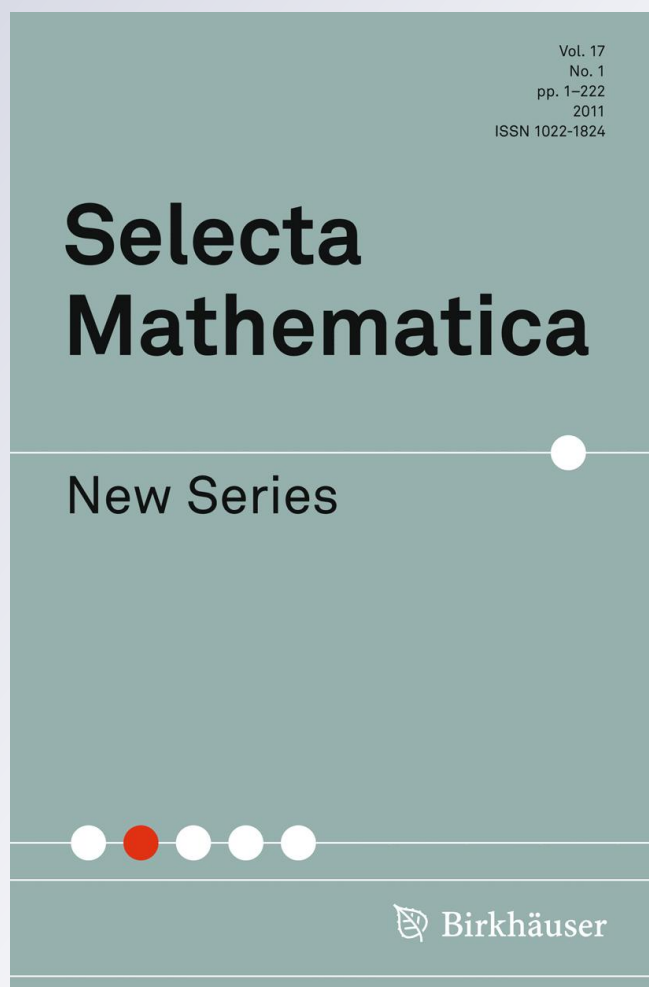


The Weil-étale fundamental group of a number field II

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The Weil-étale fundamental group of a number field II

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Abstract We define the fundamental group underlying the Weil-étale cohomology of number rings. To this aim, we define the Weil-étale topos as a refinement of the Weil-étale sites introduced by Lichtenbaum (Ann Math 170(2):657–683, 2009). We show that the (small) Weil-étale topos of a smooth projective curve defined in this paper is equivalent to the natural definition. Then we compute the Weil-étale fundamental group of an open subscheme of the spectrum of a number ring. Our fundamental group is a projective system of locally compact topological groups, which represents first degree cohomology with coefficients in locally compact abelian groups. We apply this result to compute the Weil-étale cohomology in low degrees and to prove that the Weil-étale topos of a number ring satisfies the expected properties of the conjectural Lichtenbaum topos.

Keywords Weil-étale cohomology · Topos · Fundamental group · Dedekind zeta function

Mathematics Subject Classification (2010) 14F20 · 14F35 · 11R42

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1 Introduction

Lichtenbaum has defined in [8] the Weil-étale cohomology of a number ring $X = \text{Spec}(\mathcal{O}_F)$. He has shown that the resulting cohomology groups with compact support $H_{Wc}^i(X, \mathbb{Z})$ for $i \leq 3$ are related to the special value of the Dedekind zeta function $\zeta_F(s)$ at $s = 0$. In this paper, we refine Lichtenbaum’s construction in order to define and compute the Weil-étale fundamental group.

As observed in [11], the Weil-étale cohomology introduced in [8] is not defined as the cohomology of a Grothendieck site (i.e. of a topos). More precisely, Lichtenbaum defined in [8] a family of sites $T_{F/K,S}$ for any finite Galois extension K/F and any suitable finite set S of primes of F . Then, he defined the Weil-étale cohomology as the direct limit $\varinjlim H^*(T_{L/K,S}, -)$. In this paper, we define a single Weil-étale topos \bar{X}_W which recovers Lichtenbaum’s computations. Here, \bar{X} denotes the Arakelov compactification of $X = \text{Spec}(\mathcal{O}_F)$. The topos \bar{X}_W is endowed with a morphism to the Artin–Verdier étale topos \bar{X}_{et} . This point of view has some technical advantages. For example, the same definition is used in [4] to define the Weil-étale topos of an arithmetic scheme as a fiber product.

Motivated by a question asked by Lichtenbaum (see the introduction of [8]), we show in Sect. 3.5 that our definition of the (small) Weil-étale topos of a function field is equivalent to the natural definition given in [7]. The same result is actually false with the original definition of [8]. More precisely, let Y be an open subscheme of a

smooth projective curve over a finite field k , and denote by $\mathcal{S}_{et}(W_k, \bar{Y})$ the topos of W_k -equivariant étale sheaves on the geometric curve $\bar{Y} = Y \otimes_k \bar{k}$.

Theorem 1.1 *There is an equivalence*

$$Y_W^{sm} \simeq \mathcal{S}_{et}(W_k, \bar{Y})$$

where Y_W^{sm} is the (small) Weil-étale topos defined in this paper.

