty = \emptyset$, $D_p = h^*(X, X_p)$, i and j are induced by inclusions, and k_{pq} is the connecting homomorphism in the long exact sequence of the triple (X, X_q, X_p). If we use the exact couple system definition from Remark 4.3, we have $D^p = h^*(X_p)$.

Example 4.6 (Perverse sheaves and Fáry functors)**.** Perverse sheaves and more generally Fáry functors have naturally the structure of an exact couple systems (they have more properties, and I is a set of open sets of a space X), see Beĭlinson–Bernstein–Deligne [BBD82] and Grinberg–MacPherson [GM99].

Example 4.7 (Exact couples)**.** An exact couple system E over $\mathbb{Z}$ naturally contains an exact couple, namely the collection of all exact triangles from Axiom 1 with $p = q + 1$. All (columns of the) pages of the induced spectral sequence are particular S-terms of E, as in Example 2.5.

4.1 Basic properties of exact couple systems

Now consider an exact couple system E over I. We define differentials

$$d_{pqz} : E^p_q \to E^q_z.$$

for all $(p, q, z) \in I_3$ by setting $d_{pqz} := j_{qz} \circ k_{pq}$. Composing two such differentials yields the zero map since $k \circ j = 0$ by Axiom 1. Moreover we have $d \circ \ell = \ell \circ d$.

Lemma 4.8 (Exact triangles). *For any $p_1 \leq p_2 \leq p_3$ in I there is an exact triangle*

$$\begin{tikzcd}
E^{p_2}_{p_1} \arrow[r, "\ell"] & E^{p_3}_{p_1} \arrow[r, "\ell"] & E^{p_3}_{p_2}. \arrow[ll, bend left, "d"]
\end{tikzcd}$$

The lemma is analog to the octahedral axiom in triangulated categories, since it says that the center triangle in the following diagram is exact if all three outer triangles are.

$$\begin{tikzcd}
& & D_{p_1} \arrow[lldd, bend right, "i"'] \arrow[rrdd, bend left, "i"] & & \\
& E^{p_2}_{p_1} \arrow[ur, "k"] \arrow[rr, "\ell"'] & & E^{p_3}_{p_1} \arrow[ul, "k"'] \arrow[ld, "\ell"] & \\
D_{p_2} \arrow[ur, "j"] \arrow[rrrr, bend right, "i"'] & & E^{p_3}_{p_2} \arrow[ll, "k"] \arrow[lu, "d"] & & D_{p_3} \arrow[ll, "j"] \arrow[lu, "j"']
\end{tikzcd}$$

Proof. For the exactness at $E^{p_2}_{p_1}$ we chase the diagram

$$\begin{tikzcd}
& & D_{p_3} \arrow[rd, "j"] & \\
E^{p_3}_{p_2} \arrow[r, "k"] & D_{p_2} \arrow[ru, "i"] \arrow[r, "j"] & E^{p_2}_{p_1} \arrow[r, "\ell"] \arrow[d, "k"] & E^{p_3}_{p_1} \arrow[ld, "k"] \\
& & D_{p_1}. \arrow[lu, "i"] &
\end{tikzcd}$$

Let $x \in E^{p_2}_{p_1}$ with $\ell(x) = 0 \in E^{p_3}_{p_1}$. Then $k(x) = 0 \in D_{p_1}$, so $x = j(b)$ for some $b \in D_{p_2}$. Let $c := i(b) \in D_{p_3}$. Now, $j(c) = j(i(b)) = \ell(j(b)) = \ell(x) = 0 \in E^{p_3}_{p_1}$. Thus there is an $a \in D_{p_1}$ with $i(i(a)) = c$. Hence $i(b - i(a)) = 0 \in D_{p_3}$. So there is a $z \in E^{p_3}_{p_2}$ with $k(z) = b - i(a)$. Therefore $d(z) = j(k(z)) = j(b) - j(i(a)) = j(b) = x$.

On the other hand, if $x = d(z) \in E^{p_2}_{p_1}$ with $z \in E^{p_3}_{p_1}$, then $\ell(x) = \ell(d(z)) = j(i(k(z)) = 0 \in E^{p_3}_{p_1}$ since $i \circ k = 0$.

The proof for the exactness at $E^{p_3}_{p_2}$ works similarly in the diagram

$$\begin{tikzcd}
& D_{p_1} \arrow[rd, "i"] & & \\
E^{p_3}_{p_1} \arrow[ru, "k"] \arrow[r, "\ell"] & E^{p_3}_{p_2} \arrow[r, "k"] & D_{p_2} \arrow[r, "j"] \arrow[ld, "i"] & E^{p_2}_{p_1} \\
& D_{p_3}. \arrow[lu, "j"] \arrow[u, "j"] & &
\end{tikzcd}$$

For the exactness at $E^{p_3}_{p_1}$ we chase the commutative diagram

$$\begin{tikzcd}
D_{p_2} \arrow[r, "i"] \arrow[d, "j"'] & D_{p_3} \arrow[d, "j"] \arrow[dr, "j"] & \\
E^{p_2}_{p_1} \arrow[r, "\ell"] \arrow[dr, "k"'] & E^{p_3}_{p_1} \arrow[r, "\ell"] \arrow[d, "k"] & E^{p_3}_{p_2} \arrow[d, "k"] \\
& D_{p_1} \arrow[r, "i"] & D_{p_2}.
\end{tikzcd}$$

Let $y \in E^{p_3}_{p_1}$ with $\ell(y) = 0 \in E^{p_3}_{p_2}$. Let $a = k(y) \in D_{p_1}$. Then $i(a) = k(\ell(y)) = 0 \in D_{p_2}$. Thus $a = k(x)$ for some $x \in E^{p_2}_{p_1}$, and $k(\ell(x) - y) = 0 \in D_{p_1}$. Hence there is a $c \in D_{p_3}$ with $j(c) = \ell(x) - y$, which satisfies $\ell(j(c)) = 0 \in E^{p_3}_{p_2}$. Thus there is a $b \in D_{p_2}$ with $i(b) = c$, and we have $j(i(b)) = j(c) = \ell(x) - y$. Let $x' := j(b) \in E^{p_2}_{p_1}$. Then $\ell(x') = j(i(b)) = \ell(x) - y$. Therefore $y = \ell(x - x')$.

On the other hand, let $y = \ell(x) \in E^{p_3}_{p_1}$ with $x \in E^{p_2}_{p_1}$. Let $z := \ell(y) \in E^{p_3}_{p_2}$. Then $k(z) = i(k(x)) = 0 \in D_{p_2}$. Thus there exists $c \in D_{p_3}$ with $j(c) = z \in E^{p_3}_{p_2}$. Let $y' := j(c) \in E^{p_3}_{p_1}$. Then $\ell(y' - y) = 0 \in E^{p_3}_{p_2}$. From above it follows that $y' - y$ and hence also y' is the image of some element in $E^{p_2}_{p_1}$, say $y' = \ell(x')$. Then $k(x') = k(y') = k(j(c)) = 0 \in D_{p_1}$, and $x' = j(b)$ for some $b \in D_{p_2}$. Let $c' := c - i(b) \in D_{p_3}$. Then $j(c') = 0 \in E^{p_3}_{p_1}$, hence $j(c') = 0 \in E^{p_3}_{p_2}$. Therefore $z = j(c) = j(c') - j(i(b)) = j(c') = 0$. □

We define naturally associated S-terms for all $(b, p, q, z) \in I_4$ by

$$S^{pz}_{bq} := \frac{\ker(d_{pqz} : E^p_q \to E^q_z)}{\operatorname{im}(d_{bpq} : E^b_p \to E^p_q)}. \tag{23}$$

If E is the exact couple system of an I-filtered chain complex C, then both definitions for S-terms, (3) and (23), coincide for all $(b, p, q, z) \in I_4$.

By Axiom 1, $E^p_p = 0$ for all $p \in I$, and hence

$$S^{pq}_{pq} = E^p_q.$$

We call the collection of these S-terms the 1-*page*.

For all $(b, p, q, z) \leq (b', p', q', z')$ in I_4, $\ell^{pq}_{p'q'}$ induces maps

$$S^{pz}_{bq} \to S^{p'z'}_{b'q'}.$$

For a proof, simply chase the diagram

$$\begin{tikzcd}
E^b_p \arrow[r, "d"] \arrow[d, "\ell"] & E^p_q \arrow[r, "d"] \arrow[d, "\ell"] & E^q_z \arrow[d, "\ell"] \\
E^{b'}_{p'} \arrow[r, "d"] & E^{p'}_{q'} \arrow[r, "d"] & E^{q'}_{z'}.
\end{tikzcd}$$

which is commutative by Axiom 2 and functoriality of E. We say that these maps between S-terms are *induced by inclusions*.

Lemma 4.9 (Extensions)**.** *For any $z \leq p_1 \leq p_2 \leq p_3 \leq b$ in I, we have a short exact sequence of maps induced by inclusion,*

$$0 \to S^{p_2,z}_{b,p_1} \to S^{p_3,z}_{b,p_1} \to S^{p_3,z}_{b,p_2} \to 0. \tag{24}$$

Proof. The exactness can be proved using the following diagram.

$$\begin{tikzcd}
E^b_{p_2} \arrow[r, "d"] \arrow[dd, "\ell"'] & E^{p_2}_{p_1} \arrow[r, "d"] \arrow[d, "\ell"'] & E^{p_1}_z \arrow[dd, "\ell"] \\
 & E^{p_3}_{p_1} \arrow[d, "\ell"] \arrow[ur, "d"] & \\
E^b_{p_3} \arrow[r, "d"'] \arrow[ur, "d"] & E^{p_3}_{p_2} \arrow[r, "d"'] \arrow[uul, "\ell"'] \arrow[uu, bend right=90, looseness=2.5, "d"'] & E^{p_2}_z . \arrow[uul, "\ell"]
\end{tikzcd}$$

The diagonal and the two horizontal compositions are the defining maps for the S-terms in (24). The three directed 3-cycles are exact triangles. First we show injectivity of the map $S^{p_2,z}_{b,p_1} \to S^{p_3,z}_{b,p_1}$:

Let $x \in E^{p_2}_{p_1}$ with $d(x) = 0$ and $\ell(x) = d(y) \in E^{p_3}_{p_1}$ for some $y \in E^b_{p3}$. Then $d(y)$ is zero in $E^{p_3}_{p_2}$. Thus there exists $z \in E^b_{p_2}$ with $\ell(z) = y$. Hence $\ell(d(z) - x) = 0 \in E^{p_3}_{p1}$. Thus there exists $a \in E^{p_3}_{p_2}$ with $d(a) = d(z) - x$. Therefore $z' := z - \ell(a) \in E^b_{p_2}$ has the property that $d(z') = d(z) - \ell(d(a)) = d(z) - (d(z) - x) = x$. This means that x represents zero in $S^{p_2,z}_{b,p_1}$.

Surjectivity of $S^{p_3,z}_{b,p_1} \to S^{p_3,z}_{b,p_2}$ can be proved similarly.

It remains to prove exactness at $S^{p_3,z}_{b,p_1}$: Lemma 4.8 shows that the composition of the two maps (24) is zero. On the other hand, let $x \in E^{p_3}_{p_1}$ with $d(x) = 0 \in E^{p_1}_z$ and $\ell(x) = d(y) \in E^{p_3}_{p_2}$ for some $y \in E^b_{p_3}$. Then $x' := x - d(y) \in E^{p_3}_{p_1}$ represents the same element as x in $S^{p_3,z}_{b,p_1}$, and $\ell(x') = 0 \in E^{p_3}_{p_2}$. Thus by Lemma 4.8, there exists $z \in E^{p_2}_{p_1}$ with $\ell(z) = x'$. We have $d(z) = d(x') = d(d(y)) = 0 \in E^{p_1}_z$. Therefore z represents an element in $S^{p_2,z}_{b,p_1}$ that maps to the element in $S^{p_3,z}_{b,p_1}$ that x represents. □

As for the spectral system of an I-filtration we have differentials between S-terms of an exact couple system with the same properties:

Lemma 4.10 (Differentials)**.** *For any $(b, p, q, z), (b', p', q', z') \in I_4$ with $z \leq p'$ and $q \leq b'$ there are natural differentials*

$$d : S^{pz}_{bq} \to S^{p'z'}_{b'q'} \tag{25}$$

that commute with ℓ, that is, $\ell \circ d = d \circ \ell$.

Proof. Chasing the commutative diagram

$$\begin{tikzcd}
 & E_z^q & & & \\
E_q^p \arrow[ur, "d"] \arrow[r, "k"'] & D_q \arrow[u, "j"'] & & D_{q'} \arrow[r, "j"] & E_{z'}^{q'} \\
\ker(d_{pqz}) \arrow[u, hook] \arrow[r, dotted] & D_z \arrow[u, "i"] \arrow[r, "i"] & D_{p'} \arrow[ur, "0"] \arrow[r, "j"] & E_{q'}^{p'} \arrow[u, "k"] \arrow[ur, "d"'] & \\
E_p^b \arrow[u, "d"] \arrow[r, dotted] & E_z^q \arrow[u, "k"] \arrow[r, "\ell"] & E_{p'}^{b'} \arrow[u, "k"] \arrow[ur, "d"'] & &
\end{tikzcd} \tag{26}$$

shows that there is a natural and well-defined d. Dotted arrows means that we can choose these maps element-wise such that the diagram commutes.

□

Lemma 4.11 (Kernels and cokernels)**.** *For any* $(b, p, q, z), (b', p', q', z') \in I_4$ *with* $z = p'$ *and* $q = b'$ *we have*

$$\ker\left(d : S_{bq}^{pz} \to S_{b'q'}^{p'z'}\right) = S_{bq}^{pq'}$$

and

$$\operatorname{coker}\left(d : S_{bq}^{pz} \to S_{b'q'}^{p'z'}\right) = S_{pq'}^{p'z'}$$

Proof. We give only the proof of the first statement, the second is symmetric. Let $x \in E_q^p$ represent an element $[x] \in S_{bq}^{pz}$. $[x]$ lies in $S_{bq}^{pq'}$ if and only if $x \in \ker(d_{pqq'} : E_q^p \to E_{q'}^q)$. By Axiom 1, this is if and only if $k(x) = i(y) \in D_q$ for some $y \in D_{q'}$.

$$\begin{tikzcd}
 & E_{q'}^q \arrow[r, "\ell"] & E_{q'}^q \\
E_q^p \arrow[ur, "d"] \arrow[r, "k"] & D_q \arrow[u, "j"] \arrow[ur, "j"] \arrow[r, "i"] & D_p \\
D_{q'} \arrow[ur, "i"] \arrow[r, "i"] & D_{p'} \arrow[u, "i"] \arrow[ur, "i"] \arrow[r, "j"] & E_{q'}^{p'} \\
E_{p'}^p \arrow[ur, "k"] \arrow[r, "\ell"] & E_{p'}^q \arrow[u, "k"] \arrow[ur, "d"'] &
\end{tikzcd}$$

In this case, the construction (26) of the differential (25) and the triviality of the composition $D_{q'} \to D_{p'} \to E_{q'}^{p'}$ show that (25) sends $[x]$ to zero.

