Higher spectral sequences

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1 Introduction

Consider a chain complex (C, d) that is filtered in n different compatible[1] ways over the integers, or alternatively, a space X that is filtered in n different ways over the integers together with a generalized homology theory h_*. Then, according to [Mat13], there is an associated "spectral system over $D(\mathbb{Z}^n)$" whose limit is $H_*(C)$ respectively $h_*(X)$. For $n = 1$ this contains the usual spectral sequence of C respectively X as a substructure.

In that paper several connections between the first page and the limit of this spectral system have been given (via differentials, extensions, and natural isomorphisms), which have been in particular useful for unifying several spectral sequences that one would usually apply one after another. It was conjectured that there should be many more differentials to choose from. In this paper, we prove this conjecture. With these new differentials it seems justified to give spectral system over $D(\mathbb{Z}^n)$ the new name *higher spectral sequences*.

In particular, we construct for any admissible word $\omega \in L_a^*$ (see Definition 3.2) over the alphabet

$$L := \{1, \ldots, n, 1^\infty, \ldots, n^\infty, \mathrm{x}\}.$$

a so-called ω-page, which is a collection of abelian groups $S(P; \omega)$. Here P ranges over a quotient $\mathbb{Z}^n/V_\omega \cong \mathbb{Z}^{n-k}$, where k is the number of letters x in ω. In the alphabet L, a letter $j \in [n]$ stands for taking homology with respect to the j'th differential, j^∞ denotes the same but infinitely often, and x stands for a group extension process.

In ordinary spectral sequences we have $n = 1$, and for $\omega = 1^{r-1}$ the ω-page consists of the columns in E^r_{**}, which are indexed over $P \in \mathbb{Z}$. The letter 1 stands the connection between some E^r_{**} and E^{r+1}_{**}, 1^∞ stands for the connection between some E^r_{**} and E^∞_{**}, and x for the connection between E^∞_{**} and the "limit" of the spectral sequence, e.g. $H(C)$ if the spectral sequence comes from a $\mathbb{Z}$-filtered chain complex C.

In Section 3, we define certain vectors $r^j_\omega, \delta^j_\omega \in \mathbb{Z}^n$, where r^i_ω will be the negated direction of the i'th differential at the ω-page, and δ^i_ω is the negated change of direction for the i'th differential that occurs when taking homology with respect to it. In ordinary spectral sequences, for $\omega = 1^{r-1}$ we have $r^1_\omega = r$, and $\delta^1_\omega = 1$.

Theorem 3.6 (Main theorem)**.** Let $\omega \in L_a^*$ and $j \in [n]$ such that $\omega * j$ is admissible. Then the following holds.

a) There are natural differentials

$$\ldots \longrightarrow S(P + r^j_\omega; \omega) \longrightarrow S(P; \omega) \longrightarrow S(P - r^j_\omega; \omega) \to \ldots . \quad (1)$$

[1]Compatibility means that the associated exact couple system (2) is excisive. This occurs for example if C (the underlying abelian group) is of the form $\bigoplus_{P \in \mathbb{Z}^n} C_P$ with the n canonical $\mathbb{Z}$-filtrations.

Taking homology at $S(P;\omega)$ yields $S(P;\omega * j)$.

b) $S(P;\omega * j^\infty)$ is a natural subquotient of $S(P;\omega * j^k)$ for all $k \geq 0$.

c) There exists a natural $\mathbb{Z}$-filtration $(F_i)_{i\in\mathbb{Z}}$ of $S(P;\omega * j^\infty \mathrm{x})$,

$$0 \subseteq \ldots \subseteq F_i \subseteq F_{i+1} \subseteq \ldots \subseteq S(P;\omega * j^\infty \mathrm{x}),$$

such that $S(P + i \cdot \delta_\omega^j; \omega * j^\infty) \cong F_i/F_{i-1}$, for all $i \in \mathbb{Z}$.

How is this useful: (a) gives a connection between the first page and arbitrary ω-pages for $\omega \in [n]^*$. In the S_{bq}^{pz}-description of $S(P;\omega)$ in the proof below, b, p, q, z are lexicographic downsets. Thus one can proceed with the small-step and/or big-step lexicographic connections from [Mat13, 3.2.1] in order to connect the ω-page to the limit $S_{\infty,-\infty}^{\infty,-\infty}$ via further differentials and extensions.

Alternatively, we can proceed with (b) and take homology infinitely often in one direction. Note that as with usual spectral sequences, $S(P;\omega * j^\infty)$ may indeed be a proper subquotient of the limit of the $S(P;\omega * j^k)$, compare with Weibel [Wei94], Boardman [Boa99], McCleary [McC01].

Then one can proceed with (c), which connects to $S(P;\omega * j^\infty \mathrm{x})$. Again as with usual spectral sequences, the filtration (F_i) may be neither Hausdorff nor exhaustive, and even if they are, $S(P;\omega * j^\infty \mathrm{x})$ may not be complete with respect to (F_i).

As usual these two problems in (b) and (c) can be serious, but they are the standard ones in spectral sequences.

Arriving at $S(P;\omega * j^\infty \mathrm{x})$ we can start again at (a) until ω is final.

Outline. This paper is organized as follows. In Section 2 we review the notions and basic properties of exact couple systems ("higher exact couples") and their associated spectral systems ("higher spectral sequences").

In Section 3 the necessary new notation is developed, and the main theorem is precisely restated and proved. During the proof, several equivalent descriptions for the ω-pages are given, some of which are more intuitive — the S_B^Z-descriptions — and some of which are much easier to work with — the S_{bq}^{pz}-descriptions. At the end we discuss some more properties of the $n = 2$ case, which may also serve as an example that gives more intuition.

2 Preliminaries

Let $n \geq 1$ and $[n] := \{1, \ldots, n\}$. Let $e_1, \ldots, e_n$ be the standard basis vectors in $\mathbb{Z}^n$, and $\mathbb{1} := (1, \ldots, 1)^t \in \mathbb{Z}^n$. $\mathbb{Z}^n$ is a poset via $(x_1, \ldots, x_n) \leq (x'_1, \ldots, x'_n)$ if and only if $x_i \leq x'_i$ for all i.