Section 4 is devoted to the computation of the Weil-étale fundamental group. Let \bar{U} be a connected étale \bar{X} -scheme. We define the Weil-étale topos of \bar{U} as the slice topos $\bar{U}_W := \bar{X}_W / \gamma^* \bar{U}$. Let K be the number field corresponding to the generic point of \bar{U} , and let $q_{\bar{U}} : \text{Spec}(\bar{K}) \rightarrow \bar{U}$ be a geometric point. The étale fundamental group $\pi_1(\bar{U}_{et}, q_{\bar{U}})$ is a (strict) projective system of finite quotients of the Galois group G_K . Replacing Galois groups with Weil-groups, we define the analogous (strict) projective system $\underline{W}(\bar{U}, q_{\bar{U}})$ of locally compact quotients of the Weil group W_K . The following theorem gives a computation of the fundamental group of \bar{U}_W .

Theorem 1.2 *The Weil-étale topos \bar{U}_W is connected and locally connected over the topos \mathcal{T} of locally compact spaces. The geometric point $q_{\bar{U}}$ defines a \mathcal{T} -valued point $p_{\bar{U}}$ of the topos \bar{U}_W , and we have an isomorphism*

$$\pi_1(\bar{U}_W, p_{\bar{U}}) \simeq \underline{W}(\bar{U}, q_{\bar{U}})$$

of topological pro-groups.

The consequences of this result are given in Sect. 6. We denote by $C_{\bar{U}} := C_{K,S}$ the S -idèle class group associated to \bar{U} (here S is the set of places of K not corresponding to a point of \bar{U}).

Corollary 1.3 *For any connected étale \bar{X} -scheme \bar{U} , we have an isomorphism of topological groups $\pi_1(\bar{U}_W, p_{\bar{U}})^{ab} \simeq C_{\bar{U}}$. In particular, for any locally compact abelian group A , we have*

$$H^1(\bar{U}_W, A) = \text{Hom}_{cont}(C_{\bar{U}}, A)$$

In particular $\pi_1(\bar{X}_W)^{ab}$ is topologically isomorphic to the Arakelov Picard group $\text{Pic}(\bar{X})$ of the number field F , and the canonical class is the canonical continuous morphism

$$\theta \in H^1(\bar{X}_W, \tilde{\mathbb{R}}) = \text{Hom}_{cont}(\text{Pic}(\bar{X}), \mathbb{R}).$$

The previous corollary allows one to compute the cohomology of the Weil-étale topos in low degrees and to recover Lichtenbaum's computations.

Matthias Flach has shown in [3] that the current definition of the Weil-étale cohomology is not yet the right one. More precisely, the groups $H_W^i(\bar{X}, \mathbb{Z})$ are infinitely generated for any $i \geq 4$ even. But the conjectural picture still stands. Indeed, Lichtenbaum conjectures in [8] the existence of a Grothendieck topology for an arithmetic

scheme X such that the Euler characteristic of the cohomology groups of the constant sheaf \mathbb{Z} with compact support at infinity gives, up to sign, the leading term of the zeta-function $\zeta_X(s)$ at $s = 0$. In [12], we gave a list of axioms that should be satisfied by the category of sheaves on this conjectural Grothendieck topology for $X = \text{Spec}(\mathcal{O}_F)$. We denote by \bar{X}_L this conjectural category of sheaves, and we refer to the list of axioms that must be satisfied by \bar{X}_L as Axioms (1) – (9). We also showed in [12] that any topos satisfying these axioms gives rise to complexes of étale sheaves computing the expected Lichtenbaum cohomology. The author's main motivation for the present work was to provide an example of a topos satisfying Axioms (1) – (9).

Corollary 1.4 *The Weil-étale topos \bar{X}_W satisfies Axioms (1) – (9).*

This result shows that Axioms (1)–(9) are consistent. Moreover, it gives a natural computation of the base change from the Weil-étale cohomology to the étale cohomology (see Corollary 6.13). More precisely, let $\gamma : \bar{X}_W \rightarrow \bar{X}_{\text{ét}}$ be the canonical map, and let $\varphi : X_W \rightarrow \bar{X}_W$ be the open embedding. For any abelian sheaf \mathcal{A} , we denote by $\tau_{\leq 2} R\gamma_* \mathcal{A}$ the truncated complex.

Corollary 1.5 *Assume that F is totally imaginary. Then, the Euler characteristic of the hypercohomology groups of the complex of étale sheaves $\tau_{\leq 2} R\gamma_*(\varphi_* \mathbb{Z})$ gives, up to sign, the leading term of the Dedekind zeta-function $\zeta_F(s)$ at $s = 0$.*

2 Preliminaries

2.1 Left exact sites

The category of sheaves of sets on a Grothendieck site $(\mathcal{C}, \mathcal{J})$ is denoted by $\widetilde{(\mathcal{C}, \mathcal{J})}$ while the category of presheaves on \mathcal{C} is denoted by $\widehat{\mathcal{C}}$. A Grothendieck topology \mathcal{J} on a category \mathcal{C} is said to be *sub-canonical* if \mathcal{J} is coarser than the canonical topology. This is the case precisely when any representable presheaf on \mathcal{C} is a sheaf for the topology \mathcal{J} . A family of morphisms $\{X_i \rightarrow X\}$ in \mathcal{C} is said to be a *covering family* for the topology \mathcal{J} when the sieve generated by this family of morphisms belongs to $\mathcal{J}(X)$. A category \mathcal{C} is said to be *left exact* when finite projective limits exist in \mathcal{C} , i.e. when \mathcal{C} has a final object and fiber products. A functor between left exact categories is said to be left exact if it commutes with finite projective limits.