Conversely, assume that d (25) sends $[x]$ to zero. Let z be the choice of the element in $D_{p'} = D_z$ that we made in diagram (26) in order to construct $d([x])$. Since $d([x]) = 0$, there is an $r \in E^{b'}_{p'} = E^{q}_{p'}$ such that $j(k(r)) = j(z) \in E^{p'}_{q'}$. Since $i(k(r)) = 0 \in D_q$, we could have equally well chosen $z' := z - k(r)$ instead of z. Note that $j(z') = 0 \in E^{p'}_{q'}$. Thus z' has a preimage $y \in D_{q'}$. Therefore $k(x) = i(z) = i(z') = i(i(y)) \in D_q$. ☐

Lemma 4.12 (∞-page as filtration quotients)**.** *D_∞ can be I-filtered by*

$$G_p := \operatorname{im}(i_{p,\infty} : D_p \to D_\infty) \cong S^{p,-\infty}_{\infty,-\infty}, \quad p \in I.$$

Furthermore the S-terms on the ∞-page are filtration quotients

$$S^{p,-\infty}_{\infty,q} \cong G_p/G_q.$$

Proof. By definition and Axiom 1,

$$S^{p,-\infty}_{\infty,-\infty} = \frac{\ker(d : D_p \to D_{-\infty})}{\operatorname{im}(d : E^{\infty}_{p} \to D_p)} = \frac{D_p}{\ker(i : D_p \to D_\infty)} \cong G_p.$$

Furthermore, Lemma 4.9 shows that $0 \to G_q \to G_p \to S^{p,-\infty}_{\infty,q} \to 0$ is exact. ☐

Lemma 4.13 (∞-page as quotient kernels)**.** *D_∞ has quotients*

$$Q_p := \frac{D_\infty}{\ker(j : D_\infty \to E^{\infty}_{p})} \cong S^{\infty,-\infty}_{\infty,p}, \quad p \in I.$$

Furthermore the S-terms on the ∞-page are quotient kernels

$$S^{p,-\infty}_{\infty,q} \cong \ker(Q_q \to Q_p).$$

Proof. By definition and Axiom 1,

$$S^{\infty,-\infty}_{\infty,p} = \frac{\ker(d : E^{\infty}_{p} \to D_p)}{\operatorname{im}(d : E^{\infty}_{\infty} \to E^{\infty}_{p})} = \operatorname{im}(j : D_\infty \to E^{\infty}_{p}) \cong Q_p.$$

Furthermore, Lemma 4.9 shows that $0 \to S^{p,-\infty}_{\infty,q} \to Q_q \to Q_p \to 0$ is exact. ☐

4.2 Natural isomorphisms

A lattice is complete if arbitrary meets ('∩') and joins ('∪') exist. A *closed set system* is a family of sets that is closed under taking arbitrary unions and intersections. In particular, a closed set system is a complete distributive lattice.

Definition 4.14 (Excision)**.** An exact couple system E over a complete distributive lattice I is called *excisive* if for all $a, b \in I$,

$$E^{a}_{a\cap b} \xrightarrow{\ell} E^{a\cup b}_{b}$$

is an isomorphism.

Example 4.15. An exact couple system E over a closed set system I is excisive if and only if for all $(p, q) \le (p', q')$ in I_2 with $p\backslash q = p'\backslash q'$ the map $E^p_q \xrightarrow{\ell} E^{p'}_{q'}$ is an isomorphism.

The exact couple system of a filtered space 4.5 is excisive by excision of h_* if $(X_i)_{i\in I}$ is a family of open subsets that is closed under taking arbitrary unions and intersections.

Also, the exact couple system of an I-filtered chain complex 4.4 is excisive if the filtration is distributive and I is complete.

Lemma 4.16 (Natural isomorphisms 1)**.** *In an excisive exact couple system E over a closed set system I, S^{pz}_{bq} is uniquely determined up to natural isomorphism by $b\backslash p$, $p\backslash q$, and $q\backslash z$.*

Proof. Let $(b, p, q, z), (b', p', q', z') \in I_4$ with $b\backslash p{=}b'\backslash p'$, and so on. Let $b'' := b \cup b'$, and so on. Then the vertical maps in

$$\begin{tikzcd}
E^b_p \arrow[r, "d"] \arrow[d, "\ell"] & E^p_q \arrow[r, "d"] \arrow[d, "\ell"] & E^q_z \arrow[d, "\ell"] \\
E^{b''}_{p''} \arrow[r, "d"] & E^{p''}_{q''} \arrow[r, "d"] & E^{q''}_{z''}
\end{tikzcd}$$

are isomorphisms since E is excisive. Thus $S^{pz}_{bq} \xrightarrow{\cong} S^{p''z''}_{b''q''}$ and analogously $S^{p''z''}_{b''q''} \xleftarrow{\cong} S^{p'z'}_{b'q'}$, both maps being induced by inclusion. ☐

Lemma 4.17 (Splitting principle for 1-page)**.** *In an excisive exact couple system E, for any $a, b \in I$ we have a commutative triangle of natural isomorphisms*

$$\begin{tikzcd}
 & E^{a\cup b}_{a\cap b} \arrow[dr, "(\ell,\ell)"] & \\
E^a_{a\cap b} \oplus E^b_{a\cap b} \arrow[ur, "\ell+\ell"] \arrow[rr, "\ell\oplus\ell"] & & E^{a\cup b}_b \oplus E^{a\cup b}_a .
\end{tikzcd} \tag{27}$$

Proof. By Axiom 1, both compositions $E^a_{a\cap b} \to E^{a\cup b}_{a\cap b} \to E^{a\cup b}_a$ and $E^b_{a\cap b} \to E^{a\cup b}_{a\cap b} \to E^{a\cup b}_b$ are zero, which implies that diagram (27) commutes. Since E is excisive the bottom map is an isomorphism, we find a section $E^{a\cup b}_a \cong E^b_{a\cap b} \to E^{a\cup b}_{a\cap b}$, which implies that the exact triangle

$$\begin{tikzcd}
E^a_{a\cap b} \arrow[r, "\ell"] & E^{a\cup b}_{a\cap b} \arrow[r, "\ell"] & E^{a\cup b}_a \arrow[ll, bend left, "d"]
\end{tikzcd}$$

splits, and this splitting coincides with the top left map $\ell + \ell$ in (27). ☐

Lemma 4.18 (General splitting principle)**.** *In an excisive exact couple system E, for any $z, q, p, b, x, y \in I$ with $x \cap y \subseteq z \subseteq q \subseteq p \subseteq b \subseteq x \cup y$ we have a commutative* *triangle of natural isomorphisms*

$$\begin{tikzcd}[column sep=small] & S^{pz}_{bq} \arrow[dr, "{(\ell,\ell)}"] & \\ S^{p\cap x,z\cap x}_{b\cap x,q\cap x} \oplus S^{p\cap y,z\cap y}_{b\cap y,q\cap y} \arrow[ur, "\ell+\ell"] \arrow[rr, "\ell\oplus\ell"] & & S^{p\cup x,z\cup x}_{b\cup x,q\cup x} \oplus S^{p\cup y,z\cup y}_{b\cup y,q\cup y}. \end{tikzcd} \tag{28}$$

Proof. Note that $(p \cap x) \cup q = p \cap (x \cup q)$. By excision, Lemma 4.17, and then again excision, we have

$$E^{p\cap x}_{q\cap x} \oplus E^{p\cap y}_{q\cap y} \xrightarrow{\cong} E^{p\cap x\cup q}_{q} \oplus E^{p\cap y\cup q}_{q} \xrightarrow{\cong} E^{p}_{q} \xrightarrow{\cong} E^{p}_{p\cap x\cup q} \oplus E^{p}_{p\cap x\cup q} \xrightarrow{\cong} E^{p\cup x}_{q\cup x} \oplus E^{p\cup y}_{q\cup y}.$$

We do the same with (b, p) and (q, z) in place of (p, q) and put that together with maps induced by inclusion into a 3×5-diagram, which proves that $\ell + \ell$ and (ℓ, ℓ) in (28) are isomorphisms by definition of the S-terms. Commutativity of the diagram follows from commutativity of (27). ☐

Let us now suppose that $I = D(J)$ is the complete distributive lattice of downsets of some arbitrary poset J. As in Section 3.1, we think of J as an undirected graph, whose vertices are the elements of J, and $x, y \in J$ are adjacent if they are related, i.e. $x > y$ or $x < y$. For $(b, p, q, z) \in I_4$, let $Z(z, q, p, b) \subseteq J$ denote the union of all connected components of $p\backslash z$ that intersect $p\backslash q$, and let $B(z, q, p, b) \subseteq I$ denote the union of all connected components of $b\backslash q$ that intersect $p\backslash q$.

Lemma 4.19 (Natural isomorphisms 2)**.** *In an excisive exact couple system E over $I = D(J)$, S^{pz}_{bq} is uniquely determined up to natural isomorphism by $Z := Z(z, q, p, b)$ and $B := B(z, q, p, b)$.*

Proof. We have $p\backslash q = Z \cap B$. First we prove that we can change z arbitrarily without changing S^{pz}_{bq} as long as Z stays invariant: Let $z^* := p\backslash Z$ and $z_* := \mathrm{down}(Z)\backslash Z$ be the maximal and minimal elements less or equal to q such that $Z(z_*, q, p, b) = Z = Z(z^*, q, p, b)$ (here we need the completeness of I). Then $z_* \le z \le z^*$ implies that the two maps induced by inclusion $S^{pz_*}_{bq} \hookrightarrow S^{pz}_{bq} \hookrightarrow S^{pz^*}_{bq}$ are injections. We claim that their composition is surjective.

$$\begin{tikzcd} E^p_{z^*} \arrow[r, "\ell"] \arrow[d, "d"'] & E^p_q \arrow[r, "d"] \arrow[d, "d"] & E^q_{z^*} \\ E^{z^*}_{z_*} \arrow[r, "\ell"] & E^q_{z_*}. & \end{tikzcd}$$

Let $x \in E^p_q$ with $d(x) = 0 \in E^q_{z^*}$. Then there exists $x' \in E^p_{z^*}$ with $\ell(x') = x$. In order to show that $d(x) = 0 \in E^q_{z_*}$, it suffices to show that the map $d : E^p_{z^*} \to E^{z^*}_{z_*}$ is zero. Let $y := \mathrm{down}(Z)$. Then $p = z^* \cup y$ and $z_* = z^* \cap y$. Hence Lemma 4.17 implies that the two maps induced by inclusion $E^{z^*}_{z_*} \hookrightarrow E^p_{z_*} \twoheadrightarrow E^p_{z^*}$ are an injection followed by a surjection. From the exact triangle for (p, z^*, z_*) we deduce that $d : E^p_{z^*} \to E^{z^*}_{z_*}$ is indeed zero.

Similarly one shows that changing b does not change S^{pz}_{bq} as long as B stays invariant. Thus we may assume $z = p\backslash Z$ and $b = q \cup B$. The rest follows from Lemma 4.16. ☐

4.3 Connections

From what we proved so far about kernels, cokernels, and natural isomorphisms it follows that both lexicographic connections and the secondary connection from Section 3.2 apply to the spectral systems of excisive exact couple systems over $D(\overline{\mathbb{Z}}^n)$ as well.

The only differences are that such exact couple systems do not give rise to 0-pages as in Lemmas 3.11(1), 3.12(1), and (17). They start from the 1-page

$$S_{bq}^{pz}(P;1,\varphi,0) = S_{bq}^{pz}(P;0,\varphi,e_n) = E_{A(P-e_n;\varphi)}^{A(P;\varphi)} \cong E_{A(\varphi^{-1}(P)-e_n)}^{A(\varphi^{-1}(P))},$$

and respectively

$$S_{b^*q}^{pz^*}(P;0) = E_{A(P-e_n)}^{A(P)}.$$

Example 4.20 (Spectral system for n open subsets)**.** Suppose X is a topological space with n open subsets X_i, and h is a generalized homology theory. Then spectral system of n independent subcomplexes from Section 3.3 generalizes to this setting: It converges to $h(X)$, and it's $S_{bq}^{pz}(P,1)$ terms for $P \in \{0,1\}^n$ are given by

$$S_{bq}^{pz}(P,1) = h\big(X_P, \bigcup_{Q<P} X_Q\big) \cong h\big(\bigcap_{Q>P} X^Q, X^P\big),$$

where $X_P := \bigcap_{i:p_i=0} X_i$, $X^P := \bigcup_{i:p_i=1} X_i$, Q runs in $\{0,1\}^n$, and empty intersections mean X. If we denote the collection of all $S_{bq}^{pz}(P,k)$, $P \in \{0,1\}^n$, by $S_{bq}^{pz}(\{0,1\}^n,k)$, and similarly with $S_{b^*q}^{pz^*}(P,k)$, then the 2-page is given by

$$S_{b^*q}^{pz^*}(\{0,1\}^n,n) = H(\ldots H(S_{bq}^{pz}(\{0,1\}^n,1),d_1),\ldots,d_n),$$

where d_i denotes the natural differential in direction $-e_i$.

Remark 4.21 (Spectral systems over $D(\overline{\mathbb{Z}}^n)^*$)**.** Let E be an exact couple system over $D(\overline{\mathbb{Z}}^n)^*$ (e.g. from Example 4.5). The most natural analog of the lexicographic and secondary connection in the associated spectral system S comes from regarding E as an exact couple system over $D(\overline{\mathbb{Z}}^n)$ using the identification of $D(\overline{\mathbb{Z}}^n)^*$ with $D(\overline{\mathbb{Z}}^n)$ via $p \mapsto -(\overline{\mathbb{Z}}^n \backslash p)$.

4.4 Multiplicative structure

This section about products is not most general, but it is hoped to be sufficient for many situations where spectral systems appear. Depending on the exact couple system, it may be useful to restrict I to a subposet in order to define a reasonable multiplicative structure. More general construction methods for products in usual spectral sequences are given by Massey [Mas54].