Throughout the paper, let $I := D(\mathbb{Z}^n)$ denote the lattice of downsets of $\mathbb{Z}^n$. (Everything in this paper can also be done for filtrations over $D(\overline{\mathbb{Z}}^n)$, where $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{\pm\infty\}$; here we consider only $D(\mathbb{Z}^n)$ because it makes the presentation cleaner.)

I has minimum $-\infty := \emptyset$ and maximum $\infty := \mathbb{Z}^n$. We write $I_k := \{(p_1, \ldots, p_k) \in I^k \mid p_1 \geq \ldots \geq p_k\}$, which is again a poset via $(p_1, \ldots, p_k) \leq (p'_1, \ldots, p'_k)$ if and only if $p_i \leq p'_i$ for all i.

For us a chain complex is an abelian group C together with an endomorphism $d : C \to C$ with $d \circ d = 0$, and its homology is $H(C, d) := \ker(d)/\mathrm{im}(d)$; the grading is not of importance for us. An I-filtration of C is a family of subchain complexes $(F_p)_{p\in I}$ such that $F_q \subseteq F_p$ whenever $q \leq p$.

Similarly, if X is a topological space then an I-filtration of X is a family of open subspaces $(X_p)_{p\in I}$ such that $X_q \subseteq X_p$ whenever $q \leq p$.

Whenever we have an I-filtered chain complex (C, d), or an I-filtered space X together with a generalized homology theory h_*, we can associate a so-called exact couple system via

$$E_q^p := H(F_p/F_q) \tag{2}$$

or

$$E_q^p := h_*(X_p, X_q), \tag{3}$$

respectively, which is defined as follows.

Definition 2.1 (Exact couple system)**.** And *exact couple system over I* is a collection of abelian groups $(E_q^p)_{(p,q)\in I_2}$ together with homomorphisms $\ell_{p',q'}^{p,q} : E_q^p \to E_{q'}^{p'}$ for any $(p,q) \le (p',q')$ and homomorphisms $k_{p,q} : E_q^p \to E_{-\infty}^q$ for any $(p,q) \in I_2$, such that the following properties are satisfied:

1. $\ell_{p'',q''}^{p',q'} \circ \ell_{p',q'}^{p,q} = \ell_{p'',q''}^{p,q}$.
2. The triangles

$$\begin{array}{ccc} E_{-\infty}^q & \xrightarrow{\ell_{p,-\infty}^{q,-\infty}} & E_{-\infty}^p \\ & \nwarrow^{k_{pq}} \quad \swarrow_{\ell_{p,q}^{p,-\infty}} & \\ & E_q^p & \end{array}$$

   are exact.

3. The diagrams

$$\begin{array}{ccc} E_q^p & \xrightarrow{k_{pq}} & E_{-\infty}^q \\ \downarrow{\scriptstyle \ell_{p'q'}^{pq}} & & \downarrow{\scriptstyle \ell_{q',-\infty}^{q,-\infty}} \\ E_{q'}^{p'} & \xrightarrow[k_{p'q'}]{} & E_{-\infty}^{q'} \end{array}$$

   commute.

Let E be an exact couple system over I. There is a natural differential $d_{pqz} : E_q^p \to E_z^q$ for any $(p,q,z) \in I_3$ defined by $d_{pqz} := \ell_{q,z}^{q,-\infty} \circ k_{pq}$. With this we define an associated *spectral system over I* via

$$S_{bq}^{pz} := \frac{\ker(d_{pqz} : E_q^p \to E_z^q)}{\operatorname{im}(d_{bpq} : E_p^b \to E_q^p)}, \quad (b,p,q,z) \in I_4. \tag{4}$$

At a first glance this is just a collection of abelian groups, one for each element in I_4, however there are many connections between them:

First note that the usual goal of computation, $E_{-\infty}^{\infty}$, appears as $S_{\infty,-\infty}^{\infty,-\infty}$. It is called the *limit* of this spectral system (this is just a name; it does not imply any convergence or comparison property). Moreover, terms of the form $S_{pq}^{pq} = E_q^p$ are usually known when q covers p, that is, when $|q \setminus p| = 1$.

The following facts are proved in [Mat13]. For any $(b,p,q,z) \le (b',p',q',z')$ in I_4, $\ell_{p'q'}^{pq}$ induces maps

$$S_{bq}^{pz} \to S_{b'q'}^{p'z'},$$

which we call maps induced by inclusion. When there is no confusion, we abbreviate all of them as ℓ.

Lemma 2.2 (Extensions)**.** *For any $z \le p_1 \le p_2 \le p_3 \le b$ in I, we have a short exact sequence of maps induced by inclusion,*

$$0 \to S_{b,p_1}^{p_2,z} \to S_{b,p_1}^{p_3,z} \to S_{b,p_2}^{p_3,z} \to 0. \tag{5}$$

Lemma 2.3 (Differentials)**.** *For any $(b,p,q,z), (b',p',q',z') \in I_4$ with $z \le p'$ and $q \le b'$ there are natural differentials*

$$d : S_{bq}^{pz} \to S_{b'q'}^{p'z'}, \tag{6}$$

which commute with ℓ, that is, $\ell \circ d = d \circ \ell$.