Definition 2.1 A Grothendieck site $(\mathcal{C}, \mathcal{J})$ is said to be *left exact* if \mathcal{C} is a left exact category endowed with a subcanonical topology \mathcal{J} . A *morphism of left exact sites* $(\mathcal{C}', \mathcal{J}') \rightarrow (\mathcal{C}, \mathcal{J})$ is a continuous left exact functor $\mathcal{C}' \rightarrow \mathcal{C}$.

Note that any Grothendieck topos, i.e. any category satisfying Giraud's axioms, is equivalent to the category of sheaves of sets on a left exact site. Note also that a Grothendieck site $(\mathcal{C}, \mathcal{J})$ is left exact if and only if the canonical functor (given in general by Yoneda and sheafification) $y : \mathcal{C} \rightarrow \widetilde{(\mathcal{C}, \mathcal{J})}$ identifies \mathcal{C} with a left exact

full subcategory of $(\widetilde{\mathcal{C}}, \widetilde{\mathcal{J}})$. A *morphism of left exact sites* $f^* : (\mathcal{C}', \mathcal{J}') \rightarrow (\mathcal{C}, \mathcal{J})$ induces a morphism of topoi $f : (\widetilde{\mathcal{C}}, \widetilde{\mathcal{J}}) \rightarrow (\widetilde{\mathcal{C}'}, \widetilde{\mathcal{J}'})$, such that the following diagram is commutative

$$\begin{array}{ccc} (\widetilde{\mathcal{C}}, \widetilde{\mathcal{J}}) & \xleftarrow{f^*} & (\widetilde{\mathcal{C}'}, \widetilde{\mathcal{J}'}) \\ \uparrow & & \uparrow \\ \mathcal{C} & \xleftarrow{f^*} & \mathcal{C}' \end{array}$$

where the vertical arrows are the fully faithful Yoneda functors.

Finally, recall that for any object X of \mathcal{C} , one has a canonical equivalence

$$(\widetilde{\mathcal{C}}, \widetilde{\mathcal{J}})/yX \simeq (\mathcal{C}/X, \widetilde{\mathcal{J}}_{ind})$$

where \mathcal{J}_{ind} is the topology on \mathcal{C}/X induced by \mathcal{J} via the forgetful functor $\mathcal{C}/X \rightarrow \mathcal{C}$ (forget the map to X).

2.2 Basic properties of geometric morphisms

Let \mathcal{S} and \mathcal{S}' be two Grothendieck topoi. A (*geometric*) *morphism of topoi*

$$f := (f^*, f_*) : \mathcal{S}' \longrightarrow \mathcal{S}$$

is defined as a pair of functors (f^*, f_*) , where $f^* : \mathcal{S} \rightarrow \mathcal{S}'$ is left adjoint to $f_* : \mathcal{S}' \rightarrow \mathcal{S}$ and f^* is left exact (i.e. f^* commutes with finite projective limits). One can also define such a morphism as a left exact functor $f^* : \mathcal{S} \rightarrow \mathcal{S}'$ commuting with arbitrary inductive limits. Indeed, in this case, f^* has a uniquely determined right adjoint f_* .

If X is an object of \mathcal{S} , then the slice category \mathcal{S}/X , of objects of \mathcal{S} over X , is a topos as well. The base change functor

$$\begin{array}{l} \mathcal{S} \longrightarrow \mathcal{S}/X \\ Y \longmapsto Y \times X \end{array}$$

is left exact and commutes with arbitrary inductive limits, since inductive limits are universal in a topos. We obtain a morphism

$$\mathcal{S}/X \longrightarrow \mathcal{S}.$$

Such a morphism is said to be a *localization morphism* or a *local homeomorphism* (the term local homeomorphism is inspired by the case when \mathcal{S} is the topos of sheaves on some topological space). For any morphism $f : \mathcal{S}' \rightarrow \mathcal{S}$ and any object X of \mathcal{S} , there is a natural morphism

$$f|_X : \mathcal{S}'/f^*X \longrightarrow \mathcal{S}/X.$$

The functor $f_{/X}^*$ is defined in the obvious way: $f_{/X}^*(Y \rightarrow X) = (f^*Y \rightarrow f^*X)$. The direct image functor $f_{/X,*}$ sends $Z \rightarrow f^*X$ to $f_*Z \times_{f_*f^*X} X \rightarrow X$, where $X \rightarrow f_*f^*X$ is the adjunction map. The morphism $f_{/X}$ is a pull-back of f , in the sense that the square

$$\begin{array}{ccc} S'/f^*X & \xrightarrow{f_{/X}} & S/X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

is commutative and 2-cartesian. In other words, the 2-fiber product $S' \times_S S/X$ can be defined as the slice topos S'/f^*X .

A morphism $f : S' \rightarrow S$ is said to be *connected* if f^* is fully faithful. It is *locally connected* if f^* has an S -indexed left adjoint $f_!$ (see [6, C3.3]). These definitions generalize the usual ones for topological spaces: if T is a topological space, consider the unique morphism $Sh(T) \rightarrow \underline{Sets}$ where $Sh(T)$ is the category of étalé spaces over T . For example, a localization morphism $S/X \rightarrow S$ is always locally connected (here $f_!(Y \rightarrow X) = Y$), but is connected if and only if X is the final object of S .