Definition 4.22. A *multiplicative structure* on an exact couple system E over a distributive lattice I consists of a binary operation

$$+ : I^2 \to I \tag{29}$$

and homomorphisms

$$\cup : E^{p_1}_{q_1} \otimes E^{p_2}_{q_2} \to E^{p_1+p_2}_{(p_1+q_2)\cup(q_1+p_2)} \tag{30}$$

for any $(p_1, q_1), (p_2, q_2) \in I_2$ such that the following properties hold:

1. (Monotonicity of $+$) If $p \subseteq p'$ and $q \subseteq q'$ then $p + q \subseteq p' + q'$.
2. (Functoriality of $\cup$) If $(p_1, q_1) \leq (p'_1, q'_1)$ and $(p_2, q_2) \leq (p'_2, q'_2)$ in I_2 then the following diagram commutes:

$$\begin{tikzcd} E^{p_1}_{q_1} \otimes E^{p_2}_{q_2} \arrow[r, "\cup"] \arrow[d, "\ell"'] & E^{p_1+p_2}_{(p_1+q_2)\cup(q_1+p_2)} \arrow[d, "\ell"] \\ E^{p'_1}_{q'_1} \otimes E^{p'_2}_{q'_2} \arrow[r, "\cup"] & E^{p'_1+p'_2}_{(p'_1+q'_2)\cup(q'_1+p'_2)}. \end{tikzcd} \tag{31}$$

3. (Leibniz rule) For all $(p_1, q_1), (p_2, q_2) \in I_2$ the following diagram commutes:

$$\begin{tikzcd} E^{p_1}_{q_1} \otimes E^{p_2}_{q_2} \arrow[r, "\cup"] \arrow[d, "d\otimes\ell\pm\ell\otimes d"'] & E^{p_1+p_2}_{(p_1+q_2)\cup(q_1+p_2)} \arrow[d, "d"] \\ D_{q_1} \otimes E^{p_2}_{q_2} \oplus E^{p_1}_{q_1} \otimes D_{q_2} \arrow[r, "\ell\circ\cup+\ell\circ\cup"] & E^{(p_1+q_2)\cup(q_1+p_2)}_{(q_1+q_2)\cup(p_1+\emptyset)\cup(\emptyset+p_2)}. \end{tikzcd}$$

By functoriality of $\cup$ there is no danger in abbreviating any $\ell\circ\cup\circ\ell$ by $\cup$. Let us omit discussing the sign in the Leibniz rule, since this is not where the meat is in this paper; the reader is invited to use his or her favorite sign convention. Often in applications, I is a closed system of subsets of a semigroup and the operation $+$ is Minkowski sum, which then implies $p + \emptyset = \emptyset + p = \emptyset$.

Example 4.23 (Filtered differential algebras)**.** Let C be a differential algebra, distributively filtered by $(F_i)_{i\in I}$ over a complete distributive lattice I, which has a monotone binary operation $+$ on it. Suppose the product on C sends $F_i \otimes F_j$ into F_{i+j} for all $i, j \in I$. Then this induces naturally a multiplicative structure on the associated exact couple system 4.4.

Example 4.24 (Filtered spaces)**.** Let X be filtered by a family of subsets $(X_i)_{i\in I}$ that is closed under taking unions and intersections. If h_* is a generalized homology theory, then for a multiplicative structure on the associated exact couple system of $(X_i)_{i\in I}$ we need products

$$\cup : h_*(X_{p_1}, X_{q_1}) \otimes h_*(X_{p_2}, X_{q_2}) \to h_*(X_{p_1+p_2}, X_{(p_1+q_2)\cup(q_1+p_2)})$$

for some $+ : I^2 \to I$. If h^* is a generalized cohomology theory, then for a multiplicative structure on the associated exact couple system of $(X_i)_{i\in I}$ we need products

$$\cup : h^*(X_{q_1}, X_{p_1}) \otimes h^*(X_{q_2}, X_{p_2}) \to h^*(X_{(p_1+q_2)\cup(q_1+p_2)}, X_{p_1+p_2})$$

for some $+ : I^2 \to I$. These multiplicative structures only exist in suitable cases and then possibly only on (families of) sublattices $J \subset I$. An example are the Leray–Serre spectral systems of Section 5, when h^* is a generalized cohomology theory with products.

Lemma 4.25 (Multiplication)**.** *If E is an exact couple with multiplicative structure, then there is a natural multiplication*

$$S^{p_1,z_1}_{b_1,q_1} \otimes S^{p_2,z_2}_{b_2,q_2} \xrightarrow{\cup} S^{p',z'}_{b',q'} \tag{32}$$

for all $(b_1,p_1,q_1,z_1),(b_2,p_2,q_2,z_2),(b',p',q',z') \in I_4$ that satisfy

$$\begin{aligned} b' &\supseteq (p_1+b_2)\cup(b_1+p_2),\\ p' &\supseteq p_1+p_2,\\ q' &\supseteq (p_1+q_2)\cup(q_1+p_2)\cup(b_1+z_2)\cup(z_1+b_2), \text{ and}\\ z' &\supseteq (p_1+z_2)\cup(z_1+p_2). \end{aligned}$$

Moreover the product commutes with maps between S-terms induced by inclusion. It also satisfies a Leibniz rule: For all indices as above and analogs with a bar, such that $(q_j,z_j)\le(\bar b_j,\bar p_j)$ for all $j\in\{1,2,'\}$, the following diagram commutes

$$\begin{tikzcd} S^{p_1,z_1}_{b_1,q_1} \otimes S^{p_2,z_2}_{b_2,q_2} \arrow[rrr, "\cup"] \arrow[d, "d\otimes\ell\pm\ell\otimes d"'] & & & S^{p',z'}_{b',q'} \arrow[d, "d"] \\ S^{\bar p_1,\bar z_1}_{\bar b_1,\bar q_1} \otimes S^{p_2,z_2}_{b_2,q_2} \oplus S^{p_1,z_1}_{b_1,q_1} \otimes S^{\bar p_2,\bar z_2}_{\bar b_2,\bar q_2} \arrow[rrr, "\cup+\cup"] & & & S^{\bar p',\bar z'}_{\bar b',\bar q'} \end{tikzcd} \tag{33}$$

whenever

$$\begin{aligned} \bar b' &\supseteq (\bar p_1+b_2)\cup(\bar b_1+p_2)\cup(p_1+\bar b_2)\cup(b_1+\bar p_2),\\ \bar p' &\supseteq (\bar p_1+p_2)\cup(p_1+\bar p_2),\\ \bar q' &\supseteq (\bar p_1+q_2)\cup(\bar q_1+p_2)\cup(\bar b_1+z_2)\cup(\bar z_1+b_2)\cup\\ &\qquad (p_1+\bar q_2)\cup(q_1+\bar p_2)\cup(b_1+\bar z_2)\cup(z_1+\bar b_2), \text{ and}\\ \bar z' &\supseteq (\bar p_1+z_2)\cup(\bar z_1+p_2)\cup(p_1+\bar z_2)\cup(z_1+\bar p_2). \end{aligned}$$

Proof. The map $\cup : E^{p_1}_{q_1}\otimes E^{p_2}_{q_2} \to E^{p'}_{q'}$ induces (32): It sends pairs of cycles to cycles as we deduce from the commutative diagram

$$\begin{tikzcd} E^{p_1}_{z_1}\otimes E^{p_2}_{z_2} \arrow[r, "\cup"] \arrow[d, "\ell\otimes\ell"'] & E^{p'}_{z'} \arrow[d, "\ell"] \arrow[dd, bend left=60, "0"] \\ E^{p_1}_{q_1}\otimes E^{p_2}_{q_2} \arrow[r, "\cup"] & E^{p'}_{q'} \arrow[d, "d"] \\ & E^{q'}_{z'} \end{tikzcd}$$

and Lemma 4.8. It is well-defined because of the commutative diagram

$$\begin{tikzcd}
E^{b_1}_{p_1} \otimes E^{p_2}_{z_2} \arrow[r, "\cup"] \arrow[d, "d\otimes\ell\pm\ell\otimes d"'] & E^{b'}_{p'} \arrow[d, "d"] \\
E^{p_1}_{q_1} \otimes E^{p_2}_{q_2} \oplus E^{b_1}_{p_1} \otimes D_{z_2} \arrow[r, "\cup+\cup"] & E^{p'}_{q'}
\end{tikzcd}$$

and its symmetric analog with indices 1 and 2 and the tensor order exchanged. We need the assumptions on the indices (b', p', q', z') in order to assure that the second cup on the bottom is the zero map and that the top map is well-defined. Naturality and the Leibniz rule (33) follow directly from the same properties of E. $\square$

Remark 4.26 (Compatibility with natural isomorphisms)**.** Suppose E is an excisive exact couple system over a closed set system I.

Further suppose that E has a two *compatible* multiplicative structures over sublattices I' and I'' in the following sense: If $(p_1', q_1'), (p_2', q_2') \in I_2'$ and $(p_1'', q_1''), (p_2'', q_2'') \in I_2''$ satisfy $p_1' \backslash q_1' = p_1'' \backslash q_1''$ and $p_2' \backslash q_2' = p_2'' \backslash q_2''$, then also $(p_1' + p_2') \backslash ((p_1' + q_2') \cup (p_2' + q_1')) = (p_1'' + p_2'') \backslash ((p_1'' + q_2'') \cup (p_2'' + q_1''))$ and the following diagram with vertical maps being excision commutes,

$$\begin{tikzcd}
E^{p_1'}_{q_1'} \otimes E^{p_2'}_{q_2'} \arrow[r, "\cup'"] \arrow[d, "\cong"'] & E^{p_1'+p_2'}_{(p_1'+q_2')\cup(q_1'+p_2')} \arrow[d, "\cong"] \\
E^{p_1''}_{q_1''} \otimes E^{p_2''}_{q_2''} \arrow[r, "\cup''"] & E^{p_1''+p_2''}_{(p_1''+q_2'')\cup(q_1''+p_2'')}.
\end{tikzcd}$$

Then also the corresponding products on S-terms (32) for I' and I'' commute with respect to the natural isomorphisms given by Lemma 4.16 (and Lemma 4.19 in case $I = D(J)$ for some poset J), since (32) is induced by (30).

Remark 4.27 (Compatibility with extensions)**.** Consider three extension sequences (24) from Lemma 4.9, which we write for short as $0 \to A_i \to B_i \to C_i \to 0$ with $i \in \{1, 2, '\}$, such that products $X_1 \otimes X_2 \to X'$ with $X \in \{A, B, C\}$ are defined via Lemma (4.25). Then the diagram

$$\begin{tikzcd}
A_1 \otimes A_2 \arrow[r, "\ell\otimes\ell"] \arrow[d, "\cup"] & B_1 \otimes B_2 \arrow[r, "\ell\otimes\ell"] \arrow[d, "\cup"] & C_1 \otimes C_2 \arrow[d, "\cup"] \\
A' \arrow[r, "\ell"] & B' \arrow[r, "\ell"] & C'
\end{tikzcd}$$

commutes, as the horizontal maps are induced by inclusion.

Example 4.28 (Several $\mathbb{Z}$-filtrations)**.** Suppose we are given n different $\mathbb{Z}$-filtrations of a differential algebra C as in (13), such that $F_k^{(i)} \cdot F_\ell^{(i)} \subseteq F_{k+\ell}^{(i)}$ for all i, k, ℓ, and such that the induced $D(\overline{\mathbb{Z}}^n)$-filtration is distributive. Fix a shearing matrix $\varphi \in \mathrm{GL}(n, \mathbb{Z})$ with non-negative entries. Let I be the subposet of $D(\overline{\mathbb{Z}}^n)$ consisting of all $A(P; \varphi)$, $P \in \overline{\mathbb{Z}}^n$. We define (29) as the Minkowski sum $A(P; \varphi) + A(P'; \varphi) := A(P + P'; \varphi)$, using the convention $\infty - \infty := \infty$ in $\overline{\mathbb{Z}}$ (the reason for this convention is similar to footnote 5 on page 9). The product of C induces a multiplicative structure (30) on the 1-page. Notice that in Lemma 4.25 the unions of elements in I are all trivial (that is, of the form $A \cup \ldots \cup A = A$) for all lexicographic connections and the secondary connection. We obtain cup products

$$S^{pz}_{bq}(p_1, \ldots, p_k; r, \varphi, Q) \otimes S^{pz}_{bq}(p'_1, \ldots, p'_k; r, \varphi, Q) \to S^{pz}_{bq}(p_1 + p'_1, \ldots, p_k + p'_k; r, \varphi, Q)$$

and

$$S^{pz^*}_{b^*q}(P; k) \otimes S^{pz^*}_{b^*q}(P'; k) \to S^{pz^*}_{b^*q}(P + P'; k),$$

which satisfy the obvious Leibniz rules. Also, the multiplicative structure is compatible with itself and is hence compatible with natural isomorphisms of S-terms (Remark 4.26). Thus these products induce the products on the subsequent pages of the secondary and lexicographic connections.

Remark 4.29 (Weak multiplicative structure for $I = D(\overline{\mathbb{Z}}^n)$)**.** For general exact couple systems over $D(\overline{\mathbb{Z}}^n)$ a multiplicative structure as in Definition 4.22 might be too much to ask for. In order to have a cup product along the lexicographic connections for some fixed shearing matrix φ, it suffices to have a product (30) only for pairs $(p_1, q_1) = (A(P_1; \varphi), A(Q_1; \varphi))$ and $(p_2, q_2) = (A(P_2; \varphi), A(Q_2; \varphi))$ with $P_1 + Q_2 = Q_1 + P_2$. Here the Leibniz rule needs to be slightly weakened accordingly by replacing D_{q_1} with $E^{q_1}_{A(Q_1+Q_2-P_2;\varphi)}$ and D_{q_2} with $E^{q_2}_{A(Q_1+Q_2-P_1;\varphi)}$.

In order to have a cup product along the secondary connection, we further require the products (30) for successive φ_k to be compatible; see Remark 4.26.

We call these products along the secondary and lexicographic connections a weak multiplicative structure.

Remark 4.30 (Cross product)**.** Sometimes one has a more general cross product for three exact couple systems E, E', and E'' over I,

$$\cup : E^{p_1}_{q_1} \otimes E'^{p_2}_{q_2} \to E''^{p_1+p_2}_{(p_1+q_2)\cup(q_1+p_2)}.$$

This section generalizes to this setting without difficulty.

5 Successive Leray–Serre spectral sequences

Leray [Ler46] and Serre [Ser51] constructed a spectral sequence that relates the homologies of the base, the fiber, and the total space of a fibration. Here we study the situation of towers of fibrations.