Lemma 2.4 (Kernels and cokernels)**.** *For any* $(b,p,q,z),(b',p',q',z')\in I_4$ *with* $z=p'$ *and* $q=b'$ *we have*

$$\ker\left(d: S_{bq}^{pz}\to S_{b'q'}^{p'z'}\right)=S_{bq}^{pq'}$$

and

$$\operatorname{coker}\left(d: S_{bq}^{pz}\to S_{b'q'}^{p'z'}\right)=S_{pq'}^{p'z'}$$

Lemma 2.5 (∞-page as filtration quotients)**.** $E_{-\infty}^{\infty}$ *can be* I*-filtered by*

$$G_p:=\operatorname{im}(\ell: E_{-\infty}^{p}\to E_{-\infty}^{\infty})\cong S_{\infty,-\infty}^{p,-\infty},\quad p\in I.$$

Furthermore the S*-terms on the* ∞*-page are filtration quotients*

$$S_{\infty,q}^{p,-\infty}\cong G_p/G_q.$$

Lemma 2.6 (∞-page as quotient kernels)**.** $E_{-\infty}^{\infty}$ *has quotients*

$$Q_p:=\frac{E_{-\infty}^{\infty}}{\ker(\ell: E_{-\infty}^{\infty}\to E_p^{\infty})}\cong S_{\infty,p}^{\infty,-\infty},\quad p\in I.$$

Furthermore the S*-terms on the* ∞*-page are quotient kernels*

$$S_{\infty,q}^{p,-\infty}\cong\ker(Q_q\to Q_p).$$

Definition 2.7 (Excision)**.** An exact couple system E over I is called *excisive* if for all $a,b\in I$,

$$E_{a\cap b}^{a}\xrightarrow{\ \ell\ }E_b^{a\cup b}$$

is an isomorphism.

The exact couple system (3) is automatically excisive by the excision axiom of h_*. Note however that (2) is in general not excisive, though in many applications it is, for example when $C=\bigoplus_{P\in\mathbb{Z}^n}C_p$ (as abelian group) and $(F_p)_I$ is the canonical I-filtration given by $F_p=\bigoplus_{P\in p}C_p$.

Let us think of $J:=\mathbb{Z}^n$ as an undirected graph, whose vertices are the elements of J, and $x,y\in J$ are adjacent if they are related, i.e. $x\geq y$ or $x\leq y$ (coordinate-wise). For $(b,p,q,z)\in I_4$, let $Z(z,q,p,b)\subseteq J$ denote the union of all connected components of $p\setminus z$ that intersect $p\setminus q$, and let $B(z,q,p,b)\subseteq I$ denote the union of all connected components of $b\setminus q$ that intersect $p\setminus q$.

Lemma 2.8 (Natural isomorphisms)**.** *In an excisive exact couple system* E *over* $I=D(J)$, S_{bq}^{pz} *is uniquely determined up to natural isomorphism by* $Z:=Z(z,q,p,b)$ *and* $B:=B(z,q,p,b)$.

We also write S_B^Z for S_{bq}^{pz}, which is only defined up to natural isomorphisms. A word of warning: This B-Z-description of S_{bq}^{pz} looks quite appealing. However it may be combinatorially non-trivial to check whether some given B and Z come from some (b,p,q,z), and if so there might be several good choices. Moreover, it can be quite challenging to see whether there is a differential from $S_{B_1}^{Z_1}$ to $S_{B_2}^{Z_2}$ and what the resulting kernels and cokernels are in this case.

3 New differentials

Throughout this section let us fix an excisive exact couple system E over $I=D(\mathbb{Z}^n)$.

Define an alphabet L,

$$L:=\{1,\dots,n,1^{\infty},\dots,n^{\infty},\mathrm{x}\}.$$

Remark 3.1 (Some intuition)**.** Here, a letter $j \in [n]$ stands for taking homology with respect to the j'th differential, j^∞ denotes the same but infinitely often, and x stands for a group extension process. In ordinary spectral sequences, $n = 1$, and the letter 1 stands the connection between some E^r_{**} and E^{r+1}_{**}, 1^∞ stands for the connection between some E^r_{**} and E^∞_{**}, and x for the connection between E^∞_{**} and the "limit" of the spectral sequence, e.g. $H(C)$ if the spectral sequence comes from a $\mathbb{Z}$-filtration of a chain complex C.

Let L^* denote the monoid of words of finite length with letters in L. Denote the empty word by ε, the concatenation of two words ω and ω' by $\omega * \omega'$, $\omega^n := \omega * \ldots * \omega$ (n times), and the length of ω by $|\omega|$. L^*_a becomes a poset via $\tau \leq \omega$ if and only if τ is a prefix of ω, that is, a subword that starts from the beginning ($\tau = \varepsilon$ and $\tau = \omega$ are allowed).

Definition 3.2 (Admissible words)**.** Call a finite word $\omega \in L^*$ *admissible* if the following holds:

1. if j^∞ appears, the subsequent subword of ω contains neither j nor j^∞,
2. the only letter allowed directly after j^∞ is x,
3. any x occurring in ω comes directly after some j^∞.

If furthermore ω contains subwords j^∞x for all $j \in [n]$ then ω is called *final*.

An exemplary final word for $n = 3$ is 123122^∞x133313^∞x111^∞x and any prefix of a final word is admissible. Let L^*_a denote the set of all admissible words in L^*. Define $X(\omega) \subseteq [n]$ as the set of $j \in [n]$ such that j^∞x is a subword of ω, and $Y(\omega) := [n] \setminus X(\omega)$. $X(\omega)$ is so to speak the set of indices along which the extension process has been already made, and $Y(\omega)$ is the set of indices along which we still have differentials.

For $\omega \in L^*_a$, $i, j \in [n]$, we inductively define $r^i_\omega, \delta^i_\omega \in \mathbb{Z}^n$ and $B_\omega, Z_\omega \subset \mathbb{Z}^n$ as follows. Put $r^i_\varepsilon := e_i$, $r^i_{\omega * j^\infty} := r^i_\omega$, $r^i_{\omega * \mathrm{x}} := r^i_\omega$, and

$$r^i_{\omega * j} := \begin{cases} r^i_\omega & \text{if } i \neq j, \\ r^i_\omega + \delta^i_\omega & \text{if } i = j, \end{cases}$$

where $\delta^i_\varepsilon := e_i$, $\delta^i_{\omega * j^\infty} := \delta^i_\omega$, $\delta^i_{\omega * \mathrm{x}} := \delta^i_\omega$, and

$$\delta^i_{\omega * j} := \begin{cases} \delta^i_\omega & \text{if } i \in X(\omega) \cup \{j\}, \\ \delta^i_\omega - \delta^j_\omega & \text{if } i \in Y(\omega) \setminus \{j\}. \end{cases}$$

For $\omega \in [n]^*$, $\delta^i_\omega = \mathbb{1} - \sum_{k \in [n] \setminus i} r^k_\omega$.