A morphism $f : S' \rightarrow S$ is said to be an *embedding* when f_* is fully faithful. It is an *open embedding* if f factors through $f : S' \simeq S/X \rightarrow S$, where X is a subobject of the final object of S . Then, the essential image \mathcal{U} of the functor f_* is said to be an *open subtopos* of S . The *closed complement* \mathcal{F} of \mathcal{U} is the strictly full subcategory of S consisting in objects Y such that $Y \times X$ is the final object of \mathcal{U} (i.e. f^*Y is the final object of S'). A *closed subtopos* \mathcal{F} of S is a strictly full subcategory which is the closed complement of an open subtopos. A morphism of topoi $i : \mathcal{E} \rightarrow S$ is said to be a *closed embedding* if i factors through $i : \mathcal{E} \simeq \mathcal{F} \rightarrow S$ where \mathcal{F} is a closed subtopos of S .

A *subtopos* of S is a strictly full subcategory S' of S such that the inclusion functor $i : S' \hookrightarrow S$ is the direct image of a morphism of topoi (i.e. i has a left exact left adjoint). A morphism $f : S' \rightarrow S$ is said to be *surjective* if f^* is faithful. Any morphism $f : \mathcal{E} \rightarrow S$ can be decomposed as a surjection $\mathcal{E} \rightarrow Im(f)$ followed by an embedding $Im(f) \rightarrow S$, where $Im(f)$ is a subtopos of S , which is called the *image of f* (see [5, IV. 9.1.7.2]).

2.3 The topos \mathcal{T} of locally compact topological spaces

In this paper, we denote by Top the category of locally compact topological spaces and continuous maps. Locally compact spaces are assumed to be Hausdorff. This category is endowed with the open cover topology \mathcal{J}_{op} , which is generated by the following pretopology: a family of morphisms $(X_\alpha \rightarrow X)_{\alpha \in A}$ is in $Cov(X)$ if and only if $(X_\alpha \rightarrow X)_{\alpha \in A}$ is an open cover in the usual sense. We denote by \mathcal{T} the topos of sheaves of sets on this left exact site:

$$\mathcal{T} := (\widehat{Top}, \widetilde{\mathcal{J}_{op}}).$$

The family of compact spaces is a topologically generating family for the site (Top, \mathcal{J}_{op}) . Indeed, if X is a locally compact space, then any $x \in X$ has a compact neighborhood $K_x \subseteq X$, so $(K_x \hookrightarrow X)_{x \in X}$ is a local section cover, hence a covering family for \mathcal{J}_{op} . In particular, if we denote by Top^c the category of compact spaces, then the canonical morphism $\mathcal{T} \rightarrow (\widetilde{Top^c}, \mathcal{J}_{op})$ is an equivalence.

The Yoneda functor

$$\begin{aligned} y : Top &\longrightarrow \mathcal{T} \\ X &\longmapsto y(X) = Hom_{Top}(-, X) \end{aligned}$$

sending a locally compact space to the sheaf represented by this space is fully faithful (since \mathcal{J}_{op} is subcanonical) and commutes with arbitrary projective limits. Since the Yoneda functor is left exact, any locally compact topological group G represents a group object of \mathcal{T} . In what follows we consider Top as a (left exact) full subcategory of \mathcal{T} . For example, the sheaf of \mathcal{T} represented by a locally compact space Z is sometimes also denoted by Z .

Remark 2.2 In this paper, we consider topoi defined over the topos \mathcal{T} of locally compact spaces since all sheaves, cohomology groups and fundamental groups that we use are defined by locally compact spaces. In order to use non-locally compact coefficients, one can consider the topos

$$\mathcal{T}' := (\widetilde{Top^h}, \mathcal{J}_{op})$$

where Top^h is the category of Hausdorff spaces. Then for any topos \mathcal{E} (connected and locally connected) over \mathcal{T} , we consider the base change $\mathcal{E} \times_{\mathcal{T}} \mathcal{T}'$ to obtain a (connected and locally connected) topos over \mathcal{T}' .

2.4 Topological pro-groups

In this paper, a *filtered category* I is a non-empty small category such that the following holds. For any objects i and j of I , there exists a third object k and maps $i \leftarrow k \rightarrow j$. For any pair of maps $i \rightrightarrows j$, there exists a map $k \rightarrow i$ such that the diagram $k \rightarrow i \rightrightarrows j$ is commutative. Let C be any category. A *pro-object* of C is a functor $\underline{X} : I \rightarrow C$, where I is a filtered category. One can see a pro-object in C as a diagram in C . One can define the *category* $Pro(C)$ of *pro-objects in* C (see [5, I. 8.10]). The morphisms in this category can be made explicit as follows. Let $\underline{X} : I \rightarrow C$ and $\underline{Y} : J \rightarrow C$ be two pro-objects in C . Then one has

$$Hom_{Pro(C)}(\underline{X}, \underline{Y}) := \lim_{\leftarrow j \in J} \lim_{\rightarrow i \in I} Hom(X_i, Y_j).$$

A pro-object $\underline{X} : I \rightarrow C$ is *constant* if it is a constant functor, and *essentially constant* if \underline{X} is isomorphic (in the category $Pro(C)$) to a constant pro-object.

