Flag algebras

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Asymptotic extremal combinatorics studies densities of small combinatorial structures in large ones of the same type. This talk surveys Razborov's so called flag algebras [2], which formalize common proof and calculation methods in that area.

For simplicity we restrict to the category of simple undirected graphs (and some subcategories). Razborov treats more generally any finite model theory.

1. Definitions

Definition 1.1. A **type** σ is a graph with labeled vertices $1, \dots, |\sigma|$. A **flag** F over σ is a pair of graphs (G, σ), σ being an induced subgraph of G (G is unlabeled). We write $|F| := |V(G)|$. A **morphism** between two flags $F = (G, \sigma)$ and $F' = (G', \sigma)$ is an injective graph homomorphism $m : G \to G'$ that is the identity on σ. This also clarifies what we mean by isomorphisms. We define the sum $F \cup_\sigma F'$ as $(G \cup_\sigma G', \sigma)$. A **sunflower** over σ is a sum of σ-flags $F_1 \cup_\sigma \dots \cup_\sigma F_n$; here $F_1, \dots, F_n$ are called **petals**. Let $\mathcal{F}^\sigma_\ell$ be the set of flags with $|F| = \ell$, and $\mathcal{F}^\sigma := \bigcup_\ell \mathcal{F}^\sigma_\ell$. For $F_1, \dots, F_n, F \in \mathcal{F}^\sigma$, define

$$p(F_1, \dots, F_n; F)$$

as the probability that a uniformly randomly chosen injective map $V(F_1 \cup_\sigma \dots \cup_\sigma F_n) \to V(F)$ extending id_σ yields an induced subgraph of F whose restriction to $V(F_i)$ is isomorphic to F_i, for all i.

Lemma 1.2 (Chain rule)**.** *If $|F_1 \cup_\sigma \dots \cup_\sigma F_n| \leq \ell \leq |F|$,*

$$p(F_1, \dots, F_n; F) = \sum_{\widetilde{F} \in \mathcal{F}^\sigma_\ell} p(F_1, \dots, F_n; \widetilde{F}) p(\widetilde{F}; F).$$

Definition 1.3. For a type σ, we define the **flag algebra**

$$\mathcal{A}^\sigma := (\mathbb{R}\mathcal{F}^\sigma)/K^\sigma,$$

where $K^\sigma := \langle F - \sum_{\widetilde{F} \in \mathcal{F}^\sigma_\ell} p(F; \widetilde{F}) \widetilde{F} \mid F \in \mathcal{F}^\sigma \text{ and } \ell \geq |F| \rangle$.

It is instructive to think of the basis elements $F \in \mathbb{R}\mathcal{F}^\sigma$ as densities of F in some large fixed σ-flag X. Modding out K^σ then implements the chain rule.

Lemma 1.4 (Product)**.** *There is a product $\mathcal{A}^\sigma \otimes \mathcal{A}^\sigma \to \mathcal{A}^\sigma$ defined by*

$$F_1 \cdot F_2 := \sum_{F \in \mathcal{F}^\sigma_\ell} p(F_1, F_2; F)$$

for any $\ell \geq |F_1 \cup_\sigma F_2|$. This makes $\mathcal{A}^\sigma$ into a commutative $\mathbb{R}$-algebra with unit $1_\sigma := (\sigma, \sigma)$.

Lemma 1.5. *A flag $F = (G, \sigma)$ is called connected if $G \backslash \sigma$ is a connected graph. Fix a connected flag $F_0 \in \mathcal{F}^\sigma_{|\sigma|+1}$. Then $\mathcal{A}^\sigma$ is a polynomial algebra over $\mathbb{R}$, freely generated by all connected flags except for 1_σ and F_0.*

2. Motivation

Let $\mathrm{Hom}(\mathcal{A}^\sigma;\mathbb{R})$ denote all algebra homomorphisms from $\mathcal{A}^\sigma$ to $\mathbb{R}$. Define

$$\mathrm{Hom}^+(\mathcal{A}^\sigma;\mathbb{R}) := \{\varphi \in \mathrm{Hom}(\mathcal{A}^\sigma;\mathbb{R}) \mid \varphi(F) \geq 0 \text{ for all } F \in \mathcal{F}^\sigma\}.$$

We define the **semantic cone** as

$$C_{sem}(\mathcal{A}^\sigma) := \{f \in \mathcal{A}^\sigma \mid \varphi(f) \geq 0 \text{ for all } \varphi \in \mathrm{Hom}^+(\mathcal{A}^\sigma;\mathbb{R})\}.$$

Thus, $C_{sem}(\mathcal{A}^\sigma)$ is obtained by polarizing twice the cone in $\mathcal{A}^\sigma$ spanned by all σ-flags. We write $f \succeq_\sigma g$ if $f - g \in C_{sem}(\mathcal{A}^\sigma)$.

The following theorem is Razborov's version of a theorem of Lovász and Szegedy [1]. It follows from the fact that $\mathrm{Hom}^+(\mathcal{A}^\sigma;\mathbb{R}) \subseteq [0,1]^{\mathcal{F}^\sigma}$ is the set of all limit point (with respect to the product topology in $[0,1]^{\mathcal{F}^\sigma}$) of sequences $(p(\_;F_i))_{i\in\mathbb{N}}$.