Remark 3.3 (Some intuition 2)**.** r^i_ω will be the negated direction of the i'th differential at the ω-page, and δ^i_ω is the negated change of direction for the i'th differential that occurs when taking homology with respect to it. In ordinary spectral sequences, $n = 1$, and for $\omega = 1^{r-1}$ the ω-page consists of the columns in E^r_{**}, with $r^1_\omega = r$, and $\delta^1_\omega = 1$.

Further put $B_\varepsilon := \{0\}$,

$$B_{\omega * j} := B_\omega + \{0, \delta^j_\omega\}, \tag{7}$$

$$B_{\omega * j^\infty} := B_\omega + \mathbb{Z}_{\geq 0} \cdot \delta^j_\omega, \tag{8}$$

$$B_{\omega * j^\infty \mathrm{x}} := B_\omega + \mathbb{Z} \cdot \delta^j_\omega. \tag{9}$$

Here plus denotes a Minkowski sum. Thus for $\omega \in [n]^*$, B_ω can be regarded as a discrete zonotope, that is, an affine image of the vertices of an $|\omega|$-dimensional cube. See Figures 1 and 2. Define $Z_\omega := -B_\omega$, and for $P \in \mathbb{Z}^n$,

$$S(P; \omega) := S^{P + Z_\omega}_{P + B_\omega}. \tag{10}$$

Below we show that this is indeed a well-defined S-term (only up to natural isomorphism of course) by constructing $(b, p, q, z) \in I_4$ such that S^{pz}_{bq} represents $S(P; \omega)$.

Define lattices $V_\omega \subseteq \mathbb{Z}^n$ for $\omega \in L^*_a$ inductively as follows. Put $V_\varepsilon := \{0\}$, $V_{\omega * j} := V_\omega$, $V_{\omega * j^\infty} := V_\omega$, and

$$V_{\omega * j^\infty \mathrm{x}} := V_\omega + \mathbb{Z} \cdot \delta^j_\omega.$$

Alternatively, $V_\omega = B_\omega \cap Z_\omega$. For $P, P' \in \mathbb{Z}^n$ with $P - P' \in V_\omega$, $S(P; \omega) = S(P'; \omega)$. Thus we may also think of $S(P; \omega)$ as being parametrized over $P \in \mathbb{Z}^n / V_\omega$.

Definition 3.4 (ω-page)**.** Let $\omega \in L^*_a$. We call the collection of all $S(P; \omega)$, $P \in \mathbb{Z}^n/V_\omega$, the *$\omega$-page.*

For $\omega = \varepsilon$ this was called the *first page* in [Mat13], for $\omega = 123 \dots n$ the *second page*, and for $\omega = 1^{q_1} \dots n^{q_n}$ a *generalized second page*, or the Q-page, where $Q = (q_1, \dots, q_n) \in \mathbb{Z}^n_{\geq 0}$.

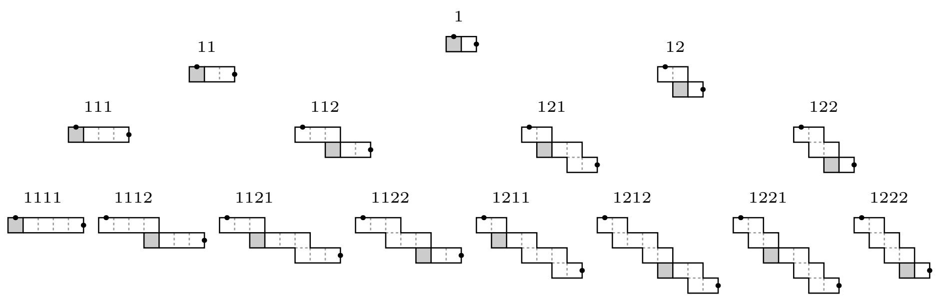

Figure 1: All B_ω with $|\omega| \leq 4$, $\omega_1 = 1$, and $n = 2$. For each B_ω, the origin is marked with a solid square, and the two points $r^i_\omega - e_i/2$ are marked with a black dot.

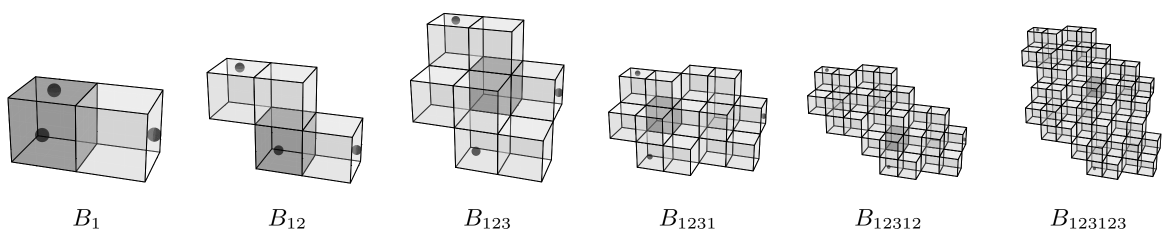

Figure 2: B_ω for $\omega = 1, \dots, 123123$ and $n = 3$. For each B_ω, the origin is marked with a dark cube, and the three points $r^i_\omega - e_i/2$ with a black dot.

Remark 3.5 (Relation between jj^∞ and j^∞)**.** Suppose $w \in L^*_a$ contains j^∞, and let w' be the same word except that j^∞ is replaced by $j^k j^\infty$ for some $k \geq 1$. Then in general, $r^i_\omega \neq r^i_{\omega'}$ and $\delta^i_\omega \neq \delta^i_{\omega'}$, but they always agree modulo $V_\omega = V_{\omega'}$. Also $B_\omega = B_{\omega'}$ and hence $S(P; \omega) = S(P; \omega')$. Moreover one can check that the differentials in the main theorem 3.6 below are the same for ω and ω'. Thus in order to speak about the ω-page it is enough to know the image of ω in the quotient semigroup $L^*/(jj^\infty \sim j^\infty)$.

