A survey on the Square Peg Problem

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Abstract

This is a short survey article on the 100 years old Square Peg Problem of Toeplitz, which is also called the Inscribed Square Problem. It asks whether every *continuous* simple closed curve in the plane contains the four vertices of a square.

1 Introduction

In 1911 Otto Toeplitz [Toe11] furmulated the following conjecture.

Conjecture 1.1 (Square Peg Problem, Toeplitz)**.** *Every continuous simple closed curve* $\gamma : S^1 \to \mathbb{R}^2$ *contains four points that are the vertices of a square.*

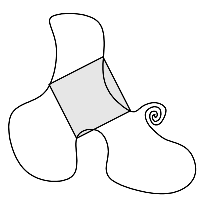

Figure 1: Example for Conjecture 1.1.

A continuous simple closed curve in the plane is also often called a Jordan curve, and it is the same as an injective map from the unit circle into the plane, or equivalently, an embedding $S^1 \hookrightarrow \mathbb{R}^2$.

In its full generality Toeplitz' problem is still open. So far it has been solved affirmatively for curves that are "smooth enough", by various authors for varying smoothness conditions, see Section 2. All of these proofs are based on the fact that smooth curves inscribe generically an odd number of squares, which can be measured in several topological ways. None of these methods however could be made work so far for the general continuous case.

One may think that the general case of the Square Peg Problem can be reduced to the case of smooth curves by approximating a given continuous curve γ by a sequence of smooth curves γ_n: Any γ_n inscribes a square Q_n, and by compactness there is a converging subsequence $(Q_{n_k})_k$, whose limit is an inscribed square for the given curve γ. However this limit

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square is possibly degenerate to a point, and so far there is no argument known that can deal with this problem.

Suppose we could show that any smooth (or equivalently, any piecewise linear) curve γ that contains in its interior a ball of radius r inscribes a square of side length at least $\sqrt{2}r$ (or at least εr for some constant $\varepsilon > 0$). Then the approximation argument would imply that any continuous curve had the same property. However it seems that we need more geometric than merely topological ideas to show the existence of *large* inscribed squares.

Other surveys are due to Klee and Wagon [KW96, Problem 11], Nielsen [Nie00], Denne [Den07], Karasev [Kar08, 2.6, 4.6], and Pak [Pak10, I.3, I.4]. Jason Cantarella's homepage [Can08] offers some animations. A java applet is available on my homepage [Mat10].

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2 History of the Square Peg Problem

The Square Peg Problem first appears in the literature in the conference report [Toe11] in 1911. Toeplitz gave a talk whose second part had the title "On some problems in topology". The report on that second part is rather short:

> *b) Ueber einige Aufgaben der Analysis situs.* [...]
> b) Der Vortragende erzählt von zwei Aufgaben der Analysis Situs, zu denen er gelangt ist, und dann von der folgenden dritten, deren Lösung ihm nur für konvexe Kurven gelungen ist: Auf jeder einfach geschlossenen stetigen Kurve in der Ebene gibt es vier Punkte, welche ein Quadrat bilden. Diskussion: Die Herren Fueter, Speiser, Laemmel, Stäckel, Grossmann.

Here is an English translation:

> *b) On some problems in topology.* [...]
> b) The speaker talks about two problems in topology that he obtained, and then about the following third one, whose solution he managed to find only for convex curves: On every simple closed continuous curve in the plane there are four points that form a square. Discussions: Messrs. Fueter, Speiser, Laemmel, Stäckel, Grossmann.

It seems that Toeplitz never published a proof. In 1913, Arnold Emch [Emc13] presented a proof for "smooth enough" convex curves. Two years later Emch [Emc15] published a further proof which requires a weaker smoothness condition. However he did not note that the special case of smooth convex curves already implies by a limit argument that all convex curves inscribe squares. In a third paper from 1916, Emch [Emc16] proved the Square Peg Problem for curves that are piecewise analytic with only finitely many inflection points and other singularities where the left and right sided tangents at the finitely many non-smooth points exist.

Emch states in his second paper [Emc15] that he had not been aware of Toeplitz' and his students' work and that the problem was suggested to him by Kempner. From 1906–1913 Toeplitz was a postdoc in Göttingen. Aubrey J. Kempner was an English mathematician who finished his Ph.D. with Edmund Landau in Göttingen in 1911. Afterwards he went to the University of Illinois in Urbana-Champaign and stayed there until 1925, according to http://www.maa.org/history/presidents/kempner.html (another biography of Kempner can be found on http://www.findagrave.com/cgi-bin/fg.cgi?page=gr&GRid=13165695, which claims different dates). Emch joined the faculty of the same university in 1911.