Definition 2.3 A *locally compact topological pro-group* \underline{G} is a pro-object in the category of locally compact topological groups. A locally compact topological pro-group is said to be *strict* if any transition map $G_j \rightarrow G_i$ has local sections.

If the category C is a topos, then a pro-object $\underline{X} : I \rightarrow C$ in C is said to be *strict* when the transition map $X_i \rightarrow X_j$ is an epimorphism in C , for any $i \rightarrow j \in Fl(I)$. In particular, a locally compact topological pro-group $\underline{G} : I \rightarrow Gr(Top)$ pro-represents a strict pro-group-object in \mathcal{T} :

$$y \circ \underline{G} : I \rightarrow Gr(Top) \rightarrow Gr(\mathcal{T})$$

where $Gr(Top)$ and $Gr(\mathcal{T})$ are the categories of group-objects in Top and \mathcal{T} , respectively. Indeed, a continuous map of locally compact spaces $X_i \rightarrow X_j$ has local sections if and only if it induces an epimorphism $y(X_i) \rightarrow y(X_j)$ in \mathcal{T} .

2.5 The classifying topos of a group-object

2.5.1 General case

For any topos \mathcal{S} and any group object \mathcal{G} in \mathcal{S} , we denote by $B_{\mathcal{G}}$ the category of (left) \mathcal{G} -object in \mathcal{S} . Then, $B_{\mathcal{G}}$ is a topos, as it follows from Giraud's axioms, and $B_{\mathcal{G}}$ is endowed with a canonical morphism $B_{\mathcal{G}} \rightarrow \mathcal{S}$, whose inverse image functor sends an object \mathcal{F} of \mathcal{S} to \mathcal{F} with trivial \mathcal{G} -action. If there is a risk of ambiguity, we denote the topos $B_{\mathcal{G}}$ by $B_{\mathcal{S}}(\mathcal{G})$. The topos $B_{\mathcal{G}}$ is said to be the classifying topos of \mathcal{G} since it classifies \mathcal{G} -torsors. More precisely, for any topos $f : \mathcal{E} \rightarrow \mathcal{S}$ over \mathcal{S} , the category $\underline{Hom}_{top_{\mathcal{S}}}(\mathcal{E}, B_{\mathcal{G}})$ is anti-equivalent to the category of $f^*\mathcal{G}$ -torsors in \mathcal{E} (see [5, IV Exercise 5.9]). It follows that the induced morphism

$$B_{\mathcal{E}}(f^*\mathcal{G}) \longrightarrow \mathcal{E} \times_{\mathcal{S}} B_{\mathcal{S}}(\mathcal{G})$$

is an equivalence (see [10, Corollary 10.14]).

2.5.2 Examples

Let G be a discrete group, i.e. a group object of the final topos \underline{Sets} . We denote the category of G -sets by $B_G^{sm} := B_{\underline{Sets}}(G)$. The topos B_G^{sm} is called the small classifying topos of the discrete group G . If G is a profinite group, then the small classifying topos B_G^{sm} is defined as the category of sets on which G acts continuously.

Let G be a locally compact topological group. Then, G represents a group object of \mathcal{T} , where $\mathcal{T} := (\widehat{Top}, \widehat{\mathcal{J}}_{op})$ is defined above. Then

$$B_G := B_{\mathcal{T}}(G)$$

is the classifying topos of the locally compact topological group G . One can define the classifying topos of an arbitrary topological group by enlarging the topos \mathcal{T} .

2.5.3 The local section site

For any locally compact topological group G , we denote by $B_{Top}(G)$ the category of G -equivariant locally compact topological spaces (elements of a given universe). The category $B_{Top}(G)$ is endowed with the local section topology \mathcal{J}_{ls} , which can be described as follows. A family of morphisms $\{X_i \rightarrow X, i \in I\}$ in $B_{Top}(G)$ is a covering family for \mathcal{J}_{ls} if and only if the continuous map $\coprod_{i \in I} X_i \rightarrow X$ has local sections. Equivalently, \mathcal{J}_{ls} is the topology induced by the open cover topology on Top via the forgetful functor $B_{Top}(G) \rightarrow Top$. The Yoneda functor yields a continuous fully faithful functor

$$B_{Top}(G) \longrightarrow B_G,$$

and the induced morphism

$$B_G \longrightarrow (B_{Top}(G), \mathcal{J}_{ls})$$

is an equivalence (see [3]).

2.6 The classifying topos of a strict topological pro-group

Topos theory provides a natural way to define the limit of a strict topological pro-group without any loss of information.

Definition 2.4 The classifying topos of a strict locally compact topological pro-group $\underline{G} : I \rightarrow Gr(Top)$ is defined as

$$B_{\underline{G}} := \varprojlim_I B_{G_i},$$

where the projective limit is computed in the 2-category of topoi.