Theorem 3.6 (Main theorem)**.** *Let $\omega \in L^*_a$ and $j \in [n]$ such that $\omega * j$ is admissible. Then the following holds.*

a) There are natural differentials

$$\dots \longrightarrow S(P + r^j_\omega; \omega) \longrightarrow S(P; \omega) \longrightarrow S(P - r^j_\omega; \omega) \to \dots. \tag{11}$$

Taking homology at $S(P; \omega)$ yields $S(P; \omega * j)$.

b) $S(P;\omega * j^\infty)$ *is a natural subquotient of* $S(P;\omega * j^k)$ *for all* $k \geq 0$*.*

c) There exists a natural $\mathbb{Z}$*-filtration* $(F_i)_{i\in\mathbb{Z}}$ *of* $S(P;\omega * j^\infty \mathrm{x})$*,*

$$0 \subseteq \ldots \subseteq F_i \subseteq F_{i+1} \subseteq \ldots \subseteq S(P;\omega * j^\infty \mathrm{x}),$$

such that $S(P + i \cdot \delta^j_\omega;\omega * j^\infty) \cong F_i/F_{i-1}$*, for all* $i \in \mathbb{Z}$*.*

Remark 3.7 (Multiplicative structure)**.** As usual, under certain assumptions on E there will be a multiplicative structure. The simplest instance is when E comes via (2) from an I-filtered differential algebra C, whose filtration $(F_p)_I$ satisfies $F_p \cdot F_q \subseteq F_{p+q}$, where $p + q$ denotes the Minkowski sum. Then for any $\omega \in L_a^*$ there is a natural product $S(P;\omega) \otimes S(Q;\omega) \to S(P + Q;\omega)$, which satisfies a Leibniz rule with respect to the differentials (a). Furthermore they are compatible with respect to (b) and (c) in the usual way, and for final ω it coincides with the product on $H(C)$. For details and a more general criterion see [Mat13, 4.4].

3.1 Proof of the main theorem

Before proving Theorem 3.6 we need some preparation. In particular it will be convenient to move to another basis spanned by the δ^i_ω, which depends on ω.

For $\omega \in L_a^*$, define

$$M_\omega := \begin{pmatrix}\delta^1_\omega & \cdots & \delta^n_\omega\end{pmatrix}^{-1} \in \mathbb{Q}^{n\times n}.$$

For $k \in [n]$ and $\omega \in L_a^*$, let T^k_ω denote the matrix

$$T^k_\omega := \mathrm{id}_{\mathbb{Z}^n} + e_k(\mathbb{1}_{X(\omega)\setminus\{k\}})^t = (\delta_{i=j \text{ or } (i=k \text{ and } j\in X(\omega))})_{ij} \in \mathrm{SL}(n,\mathbb{Z}).$$

A quick calculation shows that $M_\varepsilon = \mathrm{id}_n$, $M_{\omega*j^\infty} = M_\omega$, $M_{\omega*\mathrm{x}} = M_\omega$, and

$$M_{\omega*j} = T^j_\omega M_\omega.$$

Therefore, $M_\omega \in \mathrm{SL}(n,\mathbb{Z})$, and all its entries are non-negative. It will be very convenient to transform B_ω and V_ω by M_ω, so we define

$$B^M_\omega := M_\omega \cdot B_\omega, \quad V^M_\omega := M_\omega \cdot V_\omega \quad \subset \mathbb{Z}^n.$$

In particular,

$$B^M_{\omega*j} = T^j_\omega(B^M_\omega + \{0, e_j\}), \tag{12}$$
$$B^M_{\omega*j^\infty} = T^j_\omega(B^M_\omega + \mathbb{Z}_{\geq 0} \cdot e_j), \tag{13}$$
$$B^M_{\omega*j^\infty\mathrm{x}} = T^j_\omega(B^M_\omega + \mathbb{Z} \cdot e_j), \tag{14}$$

and $V^M_\omega = \mathbb{Z} \cdot \{e_i \mid i \in X(\omega)\}$. In particular, $\mathbb{Z}^n/V^M_\omega$ can be naturally identified with $\mathbb{Z}^{Y(\omega)}$.

Define $u_\omega \in \mathbb{Z}^n$ inductively via $u_\varepsilon := 0$, $u_{\omega*j^\infty} := u_\omega$, $u_{\omega*\mathrm{x}} := u_\omega$, and

$$u_{\omega*j} := e_j + T^j_\omega u_\omega.$$

For $n \geq 2$ and $\omega \in [n]^*$, a more explicit formula is $u_\omega = \frac{1}{n-1}(M_\omega\mathbb{1} - \mathbb{1})$.

The following lemmas about B_ω are stated and proved only in the special case when $\omega \in [n]^*$, since this simplifies the notation considerably. When working modulo V_ω and V^M_ω, respectively, analogous statements still hold for arbitrary $\omega \in L_a^*$, as long as ω does not end on some j^∞. It should be rather clear how to state and prove them.

Lemma 3.8. *For any* $\omega \in [n]^*$*,* B^M_ω *is a path in the unit-distance graph of* $\mathbb{Z}^n$ *from* 0 *to* u_ω*, which is monotone with respect to all coordinates.*

Proof. By induction, assume that the lemma holds for B_ω^M, and we want to prove it for $B_{\omega*j}^M$.

Suppose $x, x+e_k \in B_\omega^M$ are the vertices of an edge in B_ω^M. If $k \neq j$ then this edge gives rise to a path of length 2 in $B_{\omega*j}^M$ along the vertices $T_\omega^j x$, $T_\omega^j(x+e_j) = T_\omega^j x + e_j$, and $T_\omega^j(x+e_k) = T_\omega^j x + e_j + e_k$. If $k = j$ then this edge gives rise to an edge in $B_{\omega*j}^M$ whose vertices are $T_\omega^j x$ and $T_\omega^j(x+e_k) = T_\omega^j x + e_j$.