Theorem 2.1. *Let $f \in \mathbb{R}[x_1,\dots,x_n]$. Then $f(F_1,\dots,F_n) \in C_{sem}(\mathcal{F}^\sigma)$ if and only if*

$$\liminf_{F\in\mathcal{F}^\sigma} f(p(F_1,F),\dots,p(F_n,F)) \geq 0. \tag{2.2}$$

Several intersting statments in asymptotic extremal combinatorics can be written in the form (2.2). Theorem 2.1 then gives us a reformulation of that in terms of $C_{sem}(\mathcal{F}^\sigma)$. Below we review some criteria for when an element of $\mathcal{A}^\sigma$ lies in $C_{sem}(\mathcal{F}^\sigma)$.

3. Cauchy–Schwarz inequality

Definition 3.1 (Restriction operator)**.** Let $\sigma_0 \subseteq \sigma$ be a sub-type. We define a linear map (in general not an algebra homomorphism) $[\![\_]\!]_{\sigma,\sigma_0} : \mathcal{A}^\sigma \to \mathcal{A}^{\sigma_0}$ via $[\![F]\!]_{\sigma,\sigma_0} := q_{\sigma,\sigma_0}(F)F|_{\sigma_0}$, where $F = (G,\sigma)$, $F|_{\sigma_0} := (G,\sigma_0)$, and $q_{\sigma,\sigma_0}(F) \in [0,1]$ is the probability that a (uniformly) random extension $V(\sigma) \hookrightarrow G$ of the embedding $V(\sigma_0) \hookrightarrow G$ induces a flag that is isomorphic to F.

For $\sigma_0 \subseteq \sigma_1 \subseteq \sigma_2$, we have $[\![F]\!]_{\sigma_2,\sigma_0} = [\![[\![F]\!]_{\sigma_2,\sigma_1}]\!]_{\sigma_1,\sigma_0}$.

Theorem 3.2 (Cauchy–Schwarz inequality for $\mathcal{A}^\sigma$)**.** *For any $f, g \in \mathcal{A}^\sigma$ and $\sigma_0 \subseteq \sigma$,*

$$[\![f^2]\!]_{\sigma,\sigma_0} \cdot [\![g^2]\!]_{\sigma,\sigma_0} \succeq_{\sigma_0} [\![fg]\!]^2_{\sigma,\sigma_0}.$$

As an application, one obtains Goodman's bound relating the asymptotic edge- and triangle densities, which states (as flags over $\sigma = \emptyset$) that $K_3 \succeq K_2(2K_2 - K_1)$. For the proof one applies the Cauchy–Schwarz inequality for the flags (K_2,K_1) and (K_1,K_1) with $\sigma = K_1$ and $\sigma_0 = K_0$.

4. Differential method

We write the types K_0, K_1, K_2 and $\bar{K}_2$ as 0, 1, E and $\bar{E}$, respectively.

We define a linear map (in general not multiplicative) $\partial_1 : \mathcal{A}^0 \to \mathcal{A}^1$ by

$$\partial_1 G := \ell\Big(\sum_{\substack{(H,1)\in\mathcal{F}^1_{\ell+1}\\ G\cong H\setminus 1}} (H,1) - \sum_{\substack{(H,1)\in\mathcal{F}^1_\ell\\ H\cong G}} (H,1)\Big),$$

where $\ell := |G|$.

Further, define a linear map $\partial_E : \mathcal{A}^{\bar{E}} \to \mathcal{A}^E$ by

$$\partial_E(G, \bar{E}) := \binom{\ell}{2}\Big(\sum_{\substack{(H,\bar{E}) \in \mathcal{F}_\ell^{\bar{E}} \\ H \cong G}} (H \cup_{\bar{E}} E, E) - \sum_{\substack{(H,E) \in \mathcal{F}_\ell^{E} \\ H \cong G}} (H, E)\Big).$$

Theorem 4.1. *Let $G_1, \ldots, G_n$ be finite graphs. Consider $\varphi_0 \in \mathrm{Hom}^+(\mathcal{A}^0; \mathbb{R})$ and $f \in C^1(U)$ for some open subset $U \subseteq \mathbb{R}^n$, such that $\Phi : \mathrm{Hom}^+(\mathcal{A}^0; \mathbb{R}) \to \mathbb{R}$ given by $\Phi(\varphi) := f(\varphi(G_1), \ldots, \varphi(G_n))$ is maximal at φ_0 among all φ such that $(\varphi(G_1), \ldots, \varphi(G_n)) \in U$. Then, for any $g \in \mathcal{A}^1$,*

$$\varphi_0([\![g \cdot \partial_1 \langle \nabla f, (G_1, \ldots, G_n) \rangle]\!]_{1,0}) = 0.$$

Furthermore, for any $g \in C_{sem}(\mathcal{A}^E)$,

$$\varphi_0([\![g \cdot \partial_E \langle \nabla f, (G_1, \ldots, G_n) \rangle]\!]_{E,0}) \geq 0.$$

As an application, Razborov [2, 3] calculated the asymptotically minimal possible triangle density in a graph for any given edge density. Based on a similar ideas, Reiher [4] calculated more generally the minimal possible K_k-density in a graph with a given edge density, and this not only asymptotically.

References

[1] L. Lovász, B. Szegedy. *Limits of dense graph sequences*, Technical Report TR-2004-79, Microsoft Research, August 2004, 27 pages.

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[4] C. Reiher. *The Clique Density Theorem*, preprint, arXiv:1212.2454, 2012, 25 pages.