I will let the reader decide whether this is enough information on how this fits together and who considered the Square Peg Problem first. It is usually attributed to Toeplitz.

In 1929 Schnirelman proved the Square Peg Problem for a class of curves that is slightly larger than C^2. An extended version [Sch44], which corrects also some minor errors, was published posthumously in 1944. Guggenheimer [Gug65] states that the extended version still contains errors which he claims to correct. However in my point of view Schnirelman's proof is up to minor errors correct. His main idea is a bordism argument, below we give some details. Since the transversality machinery was not invented at this time, Schnirelman's proof contains many computations in explicit coordinates. Guggenheimer's main lemma on the other hand admits counter-examples, see [Mat08, Section III.9]; he was not aware that squares can vanish pairwise when one deforms the curve.

Other proofs are due to Zindler [Zin21] and Christensen [Chr50] for convex curves, Jerrard [Jer61] for analytic curves, Stromquist [Str89] for locally monotone curves, Vrećica–Živaljević [VŽ11] for Stromquist's class of curves, Pak [Pak08, 2. proof] for piecewise linear curves (his first proof unfortunately contains an error), Sagols–Marín [SM09], [SM11] for similar discretizations, Cantarella–Denne–McCleary [CDM] for curves with bounded total curvature without cusps and for C^1-curves, Makeev [Mak95] for star-shaped C^2-curves that intersect every circle in at most 4 points (more generally he proved the Circular Quad Peg Problem 4.4 for such curves, see below), [Mat09] for curves without inscribed "special trapezoids of size ε", and [Mat11, Mat12] for continuous curves in certain bounded domains. In the next section we will review some of these special cases in more detail.

3 Special cases

Let us discuss some of the above mentioned proofs in more detail.

Emch's proof. Let $\gamma : S^1 \hookrightarrow \mathbb{R}^2$ be the given piecewise analytic curve. Fixing a line τ, Emch considers all secants of γ that are parallel to τ and he calls the set of all midpoints of these secants the set of *medians* M_τ. Under some genericity assuptions he proves that for two orthogonal lines τ and $\tau^\perp$, M_τ intersects $M_{\tau^\perp}$ in an *odd* number of points. Nowadays one could write this down homologically. These intersections correspond to inscribed rhombi, where the two intersecting secants are the two diagonals of the rhombus.

Now he rotates τ continuously by $90°$ and argues that $M_\tau \cap M_{\tau^\perp}$ moves continuously, where at finitely many times, two intersection points can merge and disappear, or two new intersection points can appear. When τ is rotated by $90°$, the one-dimensional family of intersections points closes up to a possibly degenerate union of circle components.

Since $M_\tau \cap M_{\tau^\perp}$ is odd, Emch argues that an odd number of these components must be $\mathbb{Z}/4\mathbb{Z}$-invariant, meaning that if $R_1R_2R_3R_4$ is a rhombus in such a component, then $R_2R_3R_4R_1$ must also be in the same component. By the mean value theorem, when moving from $R_1R_2R_3R_4$ to $R_2R_3R_4R_1$ along a component of inscribed rhombi, at some point the diagonals must have equal length. That is, we obtain an inscribed square. This argument also implies that the number of inscribed squares is (generically) odd for Emch's class of curves.

Schnirelman's proof. Schnirelman proved the Square Peg Problem for a slightly larger class than C^2 using an early bordism argument. His idea was that the set of inscribed squares can be described as a preimage, for example in the following way: Let $\gamma : S^1 \hookrightarrow \mathbb{R}^2$ be the given curve. The space $(S^1)^4$ parameterizes quadrilaterals that are inscribed in γ. We construct a *test-map*

$$f : (S^1)^4 \to \mathbb{R}^6 \tag{1}$$

that sends a 4-tuple (x_1, x_2, x_3, x_4) of points on the circle to the mutual distances between $\gamma(x_1), \ldots, \gamma(x_4) \in \mathbb{R}^2$. Let V be the 2-dimensional linear subspace of $\mathbb{R}^6$ that corresponds to the points where all four edges are of equal length and the two diagonal are of equal length. The preimage $f^{-1}(V)$ is parameterizing the set of inscribed squares, plus a few 'degenerate components'. The degenerated components consist of points where $x_1 = x_2 = x_3 = x_4$, these are the degenerate squares, and more generally of 4-tuples where $x_1 = x_3$ and $x_2 = x_4$.