Moreover $u_{\omega*j} = T_\omega^j(u_\omega + e_j)$. □

Corollary 3.9. *For any $\omega \in [n]^*$, $B_\omega^M \subseteq \{x \in \mathbb{Z}^n \mid 0 \leq x \leq u_\omega\}$. Moreover, $B_\omega^M = u_\omega - B_\omega^M$.*

Lemma 3.10. *For any $\omega \in [n]^*$, $i, j \in [n]$, $x \in B_\omega^M$, the following holds:*

1. *Either $x - M_\omega e_i \in B_\omega^M$, or $x - M_\omega e_i \leq 0$, or both.*
2. *Either $x + M_\omega e_i \in B_\omega^M$, or $x + M_\omega e_i \geq u_\omega$, or both.*

Proof. Suppose the lemma holds for B_ω^M, and we want to prove it for $B_{\omega*j}^M$. By Corollary 3.9 it is enough to prove the first statement. By (12), any element of $B_{\omega*j}^M$ is of the form $T_\omega^j x$ or $T_\omega^j(x+e_j) = T_\omega^j x + e_j$ for some $x \in B_\omega^M$. If now $z := x - M_\omega e_i \in B_\omega^M$, then also $z' := T_\omega^j x - M_{\omega*j} e_i = T_\omega^j z$ and $z'' := T_\omega^j(x+e_j) - M_{\omega*j} e_i = T_\omega^j(z+e_j)$ lie in $B_{\omega*j}^M$.

Thus it remains to check the case when $z \leq 0$. Then clearly $z' = T_\omega^j z \leq 0$ since all entries of T_ω^j are non-negative. Similarly $z'' = T_\omega^j z + e_j \leq e_j$. If $z'' \leq 0$ does not hold, then $1 = z''_j = 1 + \sum_k z_k$, hence $z = 0$, thus $z'' = e_j$, which lies in $B_{\omega*j}^M$ since $0 \in B_\omega^M$. □

Let us regard any subset of $\mathbb{Z}^n$ as a graph by connecting any two elements with distance 1 by an edge. In particular we can then talk about connected components of such subsets.

Lemma 3.11. *For any $\omega \in [n]^*$, B_ω is connected.*

Proof. By induction one immediately sees that

$$r_\omega^j - e_j \in B_\omega. \tag{15}$$

Also by induction,

$$M_\omega r_\omega^j = u_\omega + e_j \in e_j + B_\omega^M, \tag{16}$$

from which we get

$$r_\omega^j \in \delta_\omega^j + B_\omega. \tag{17}$$

Thus if B_ω is connected then so is $B_{\omega*j}$ by (7), (15), and (17). □

For any subset $X \subseteq \mathbb{Z}^n$, let $\mathrm{Comp}_0(X)$ denote the connected component that contains 0.

From Lemmas 3.10 and 3.11 it follows that B_ω is $\mathrm{Comp}_0(X)$ where X is the intersection of the n "discrete hyperplanes"

$$\{x \in \mathbb{Z}^n \mid 0 \leq e_i^t M_\omega x \leq e_i^t u_\omega\}, \quad 1 \leq i \leq n.$$

For our purposes the following similar description of B_ω is more useful. The symmetric group $\mathfrak{S}_n$ acts on $\mathbb{Z}^n$ by permutation of the coordinates, $\sigma \cdot x := (x_{\sigma^{-1}(i)})_{i \in [n]}$. Let $\leq_{\text{lex}}$ denote the lexicographic relation on $\mathbb{Z}^n$. Define a new relation $\leq_{\sigma\text{-lex}}$ by setting $x \leq_{\sigma\text{-lex}} y$ if and only if $\sigma x \leq_{\text{lex}} \sigma y$.

For any $P \in (\mathbb{Z} \cup \{\infty\})^n$, $M \in \mathbb{Z}_{\geq 0}^{n \times n}$ with $\det M = 1$, and $\sigma \in \mathfrak{S}_n$, define

$$D(P; M, \sigma) := \{x \in \mathbb{Z}^n \mid Mx \leq_{\sigma\text{-lex}} MP\}$$

and

$$D^\circ(P; M, \sigma) := \{x \in \mathbb{Z}^n \mid Mx <_{\sigma\text{-lex}} MP\}.$$

Lemma 3.12. *For any $\omega \in [n]^*$, $j \in [n]$, and $\sigma \in \mathfrak{S}_n$ with $\sigma(j) = n$, the following two equations hold.*

1. $B_\omega = \mathrm{Comp}_0\big(D^\circ(r_\omega^j; M_\omega, \sigma) \setminus D^\circ(0; M_\omega, \sigma)\big)$.

2. $B_{\omega*j} = \mathrm{Comp}_0\big(D(r^j_\omega; M_\omega, \sigma) \setminus D^\circ(0; M_\omega, \sigma)\big)$.

Of course analogous formulas also hold for Z_ω and $Z_{\omega*j}$.

Proof. Since $\sigma(j) = n$, the only $z \in \mathbb{Z}^n$ with $u_\omega <_{\sigma\text{-lex}} M_\omega z \leq_{\sigma\text{-lex}} M_\omega r^j_\omega$ is r^j_ω. Now the first equation follows readily from Lemmas 3.10 and 3.11.

As for the second equation, "$\subseteq$" follows from $M_\omega B_{\omega*j} = M_\omega B_\omega + \{0, e_j\}$ and the connectivity of $B_{\omega*j}$. It remains to check "$\supseteq$". By Lemma 3.10 and (16) the only neighbor y of $x \in B_\omega$ such that $y \notin B_\omega$ and $y \in D(r^j_\omega; M_\omega, \sigma) \setminus D^\circ(0; M_\omega, \sigma)$ is $y = r^j_\omega$. Similarly (or by symmetry), the only neighbor y of $x \in B_\omega + \delta^j_\omega$ such that $y \notin B_\omega + \delta^j_\omega$ and $y \in D(r^j_\omega; M_\omega, \sigma) \setminus D^\circ(0; M_\omega, \sigma)$ is $y = 0$. Both, 0 and r^j_ω, lie in $B_{\omega*j}$, which proves the claimed equality. ☐

Proof of Theorem 3.6. We may assume $P = 0$, otherwise translate everything.

