Motivic Periods, Coleman Functions, and the Unit Equation

An Ongoing Project

D. Corwin[1] I. Dan-Cohen[2]

[1]MIT/ENS Paris

[2]Ben Gurion University of the Negev

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Table of Contents

Table of Contents

Motivation: The Unit Equation

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Refined Problem (Chabauty-Kim Theory)

Find p-adic analytic (Coleman) functions on $X(\mathbb{Q}_p)$ that vanish on $X(R)$.

Table of Contents

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- One may deduce relations between periods using rules for linearity, products, algebraic changes of variables, and Stokes' Theorem.
- For example, one can theoretically deduce $6\zeta(2) = \pi^2$ in this way.

Motivic Periods

Defintion

The ring P of *effective motivic periods* is the formal $\mathbb{Q}$-algebra generated by tuples (X, D, ω, γ) as in the previous slide, modulo relations coming from linearity, algebraic change of variables, and Stokes' Theorem.

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Examples

Applying I_{BC} to each, we obtain $\zeta^{p}(n)$, $\log^{p}(n)$, $\mathrm{Li}_k^{p}(n)$, $\cdots$

De Rham Periods

Coleman integrals use de Rham cohomology (specifically, the Frobenius and Hodge filtration) but not Betti cohomology. We therefore need:

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The ring $\mathrm{P}^{\mathfrak{dr}}$ of *effective de Rham periods* is a variant of P in which γ represents a de Rham homology class. For each p, there is a map $I_{BC}\colon \mathrm{P}^{\mathfrak{dr}} \to \mathbb{Q}_p$ given by Coleman integration.

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Examples

We similarly write $\zeta^{\mathfrak{dr}}(n)$, $\log^{\mathfrak{dr}}(n)$, $\mathrm{Li}_k^{\mathfrak{dr}}(n)$, $\cdots$

Mixed Tate Periods

- We will focus on a subring $\mathcal{P}^{\mathfrak{dr},+}(R) \subseteq \mathrm{P}^{\mathfrak{dr}}$ of effective mixed Tate de Rham periods over R. These contain all periods coming from unirational pairs (X, D) with good reduction over R.

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Our Motivic Periods

We will therefore work with $\mathcal{P}^{\mathfrak{u}}(R) : = \mathcal{P}^{\mathfrak{dr},+}(R)/\zeta(2)$ and the integration map $I_{BC} : \mathcal{P}^{\mathfrak{u}}(R) \to \mathbb{Q}_p$ for $p \in Spec(R)$.

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We note that an inclusion $R \subseteq R'$ gives rise to an inclusion $\mathcal{P}^{\mathfrak{u}}(R) \subseteq \mathcal{P}^{\mathfrak{u}}(R')$ (e.g., $\mathcal{P}^{\mathfrak{u}}(\mathbb{Q})$ contains $\mathcal{P}^{\mathfrak{u}}(\mathbb{Z}[1/S])$ for all S).

Why Motivic Periods

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Theorem (Deligne, Goncharov, Borel, ...)

$\mathcal{P}^u(R)$ has the structure of a graded Hopf algebra, and as such is abstractly isomorphic to an explicit free shuffle algebra. Assuming $\mathrm{Frac}(R) = \mathbb{Q}$, it is the free shuffle algebra

$$\mathbb{Q}\langle\{\{g_p\}_{p\in S}, \{f_{2n+1}\}_{n\geq 1}\}\rangle,$$

where each g_p has degree 1, and f_{2n+1} has degree $2n+1$.

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As a graded vector space, it's the free non-commutative algebra in these generators. However, it's equipped with a commutative product denoted by Ш.

Table of Contents

Polylogarithms and Integral Points

Definition

Let $\mathcal{O}(\Pi_X)^{PL} = \mathbb{Q}[\mathrm{Li}_0^{\mathfrak{u}} = \log^{\mathfrak{u}}, \mathrm{Li}_1^{\mathfrak{u}}, \mathrm{Li}_2^{\mathfrak{u}}, \cdots]$ as a $\mathbb{Q}$-algebra.

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- As before, let $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Let $z \in X(\mathbb{Q})$. For each integer k, one can define a motivic period $\mathrm{Li}_k^{\mathfrak{u}}(z) \in \mathcal{P}^{\mathfrak{u}}(\mathbb{Q})$.

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- It follows that each $z \in X(\mathbb{Q})$ defines a homomorphism $\kappa(z)\colon \mathcal{O}(\Pi_X)^{PL} \to \mathcal{P}^{\mathfrak{u}}(\mathbb{Q})$ sending $\mathrm{Li}_k^{\mathfrak{u}}$ to $\mathrm{Li}_k^{\mathfrak{u}}(z)$.

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Fact

$z \in X(R)$ iff $\mathrm{Image}(\kappa(z)) \subseteq \mathcal{P}^{\mathfrak{u}}(R)$

Hopf Algebra Structure

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Motivic Kim's Cutter

For a prime p, this gives us a diagram:

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- Dimension counts show that this arrow is non-dominant, which is what proves Siegel's theorem.

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- Dimension counts show that this arrow is non-dominant, which is what proves Siegel's theorem.
- Therefore, there is a nonzero ideal $I_{CK} \subseteq \mathcal{O}(\Pi_X)^{PL}$ vanishing on the image of the bottom arrow, known as the *Chabauty-Kim ideal*.

Table of Contents

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Theorem (Dan-Cohen, Wewers, 2013)

For $R = \mathbb{Z}[1/2]$, the following Coleman function is in I_{CK}:

$$\det \begin{pmatrix} \mathrm{Li}_4^p(z) & \log^p(z)\mathrm{Li}_3^p(z) & (\log^p(z))^3\mathrm{Li}_1^p(z) \\ \mathrm{Li}_4^p(\frac{1}{2}) & \log^p(\frac{1}{2})\mathrm{Li}_3^p(\frac{1}{2}) & (\log^p(\frac{1}{2}))^3\mathrm{Li}_1^p(\frac{1}{2}) \\ \frac{1}{24} & \frac{1}{6} & 1 \end{pmatrix}$$

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In 2015, Dan-Cohen posted a preprint showing that, assuming certain well-known conjectures, this could be made into an algorithm.

Our Current Work

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- More specifically, we are working on $\mathbb{Z}[1/6]$.
- To do this, we need to compute a basis for $\mathcal{P}^{\mathfrak{u}}(\mathbb{Z}[1/6])$ (up to a certain degree) as linear combinations of explicit polylogarithms of the form $\mathrm{Li}^{\mathfrak{u}}(z)$ for $z \in X(\mathbb{Q})$.

Computations for $\mathbb{Z}[1/6]$

- To simplify notation, we let A denote $\mathcal{P}^{\mathfrak{u}}(\mathbb{Z}[1/6])$. We let A_n denote the nth graded piece.

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- The non-injectivity of d' for $k = 3$ requires use of p-adic approximation to determine rational multiples of $\zeta^{\mathfrak{u}}(3)$.

Useful References

The following are on arXiv:

- Mixed Tate Motives and the Unit Equation, Ishai Dan-Cohen and Stefan Wewers
- Mixed Tate Motives and the Unit Equation II, Ishai Dan-Cohen
- Single-Valued Motivic Periods, Francis Brown
- Motivic Periods and $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, Francis Brown
- Notes on Motivic Periods, Francis Brown
- Integral Points on Curves and Motivic Periods, Francis Brown

Our definition of motivic periods comes from Periods, Kontsevich and Zagier (http://www.maths.ed.ac.uk/ aar/papers/kontzagi.pdf).

Thank You!