5.1 The spectral system

Suppose we are given a tower of fibrations (always in the sense of Serre)

$$\begin{tikzcd}
F_{i-1} \arrow[r, hook, "i_{i-1}"] & E_{i-1} \arrow[d, "f_i"] & \\
 & E_i & (1 \leq i \leq n),
\end{tikzcd} \tag{34}$$

such that $E_1, \dots, E_n$ have the homotopy type of a CW-complex. We denote the tower with E_*. Let's write $F_n := E_n$, $E_{n+k} := \mathrm{pt}$, $E_{-k} := E_0$, and $F_{-k} := \mathrm{pt}$ for $k \geq 1$.

Theorem 5.1. *Let h be a generalized homology theory. Associated to the fibration tower* (34) *there is a spectral system over $I = D(\mathbb{Z}^n)$ with 2-page*

$$S_{b^*q}^{pz^*}(P; n) = H_{p_n}(F_n; H_{p_{n-1}}(F_{n-1}; \dots H_{p_1}(F_1; h(F_0))))) \tag{35}$$

and limit $S_{\infty,-\infty}^{\infty,-\infty} = h(E_0)$.

Later we will use the shorter notation $H_\bullet(F_n; F_{n-1}; \dots; F_1; h(F_0))$ for the direct sum of (35) over all P.

Remark 5.2 (Local coefficients)**.** Of course in (35) we have local coefficients everywhere: For $i \leq k$, we write $f_{i,k} := f_k \circ \dots \circ f_{i+1} \circ \mathrm{id}_{E_i} : E_i \to E_k$, and $F_{i,k} := f_{i,k}^{-1}(\mathrm{pt})$. They also form fibrations $F_{i,k} \hookrightarrow E_i \to E_k$, and more generally, $F_{i,k} \hookrightarrow F_{i,\ell} \to F_{k,\ell}$ for $i \leq k \leq \ell$.

An element $\gamma \in \pi_1(F_k)$ induces a map $m_{k-1,k}^\gamma : F_{k-1,k} \to F_{k-1,k}$, and over it a map $m_{k-2,k}^\gamma : F_{k-2,k} \to F_{k-2,k}$, and over it a map $m_{k-3,k}^\gamma : F_{k-3,k} \to F_{k-3,k}$, and so on. We regard them as a self-map $\widetilde{m}_k^\gamma$ on the fibration tower $F_i \hookrightarrow F_{i,k} \to F_{i+1,k}$, $i < k$, and $\widetilde{m}_k^\gamma$ is uniquely given up to fiber homotopy.

For $k = 1$, this makes $h(F_0)$ into a local coefficient system over F_1, as usual. For $k = 2$, $\gamma \in \pi_1(F_2)$ induces a map $F_1 \to F_1$ that respects the local coefficient system $h(F_0)$ over it; thus $H_*(F_1; h(F_0))$ becomes a local coefficient system over F_2. And so on.

Remark 5.3 (Naturality)**.** Any map $m : E_0 \to E_0'$ between two such fibration towers (that is, it induces well-defined quotient maps $E_i \to E_i'$) naturally induces a morphisms between the associated spectral systems.

Remark 5.4 (Cross product)**.** Suppose h is a multiplicative generalized homology theory, that is it comes from a ring spectrum. The Cartesian product of two fibration towers E_* and E_*' is again a fibration tower E_*''. As usual (compare with Switzer [Swi75, p. 352–353]) the composition $h(X_p, X_q) \otimes h(Y_{p'}, Y_{q'}) \xrightarrow{\times} h((X_p, X_q) \times (Y_{p'}, Y_{q'})) \xrightarrow{i_*} h((X \times X)_{p+p'}, (X \times Y)_{p+q' \cup p'+q})$ induces a cross product between the spectral systems of E_* and E_*' to the one of E_*'' for all pages in the lexicographic connections for arbitrary shearing matrix (as in Examples 4.28, 4.30), and it commutes with the differentials

Proof. Our following construction follows the idea of the Fadell–Hurewicz construction [FH58] of the Leray–Serre spectral sequence using singular prisms (very similar to Dress' construction [Dre67]; also compare with Brown [Bro59, Sect. 7]) and Dold's construction of the Leray–Serre spectral sequence for a generalized (co-)homology theory [Dol62].

For $P = (p_1, \ldots, p_n) \in \mathbb{Z}^n$, let $\Delta_P := \Delta_{p_1} \times \ldots \times \Delta_{p_n}$ be the product of p_i-dimensional simplices. Let $\Delta_{P,*}$ denote the trivial fibration tower $\Delta_P \to \Delta_{P_{2\ldots n}} \to \ldots \to \Delta_{p_n}$. Similarly, let E^k_* denote the subtower $E_k \to \ldots \to E_n$ of (34). We define $K_P(E^1_*)$ as the set of all fibration tower maps $\Delta_{P,*} \to E^1_*$, that is, collections of maps $\Delta_{P_{i\ldots n}} \to E_i$ $(1 \leq i \leq n)$ that commute with the projections. $K := K_\bullet(E^1_*)$ forms in a natural way an n-fold simplicial set (even coalgebra), that is, a simplicial set in n ways with commuting face and degeneracy maps, and it has a geometric realization $|K|$. Let $X \to |K|$ denote the pullback of f_1 along the natural map $|K| \to E_1$,

$$\begin{tikzcd} X \arrow[r] \arrow[d] & E_0 \arrow[d, "f_1"] \\ {|K|} \arrow[r] & E_1. \end{tikzcd} \tag{36}$$

We claim that $\mathrm{Tot}(K)$ is chain homotopy equivalent to the singular chain complex $C_*(E_1)$. This can be proved similarly to the Eilenberg–Zilber theorem [EZ53] using acyclic models, compare with [FH58, Sect. 2.3.1 and 6.1] for the case $n = 2$. A chain homotopy equivalence $\alpha : \mathrm{Tot}(K) \to C_*(E_1)$ is obtained from the standard triangulation of Δ_P.[7]

Alternatively, one can use the argument in Dress [Dre67, Sect. 2] inductively to show that $H_{p_1}(\ldots H_{p_n}(K, d_n) \ldots, d_1)$ is zero except when $p_1 = \ldots = p_{n-1} = 0$, in which case it is naturally isomorphic $H_{p_n}(E_1)$ via α. Thus, a spectral system argument for the n-complex K shows that α is a quasi-isomorphism, and hence a homotopy equivalence since both complexes are free and bounded below.

Thus (36) is a fiber homotopy equivalence and $h(X) = h(E_0)$.

Every $p \in I := D(\overline{\mathbb{Z}}^n)$ indexes a skeleton of K and thus a subspace $X_p \subseteq X$. We define S to be the spectral system of the I-filtered space X with respect to h as in Example 4.5.

The secondary connection for S starts with abelian groups

$$S^{pz}_{bq}(P; 1) = h(X_{A(P)}, X_{A(P)}) \cong K_P(E^1_*) \otimes h(F_0).$$

Taking k times homology in directions $-e_1, \ldots, -e_k$, we obtain with the usual arguments (see Dress [Dre67, Sect. 3, 4], McCleary [McC01, Thm. 6.47])

$$S^{pz^*}_{b^*q}(P; k) \cong K_{P_{k+1,\ldots,n}}(E^{k+1}_*) \otimes H_{p_k}(F_k; \ldots H_{p_1}(F_1; h(F_0)))))$$

---

[7] Δ_k coincides with the order complex of the poset $\{0, \ldots, k\}$ with the usual order. Thus we obtain a triangulation of Δ_P by taking the order complex of the product poset $\prod_i \{0, \ldots, p_i\}$, which is called the standard triangulation of Δ_P.

as abelian groups. In particular for $k = n$ we get that the second page is naturally isomorphic to (35). $\square$

5.2 Cohomology version

As usual there is an analogous version for generalized cohomology theories h, using the exact couple $E^p_q = h(X_q, X_p)$ over I^*, where $I := D(\overline{\mathbb{Z}}^n)$ (Example 4.5). We identify $I^* \xrightarrow{\cong} I$ by $p \mapsto -(\overline{\mathbb{Z}}^n \backslash p)$ (Remark 4.21) in order to speak about secondary and lexicographic connections over I^*. The second page is given by

$$S^{pz^*}_{b^*q}(-P;n) = H^{p_n}(F_n; H^{p_{n-1}}(F_{n-1}; \dots H^{p_1}(F_1; h(F_0))))) \tag{37}$$

If h is multiplicative, then the spectral system S has indeed a natural product structure along the secondary and lexicographic connections as in Example 4.28 and Remark 4.29. For this we need to show that E has a multiplicative structure with respect to the subposet of I consisting of all $A(P;\varphi)$, $P \in \overline{\mathbb{Z}}^n$, for certain fixed φ:

Let $(p_i, q_i) = (A(\varphi(-P_i);\varphi), A(\varphi(-Q_i);\varphi)) \in I_2$ for $i = 1, 2$, and define M (possibly not uniquely) by the equation $A(\varphi(-M);\varphi) = A(\varphi(-Q_1 - P_2);\varphi) \cup A(\varphi(-P_1 - Q_2);\varphi)$. Define $A^\circ_P := -(\overline{\mathbb{Z}}^n \backslash A(\varphi(-P);\varphi) \in I^*$. For $P \in \mathbb{Z}^n$, this equals $A(\varphi(P);\varphi)\backslash\{-P\}$. We define the sum (29) in I as the Minkowski sum in I, *not* in I^*. Thus the desired pairing (30) reads

$$h(X_{A^\circ(Q_1)}, X_{A^\circ(P_1)}) \otimes h(X_{A^\circ(Q_2)}, X_{A^\circ(P_2)}) \to h(X_{A^\circ(M)}, X_{A^\circ(P_1+P_2)}). \tag{38}$$

Let $D : X \to X \times X$ denote a diagonal approximation (as the vertices of each face of X are consistently ordered, a canonical choice is the Alexander–Whitney diagonal approximation). The key is that D restricts to a map

$$(X_{A^\circ(M)}, X_{A^\circ(P_1+P_2)}) \to (X_{A^\circ(Q_1)} \times X_{A^\circ(Q_2)} \cup Y, Y), \tag{39}$$

where $Y := X \times X_{A^\circ(P_2)} \cup X_{A^\circ(P_1)} \times X$. By excision,

$$h(X_{A^\circ(Q_1)} \times X_{A^\circ(Q_2)} \cup Y, Y) \xrightarrow{\cong} h(X_{A^\circ(Q_1)} \times X_{A^\circ(Q_2)}, X_{A^\circ(Q_1)} \times X_{A^\circ(P_2)} \cup X_{A^\circ(P_1)} \times X_{A^\circ(Q_2)}). \tag{40}$$

We define the pairing (38) via the cross product of h composed with the inverse of (40) and the map induced by (39).

Clearly the product of S agrees in the limit $S^{\infty,-\infty}_{\infty,-\infty} = h(E_0)$ with the one in $h(E_0)$. The multiplicative structures for different φ are compatible, thus the products along the secondary connection are compatible (Remark 4.26). Therefore it agrees on the second page with the cup product of the right hand side of (37) (up to signs, depending on the convention).

5.3 Edge homomorphisms

Let E_* be the fibration tower (34). We use the notation from Remark 5.2. Clearly, for any $(b, a) \le (d, c) \in \mathbb{Z}_2$, we have a natural map

$$F_{a,b} \to F_{c,d}. \tag{41}$$

If $b = d$ this map is a fibration with fiber $F_{a,c}$.

Let $x, y : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be increasing maps with $x(i) \leq i \leq y(i)$ for all $i \geq 0$, and $x(n+k) = x(n+1)$ for all $k \geq 1$ We define fibration towers X_* and Y_* by $X_i := F_{x(i),x(n+1)}$ and $Y_i := F_{y(i),y(n+1)} = E_{y(i)}$, $0 \leq i \leq n$, using the maps (41) as the projections. By definition we get a composition of two fibration tower maps

$$X_* \to E_* \to Y_*.$$

It induces a map of spectral systems, which on the second page yields the maps

$$H_\bullet(F_{x(n),x(n+1)}; \ldots ; h(F_{x(0),x(1)})) \to H_\bullet(F_n; \ldots ; h(F_0)) \to H_\bullet(F_{y(n),y(n+1)}; \ldots ; h(F_{y(0),y(1)})),$$

and in the limit $h(F_{0,x(n+1)}) \to h(E_0) \to h(E_{y(0)})$.

For a useful special case, choose $0 \leq k \leq n$, set $x(i) = 0$ and $y(i) = k$ for $i \leq k$, and set $x(i) = k + 1$ and $y(i) = n + 1$ for $i \geq k + 1$. Then X_* and Y_* have only one non-trivial fiber at $* = k$, and on the second pages we get

$$H_\bullet(\mathrm{pt}; \ldots ; F_{0,k+1}; \ldots ; h(\mathrm{pt})) \to H_\bullet(F_n; \ldots ; F_k; \ldots ; h(F_0)) \to H_\bullet(\mathrm{pt}; \ldots ; E_k; \ldots ; h(\mathrm{pt})).$$

In case h is ordinary homology with coefficients in some abelian group, or if $k = 0$, then the spectral systems for X_* and Y_* clearly collapse at that second page. For the limit we get $h(F_{0,k+1}) \to h(E_0) \to h(E_k)$.

There are more x and y for which X_* and Y_* have only one non-trivial fiber, however the corresponding maps to and from E_* factor through the above ones.

For a generalized cohomology theory we get the analogous edge homomorphisms but in reversed direction.

5.4 Rotating fibration towers

In the introduction we saw that a vertical sequence of fibrations (1) induces a sequence of horizontal fibrations (2). Under certain conditions we can also go the other way around:

Lemma 5.5. *If a horizontal sequence of fibrations up to homotopy* (2) *can be delooped, then there is an associated vertical sequence of fibrations up to homotopy* (1).

Proof. Suppose $F = \Omega F'$, $P = \Omega P'$, $i_{FP} = \Omega i'_{FP}$, and so on. Let N be the homotopy fiber of $F' \hookrightarrow E'$. Then $F \hookrightarrow E \to N$ is a fibration up to homotopy. Let $N \to B = \Omega B'$ be the map that sends (f, γ) to $p'_{EB} \circ \gamma$. The preimage of the constant loop $\mathrm{const}_{b'_0} \in \Omega B'$ under this map is the homotopy fiber of i'_{FP}, which is homotopy equivalent to $\Omega(M) = M$. $\square$

More generally we can consider any proper binary tree of fibrations, in which every node is the total space of a fibration whose left child is fiber and whose right child is the base space: The root E is the space whose homology we are interested in. The leaves $F_1, \ldots, F_n$ are the fibers and base spaces whose homology we know. At every non-leaf we can use a Leray–Serre spectral sequence that calculates the homology of that node. Thus the tree gives us one way to compute $H_*(E)$ by successive spectral sequences from $H_*(F_1), \ldots, H_*(F_n)$.