By ([5, VI.8.2.3]), a site for the projective limit topos $B_{\underline{G}}$ is given by $(\varinjlim_I B_{Top}G_i, \mathcal{J})$, where $\varinjlim_I B_{Top}G_i$ is the direct limit category and \mathcal{J} is the coarsest topology such that all the functors

$$(B_{Top}G_i, \mathcal{J}_{ls}) \longrightarrow (\varinjlim_I B_{Top}G_i, \mathcal{J})$$

are continuous. The direct limit category

$$B_{Top}\underline{G} := \varinjlim_I B_{Top}G_i$$

can be made explicit as follows. An object of this category is given by a locally compact topological space on which G_i acts continuously for some $i \in I$. Let Z_i and Z_j be two objects of $B_{Top}\underline{G}$, such that Z_i and Z_j are given with an action of G_i and G_j , respectively. Then, there exists an index $k \in I$ and maps $G_k \rightarrow G_i$ and $G_k \rightarrow G_j$

admitting local sections. Then, a morphism $Z_j \rightarrow Z_i$ is a G_k -equivariant continuous map $Z_j \rightarrow Z_i$. Consider the forgetful functor

$$B_{Top}\underline{G} \longrightarrow Top.$$

One can prove that the topology \mathcal{J} on $B_{Top}\underline{G}$ is induced by the local section topology on Top via this forgetful functor, so that the topology \mathcal{J} can be denoted by \mathcal{J}_{ls} . We have obtained the following result.

Proposition 2.5 *The site $(B_{Top}\underline{G}, \mathcal{J}_{ls})$ is a site for the classifying topos of the strict topological pro-group \underline{G} . In other words, the natural morphism*

$$B_{\underline{G}} \longrightarrow (B_{Top}\underline{G}, \widetilde{\mathcal{J}}_{ls})$$

is an equivalence.

3 The Weil-étale topos

In this section, we define a topos satisfying the expected properties of the conjectural Lichtenbaum topos (see [12]). This yields a new computation of the Weil-étale cohomology. Our construction is a suitable refinement of the family of Weil-étale sites introduced by Lichtenbaum in [8]. We denote by $\bar{X} = (Spec \mathcal{O}_F, X_\infty)$ the Arakelov compactification of the ring of integers in a number field F (i.e. X_∞ is the set of archimedean places of F).

3.1 The Weil-étale topos

As an illustration of the artificiality of the following construction, we start by making several non-canonical choices.

Data 3.1 1. We choose an algebraic closure \bar{F}/F .

2. We choose a Weil group W_F .

3. For any place v of F , we choose an algebraic closure \bar{F}_v/F_v .

4. For any place v of F , we choose a local Weil group W_{F_v} .

5. For any place v of F , we choose an embedding $\bar{F} \rightarrow \bar{F}_v$ over F .

6. For any place v of F , we choose a Weil map $\theta_v : W_{F_v} \longrightarrow W_F$.

These choices are required to be compatible in the sense that the diagram

$$\begin{array}{ccc} W_{F_v} & \longrightarrow & G_{F_v} \\ \theta_v \downarrow & & \downarrow \\ W_F & \longrightarrow & G_F \end{array}$$

is commutative for any place v .

Recall that if \bar{F}/F is an algebraic closure and $\bar{F}/K/F$ a finite Galois extension then the relative Weil group $W_{K/F}$ is defined by the extension of topological groups

$$1 \rightarrow C_K \rightarrow W_{K/F} \rightarrow G_{K/F} \rightarrow 1$$

corresponding to the fundamental class in $H^2(G_{K/F}, C_K)$ given by class field theory, where C_K is the idèle class group of K . A Weil group of F is then defined as the projective limit $W_F := \varprojlim W_{K/F}$, computed in the category of topological groups. A Weil group for the local field F_v is defined as earlier, replacing the idèle class group C_K with the multiplicative group K_w^\times where K_w/F_v is finite and Galois.

Definition 3.2 Let $W_{F_v}^1$ be the maximal compact subgroup of W_{F_v} . The Weil group of the “residue field” at $v \in \bar{X}$ is defined as $W_{k(v)} := W_{F_v} / W_{F_v}^1$. We denote by

$$q_v : W_{F_v} \longrightarrow W_{F_v} / W_{F_v}^1 =: W_{k(v)}$$

the map from the local Weil group W_{F_v} to the Weil group of the residue field at v .

Lichtenbaum defined in [8] a family of sites $T_{K/F,S}$ for K/F Galois and S a suitable finite set of primes of F . Then, he defined the Weil-étale cohomology as the direct limit of the cohomologies of the sites $T_{K/F,S}$. Here, we define a single site $T_{\bar{X}}$ inspired by a closer look at the étale site. This allows us to define a Weil-étale topos, over the Artin-Verdier étale topos, giving rise to the Weil-étale cohomology without the direct limit process.

Definition 3.3 Let $T_{\bar{X}}$ be the category of objects (Z_0, Z_v, f_v) defined as follows. The topological space Z_0 is endowed with a continuous W_F -action. For any place v of F , Z_v is a topological space endowed with a continuous $W_{k(v)}$ -action. The continuous map $f_v : Z_v \rightarrow Z_0$ is W_{F_v} -equivariant, when Z_v and Z_0 are seen as W_{F_v} -spaces via the maps $\theta_v : W_{F_v} \rightarrow W_F$ and $q_v : W_{F_v} \rightarrow W_{k(v)}$. Moreover, we require the following facts.

- The map f_v is an homeomorphism for almost all places v of F and a continuous injective map for all places.
- For any valuation v , the space Z_v is locally compact.
- The action of W_F on Z_0 factors through $W_{K/F}$, for some finite Galois subextension $\bar{F}/K/F$.