(a) We first consider the case $\omega \in [n]^*$. Let $\sigma \in \mathfrak{S}_n$ be any permutation with $\sigma(j) = n$. Define

$$\begin{aligned} p_\omega &:= D(0; M_\omega, \sigma), \\ q_\omega &:= D^\circ(0; M_\omega, \sigma) = p_\omega \setminus 0, \\ b_\omega &:= D^\circ(r^j_\omega; M_\omega, \sigma), \\ z_\omega &:= D(-r^j_\omega; M_\omega, \sigma). \end{aligned}$$

Then Lemma 3.12 implies that $S^{p_\omega z_\omega}_{b_\omega q_\omega}$ represents $S^{Z_\omega}_{B_\omega}$, which by definition is $S(0;\omega)$. One can describe $S(r^j_\omega;\omega)$ and $S(-r^j_\omega;\omega)$ similarly by translating all downsets by $\pm r^j_\omega$. Now Lemma 2.3 implies the claimed differentials in (11). Lemma 2.4 shows that taking homology in (11) at $S(0;\omega)$ yields $S^{p_\omega z^*_\omega}_{b^*_\omega q_\omega}$ where

$$\begin{aligned} b^*_\omega &:= D(r^j_\omega; M_\omega, \sigma) = b_\omega \cup \{r^j_\omega\}, \\ z^*_\omega &:= D^\circ(-r^j_\omega; M_\omega, \sigma) = z_\omega \setminus \{-r^j_\omega\}. \end{aligned}$$

By Lemma 3.12, $S^{p_\omega z^*_\omega}_{b^*_\omega q_\omega}$ represents $S^{Z_{\omega*j}}_{B_{\omega*j}} = S(0;\omega*j)$.

The general case $\omega \in L^*_a$ only needs minor modifications: In particular, the downsets b_ω, p_ω, q_ω, z_ω need to be replaced by the sum of V_ω with the analogous downsets in $\mathbb{Z}^n/V_\omega$. Explicitly, let $k := |X(\omega)|$, and choose a $\sigma \in \mathfrak{S}_n$ that satisfies $\sigma(j) = n-k$ and $\sigma(i) > n-k$ for all $i \in X(\omega)$. Then p_ω can be defined as

$$p_\omega := \{x \in \mathbb{Z}^n \mid M_\omega x \leq_{\sigma\text{-lex}} (0^{n-k}, \infty^k)\},$$

and put $q_\omega := p_\omega \setminus V_\omega$, $b_\omega := q_\omega + r^j_\omega$, $z_\omega := p_\omega - r^j_\omega$, $b^*_\omega := p_\omega + r^j_\omega = b_\omega \cup (r^j_\omega + V_\omega)$ and $z^*_\omega := q_\omega - r^j_\omega = z_\omega \setminus (-r^j_\omega + V_\omega)$. Note that $V^M_\omega = \mathbb{Z}^{X(\omega)}$. Now one can repeat the previous argument in the quotient space $\mathbb{Z}^n/V_\omega$.

(b) In M_ω-coordinates,

$$M_\omega B_{\omega*j^i} = B^M_\omega + \{0,\ldots,i\}.$$

Hence as in the proof of Lemma 3.12 one can show that

$$B_{\omega*j^i} = \mathrm{Comp}_0\big((b_\omega + i\cdot\delta^i_\omega) \setminus q_\omega\big),$$

where b_ω and q_ω are as above. Therefore $S(0;\omega*j^i)$ can be represented by $S^{p_\omega z^i_\omega}_{b^i_\omega q_\omega}$, where

$$\begin{aligned} b^i_\omega &:= b_\omega + i\cdot\delta^j_\omega = \{x \in \mathbb{Z}^n \mid M_\omega x \leq_{\sigma\text{-lex}} u_\omega + (0^{n-k-1}, i, \infty^k)^t\}, \\ z^i_\omega &:= z_\omega - i\cdot\delta^j_\omega = \{x \in \mathbb{Z}^n \mid M_\omega x \leq_{\sigma\text{-lex}} -u_\omega + (0^{n-k-1}, -i-1, \infty^k)^t\}. \end{aligned}$$

Let $b^\infty_\omega := \bigcup_i b^i_\omega$ and $z^\infty_\omega := \bigcap_i z^i_\omega$. Then $S(0;\omega*j^\infty) = S^{p_\omega z^\infty_\omega}_{b^\infty_\omega, q_\omega}$.

(c) Let $p_\omega, b^\infty_\omega, z^\infty_\omega$ as above. Put $p^i_\omega := i\cdot\delta^j_\omega + p_\omega$, $p^{-\infty}_\omega := \bigcap_{i\in\mathbb{Z}} p^i_\omega$, and $p^\infty_\omega := \bigcup_{i\in\mathbb{Z}} p^i_\omega$. Define

$$F_i := \mathrm{im}\Big(\ell : S^{p^i_\omega z^\infty_\omega}_{b^\infty_\omega p^{-\infty}_\omega} \to S^{p^\infty_\omega z^\infty_\omega}_{b^\infty_\omega p^{-\infty}_\omega}\Big).$$

Then the assertion follows from Lemma 2.2 (or from the proof of Lemma 2.5). ☐

3.2 The 2-dimensional case

The probably most frequent case (apart from the classical one, $n = 1$) is $n = 2$. A few more things can be said about this case:

Every final $\omega \in L_a^*$ is of the form

$$\omega = \tau * j_1^\infty \mathrm{x} j_2^k j_2^\infty \mathrm{x}, \tag{18}$$

for some $\tau \in [2]^*$, $\{j_1, j_2\} = [2]$, and $k \geq 0$. Any such ω gives a recipe to connect the first page to the limit of the spectral system. This recipe is therefore already determined by τ and j_1.

Note that for all prefixes $\tau \leq \omega' \leq \omega$, $M_\tau = M_{\omega'} = M_\omega$. Let's define

$$N_\omega := (e_{j_2})^t \cdot M_\omega,$$

which is the "normal vector" along which the downsets b, p, q, z grow respectively shrink during $j_2^k j_2^\infty \mathrm{x}$. Clearly $N_\omega \geq 0$ and it is primitive (i.e. its entries are coprime), and $N = (e_i)^t$ can happen only if $i = j_2$. Also, N_ω is invariant under the relation $jj^\infty \sim j^\infty$, compare with Remark 3.5.