Now Schnirelman argues as follows: 1. An ellipse inscribes exactly one square up to symmetry. 2. Deforming the curve smoothly from an ellipse to the given curve moves the preimages such that some squares can disappear pairwise and appear also pairwise (this is the bordism argument). 3. By the smoothness condition these inscribed squares do not come close to the degenerate components.

Thus, any smooth curve inscribes generically an odd number of squares. The degenerated quadrilaterals in $f^{-1}(V)$ are the basic reason why the Square Peg Problem is so difficult for general curves.

Stromquist's criterion. Stromquist's class of curves for which he proved the Square Peg Problem is very beautiful and it is the second strongest one: A curve $\gamma : S^1 \hookrightarrow \mathbb{R}^2$ is called **locally monotone** if every point of $x \in S^1$ admits a neighborhood U and a linear functional $\ell : \mathbb{R}^2 \to \mathbb{R}$ such that $\ell \circ \gamma|_U$ is strictly monotone.

Theorem 3.1 (Stromquist)**.** *Any locally monotone embedding $\gamma : S^1 \hookrightarrow \mathbb{R}^2$ inscribes a square.*

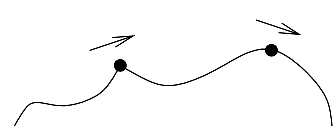

Figure 2: Example of a piece of a locally monotone curve. Note that Figure 1 is not locally monotone because of the spiral.

In his proof Stromquist also considers the set of inscribed rhombi first.

Fenn's table theorem. A beautiful proof for convex curves is due to Fenn [Fen70]. It follows as an immediate corollary from his table theorem.

Theorem 3.2 (Fenn). *Let $f : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be a non-negative function that is zero outside of a compact disc D and let $a > 0$ be an arbitrary real number. Then there exists a square in the plane with side length a and whose center point belongs to D such that f takes the same value on the vertices of the square.*

As the reader might guess Fenn's proof basically uses a mod-2 argument, showing that the number of such tables is generically odd.

The table theorem implies the Square Peg Problem for convex curves γ by constructing a height function $f : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ whose level sets $f^{-1}(x)$ are similar to γ for all $x > 0$.

Zaks [Zak71] found an analogous "chair theorem", where instead of a square table he considers triangular chairs with a fixed direction. Kronheimer–Kronheimer [KK81] found conditions on ∂D such that the table/chair can be chosen such that all four/three vertices lie in D: Namely, ∂D should not inscribe a table/triangle of a smaller size. More table theorems are due to Meyerson [Mey81a], see also [Mey81b, Mey84] for interesting examples.

No special trapezoids of size ε. In [Mat09] the Square Peg Problem was proved for the so far weakest smoothness condition. The proof extends to curves in arbitrary metric spaces, so let's state it in this more general form.

Theorem 3.3. *Let $\gamma : S^1 \to X$ be an embedded circle in a metric space (X, d). Assume that there is an $0 < \varepsilon < 2\pi$ such that γ contains no (or generically an even number of) special trapezoids of size ε. Then γ inscribes a square, in the sense that there exist four pairwise distinct points $P_1, \ldots, P_4 \in \gamma$ such that*

$$d(P_1, P_2) = d(P_2, P_3) = d(P_3, P_4) = d(P_4, P_1) \text{ and } d(P_1, P_3) = d(P_2, P_4).$$

Here a **special trapezoid** on a curve γ is a 4-tuple of pairwise distinct points $x_1, \ldots, x_4 \in S^1$ lying clockwise on S^1 such that the points $P_i := \gamma(x_i)$ satisfy

$$d(P_1, P_2) = d(P_2, P_3) = d(P_3, P_4) > d(P_4, P_1) \text{ and } d(P_1, P_3) = d(P_2, P_4).$$

The **size** of this special trapezoid is defined as the length of the clockwise arc in S^1 from x_1 to x_4.

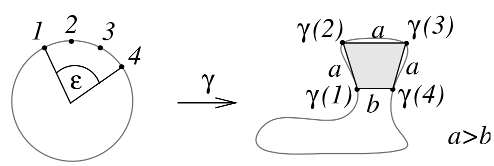

Figure 3: A special trapezoid of size ε.

