Curvature and rational homotopy I – Many manifolds with bounded curvature and diameter

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1. Introduction

Bounding geometric invariants of Riemannian manifolds (such as sectional curvature, diameter or volume) often puts strong conditions on the diffeomorphism types of manifolds that fulfill these bounds.

The first talk of this conference gave already several instances, valid for all complete Riemannian manifolds M:

Theorem (von Mangoldt–Hadarmard–Cartan)**.** *If* $\sec(M) \leq 0$ *then the universal cover of* M *is diffeomorphic to* $\mathbb{R}^n$*. In particular,* $\pi_{*\geq 2}(M) = 0$*.*

Theorem (Bonnet–Myers)**.** *If* $\sec(M) \geq \delta > 0$ *then* $\operatorname{diam}(M) \leq \pi/\sqrt{\delta}$*. In particular,* M *and its universal cover are compact and* $\pi_1(M)$ *is finite.*

Theorem (Sphere Theorem, Rauch–Berger–Klingenberg, Brendle–Schoen)**.** *If* $1/4 < \sec(M) \leq 1$ *and* $\pi_1(M) = 0$ *then* M *is diffeomorphic to a sphere.*

Theorem (Cheeger–Peters)**.** *For all* $n, C, C', D, V > 0$ *there are only finitely many diffeomorphism types of closed smooth* n*-manifolds admitting a Riemannian metric such that* $C \leq \sec(M) \leq C'$*,* $\operatorname{diam}(M) \leq D$*, and* $\operatorname{vol}(M) > V$*.*

In a similar manner, Grove asked the following question (see [2]):

Question. *Does the class* $M^{\leq D}_{-1\leq \sec\leq +1}(n)$ *of simply connected* n*-manifolds of diameter at most* D *and sectional curvature bounded by* -1 *and* $+1$ *contain only finitely many rational homotopy types?*

In this talk we present the solution of Fang & Rong [1] and Totaro [3] who showed that the perhaps surprising answer to Grove's question is in general No.

Theorem 1.1 (Fang–Rong)**.** *For all* $n \geq 22$ *there exists a* $D > 0$ *such that* $M^{\leq D}_{-1\leq \sec\leq +1}(n)$ *contains infinitely many rational homotopy types.*

Remark. Their proof can be easily extended: Their examples already work in dimension $n \geq 20$, and they have already pairwise non-isomorphics rational cohomology rings.

Theorem 1.2 (Totaro)**.** *There exists a* $D > 0$ *such that* $M^{\leq D}_{-1\leq \sec\leq +1}(7)$ *contains infinitely many rational homotopy types.*

Again, Totaro's examples have already pairwise different rational cohomology rings.

2. Fang & Rong's approach

The construction of Fang & Rong's examples M_i, $i \in \mathbb{N}$, works as follows.

(1) Find a suitable principal T^3-bundle of manifolds $M \to B$, where T^3 is the 3-torus, and give M a T^3-invariant metric g.
(2) For suitable two-dimensional subtori $T_i \subseteq T^3$, let $M_i := M/T_i$.
(3) Do the construction in such a way that M_i have pairwise distinct rational homotopy.

Since the quotient maps $M \to M_i$ are Riemannian submersions and the 2-dimensional subtori T^2 of T^3 can be parametrized by the compact Stiefel manifold $V_{3,2}$, one can deduce that the sectional curvatures of all such quotients M/T^2 are uniformly bounded from above and below. Hence by scaling the metric g we can assume that $-1 \leq \sec(M_i) \leq +1$ for all i. Let $D := \operatorname{diam}(M)$. Then $\operatorname{diam}(M_i) \leq D$. In the construction we still need to take care of (3).

Fang and Rong start by constructing models $\mathcal{M}_i$ that will be part of the minimal models of M_i. First, let

$$\mathcal{M}_8 := (\Lambda(x_1, x_2, x_3, y, z), d),$$

where $|x_i| = 0$, $d(x_i) = 0$, $|y| = 5$, $d(y) = x_1^2 x^2$, $|z| = 7$, $d(z) = x_1^4 + x_2^4 + x_3^4$.

Lemma. *There is a CW-complex X whose minimal model $\mathcal{M}_E$ is $\mathcal{M}_8$.*

For this, first take $K(\mathbb{Z}^3, 2) = (\mathbb{C}P^\infty)^3$ whose minimal model is $(\Lambda(x_1, x_2, x_3), 0)$. Then take the pullback of the path-loop fibration $PK(\mathbb{Z}, 6) \to K(\mathbb{Z}, 6)$ along the map $K(\mathbb{Z}^3, 2) \to K(\mathbb{Z}, 6)$ that corresponds to the cohomology class $x_1^2 x_2 \in H^6(K(\mathbb{Z}^3, 2), \mathbb{Z})$. This adds y with $d(y) = x_1^2 x_2$ to the minimal model. Then add z in an analogous way.