A morphism

$$\phi : (Z_0, Z_v, f_v) \longrightarrow (Z'_0, Z'_v, f'_v)$$

in the category $T_{\bar{X}}$ is a continuous W_F -equivariant map $\phi : Z_0 \rightarrow Z'_0$ inducing a continuous map $\phi_v : Z_v \rightarrow Z'_v$ for any place v . Then, ϕ_v is necessarily $W_{k(v)}$ -equivariant.

The category $T_{\bar{X}}$ is endowed with the local section topology \mathcal{J}_{I_S} , i.e. the topology generated by the pretopology for which a family

$$\{\varphi_i : (Z_{i,0}, Z_{i,v}, f_{i,v}) \rightarrow (Z_0, Z_v, f_v), i \in I\}$$

is a covering family if the continuous map $\coprod_{i \in I} Z_{i,v} \rightarrow Z_v$ has local sections, for any place v .

Definition 3.4 We define the *Weil-étale topos* \bar{X}_W as the topos of sheaves of sets on the site defined earlier:

$$\bar{X}_W := (\widetilde{T_{\bar{X}}}, \mathcal{J}_{I_S}).$$

Remark 3.5 One can extend the previous definition to any étale \bar{X} -scheme. If one does so, the Weil-étale topos is no longer functorial (see Sect. 3.4).

Lemma 3.6 *The site $(T_{\bar{X}}, \mathcal{J}_{I_S})$ is a left exact site.*

Proof The category $T_{\bar{X}}$ has fiber products and a final object; hence, finite projective limits are representable in $T_{\bar{X}}$. It remains to show that \mathcal{J}_{I_S} is subcanonical. But for any topological group G , the local section topology $\mathcal{J}_{I_S} = \mathcal{J}_{op}$ on $B_{Top}G$ is nothing else than the open cover topology (see [3, Corollary 2]), which is easily seen to be subcanonical. The result follows easily from this fact. \square

Proposition 3.7 *We have a morphism of topoi*

$$j : B_{W_F} \longrightarrow \bar{X}_W.$$

Proof The classifying topos B_{W_F} is defined as the topos of $y(W_F)$ -objects of \mathcal{T} and the site $(B_{Top}W_F, \mathcal{J}_{I_S})$ is a defining site for B_{W_F} (see Sect. 2.5.3). We have a morphism of left exact sites

$$\begin{aligned} j^* : (T_{\bar{X}}, \mathcal{J}_{I_S}) &\longrightarrow (B_{Top}W_F, \mathcal{J}_{I_S}) \\ (Z_0, Z_v, f_v) &\longmapsto Z_0. \end{aligned}$$

inducing the morphism of topoi j . \square

We have a morphism of left exact sites

$$\begin{aligned} t^* : (Top, \mathcal{J}_{op}) &\longrightarrow (T_{\bar{X}}, \mathcal{J}_{I_S}) \\ Z &\longmapsto (Z, Z, Id_Z) \end{aligned} \tag{3.1}$$

Definition 3.8 The canonical morphism from \bar{X}_W to \mathcal{T} is the morphism of topoi

$$t : \bar{X}_W \longrightarrow \mathcal{T}$$

induced by the morphism of left exact sites (3.1).

Consider the functor $u^* : \mathcal{T} \rightarrow B_{W_F}$ sending an object \mathcal{L} of \mathcal{T} to \mathcal{L} with trivial $y(W_F)$ -action. This functor commutes with arbitrary inductive and arbitrary projective limits. Therefore, we have a sequence of three adjoint functors $u_! , u^* , u_*$. More explicitly, one has $u_! \mathcal{F} = \mathcal{F}/y(W_F)$ and $u_*(\mathcal{F}) = \underline{Hom}_{W_F}(\{*\}, \mathcal{F})$, where $\{*\}$ is

the final object of B_{W_F} . We have a connected (u^* is clearly fully faithful) and locally connected morphism

$$u : B_{W_F} \longrightarrow \mathcal{T}.$$

The topos B_{W_F} has a canonical point q over \mathcal{T} . In other words, there exists a section $q : \mathcal{T} \rightarrow B_{W_F}$ of the structure map $u : B_{W_F} \rightarrow \mathcal{T}$. Indeed, the inverse image of the morphism q is the functor $q^* : B_{W_F} \rightarrow \mathcal{T}$ sending a $y(W_F)$ -object \mathcal{F} to \mathcal{F} with no action. Therefore, we have a canonical isomorphism of functor $Id \simeq q^* \circ u^*$, hence an isomorphism of morphisms of topoi:

$$Id \simeq u \circ q : \mathcal{T} \rightarrow B_{W_F} \rightarrow \mathcal{T}. \tag{3.2}$$

Of course everything above is valid for any topological group G (and more generally for any group object \mathcal{G} in any topos \mathcal{E}).

Proposition 3.9 *One has a canonical isomorphism*

$$u \simeq t \circ j : B_{W_F} \longrightarrow \bar{X}_W \longrightarrow \mathcal{T}$$

In particular the morphism $j \circ q$ is a point of \bar{X}_W over \mathcal{T} , i.e. the following diagram is commutative.

$$\begin{array}{ccc} B_{W_F} & \xrightarrow{j} & \bar{X}_W \\ q \uparrow & \nearrow p & \downarrow t \\ \mathcal{T} & \xrightarrow{Id} & \mathcal{T} \end{array}$$

If there is a risk of ambiguity, the point p of \bar{X}_W over \mathcal{T} will be denoted by $p_{\bar{X}}$.