The set of curves without inscribed special trapezoids of a fixed size ε is open and dense in the space of embeddings $S^1 \hookrightarrow X$ with respect to the compact-open topology. This theorem is basically the exact criterion that one obtains by applying equivariant obstruction theory to the test-map (1). Vrećica and Živaljević [VŽ11] have been the first who applied obstruction theory to the Square Peg Problem and they proved it for Stromquist's class of locally monotone curves.

An open set of curves. All previous criterions on curves for which the Square Peg Problem was proved are defined by local smoothness conditions. The following criterion from [Mat11, Mat12] is a global one, which yields an open set of not necessarily injective curves in $C^0(S^1, \mathbb{R}^2)$ with respect to the C^0-topology, or equivalently, the compact-open topology.

Theorem 3.4. *Let A denote the annulus $\{x \in \mathbb{R}^2 \mid 1 \leq ||x|| \leq 1 + \sqrt{2}\}$. Suppose that $\gamma : S^1 \to A$ is a continuous closed curve in A that is non-zero in $\pi_1(A) = \mathbb{Z}$. Then γ inscribes a square of side length at least $\sqrt{2}$.*

It is open whether the outer radius $1 + \sqrt{2}$ of A can be increased by any small $\varepsilon > 0$.

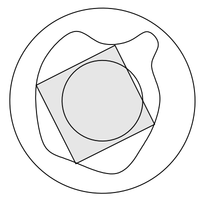

Figure 4: Example for Theorem 3.4.

The proof idea is very simple: If the annulus A is thin enough, then the set of squares with all vertices in A splits into two connected components: big squares and small squares. A generic curve that represents a generator of $\pi_1(A)$ inscribes an odd number of big squares (and an even number of small squares).

4 Related problems

More squares on curves. Popvassilev [Pop08] constructed for any $n \geq 1$ a smooth *convex* curve that has exactly n inscribed squares, every square being counted exactly once and not with multiplicity. All but one of the n squares in his construction are non-generic. They will disappear immediately after deforming the curve by a suitable C^∞-isotopy. An analog piecewise linear example was given by Sagols–Marín [SM11].

Van Lamoen [vL04] studied the geometric relationship between the inscribed squares in the union of three lines (an extended triangle).

In [Mat11] the parity of the number of squares on generic smooth *immersed* curves in the plane was given, which depends not only on the isotopy type of the immersion but also on the intersection angles.

Triangles on curves. It is not hard to show that any smooth embedding $\gamma : S^1 \to \mathbb{R}^2$ inscribes arbitrary triangles, even if we prescribe where one of the vertices has to sit. Moreover the set of all such inscribed triangles determines a homology class $\alpha \in H_1(P_3, \mathbb{Z}) = \mathbb{Z}$, where P_3 is the set of 3-tuples of points on γ that lie counter-clockwise on the curve. The class α turns out the be a generator, as one sees from inspecting the situation for the circle.

For continuous curves Nielsen [Nie92] proved the following version of that.

Theorem 4.1 (Nielsen)**.** *Let T be an arbitrary triangle and $\gamma : S^1 \longrightarrow \mathbb{R}^2$ an embedded circle. Then there are infinitely many triangles inscribed in γ which are similar to T, and if one fixes a vertex of smallest angle in T then the set of the corresponding vertices on γ is dense in γ.*

Rectangles on curves. Instead of squares one may ask whether any embedded circle in the plane inscribes a rectangle. If one does not prescribe the aspect ratio then the answer is affirmative.

Theorem 4.2 (Vaughan)**.** *Any continuous embedding $\gamma : S^1 \hookrightarrow \mathbb{R}^2$ inscribes a rectangle.*

Vaughan's proof, which appeared in Meyerson [Mey81a], is very beautiful: $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ acts on the torus $(S^1)^2$ by permuting the coordinates, and the quotient space $(S^1)^2/\mathbb{Z}_2$ is a Möbius strip. The proof of Theorem 4.2 uses that the map $f : (S^1)^2/\mathbb{Z}_2 \to \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ given by

$$f(x, y) = \big((\gamma(x) + \gamma(y))/2, ||\gamma(x) - \gamma(y)||\big),$$

must have a double point, otherwise it would extend to an embedding of $\mathbb{R}P^2$ into $\mathbb{R}^3$ by gluing to that Möbius strip the disc $I \times \{0\}$, where $I \subset \mathbb{R}^2$ is the interior of γ. The double point corresponds to two secants in γ having the same length and the same midpoint. Hence this forms an inscribed rectangle.

If we furthermore prescribe the aspect ratio of the rectangle, then the problem is widely open, even for smooth or piecewise linear curves.