The above operations for a diagram of two iterated fibrations can be applied also in the binary tree, giving left- and right rotations of the tree. The number of such trees and hence ways to use successive spectral sequences (assuming we can always deloop) is the Catalan number $C_{n-1} = \binom{2n-2}{n-1}/n$.

5.5 Successive Leray spectral sequences

Another version for sheaves arises from the successive Grothendieck spectral sequences in Section 6. Here we start with any sequence of maps $E_0 \xrightarrow{f_0} \ldots \xrightarrow{f_{n-1}} E_{n-1} \xrightarrow{f_n} E_n := \mathrm{pt}$ and a sheaf of abelian groups G on E_0, for example the constant $\mathbb{Z}$ sheaf. Let $\mathit{Sh}(X)$ denote the category of sheaves of abelian groups on a space X, and let $f_* : \mathit{Sh}(X) \to \mathit{Sh}(Y)$ denote the direct image functor for a map $f : X \to Y$. Then we have a sequence of left-exact additive functors

$$\mathit{Sh}(E_0) \xrightarrow{(f_1)_*} \ldots \xrightarrow{(f_{n-1})_*} \mathit{Sh}(E_{n-1}) \xrightarrow{(f_n)_*} \mathit{Sh}(\mathrm{pt}).$$

The associated spectral system over $D(\overline{\mathbb{Z}}^n)$ converges to

$$H^*(E_0; G) = R(f_n \circ \ldots \circ f_1)_*(G),$$

and its 2-page is given by

$$H^*(E_{n-1}; R(f_{n-1})_* \circ \ldots \circ R(f_1)_*(G)) = R(f_n)_* \circ \ldots \circ R(f_1)_*(G).$$

Note that $R^k(f_i)_*(S)$ coincides with the sheaf associated to the presheaf $(U \subseteq E_i) \mapsto H^k(f_i^{-1}(U); S)$.

6 Successive Grothendieck spectral sequences

Let $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{A}_2$ be abelian categories with enough projectives. Then Grothendieck's spectral sequence [Gro57] computes the left derived functors of a composition of two right-exact functors $F_1 : \mathcal{A}_0 \to \mathcal{A}_1$ and $F_2 : \mathcal{A}_1 \to \mathcal{A}_2$ from the left derived functors of F_1 and F_2, assuming that F_1 sends projective objects to F_2-acyclic objects. More precisely, the second page is given by $E^2_{pq} = L_pF_2 \circ L_qF_1(A)$ and it converges to $L_{p+q}(F_2 \circ F_1)(A)$, for any object $A \in \mathcal{A}_0$.

Now suppose we are given a sequence of n right-exact functors

$$\mathcal{A}_0 \xrightarrow{F_1} \mathcal{A}_1 \xrightarrow{F_2} \ldots \xrightarrow{F_n} \mathcal{A}_n \tag{42}$$

between abelian categories with enough projectives. We write $F_{ij} := F_j \circ \ldots \circ F_{i+1} : \mathcal{A}_i \to \mathcal{A}_j$. Assuming that for any $0 \le i < j < k \le n$, F_{ij} sends projective objects to

F_{jk}-acyclic objects, we can relate $LF_1, \ldots, LF_n$ to $L(F_n \circ \ldots \circ F_1)$ by applying $n-1$ Grothendieck spectral sequences successively. There are again $C_{n-1} = \binom{2n-2}{n-1}/n$ many ways to do that, since there are C_{n-1} ways to bracket $F_n \circ \ldots \circ F_1$.

The construction of Grothendieck's spectral sequence is based on the Cartan–Eilenberg resolution for chain complexes, see Cartan–Eilenberg [CE56]. Here we will use higher Cartan–Eilenberg resolutions for n-complexes.

6.1 Cartan–Eilenberg–Moore resolutions of n-complexes

n-Complexes. Let $\mathcal{A}$ be an abelian category. Let $\mathrm{Ch}(\mathcal{A})$ denote the category of chain complexes with objects in $\mathcal{A}$, graded over $\mathbb{Z}$, the differentials being of degree -1. For $(X, d) \in \mathrm{Ch}(\mathcal{A})$, let $C_k(X) := X_k$, $Z_k(X) := \ker(d : X_k \to X_{k-1})$, $Z'_k(X) := \mathrm{coker}(d : X_{k+1} \to X_k)$, $B_k(X) := \mathrm{im}(d : X_{k+1} \to X_k)$, and $H_k(X) := Z_k(X)/B_k(X)$ denote the graded pieces, cycles, dual cycles, boundaries, and homology groups of X.

We define the category $\mathrm{Ch}^n(\mathcal{A})$ of *n-complexes* inductively by $\mathrm{Ch}^0(\mathcal{A}) := \mathcal{A}$ and $\mathrm{Ch}^n(\mathcal{A}) := \mathrm{Ch}(\mathrm{Ch}^{n-1}(\mathcal{A}))$. Up to sign conventions, $\mathrm{Ch}^2(\mathcal{A})$ is the category of double complexes over $\mathcal{A}$. We denote the n differentials of an n-complex by $d_1, \ldots, d_n$. The order is important. They satisfy $d_i \circ d_i = 0$ and $d_i \circ d_j = d_j \circ d_i$ for all $i \neq j$. We can regard an n-complex X as chain complex over $\mathrm{Ch}^{n-1}(\mathcal{A})$ in n different ways, which we denote by (X, d_i). If we write only X, we always mean (X, d_n). The total complex $\mathrm{Tot}(X)$ of an n-complex X over $\mathcal{A}$ is the chain complex over $\mathcal{A}$ with $\mathrm{Tot}(X)_k = \oplus_{|P|=k} X_P$, $|P| := \sum p_i$, whose differential at the summand X_P is $\sum (-1)^{p_1 + \ldots + p_{i-1}} d_i$.

A homomorphism between two n-complexes $f : X \to Y$ is a $\mathbb{Z}^n$-graded homomorphism that commutes with the differentials. A homotopy between two such homomorphisms $f, g : X \to Y$ is an n-tuple $(s_1, \ldots, s_n)$ of $\mathbb{Z}^n$-graded homomorphisms of degree $e_1, \ldots, e_n$, respectively, such that $f - g = \sum_{i=1}^n s_i d_i - d_i s_i$ and $s_i d_j = d_j s_i$ for all $i \neq j$.

Relative homological algebra. Let us fix the basic definitions of relative homological algebra. (There are notational differences in the literature, see Eilenberg–Moore [EM65] and Hilton–Stammbach [HS97], but the concept is always the same.) We will need this generality for the product structure in Section 6.3.

Consider a class $\mathcal{E}$ of epimorphisms in $\mathcal{A}$ which is closed under compositions and direct sums, and which contains all isomorphisms as well as all morphisms to 0 (the standard choice is the class of all epimorphisms). An object $P \in \mathcal{A}$ is called projective with respect to $X \to Y$, if any map $f : P \to Y$ factors over X. P is called $\mathcal{E}$-projective, if it is projective with respect to all epimorphisms in $\mathcal{E}$. $\mathcal{E}$ is called a projective class of epimorphisms, if for any object $K \in \mathcal{A}$ there exists an epimorphism $P \to K$ in $\mathcal{E}$ with P being $\mathcal{E}$-projective. $\mathcal{E}$ is called closed, if any epimorphism in $\mathcal{A}$ such that all $\mathcal{E}$-projective objects are projective with respect to it lies already in $\mathcal{E}$.

A morphism $f : X \to Y$ in $\mathcal{A}$ is called $\mathcal{E}$-admissible if for the canonical factorization of f into an epimorphism and a monomorphism, the epimorphism is in $\mathcal{E}$. An exact sequence $\ldots \to X_{n+1} \xrightarrow{f_n} X_n \to \ldots$ is called $\mathcal{E}$-exact if all f_n are $\mathcal{E}$-admissible. If furthermore $X_n = 0$ for $n < 0$ then it is called an $\mathcal{E}$-acyclic resolution of X_0. If furthermore X_n are $\mathcal{E}$-projective for all $n > 0$ it is called an $\mathcal{E}$-projective resolution of X_0. $\mathcal{E}$-projective resolutions exist for any projective class $\mathcal{E}$ and have the usual properties. We denote the derived functors of an additive functor $F : \mathcal{A} \to \mathcal{B}$ with respect to $\mathcal{E}$-projective resolutions by $R_{\mathcal{E}}F$.

CEM-resolutions. For all $P \in \mathbb{Z}^n$, $J \subseteq [n]$, we have a pair of adjoint functors $Q_{P,J} \dashv Z_{P,J}$: Here, $Q_{P,J} : \mathcal{A} \to \mathrm{Ch}^n(\mathcal{A})$ denotes the "cube complex functor" with $Q_{P,J}(K)$ having the object K at all positions in $P + \{0\}^J \times \{0, -1\}^{[n]\setminus J}$ and zeros otherwise, with isomorphisms between the K-entries. $Z_{P,J} : \mathrm{Ch}^n(\mathcal{A}) \to \mathcal{A}$ denotes the functor $Z_{P,J}(X) := \{x \in X_P \mid d_j(x) = 0 \text{ for } j \in J\}$. Clearly, $\hom_{\mathrm{Ch}^n(\mathcal{A})}(Q_{P,J}(K), X) \cong \hom_{\mathcal{A}}(K, Z_{P,J}(X))$.

Now, fix a closed projective class $\mathcal{E}$ of epimorphisms in the abelian category $\mathcal{A}$. We define

$$\mathcal{E}\text{-CEM}_n$$

to be the class of all homomorphism of n-complexes $f : X \to Y$ such that the induced maps $Z_{P,J}(X) \to Z_{P,J}(Y)$ are $\mathcal{E}$-surjective for all $P \in \mathbb{Z}^n$, $J \subseteq [n]$. Below we usually omit n from the notation because it will be clear from the context; also compare with Remark 6.5. From the Eilenberg–Moore multiple adjoint theorem [EM65, Thm. 3.1] we get:

Lemma 6.1 ($\mathcal{E}$-CEM-projective objects)**.** *$\mathcal{E}$-CEM is a closed projective class. Furthermore, an n-complex $X \in \mathrm{Ch}^n(\mathcal{A})$ is $\mathcal{E}$-CEM-projective, if it is of the form $X = \bigoplus_{P\in\mathbb{Z}^n} \bigoplus_{J\subseteq[n]} Q_{P,J}(X_{P,J})$ with $X_{P,J}$ being $\mathcal{E}$-projective objects of $\mathcal{A}$.*

Remark 6.2 (Canonical $\mathcal{E}$-CEM-resolutions)**.** In case any $K \in \mathcal{A}$ admits a canonical $P \to K$ in $\mathcal{E}$ with P being $\mathcal{E}$-projective, the same holds for $\mathcal{E}$-CEM and there is a canonical $\mathcal{E}$-CEM-resolution for any $X \in \mathrm{Ch}^n(\mathcal{A})$.

Remark 6.3 (Alternative definition)**.** Equivalently we could inductively define projective classes of exact sequences $\mathcal{E}_n := \mathcal{E}_n(\mathcal{A}) \subseteq \mathrm{Ch}(\mathrm{Ch}^n(\mathcal{A}))$ as follows. Let $\mathcal{E}_0 \subseteq \mathrm{Ch}(\mathcal{A})$ be the class of $\mathcal{E}$-exact sequences in $\mathcal{A}$. For $n \geq 1$, let $\mathcal{E}_n \subseteq \mathrm{Ch}(\mathrm{Ch}^n(\mathcal{A}))$ be the class of chain complexes $X \in \mathrm{Ch}(\mathrm{Ch}^n(\mathcal{A}))$ such that $C_k(X, d_1)$ and $Z_k(X, d_1)$ lie in $\mathcal{E}_{n-1}$. (If $\mathcal{E}$ is the class of all epimorphisms, then $B_k(X, d_1)$ and $H_k(X, d_1)$ will lie in $\mathcal{E}_{n-1}$ as well; but not in general as differentials might not be $\mathcal{E}$-admissible.) It follows from [EM65, Thm. IV.2.1] that $\mathcal{E}_n$ are indeed projective classes, and that the $\mathcal{E}_n$-projective objects are exactly the $\mathcal{E}$-CEM-projective n-complexes. From their symmetry it follows that $\mathcal{E}$-CEM-resolutions are symmetric: We can reorder the differentials of an $\mathcal{E}$-CEM-resolution $Y \to X$ arbitrarily, and $Y \to X$ stays an $\mathcal{E}$-CEM-resolution.

Lemma 6.4 (Maps and homotopies between $\mathcal{E}$-CEM-resolutions)**.** *Let X and Y be n-complexes, and let $P \to X$ be an $\mathcal{E}$-CEM-projective resolution and $Q \to Y$ be a $\mathcal{E}$-CEM-acyclic resolution. Then any homomorphism $f : X \to Y$ admits an extension $P \to Q$ (as a homomorphism of $(n + 1)$-complexes). Any two such extensions are homotopic. More generally, if $F, G : P \to Q$ are extensions of two homotopic maps $f, g : X \to Y$, then F and G are homotopic.*

Therefore, $\mathcal{E}$-CEM resolutions can be regarded as a 'resolution functor' $\mathrm{hCh}^n(\mathcal{A}) \to \mathrm{hCh}^{n+1}(\mathcal{A})$ between the homotopy categories of $\mathrm{Ch}^n(\mathcal{A})$ and $\mathrm{Ch}^{n+1}(\mathcal{A})$, which is well-defined only up to natural isomorphisms.

Remark 6.5 (Stabilization)**.** In Lemma 6.7 and below we will regard and n-complex $X \in \mathrm{Ch}^n(\mathcal{A})$ also as an $(n+k)$-complex $\widetilde{X}$ that is concentrated in the first n coordinates. Thus, $\mathrm{Ch}^n(\mathcal{A})$ and $\mathrm{hCh}^n(\mathcal{A})$ are full subcategories of $\mathrm{Ch}^{n+k}(\mathcal{A})$ and $\mathrm{hCh}^{n+k}(\mathcal{A})$, respectively. In the same way, an $\mathcal{E}$-CEM_n resolution $Y \to X$ can be regarded as an $\mathcal{E}$-CEM_{n+k} resolution $\widetilde{Y} \to \widetilde{X}$ in $\mathrm{Ch}(\mathrm{Ch}^{n+k}(\mathcal{A}))$. Thus the resolution functor $\mathrm{hCh}^n(\mathcal{A}) \to \mathrm{hCh}^{n+1}(\mathcal{A}) \subseteq \mathrm{hCh}^{n+k+1}(\mathcal{A})$ coincides (again up to natural isomorphisms) with the restriction of the resolution functor $\mathrm{hCh}^{n+k}(\mathcal{A}) \to \mathrm{hCh}^{n+K+1}(\mathcal{A})$ to $\mathrm{hCh}^n(\mathcal{A})$.