Observation 3.13. *Modulo $jj^\infty \sim j^\infty$, ω is uniquely determined by N_ω and j_1. Conversely, for any primitive $N^t \in \mathbb{Z}^2_{\geq 0}$ and $j_1 \in [2]$ with $N^t \neq e_{j_1}$ there is a final $\omega \in L_a^*$ of the form* (18) *such that $N = N_\omega$.*

Thus the connection determined by ω can be equivalently described by N_ω and j_1.

Proof. In fact there is a simple algorithm that determines all possible τ from N (respectively N_ω) and $j_2 = 3 - j_1$. If $j_2 = 1$, choose $(N')^t \in \mathbb{Z}^2_{\geq 0}$ such that $M := \left(\begin{smallmatrix} N \\ N' \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z}) \cap \mathbb{Z}^{2\times 2}_{\geq 0}$. If $j_2 = 2$, choose $(N')^t \in \mathbb{Z}^2_{\geq 0}$ such that $M := \left(\begin{smallmatrix} N' \\ N \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z}) \cap \mathbb{Z}^{2\times 2}_{\geq 0}$. In any case, N' is well-defined up to adding an integral multiple of N. If N'_0 is the smallest choice, then all others are of the form $N'_k := N'_0 + kN$, $k \in \mathbb{Z}_{\geq 0}$. Now one can repetitively take one of the two rows of M and subtract it from the other one such that all entries stay non-negative until one arrives at $\mathrm{id}_{\mathbb{Z}^2}$, and there is a unique way to do that. Let $q_i \in [2]$ denote the index of the column from which the other column was subtracted during the i'th round. And say there were ℓ rounds. Then $M = M_\omega$ for $\omega = \tau j_1^\infty \mathrm{x} j_2^\infty \mathrm{x}$ and $\tau := q_\ell * \ldots * q_1$. The choice of N' correspond to how often j_1 appears at the end of τ, namely k times if $N' = N'_k$. □

The algorithm has similarities to the extended Euclidean algorithm applied to the first column of M.

Example 3.14. Consider an excisive exact couple system E over $I(\mathbb{Z}^2)$. Suppose we want to determine both ω for which $N_\omega = (3, 5)$. For $j_1 = 1$, the algorithm runs as follows:

$$\left(\begin{smallmatrix} 2 & 3 \\ 3 & 5 \end{smallmatrix}\right) \xrightarrow{2} \left(\begin{smallmatrix} 2 & 3 \\ 1 & 2 \end{smallmatrix}\right) \xrightarrow{1} \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right) \xrightarrow{2} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \xrightarrow{1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right),$$

Thus $\omega = 12121^\infty \mathrm{x} 2^\infty \mathrm{x}$ does it. Similarly, for $j_1 = 2$ one gets $\omega = 12112^\infty \mathrm{x} 1^\infty \mathrm{x}$. See Figure 3.

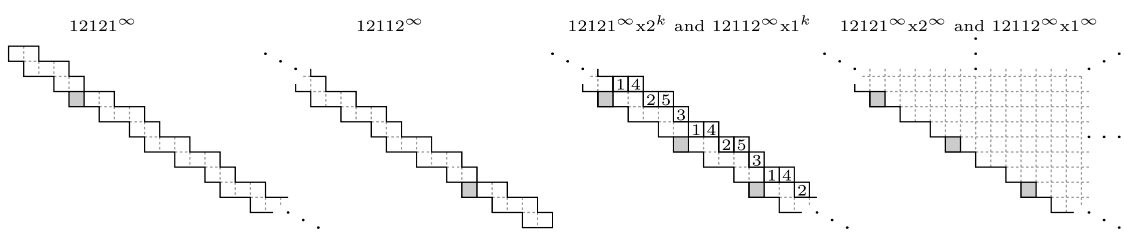

Figure 3: B_{12121^∞}, B_{12112^∞}, and $B_{12121^\infty \mathrm{x} 2^k} = B_{12112^\infty \mathrm{x} 1^k}$ for $0 \leq k \leq 5$ and for $k = \infty$. In the third figure, the squares with number i belong to $B_{12121^\infty \mathrm{x} 2^k}$ if and only if $i \leq k$. The solid squares depict V_σ.

Continued fractions. This algorithm is essentially the same as the one behind continued fractions: Write

$$(a_0, a_1, \ldots, a_\ell) := a_0 + \frac{1}{a_1 + \frac{1}{\ldots + \frac{1}{a_\ell}}}.$$

Let $N^t = (x, y) \in \mathbb{Z}^2_{\geq 0}$ be primitive, and write the slope of N as a continued fraction, $\frac{y}{x} = (a_0, \ldots, a_\ell, \infty)$, with $a_i \in \mathbb{Z}$, all positive except for possibly $a_0 = 0$. For each $N \neq e_i^t$, there are two such representations, namely one with $a_\ell \geq 2$, and one with $a_\ell = 1$: To get from the former representation to the latter, write $a_\ell = (a_\ell - 1) + \frac{1}{1}$. Comparing the above algorithm with the recursion for the successive convergents of this continued fraction, one sees immediately that $N = N_\omega$ for $\tau = 1^{a_0} 2^{a_1} 1^{a_2} 2^{a_3} \cdots$.

Fun fact 3.15. The golden ratio can be arbitrarily well approximated by the slope of N_ω using $\tau = (12)^k$, since for this τ, $M_\tau = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)^k = \left(\begin{smallmatrix} f_{2k-1} & f_{2k} \\ f_{2k} & f_{2k+1} \end{smallmatrix}\right)$, where f_k are the Fibonacci numbers. In terms of continued fractions, this is because the golden ratio satisfies $(1+\sqrt{5})/2 = (1, 1, 1, \ldots)$. However irrational slopes are not particularly useful, since one cannot connect the obtained page naturally to the limit (at least without further assumptions on E and without going backwards).

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