Conjecture 4.3 (Rectangular Peg Problem)**.** *Every C^∞ embedding $\gamma : S^1 \to \mathbb{R}^2$ contains four points that are the vertices of a rectangle with a prescribed aspect ratio $r > 0$.*

This conjecture is highly interesting since the standard topological approach does not yield a proof: The equivariant homology class of the solution set, a $\mathbb{Z}$-valued smooth isotopy invariant of the curve, turns out to be zero. For example, an ellipse inscribes a positive and a negative rectangle. Stronger topological tools fail as well. It seems again that more geometric ideas are needed.

Equivalently we could state Conjecture 4.3 for all piecewise linear curves. Proofs exist only for the case $r = 1$, which is the *smooth* Square Peg Problem, for arbitrary r in case the curve is close to an ellipse, see Makeev [Mak95] and Conjecture 4.4 below, and for $r = \sqrt{3}$ in case the curve is close to convex, see [Mat11, Mat12].

A proof for the Rectangular Peg Problem was claimed by Griffiths [Gri90], but it contains errors regarding the orientations. Essentially he calculated that the number of inscribed rectangles of the given aspect ratio counted with appropriate signs and modulo symmetry is 2, however zero is correct.

Other quadrilaterals on curves. It is natural to ask what other quadrilaterals can be inscribed into closed curves in the plane. Since the unit circle is a curve, those quadrilaterals must be circular, that is, they must have a circumcircle.

Depending on the class of curves that we look at, the following two conjectures seem reasonable.

Conjecture 4.4 (Circular Quad Peg Problem)**.** *Let Q be a circular quadrilateral. Then any C^∞ embedding $\gamma : S^1 \to \mathbb{R}^2$ admits an orientation preserving similarity transformation that maps the vertices of Q into γ.*

Makeev [Mak95] proved a first instance of this conjecture, namely for the case of star-shaped C^2-curves that intersect every circle in at most 4 points.

Furthermore, Karasev [KV10b] proved that for any smooth curve and a given $Q = ABCD$, either this conjecture holds, or one can find two inscribed triangles similar to ABC, such that the two corresponding fourth vertices D coincide (but D may not lie on γ). The proof idea is a beautiful geometric volume argument. It should be stressed that most open problems discussed here are geometric problems rather than topological ones: We understand the basic algebraic topology here quite well, but not the restrictions on the topology that the geometry dictates. New geometric ideas are needed.

Conjecture 4.5 (Trapezoidal Peg Problem)**.** *Let T be an isosceles trapezoid. Then any piecewise-linear embedding $\gamma : S^1 \to \mathbb{R}^2$ inscribes a quadrilateral similar to T.*

The reason for restricting the latter conjecture to isosceles trapezoids, that is, trapezoids with circumcircle, is that all other circular quadrilaterals cannot be inscribed into very thin triangles. This was observed by Pak [Pak08].

Other polygons on curves. For any n-gon P with $n \geq 5$ it is easy to find many curves that do not inscribe P. If we do not require all vertices to lie on γ then Makeev has some results for circular pentagons, see [Mak05a].

Alternatively, we can relax the angle conditions, that is, we require only that the edge ratios are the same as the ones in a given polygon P. Then as for the triangles above one can show that the set of such n-gons represents the generator of $H_1(P_n; \mathbb{Z}) = \mathbb{Z}$, where P_n is the set of n-tuples on γ that lie counter-clockwise on the curve; see Mayerson [Mey80], Wu [Wu04], Makeev [Mak05a], Vrećica–Živaljević [VŽ11], and [Mat08].

Higher dimensions. In higher dimensions one may ask whether any $(n-1)$-sphere that is smoothly embedded in $\mathbb{R}^n$ inscribes an n-cube in the sense that all vertices of the cube lie on the sphere. However, most smooth embeddings $S^{n-1} \hookrightarrow \mathbb{R}^n$ do not inscribe an n-cube for $n \geq 3$, in the sense that these embeddings form an open and dense subset of all smooth embeddings in the compact-open topology, a heuristic reason being that the number of equations to fulfill is larger than the degrees of freedom. An explicit example are the boundaries of very thin simplices, as was noted by Kakutani [Kak42] for $n = 3$. Hausel–Makai–Szűcs [HMS02] proved that the boundary of any centrally symmetric convex body in $\mathbb{R}^3$ inscribes a 3-cube.