In general however not every $\mathcal{E}$-CEM resolution of $\widetilde{X}$ is concentrated in the first n plus the resolution coordinates. For example take a canonical resolution $Y \to X$ (if they exist for $\mathcal{E}$), then $\widetilde{Y} \to \widetilde{X}$ is not the canonical resolution of $\widetilde{X}$, except if $X = 0$. In this sense, taking the canonical resolution is not a stable operation.

Remark 6.6 (Injective resolutions)**.** As usual one can dualize this and the next section and deal with injective classes of monomorphisms $\mathcal{M}$ instead of $\mathcal{E}$. We only remark that the corresponding class $\mathcal{M}$-CEM is then given by all maps of n-complexes $X \to Y$ such that the induced maps $Z'_{P,J}(X) \to Z'_{P,J}(Y)$ are $\mathcal{M}$-injective for all $P \in \mathbb{Z}^n$, $J \subseteq [n]$, where $Z'_{P,J}(X) := X_P / \sum_{j \in J} \mathrm{im}(d_j : X_{P+e_j} \to X_P)$. $\mathcal{M}$-CEM is indeed a closed injective class since $Z'_{P-\mathbb{1}_{[n]\setminus J},J} \dashv Q'_{P,J}$: $\mathrm{Hom}_{\mathcal{A}}(Z'_{P-\mathbb{1}_{[n]\setminus J},J}(X), K) \cong \mathrm{Hom}_{\mathrm{Ch}^n(\mathcal{A})}(X, Q_{P,J}(K))$, where $\mathbb{1}_{[n]\setminus J} := \sum_{j \in [n]\setminus J} e_j$.

If $\mathcal{E}$ is the class of all epimorphisms, then we will omit it from the notation. In this case, if $X \to A$ is a CEM-resolution, then so is $H_k(X, d_i) \to H_k(A, d_i)$ (and similarly with C_k, Z_k, B_k).

Iterated resolutions. It will be convenient to have a symbol for iterated resolutions: Let $D \in \mathrm{Ch}^{b_0}(\mathcal{B}_0)$, and let

$$G_i : \mathcal{B}_{i-1} \to \mathrm{Ch}^{b_i}(\mathcal{B}_i), \quad 1 \le i \le m, \tag{43}$$

be functors. We write

$$\mathrm{Res}(G_m, \ldots, G_1; D) \in \mathrm{Ch}^{m-1+\sum b_i}(\mathcal{B}_m)$$

to be a complex that results from taking the CEM-resolution of $G_1(D)$, applying G_2, resolving it again, applying G_3, and so on, until we applied G_m. Since the CEM-resolution depends on choices, so does $\mathrm{Res}(G_m \ldots, G_1; D)$. However a map $D \to D'$ of b_0-complexes induces a map between the chosen iterated resolutions $\mathrm{Res}(G_m \ldots, G_1; D) \to \mathrm{Res}(G_m \ldots, G_1; D')$, and homotopic maps induce homotopic maps.

$\mathrm{Res}(G_m \ldots, G_1; D)$ has $m-1$ differentials that come from CEM-resolutions, we denote them by $d_{r_2}, \ldots, d_{r_m}$.

Lemma 6.7 (Homology of Res). *Taking homology of* $\mathrm{Res}(G_m \ldots, G_1; D)$ *in direction* d_{r_i}*, for some* $2 \le i \le m$*, yields*

$$H(\mathrm{Res}(G_m \ldots, G_1; D), d_{r_i}) = \mathrm{Res}(G_m, \ldots, G_{i+1}, (LG_i) \circ G_{i-1}, G_{i-2}, \ldots, G_1; D).$$

Here, $LG_i : \mathrm{Ch}^{b_{i-1}}(\mathcal{B}_{i-1}) \to \mathrm{Ch}^{b_{i-1}+b_i+1}(\mathcal{B}_i)$ *denotes the left-derived functor of* G_i *with respect to CEM-resolutions.*

If G_{i-1} *sends projective objects to (*b_i*-complexes of)* G_i*-acyclic objects, then* $(LG_i) \circ G_{i-1}$ *simplifies to* $G_i \circ G_{i-1}$*, seen as a functor* $\mathcal{B}_{i-2} \to \mathrm{Ch}^{b_{i-1}+b_i+1}(\mathcal{B}_i)$ *that is concentrated where the last coordinate is zero.*

Proof. We have

$$\begin{aligned}
&H(\mathrm{Res}(G_m, \ldots, G_1; D), d_{r_i}) &&= \\
&H\big(\mathrm{Res}\big(G_m, \ldots, G_{i+1}, \mathrm{id}; \mathrm{Res}(G_i, G_{i-1}, \ldots, G_1; D)\big), d_{r_i}\big) &&= \\
&\mathrm{Res}\big(G_m, \ldots, G_{i+1}, \mathrm{id}; H(\mathrm{Res}(G_i, G_{i-1}, \ldots, G_1; D), d_{r_i})\big) &&= \\
&\mathrm{Res}\big(G_m, \ldots, G_{i+1}, \mathrm{id}; \mathrm{Res}((LG_i) \circ G_{i-1}, G_{i-2}, \ldots, G_1; D)\big) &&= \\
&\mathrm{Res}(G_m, \ldots, G_{i+1}, (LG_i) \circ G_{i-1}, G_{i-2}, \ldots, G_1; D).
\end{aligned}$$

The second equality uses that CEM-resolutions commute with taking homology in one of the complex directions and that CEM-injectives are sums of cube complexes by Lemma 6.1. ☐

We can repeat this lemma for different directions d_{r_i}. There are $(m-1)!$ ways to apply the lemma $m-1$ times successively.

6.2 The spectral system

Now suppose we are given a sequence of functors (42) as above, and let $X^0 \in \mathcal{A}_0$. We define

$$X := \mathrm{Res}(F_n, \ldots, F_0, X^0),$$

where $F_0 : \mathcal{A}_0 \to \mathcal{A}_0$ is the identity functor.

Let C be the total complex of X. X and C can be naturally $\mathbb{Z}$-filtered in n different ways: For $1 \le i \le n$ and $p \in \mathbb{Z}$, let $F_p^{(i)} := \bigoplus_{k \le p} (X, d_i)_k$. Thus from Section 3 we obtain a spectral system S.

Lemma 6.8 (Limit). *If* F_i *sends projective objects of* $\mathcal{A}_{i-1}$ *to* $F_{i,n}$*-acyclic objects for all* $1 \le i \le n-1$*, then the limit of* S *is*

$$S^{\infty,-\infty}_{\infty,-\infty} = H_*(C) = L_* F_{0,n}(X^0). \tag{44}$$

Proof. We apply the secondary connection from Section 3.2.2 to S with reversed coordinates, that is, we will first take differentials in direction d_n, then d_{n-1}, and so on. Let us denote the corresponding S-terms from Section 3.2.2 by $\bar{S}^{pz}_{bq}(P; k)$. We have

$\bar{S}^{pz}_{bq}(P;0) = \bar{S}^{pz}_{bq}(P;1) = X_P = \mathrm{Res}(F_n, \ldots, F_0, X^0)_P$ for $P \in \mathbb{Z}^n$. Note that the graded pieces of $\mathcal{E}_n(\mathcal{A}_i)$-projective objects are projective objects in $\mathcal{A}_i$, and F_{n-1} sends projective objects of A_{n-2} to F_n-acyclic objects in $\mathcal{A}_{n-1}$. Thus taking differentials in direction d_n and using Lemmas 3.15, 3.14, and 6.7 yields

$$\bar{S}^{pz}_{bq}(P;2) = \mathrm{Res}(F_n \circ F_{n-1}, F_{n-2}, \ldots, F_0; X^0)_P.$$

Iterating this we get

$$\bar{S}^{pz}_{bq}(P;k) = \mathrm{Res}(F_{n-k,n}, F_{n-k}, \ldots, F_0; X^0)_P$$

for $k = 1, \ldots, n$, using that F_{n-k} sends projective objects of $\mathcal{A}_{n-k-1}$ to $F_{n-k,n}$-acyclic objects. Taking homology with respect to the last differential in direction d_1 yields

$$\bar{S}^{pz^*}_{b^*q}(P;n) = \mathrm{Res}((LF_{0,n}) \circ F_0; X^0),$$

which at positions $P \in \mathbb{Z}_{\geq 0} \times \{0\}^{n-1}$ is $L_P F_{0,n}(X^0)$, and zero otherwise. These non-zero S-terms are all homogeneous in different total degree given by p_1. Thus when we proceed with the lexicographic connection, the result (44) follows. There is no problem with limits, since X has only finitely many non-zero graded pieces in every degree. □

Lemma 6.9 (Second page)**.** *The second page* (20) *of S at position P is*

$$S^{pz^*}_{b^*q}(P;n) = (L_{p_n} F_n) \circ \ldots \circ (L_{p_1} F_1)(X^0). \tag{45}$$

Proof. The secondary connection starts with

$$S^{pz}_{bq}(P;0) = S^{pz}_{bq}(P;1) = X_P = \mathrm{Res}(F_n, \ldots, F_0, X^0)_P.$$

By Lemma 6.7, taking homology in direction d_1 yields

$$S^{pz^*}_{b^*q}(P;1) = \mathrm{Res}(F_n, \ldots, F_2, LF_1; X^0)_P. \tag{46}$$

Iterating this by taking homology in directions d_2, d_3, and so on, we obtain

$$S^{pz^*}_{b^*q}(P;k) = \mathrm{Res}(F_n, \ldots, F_{k+1}, (L_* F_k) \circ \ldots \circ (L_* F_1); X^0)_P,$$

for $k = 1, \ldots, n$. For $k = n$ we obtain the second page. □

There are many more second pages given by permuting the coordinates of X.

Remark 6.10 (Naturality)**.** Lemma 6.7 yields explicit formulas for these $n!$ second pages in terms of left-derived functors only. Thus the second pages do not depend anymore on the particular choice of the CEM-resolutions that we made in order to define X.

Any morphism $f_0 : X^0 \to (X^0)'$ in $\mathcal{A}_0$ induces a well-defined and natural morphism between second pages (45) of the spectral systems for X^0 and $(X^0)'$ by Lemma 6.4. Hence all S-terms that follow the respective second page by the lexicographic connections are well-defined, and morphisms f_0 induce natural morphisms between these S-terms.

6.3 Product structure

Constructing a product structure for the Grothendieck spectral sequence is not a completely trivial matter, even if the second page has a natural one.

In this section we build upon the work of Swan [Swa99]. He constructs a product structure on the Leray–spectral sequence $E_2^{pq} = H^p(Y; R^q f_*(S)) \Rightarrow H^{p+q}(X; S)$ (where $f : X \to Y$ is continuous and S is a sheaf of rings on X), by regarding it as a hypercohomology spectral sequence for $\mathrm{Ext}(\underline{\mathbb{Z}}, f_* I^*)$, where I^* is an injective resolution of S and $\underline{\mathbb{Z}}$ is the constant $\mathbb{Z}$-sheaf. He obtains a product structure using a final system of flat resolutions of $\underline{\mathbb{Z}}$. At the end of [Swa99, Sect. 3], he remarks that a suitable pure injective Cartan–Eilenberg resolution of $f_*(I)$ would suffice as well if one could construct it. That is exactly what we will do. (Note that in an earlier paper [Swa96] Swan defined a version of pure injective CE-resolutions, which differs from ours for $n = 2$.)

For this section, we consider the sequence of additive functors (42), this time with the following extra structure: On the abelian categories $\mathcal{A}_0, \ldots, \mathcal{A}_n$ we have biadditive functors $\otimes : \mathcal{A}_i \times \mathcal{A}_i \to \mathcal{A}_i$ with natural isomorphisms $X \otimes Y \cong Y \otimes X$ and $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$ and a right adjoint biadditive functor $\mathrm{hom} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that $\mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X, \mathrm{hom}(Y, Z))$. Furthermore we require that $\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}$ (all but $\mathcal{A}_n$) are Baer–Grothendieck categories, that is, they have generators and filtered colimits exist and are exact. By Grothendieck [Gro57, Thm. 1.10.1], $\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}$ have enough injectives. Further we require that the functors F_i commute with $\otimes$, that is, there are natural maps

$$F_i(X) \otimes F_i(Y) \to F_i(X \otimes Y), \quad X, Y \in \mathcal{A}_{i-1},\ 0 \le i \le n-1. \tag{47}$$

Note that $\otimes : \mathcal{A}_i \times \mathcal{A}_i \to \mathcal{A}_i$ extends naturally to a biadditive functor $\mathrm{Ch}^k(\mathcal{A}_i) \otimes \mathrm{Ch}^k(\mathcal{A}_i) \to \mathrm{Ch}^k(\mathcal{A}_i)$, which we also denote by $\otimes$.

Remark 6.11. More generally, we could start with three sequences of functors $\mathcal{A}_0 \xrightarrow{F_1} \ldots \xrightarrow{F_n} \mathcal{A}_n$, $\mathcal{A}_0 \xrightarrow{F_1'} \ldots \xrightarrow{F_n'} \mathcal{A}_n$, and $\mathcal{A}_0 \xrightarrow{F_1''} \ldots \xrightarrow{F_n''} \mathcal{A}_n$, together with natural maps $F_i(X) \otimes F_i'(Y) \to F_i''(X \otimes Y)$, $X, Y \in \mathcal{A}_{i-1}$, $0 \le i \le n-1$. As above, this yields a pairing between the first and the second spectral system to the third one.

A monomorphism $X \to Y$ in $\mathcal{A}_i$ is called pure if $X \otimes Z \to Y \otimes Z$ is a monomorphism for all $Z \in \mathcal{A}_i$. Let $\mathcal{M}_i$ be the class of all pure monomorphisms in $\mathcal{A}_i$. From [Swa99, Thm. 2.1] it follows that the $\mathcal{M}_i$ are injective classes for $0 \le i \le n-1$, that is, $\mathcal{A}_i$ has enough pure injectives.