Let $X^{(9)}$ be the 9-skeleton of X.

Lemma. *There exists a closed 19-manifold B and a map $X^{(9)} \to B$ that induces an isomorphism on $\pi_{*\leq 8}$.*

For this, embed $X^{(9)}$ into $\mathbb{R}^{17}$, thicken it to an open set N, and take the double $B := N \cup_{\partial N} (-N)$. Then one can show with Morse theory that B has a handle decomposition whose i-handles with $i \leq 9$ correspond to the i-cells of X.

Now $H^2(B; \mathbb{Z}) = \mathbb{Z}^3 = \langle x_1, x_2, x_3 \rangle$. Define M_i to be the principal S^1-bundle over B with Euler class $e_i = ix_1 + x_2 - x_3$. Let $\mathcal{M}_i(8)$ be the submodel of the minimal model of M_i generated by all elements in degree less then or equal to 8, i.e. the minimal model of the 8'th space in the Postnikov tower of M_i.

Lemma. *$\mathcal{M}_i(8) = (\Lambda(x_1, x_2, y, z_i), d)$, where $d(x_i) = 0$, $d(y) = x_1^2 x_2$, $d(z_i) = x_1^4 + x_2^4 + (ix_1 + x_2)^4$.*

For this, find a suitable morphism from the above to the relative minimal model of $M_i \to B$, which is $(\mathcal{M}_B \otimes \Lambda(t), d)$, $d(t) = e_i$.

Lemma. *If $i \neq j$ then $H^{*\leq 8}(M_i; \mathbb{Q}) \ncong H^{*\leq 8}(M_i; \mathbb{Q})$ as rings.*

This is staight forward computation. Now we let M be the following pullback, where f induces $\pi_2(B) \cong \pi_2(BT^3) = \mathbb{Z}^3$,

$$\begin{array}{ccc} M & \longrightarrow & ET^3 \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & BT^3. \end{array}$$

We write $T^3 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid |z_i| = 1\}$, and let $T_i := \langle (1, z, z), (z, 1, z^i) \rangle$.

Lemma. *$M_i \cong M/T_i$.*

For this, compare the Euler classes. The lemmas complete the proof of Fang–Rong's Theorem 1.1.

3. Totaro's approach

The proof of Totaro's Theorem 1.2 relies on the deeper theorem of Berge–Sullivan, however it needs less computation and finds examples already in dimension 7. First, Totaro defines a cdga

$$H = \lambda(x_0, \ldots, x_4)/R,$$

where $|x_i| = 2$, $R = \langle x_i^2 = x_{i+1}x_{i+2}, x_i x_j = 0\ (j \neq i, i \pm 1) \rangle$. This model satisfies Poincaré duality. Hence by the Theorem of Berge–Sullivan there exists a closed 6-manifold B whose minimal model is the minimal model of H. As above we find a principal T^5-bundle $M \to B$ that is classified by the map $B \to BT^5 = K(\mathbb{Z}, 2)^5$ induced by the cohomology classes $x_0, \ldots, x_4$. Choose a T^5-invariant metric on M and consider quotients of M by 4-subtori $T^4 \subset T^5$. Let T_{a_0,a_1} be the subtorus such that $M_{a_0,a_1} = M/T_{a_0,a_1} \to B$ is the S^1-bundle whose Euler class is $e = a_0x_0 - a_1x_1 + (a_1^3/a_0^2)x_3$. We have $H^2(M_{a_0,a_1}; \mathbb{Q}) = H^2(B; \mathbb{Q})/\langle e \rangle$.

Lemma. *The map $H^2(B) \otimes H^2(B) \to H^6(B) \cong \mathbb{Q}$ that sends $a \otimes b$ to $a \cup b \cup e$ factors through $H^2(M_{a_0,a_1}) \otimes H^2(M_{a_0,a_1})$.*

This gives a quadratic form on $H^2(M_{a_0,a_1})$ whose determinant is well defined in $\mathbb{Q}/(\mathbb{Q}^*)^2$, and which can be computed to be $-a_0a_1(\mathbb{Q}^*)^2$. Hence different choices of a_0 and a_1 give infinitely many M_{a_0,a_1} with pairwise distinct rational cohomology rings.

The curvature and diameter arguments are the same as in proof of Fang–Rong, which finishes the proof of Totato's Theorem 1.2.

References

[1] F. Fang, X. Rong. *Curvature, diameter, homotopy groups, and cohomology rings*, Duke Math. J. 107 (1), 135-158.

[2] Y. Félix, J. Oprea, D. Tantré. *Algebraic Models in Geometry*, Oxford Graduate Texts in Mathematics 17, 2008.

[3] B. Totaro. *Curvature, diameter, and quotient manifolds*, Math. Res. Lett. 10 (2-3), 191-203.