If we do not want to require further symmetry on the embedding $S^{n-1} \hookrightarrow \mathbb{R}^n$, then crosspolytopes are more suitable higher analogs of squares: The regular n-dimensional crosspolytope is the convex hull of $\{\pm e_i\}$ where e_i are the standard basis vectors in $\mathbb{R}^n$.

Theorem 4.6 (Makeev, Karasev)**.** *Let n be an odd prime power. Then every smooth embedding $\Gamma : S^{n-1} \to \mathbb{R}^n$ contains the vertices of a regular n-dimensional crosspolytope.*

The $n = 3$ case was posed as Problem 11.5 in Klee & Wagon [KW96]. This was answered affirmatively by Makeev [Mak03]. Karasev [Kar09] generalized the proof to arbitrary odd prime powers. Akopyan and Karasev [AK11] proved the same theorem for $n = 3$ in case Γ is the boundary of a simple polytope by a careful and non-trivial limit argument from the smooth case.

Gromov [Gro69] proved an similar theorem for inscribed simplices.

Theorem 4.7 (Gromov)**.** *Any compact set $S \subset \mathbb{R}^d$ with C^1-boundary and non-zero Euler characteristic inscribes an arbitrary given simplex up to similarity on its boundary ∂S.*

Circumscribing problems. Instead of inscribing polytopes into surfaces we can ask which polytopes P can circumscribe a given surface S in the sense that P contains S and every facet of P touches S.

The most prominent result is the following theorem.

Theorem 4.8 (Kakutani, Yamabe–Yujobô)**.** *Any compact convex body in $\mathbb{R}^n$ has an n-dimensional circumscribing cube.*

The problem was posed by Rademacher. Kakutani [Kak42] proved the $n = 3$ case. Yamabe and Yujobô [YY50] generalized this to arbitrary dimensions. The latter proof is in particular interesting since it uses a clever and simple induction without any oddness arguments.

More generally they proved that for any map $f : S^{n-1} \to \mathbb{R}$ there exist n pairwise orthogonal points on the sphere that are mapped by F to the same value. This has been a motivation for many similar problems and theorems:

Conjecture 4.9 (Knaster [Kna47])**.** *For any map $f : S^{n-1} \to \mathbb{R}^m$ and any $n - m + 1$ points $x_0, \ldots, x_{n-m} \in S^{n-1}$ there exists a rotation $\rho \in SO(n)$ such that $f(\rho(x_0)) = \ldots = f(\rho(x_{n-m}))$.*

Hopf [Hop44] proved the special case of two points, that is, for $n - m + 1 = 2$ (he actually proved a more general version for functions on Riemannian manifolds, see also Akopyan–Karasev–Volovikov [AKV12]). Floyd [Flo55] proved it for the special case $(n, m) = (3, 1)$. For all other pairs (n, m) Knaster's conjecture is either open or false; counter-examples have been found by Makeev [Mak86], Babenko–Bogatyĭ [BB90], Chen [Che98], Kashin–Szarek [KS03], and Hinrichs–Richter [HR05]. See Hinrichs–Richter [HR05] and Liu [Liu10] for more detailed overviews of known results.

In case of particular point configurations $x_0, \ldots, x_{n-m}$ or if the function f satisfies particular symmetry properties, many more results have been established: Dyson [Dys51] showed that for $(n, m) = (3, 1)$ some level set has to contain the four vertices of a square whose midpoint is the origin. Again, this can be easily proved by a parity argument. Livesay [Liv54] generalized this to rectangles with a prescribed aspect ratio. Here the number of solutions is *even*. Livesay circumvents this difficulty by reducing the problem to the mean value theorem. Further Knaster type theorems are due to Yang [Yan57], Makeev [Mak88, Mak93, Mak06], Volovikov [Vol92, Vol01], Crabb–Jaworowski [CJ09], Karasev–Volovikov [KV10a], Karasev [Kar10], and Liu [Liu10, Liu12].

Any of those restricted special cases of Knaster's conjecture imply a corrsponding circumscribing result: The particular point configuration $(x_0, \ldots, x_{n-m})$ corresponds to the facet normals of the possibly unbounded circumscribing polyhedron, and the symmetry conditions on f also have to be put on the symmetry of the convex body that one wants to circumscribe.

Many other very interesting inscribing and circumscribing theorems and problems are due to Hausel–Makai–Szűcs [HMS97, HMS02], Kuperberg [Kup99], Makeev [Mak01, Mak03, Mak05b], and Yang [Yan54, Yan55b, Yan55a], and this list is not complete.

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