Let $X, Y \in \mathrm{Ch}^k(\mathcal{A}_i)$ for some $0 \le i \le n-1$. We regard $X \otimes Y$ also as an k-complex over $\mathcal{A}_i$. Choose $\mathcal{M}_i$-CEM-injective resolutions $X \to I$, $Y \to J$ and $X \otimes Y \to K$. We regard $I \otimes J$ as a sequence of k-complexes (these k-complexes are in general not injective anymore), together with a map $X \otimes Y \to I \otimes J$. As in [Swa99, Sect. 2, 3] one obtains:

Lemma 6.12. *$X \otimes Y \to I \otimes J$ is an $\mathcal{M}_i$-CEM-acyclic resolution.*

Thus we can choose a map $I \otimes J \to K$ over $\mathrm{id}_{X \otimes Y}$. Applying F_{i+1} and precomposing with (47) yields $F(I) \otimes F(J) \to F(I \otimes J) \to F(K)$. Taking homology yields

$$(R_{\mathcal{M}_i\text{-CEM}} F_{i+1})(X) \otimes (R_{\mathcal{M}_i\text{-CEM}} F_{i+1})(Y) \to (R_{\mathcal{M}_i\text{-CEM}} F_{i+1})(X \otimes Y).$$

If furthermore we are given a map of k-complexes $X \otimes Y \to Z$, we can take an $\mathcal{M}_i$-CEM-resolution $Z \to L$, choose an extension $K \to L$, and get $F_{i+1}(K) \to F_{i+1}(L)$ over $F_{i+1}(X \otimes Y) \to F_{i+1}(Z)$. This induces a cup product

$$(R_{\mathcal{M}_i\text{-CEM}} F_{i+1})(X) \otimes (R_{\mathcal{M}_i\text{-CEM}} F_{i+1})(Y) \to (R_{\mathcal{M}_i\text{-CEM}} F_{i+1})(Z).$$

By Lemma 6.4 it is well-defined, and homotopic maps $X \otimes Y \to Z$ define the same cup product.

More generally, suppose we are given a sequence of functors (43) with the analog extra structure: $\mathcal{B}_0, \ldots, \mathcal{B}_{m-1}$ are Baer–Grothendieck categories, $\mathcal{B}_0, \ldots, \mathcal{B}_m$ have $\otimes$ and hom, and the functors G_i preserve $\otimes$.

Suppose we are given $X, Y, Z \in \mathrm{Ch}^{b_0}(\mathcal{B}_0)$ and a map $X \otimes Y \to Z$ of b_0-complexes (which might be given only up to homotopy). Iterating the above procedure, we obtain a map of $(m - 1 + \sum b_i)$-complexes,

$$\mathrm{Res}_{\mathcal{M}}(G_m, \ldots, G_1; X) \otimes \mathrm{Res}_{\mathcal{M}}(G_m, \ldots, G_1; Y) \to \mathrm{Res}_{\mathcal{M}}(G_m, \ldots, G_1; Z), \tag{48}$$

which is well-defined up to homotopy, where the subscript $\mathcal{M}$ means that we always take $\mathcal{M}_i$-CEM-injective resolutions. Therefore we obtain a multiplicative structure on the associated spectral systems over $I = D(\overline{\mathbb{Z}}^n)$ mas in Example 4.28 (note that here the n-complexes are concentrated in the negative quadrant). We still need to discuss when this induces a multiplication on the spectral system S from the previous section.

If $\mathcal{M} \subseteq \mathcal{M}'$ are two injective classes of monomorphisms in an abelian category $\mathcal{A}$, then $\mathcal{M}$-injective resolutions are $\mathcal{M}'$-acyclic resolutions. Thus if $X \to I$ and $X \to I'$ are an $\mathcal{M}$- and an $\mathcal{M}'$-injective resolution of X, we obtain a map $I \to I'$ over id_X, which is natural up to homotopy. Iterating this, we obtain a map

$$\mathrm{Res}_M(G_m, \ldots, G_1; X) \to \mathrm{Res}(G_m, \ldots, G_1; X), \tag{49}$$

and analogous maps for Y and Z. We want to compose them with (48) in order to obtain a product structure on the spectral system from the previous section, however the maps for X and Y go in the wrong direction.

Let S denote (as in the previous section) the spectral system for $\mathrm{Res}(G_m, \ldots, G_1; X)$, and $S_{\mathcal{M}}$ the spectral system for $\mathrm{Res}_{\mathcal{M}}(G_m, \ldots, G_1; X)$. Then (49) induces maps $S_{\mathcal{M}} \to S$.

Lemma 6.13. *Suppose that all pure injective objects in $\mathcal{A}_{i-1}$ are F_i-acyclic for all $1 \leq i \leq n$. Then* (49) *induces an isomorphism between the second pages of $S_{\mathcal{M}}$ and S, both of which are thus given by* (45)*.*

Proof. The proof idea of Lemma 6.9 works here as well using the following extra facts: Any $\mathcal{M}_{i-1}$-CEM-injective resolutions $X \to D$ of any complex $D \in \mathrm{Ch}^n(\mathcal{A}_{i-1})$ is a CEM-acyclic resolution of D with F_i-acyclic entries. Let $1 \leq j \leq n$. Using the exact sequences $0 \to Z_k(\_, d_j) \to C_k(\_, d_j) \to B_{k-1}(\_, d_j) \to 0$ and $0 \to B_k(\_, d_j) \to Z_k(\_, d_j) \to H_k(\_, d_j) \to 0$, we deduce using induced long exact sequences (as with usual Cartan–Eilenberg resolutions) that $H(X, d_j) \to H(D, d_j)$ is also a CEM-acyclic resolution, whose entries are again F_i-acyclic by the special form of $\mathcal{M}_i$-injectives. And as usual, LF_i can be constructed from F_i-acyclic resolutions. □

Corollary 6.14. *If all pure injective objects in $\mathcal{A}_{i-1}$ are F_i-acyclic for all $1 \leq i \leq n$, then the product structure on $S_{\mathcal{M}}$ induces a product structure on S on the second page and all pages following it via the lexicographic connections.*

Example 6.15 (Successive Leray spectral sequences)**.** By Swan [Swa99, Lemma 3.6], in *Sh*(X) all pure injective sheaves are flasque, and flasque sheaves are f_*-acyclic for any $f : X \to Y$. Thus, Lemma 6.13 implies that the Leray spectral system from Section 5.5 has a product structure from the second page on.

7 Adams–Novikov and chromatic spectral sequences

The chromatic spectral sequence converges to the second page of the Adams–Novikov spectral sequence, which in turn converges to the p-component of the stable homotopy groups of spheres $\pi_*(S^0)$. For background see Ravenel [Rav86]. In this section we show how to unify these two spectral sequences into one spectral system over $D(\overline{\mathbb{Z}}^2)$.

Let BP be the Brown–Peterson spectrum, $\Gamma := BP_*(BP)$, and let $\mathrm{Ext}(\_) := \mathrm{Ext}_\Gamma(BP_*, \_)$. Miller, Ravenel, and Wilson [MRW77] constructed the *chromatic resolution*, which is a long exact sequence of $BP_*(BP)$-comodules,

$$0 \to BP_* \to M^0 \to M^1 \to \dots \tag{50}$$

Ravenel [Rav87] realized the chromatic resolution (50) geometrically in the following sense (note that our N_n is Ravenel's $\Sigma^{-n}N_n$): There are spectra N_n and M_n, $n \geq 0$, such that $M^n = BP_*(M_n)$, $N_0 := S^0$, and there are fibrations $N_{n+1} \to N_n \to \Sigma^{-n}M_n$ that induce short exact sequences $0 \to BP_*(N_n) \to BP_*(\Sigma^{-n}M_n) \to BP_*(\Sigma N_{n-1}) \to 0$. Thus this gives a diagram

$$\begin{tikzcd} S^0 = N_0 \arrow[d] & N_1 \arrow[l] \arrow[d] & N_2 \arrow[l] \arrow[d] & \dots \arrow[l] \\ M_0 & \Sigma^{-1}M_1 & \Sigma^{-2}M_2. & \end{tikzcd} \tag{51}$$

Let S^0 be the sphere spectrum. Let

$$\begin{tikzcd} S^0 = X_0 \arrow[d] & X_1 \arrow[l] \arrow[d] & X_2 \arrow[l] \arrow[d] & \dots \arrow[l] \\ I_0 & \Sigma^{-1}I_1 & \Sigma^{-2}I_2, & \end{tikzcd} \tag{52}$$

be the canonical BP-Adams resolution of S^0. That is,

$$\Sigma^{-n} I_n := BP \wedge X_n$$

and X_{n+1} is defined as the fiber of $X_n \to \Sigma^{-n} I_n$. We obtain the canonical BP-Adams resolution of any spectrum X by smashing (52) with X. As usual we obtain short exact sequences

$$0 \to BP_*(X \wedge X_n) \to BP_*(X \wedge \Sigma^{-n} I_n) \to BP_*(X \wedge \Sigma X_{n+1}) \to 0 \tag{53}$$

that splice together to long exact sequences of Γ-comodules

$$0 \to BP_*(X) \to BP_*(X \wedge I_0) \to BP_*(X \wedge I_1) \to \dots, \tag{54}$$

which is an Ext-acyclic Γ-resolution of $BP_*(X)$, since

$$\mathrm{Ext}^t(BP_*(X \wedge I_n)) = \begin{cases} \pi_*(X \wedge I_n), & \text{if } t = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{55}$$

.

The Adams–Novikov spectral sequence (Adams [Ada64], Novikov [Nov67]) for a spectrum X is derived from the exact couple given by

$$\begin{tikzcd} \pi_*(X \wedge X_n) \arrow[r] & \pi_*(X \wedge \Sigma^{-n} I_n) \arrow[r] & \pi_*(X \wedge \Sigma X_{n+1}) \arrow[ll, bend left, "\mathrm{deg}=-1"] \end{tikzcd}$$

and thus the E_1-page consists of terms

$$E_1^{n,*}(X, BP_*) := \pi_*(X \wedge \Sigma^{-n} I_n) = \mathrm{Ext}^0(BP_*(X \wedge \Sigma^{-n} I_n)).$$

From the resolution (54) and (55) it follows that the E_2-page is given by

$$E_2^{n,*}(X, BP_*) := \mathrm{Ext}^*(BP_*(X)).$$

Following Miller [Mil81, Section 5], we smash (51) with (52) in order to construct a "double complex of spectra": First by a telescope argument we may assume that $X_0 \supseteq X_1 \supseteq \dots$ and $N_0 \supseteq N_1 \supseteq \dots$ are decreasing filtrations of CW spectra, $X_n/X_{n+1} = \Sigma^{-n} I_n$, and $N_n/N_{n+1} = \Sigma^{-n} M_n$. We extend these definitions to negative indices simply by $X_n := X_0$ and $N_n := N_0$ for $n \leq 0$. Let $I = U(\mathbb{Z}^2)$ be the poset of upsets in $\mathbb{Z}^2$, that is, the set of all subset $p \subset \mathbb{Z}^2$ with the property: if $(a, b) \leq (c, d) \in \mathbb{Z}^2$ and $(a, b) \in p$ then $(c, d) \in p$. In I we have $-\infty = \emptyset$ and $\infty = \mathbb{Z}^2$. We define a filtration of $Z := X_0 \wedge N_0 \cong S^0$ by

$$Z_p := \bigcup_{(i,j) \in p} X_i \wedge N_j, \quad p \in I,$$

with $Z_{-\infty} = \mathrm{pt}$ and $Z_\infty = Z$. From that define an exact couple system E over I by

$$E_q^p := \pi_*(Z_p/Z_q) \quad \text{and} \quad D_p := \pi_*(Z_p).$$

Lemma 7.1. *The associated spectral system has the following properties.*

1. *Let* $(p,q) \in I_2$. *If* $p \backslash q = \{(i,j)\}$ *with* $i,j \geq 0$ *then*
$$E^p_q = \pi_*(\Sigma^{-i} I_i \wedge \Sigma^{-j} M_j) = \operatorname{Ext}^0(BP_*(\Sigma^{-i} I_i \wedge \Sigma^{-j} M_j)). \tag{56}$$

2. *Taking homology of* (56) *in direction* d_2 *yields*
$$S^{pz}_{bq} = \operatorname{Ext}^i(BP_*(\Sigma^{-j} M_j)) = \operatorname{Ext}^i(M^j[-j]).$$
whenever $b \backslash p = \{(i-1,j)\}$, $p \backslash q = \{(i,j)\}$, $q \backslash z = \{(i+1,j)\}$.

3. *Let* $f_n := \{(i,j) \mid i+j \geq n\} \in I$. *Then*
$$E^{f_n}_{f_{n+1}} = \bigoplus_{\substack{i,j \geq 0,\\ i+j=n}} \pi_*(\Sigma^{-i} I_i \wedge \Sigma^{-j} M_j) \cong E_1^{n,*}(S^0, BP_*) = \operatorname{Ext}^0(BP_*(\Sigma^{-n} I_n)). \tag{57}$$

4. *Taking homology of* (57) *in the only possible anti-diagonal direction yields*
$$S^{f_n,f_{n+2}}_{f_{n-1},f_{n+1}} \cong E_2^{n,*}(S^0, BP_*) = \operatorname{Ext}^n(BP_*).$$

5. $S^{\infty,-\infty}_{\infty,-\infty} = \pi_*(S^0)$.

More generally we can smash Z and all Z_p with a spectrum X and obtain a spectral system with limit $S^{\infty,-\infty}_{\infty,-\infty} = \pi_*(X)$. Note that this limit does not imply anything about convergence in the usual sense (e.g. the sub spectral system given by the f_n's coincides with the ordinary Adams–Novikov spectral sequence and in general it does not converge to $\pi_*(X)$, see Ravenel [Rav86, Thm. 4.4.1.(b)]).

Proof. 3.) follows from the fact that one can replace $\Sigma^{-n} I_n$ by Z_n/Z_{n+1} in (52) and X_n by Z_n and obtains another (non-standard) BP-Adams resolution of S^0. The natural inclusion from the standard resolution (52) into the new one induces the claimed isomorphism. □

Analogously to the above construction, we can also unify Miller's generalizations of May's and Mahowald's spectral sequences [Mil81] into a spectral system. This follows closely his construction in [Mil81, Section 5].

8 Successive Eilenberg–Moore spectral sequences

Eilenberg and Moore [EM66] constructed a second quadrant cohomology spectral sequence for the following setting. Suppose we are given a pullback diagram of fibrations,

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f_1'"'] & E_1 \arrow[d, "f_1"] \\
E_2 \arrow[r, "f_2"] & B,
\end{tikzcd}$$

f_1' being the pullback of the fibration f_1 along f_2. Let $H(\_) := H^*(\_; k)$ denote singular cohomology with coefficients in some field k. In this section we assume that all spaces have the homotopy type of a CW-complex and that $H(B)$ and all $H(E_i)$ are finite dimensional in every dimension. For any map $f : X \to Y$, $H(Y)$ acts on $H(X)$ via f^*. Assume that $\pi_1(B_0)$ acts trivially on $H(E_0)$. Then there is an Eilenberg–Moore spectral sequence,

$$E_2^{p,q} = \mathrm{Tor}^{p,q}_{H(B)}(H(E_1), H(E_2)) \;\Rightarrow\; H(X).$$

The index $p \leq 0$ in $\mathrm{Tor}^{p,q}_{H(B)}$ is the resolution index and $q \geq 0$ is the grading index. Smith [Smi70] and Hodgkin [Hod75] constructed a geometric Eilenberg–Moore spectral sequence, see also Hatcher [Hat04, Chapter 3]. (So far it is still open whether the geometric construction yields a spectral sequence that is isomorphic to the original one, which Eilenberg and Moore defined algebraically.) We will follow and extend their approach to the following setting.

8.1 A cube of pullbacks

Suppose we are given n fibrations $f_i : E_i \to B$, $1 \leq 1 \leq n$. For any subset $I \subseteq [n]$, let X_I be the pullback of the maps $\{f_i \mid i \in I\}$. Thus, $E_i = X_{\{i\}}$ and $B = X_\emptyset$. Let $X := X_{[n]}$. For $I \subseteq J \subseteq [n]$, $X_J \to X_I$ is a fibration. Assume that $\pi(B) = 0$ (as usual, this assumption can be weakened).

We will construct a spectral system over $D(\mathbb{Z}^n)$ with limit

$$S^{\infty,-\infty}_{\infty,-\infty} = H(E_n)$$

and second page $S^{pz^*}_{b^*q}((p_1, \ldots, p_n); n) =$

$$\mathrm{Tor}^{p_n}_{HB}\Big(\ldots \mathrm{Tor}^{p_2}_{HB}\big(\mathrm{Tor}^{p_1}_{HB}(HB, HE_1), HE_2\big) \ldots, HE_n\Big).$$

Here, $\mathrm{Tor}^{p_1}_{HB}(HB, HE_1)$ is equal to HE_1 if $p_1 = 0$ and zero otherwise. Actually there are $n!$ different second pages, one for every total order of the n coordinates of $\mathbb{Z}^n$ and the order of the E_i, which appear in the same spectral system.

8.2 The spectral system

For a fixed topological space B, let Top/B denote the category of spaces over B, whose objects are $p_X : X \to B$, and morphisms are the obvious commutative triangles. Let Top_B (Smith writes $(Top/B)_*$ instead) denote the category of ex-spaces over B, whose objects are spaces X together with a map $p_X : X \to B$ and a section $s_X : B \to X$ such that $p_X \circ s_X = \mathrm{id}_B$, and morphisms are the obvious commutative double triangles. We have a functor $Top/B \to Top_B$ that sends X to $X_+ := X \sqcup B$. Define $S^n_B := B \times S^n \in Top_B$, where the section $B \to S^n_B$ is coming from picking a basepoint of S^n. Products $X \times_B Y$, smash products $X \wedge_B Y$, suspensions $\Sigma_B X = S^n_B \wedge_B X$, homotopies $\alpha \sim_B \beta$, mapping cylinders $M_B(\alpha)$, mapping cones $C_B(\alpha)$, and quotients $X/_BY$ are all defined fiberwise; see Smith [Smi70].

In order to simplify indices quite a bit, we move into a naive category of spectra over B, Sp_B, whose objects are sequences $(X_n)_{n\in\mathbb{Z}}$ of ex-spaces over B together with maps $\Sigma_B X_n \to X_{n+1}$ over B. There is a functor $\Sigma_B^\infty : Top_B \to Sp_B$ that sends X to $X_n := \Sigma_B^n X$ for $n \geq 0$, and $X_n := B$ for $n < 0$. Let $S_B := \Sigma_B^\infty(S_B^0)$. This naive definition is all we need; for a thorough treatment of parametrized spectra see May–Sigurdsson [MS06, Section 2]. A reader who prefers spaces can proceed as in Smith [Smi70].

We construct a diagrams in Sp_B for all $1 \leq k \leq n$, similar to the Adams resolution,

$$\begin{tikzcd}
\Sigma_B^\infty(E_{k+}) =: E_k^0 \arrow[d] & E_k^{-1} \arrow[l] \arrow[d] & E_k^{-2} \arrow[l] \arrow[d] & \dots \arrow[l] \\
I_k^0 & \Sigma^{-1} I_k^{-1} & \Sigma^{-2} I_k^{-2},
\end{tikzcd} \tag{58}$$

where $\Sigma^{-i} I_k^{-i}$ is defined as $S_B \wedge_B E_k^{-i}$, and ΣE_k^{-i-1} is defined as the cofiber of $E_k^{-i} \to \Sigma^{-i} I_k^{-i}$.

Let $H_B(\_) : Sp_B \to GrAb$ denote cohomology relative to $\Sigma_B^\infty(B)$ with coefficients in k. Since $H_B(\Sigma^{-i} I_k^{-i}) = H_B(S_B) \otimes_k H_B(E_k^{-i})$, $H_B(\Sigma^{-i} I_k^{-i})$ is a free $H_B(S_B)$-module and $H_B(\Sigma^{-i} I_k^{-i}) \to H_B(E_k^{-i})$ is surjective. Thus the associated long exact sequences in homology become short exact sequences

$$0 \to H_B(\Sigma E_k^{-i-1}) \to H_B(\Sigma^{-i} I_k^{-i}) \to H_B(E_k^{-i}) \to 0$$

They splice together to a free $H_B(S_B)$-resolution of $H_B(\Sigma_B^\infty(E_{k+}))$,

$$\dots \to H(I_k^2) \to H(I_k^1) \to H(I_k^0) \to H(\Sigma_B^\infty(E_{k+})) \to 0.$$

For $i > 0$ we define $E_k^i := E_k^0$ and correspondingly $I_k^i := \Sigma_B^\infty(B)$. As in Section 7, following Miller [Mil81, Section 5], we can first assume by a telescope argument that the horizontal maps in (58) are inclusions of spectra over B, and I_k^{-i} the corresponding quotients. Then we smash the n diagrams (58): For $p \in I := D(\mathbb{Z}^n)$ define

$$Z_p := \bigcup_{P\in p} E_1^{p_1} \wedge_B \dots \wedge_B E_n^{p_n},$$

which gives an I-filtration of $Z := E_1^0 \wedge_B \dots \wedge_B E_n^0$. Let S be the exact system associated to this filtered spectrum and the cohomology theory H_B, as in Example 4.5. In the notation of Section 3.2.2, we have

$$S_{bq}^{pz}(P;1) = H_B(I_1^{p_1} \wedge_B \dots \wedge_B I_n^{p_n}).$$

Since all $H_B(I_i^{p_i})$ are of finite type (that is, finite dimensional in every dimension) and projective over $H_B(S_B)$, it follows as in [Smi70, Prop. 5.1] that

$$S_{bq}^{pz}(P;0) = H_B(I_1^{p_1}) \otimes_{H_B(S_B)} \dots \otimes_{H_B(S_B)} H_B(I_n^{p_n}).$$

Since the functors $\_ \otimes_{H_B(S_B)} H_B(I^{p_k}_k)$ are exact, taking homology with respect to the differential in direction $-e_1$ yields

$$S^{pz^*}_{b^*q}(P;1) = H_B(E^0_1) \otimes_{H_B(S_B)} H_B(I^{p_2}_2) \otimes_{H_B(S_B)} \dots \otimes_{H_B(S_B)} H_B(I^{p_n}_n)$$

if $p_1 = 0$ and zero otherwise. Continuing taking homology with respect to differentials in direction $-e_2$,…,$-e_n$ yields eventually

$$S^{pz^*}_{b^*q}(P;n) = \mathrm{Tor}^{p_2}_{H_B(S_B)}\Big(\dots \mathrm{Tor}^{p_2}_{H_B(S_B)}\big(H_B(E^0_1), H_B(E^0_2)\big) \dots, H_B(E^0_n)\Big)$$

if $p_1 = 0$ and zero otherwise.

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References

[Ada64] J. Frank Adams. *Stable homotopy theory*, volume 3 of *Lecture Notes in Mathematics*. Springer-Verlag, 1964.

[BBD82] Aleksandr A. Beĭlinson, Joseph Bernstein, and Pierre Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, 1982.

[Beh06] Mark Behrens. Root invariants in the Adams spectral sequence. *Trans. Amer. Math. Soc.*, 358(10):4279–4341, 2006.

[Beh12] Mark Behrens. The Goodwillie tower and the EHP sequence. *Mem. Amer. Math. Soc.*, 218(1026):xii+90, 2012.

[Boa99] Michael J. Boardman. Conditionally convergent spectral sequences. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 49–84. Amer. Math. Soc., Providence, RI, 1999.

[Bro59] Edgar H. Brown, Jr. Twisted tensor products, I. *Ann. of Math. (2)*, 69:223–246, 1959.

[BT82] Raul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, 1982.

[CE56] Henri Cartan and Samuel Eilenberg. *Homological algebra*. Princeton University Press, 1956.

[Del71] Pierre Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, 40:5–57, 1971.

[Dol62] Albrecht Dold. Relations between ordinary and extraordinary homology. In *Colloq. algebr. Topology*, pages 2–9. Aarhus Universitet, 1962.

[Dre67] Andreas W. M. Dress. Zur Spectralsequenz von Faserungen. *Invent. Math.*, 3:172–178, 1967.

[EM62] Samuel Eilenberg and John C. Moore. Limits and spectral sequences. *Topology*, 1:1–23, 1962.

[EM65] Samuel Eilenberg and John C. Moore. Foundations of relative homological algebra. *Mem. Amer. Math. Soc. No.*, 55:39, 1965.

[EM66] Samuel Eilenberg and John C. Moore. Homology and fibrations. I. Coalgebras, cotensor product and its derived functors. *Comment. Math. Helv.*, 40:199–236, 1966.

[EZ53] Samuel Eilenberg and Joseph A. Zilber. On products of complexes. *Amer. J. Math.*, 75:200–204, 1953.

[FH58] Edward R. Fadell and Witold Hurewicz. On the structure of higher differential operators in spectral sequences. *Ann. of Math. (2)*, 68:314–347, 1958.

[GM99] Mikhail Grinberg and Robert D. MacPherson. Euler characteristics and Lagrangian intersections. In *Symplectic geometry and topology (Park City, UT, 1997)*, volume 7 of *IAS/Park City Math. Ser.*, pages 265–293. Amer. Math. Soc., 1999.

[GM03] Sergei I. Gelfand and Yuri I. Manin. *Methods of homological algebra*. Springer Monographs in Mathematics. Springer-Verlag, second edition, 2003.

[Gro57] Alexander Grothendieck. Sur quelques points d'algèbre homologique. *Tôhoku Math. J. (2)*, 9:119–221, 1957.

[Hat04] Allen Hatcher. Spectral sequences in algebraic topology. `http://www.math.cornell.edu/~hatcher/SSAT/SSATpage.html`, 2004.

[Hod75] Luke Hodgkin. The equivariant Künneth theorem in K-theory. In *Topics in K-theory. Two independent contributions*, pages 1–101. Lecture Notes in Math., Vol. 496. Springer, 1975.

[HS97] Peter J. Hilton and Urs Stammbach. *A course in homological algebra*, volume 4 of *Graduate Texts in Mathematics*. Springer, second edition, 1997.

[Hu98] Po Hu. *The cobordism of Real manifolds and calculations with the Real Adams-Novikov spectral sequence*. PhD thesis, University of Michigan, 1998.

[Hu99] Po Hu. Transfinite spectral sequences. In *Homotopy invariant algebraic structures*, volume 239 of *Contemp. Math.*, pages 197–216. Amer. Math. Soc., Providence, RI, 1999.

[Ler46] Jean Leray. Sur la forme des espaces topologiques et sur les point fixes des répresentations; sur la position d'in ensemble fermé de points d'in espace topologique; sur les équations et les transformations. *J. Math. Pures Appl. (9)*, 24:95–167, 169–199, 201–248, 1946.

[Mas52] William S. Massey. Exact couples in algebraic topology. I, II. *Ann. of Math. (2)*, 56:363–396, 1952.

[Mas53] William S. Massey. Exact couples in algebraic topology. III, IV, V. *Ann. of Math. (2)*, 57:248–286, 1953.

[Mas54] William S. Massey. Products in exact couples. *Ann. of Math. (2)*, 59:558–569, 1954.

[Mat13] Benjamin Matschke. A parameterized version of Gromov's waist of the sphere theorem. In preparation, 2013.

[McC99] John McCleary. A history of spectral sequences: origins to 1953. In *History of topology*, pages 631–663. North-Holland, 1999.

[McC01] John McCleary. *A User's Guide to Spectral Sequences*, volume 58. Cambridge University Press, second edition edition, 2001.

[Mil81] Haynes R. Miller. On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space. *J. Pure Appl. Algebra*, 20(3):287–312, 1981.

[MRW77] Haynes R. Miller, Douglas C. Ravenel, and W. Stephen Wilson. Periodic phenomena in the Adams-Novikov spectral sequence. *Ann. of Math. (2)*, 106(3):469–516, 1977.

[MS06] J. Peter May and Johann Sigurdsson. *Parametrized homotopy theory*, volume 132 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2006.

[Nov67] Sergei P. Novikov. Methods of algebraic topology from the point of view of cobordism theory. *Izv. Akad. Nauk SSSR Ser. Mat.*, 31:855–951, 1967.

[Rav86] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press Inc., 1986.

[Rav87] Douglas C. Ravenel. The geometric realization of the chromatic resolution. In *Algebraic topology and algebraic K-theory*, volume 113 of *Ann. of Math. Stud.*, pages 168–179. Princeton Univ. Press, 1987.

[Ser51] Jean-Pierre Serre. Homologie singulière des espaces fibrés. Applications. *Ann. of Math. (2)*, 54:425–505, 1951.

[Smi70] Larry Smith. *Lectures on the Eilenberg-Moore spectral sequence*. Lecture Notes in Mathematics, Vol. 134. Springer-Verlag, 1970.

[Spa66] Edwin H. Spanier. *Algebraic topology*. Springer-Verlag, 1966. Corrected reprint 1981.

[Swa96] Richard G. Swan. Hochschild cohomology of quasiprojective schemes. *J. Pure Appl. Algebra*, 110(1):57–80, 1996.

[Swa99] Richard G. Swan. Cup products in sheaf cohomology, pure injectives, and a substitute for projective resolutions. *J. Pure Appl. Algebra*, 144(2):169–211, 1999.

[Swi75] Robert M. Switzer. *Algebraic topology – homotopy and homology*. Springer-Verlag, 1975. Die Grundlehren der mathematischen Wissenschaften, Band 212.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1994.