Proof The first claim of the proposition follows immediately from the description of these morphisms of topoi in terms of morphisms of left exact sites. The second claim then follows from (3.2). □

Proposition 3.10 *The morphism $t : \bar{X}_W \rightarrow \mathcal{T}$ is connected and locally connected.*

Proof Let us first make the inverse image functor t^* explicit. Consider the full subcategory $\mathbb{C}_{\bar{X}}$ of $T_{\bar{X}}$ consisting in objects (Z_0, Z_v, f_v) such that the quotient space Z_0/W_F is locally compact and such that the canonical morphism in T

$$y(Z_0)/y(W_F) \longrightarrow y(Z_0/W_F)$$

is an isomorphism. By Corollary 4.21, $\mathbb{C}_{\bar{X}}$ is a *topologically generating* family of $T_{\bar{X}}$ (see [5, II 3.0.1]). Hence, the sheaf $t^*\mathcal{L}$ is completely determined by its restriction to $\mathbb{C}_{\bar{X}}$, for any \mathcal{L} of \mathcal{T} .

One can prove that that one has

$$t^* \mathcal{L}(Z_0, Z_v, f_v) = \text{Hom}_{B_{W_F}}(y(Z_0), u^* \mathcal{L}) = u^* \mathcal{L}(Z_0). \quad (3.3)$$

for any object \mathcal{L} of \mathcal{T} and any $(Z_0, Z_v, f_v) \in \mathbb{C}_{\bar{X}}$. Indeed, we check that one has a bifunctorial isomorphism, in $\mathcal{L} \in \mathcal{L}$ and $\mathcal{F} \in \bar{X}_W$:

$$\text{Hom}_{\bar{X}_W}(t^* \mathcal{L}, \mathcal{F}) \simeq \text{Hom}_{\mathcal{T}}(\mathcal{L}, t_* \mathcal{F}),$$

where $t^* \mathcal{L}$ is defined as above and $t_* \mathcal{F}(Z) := \mathcal{F}(Z, Z, Id_Z)$ for any $Z \in \text{Ob}(\text{Top})$. The proof of this fact is tedious but straightforward, using the fact that $(Z_0/W_F, Z_0/W_F, Id) \in \mathbb{C}_{\bar{X}}$ and the identification $y(Z_0/W_F) = y(Z_0)/y(W_F)$.

More generally, we have

$$\text{Hom}_{\bar{X}_W}(\mathcal{F}, t^* \mathcal{L}) = \text{Hom}_{B_{W_F}}(j^* \mathcal{F}, u^* \mathcal{L}) \quad (3.4)$$

for any object \mathcal{F} of \bar{X}_W and any object \mathcal{L} of \mathcal{T} . Indeed the family of representable objects $y(Z_0, Z_v, f_v)$ is a generating family of \bar{X}_W (see [5, II Proposition 4.10]) hence any object \mathcal{F} of \bar{X}_W can be written as an (arbitrary) inductive limit of such representable objects (see [5, I Proposition 7.2]). Therefore, (3.4) follows from (3.3) and from the fact that j^* commutes with inductive limits and with the Yoneda embedding.

If \mathcal{L} and \mathcal{L}' are two objects of \mathcal{T} , then one has

$$\begin{aligned} \text{Hom}_{\bar{X}_W}(t^* \mathcal{L}', t^* \mathcal{L}) &= \text{Hom}_{B_{W_F}}(j^* t^* \mathcal{L}', u^* \mathcal{L}) \\ &= \text{Hom}_{B_{W_F}}(u^* \mathcal{L}', u^* \mathcal{L}) \\ &= \text{Hom}_{\mathcal{T}}(\mathcal{L}', \mathcal{L}) \end{aligned}$$

since $t \circ j \simeq u$ and u^* is fully faithful. Hence, t^* is fully faithful, i.e. t is connected.

Let us define the left adjoint of t^* . We consider the functor defined by

$$t_! \mathcal{F} := u_! j^* \mathcal{F} = j^* \mathcal{F} / y(W_F)$$

where the quotient $j^* \mathcal{F} / y(W_F)$ is defined in \mathcal{T} , for any object \mathcal{F} of \bar{X}_W . The following identifications show that $t_!$ is left adjoint to t^* .

$$\text{Hom}_{\mathcal{T}}(t_! \mathcal{F}, \mathcal{L}) = \text{Hom}_{\mathcal{T}}(u_! j^* \mathcal{F}, \mathcal{L}) = \text{Hom}_{B_{W_F}}(j^* \mathcal{F}, u^* \mathcal{L}) = \text{Hom}_{\bar{X}_W}(\mathcal{F}, t^* \mathcal{L}).$$

It remains to show that the functor $t_!$ is a \mathcal{T} -indexed left adjoint to t^* . This means that for any morphism $x : I \rightarrow J$ in \mathcal{T} , the natural transformation

$$t_!^I \circ (t^* x)^* \rightarrow x^* \circ t_!^J \quad (3.5)$$

defined by the square

$$\begin{array}{ccc} \bar{X}_W/t^*I & \xrightarrow{t_!^I} & \mathcal{T}/I \\ (t^*x)^* \uparrow & & \uparrow x^* \\ \bar{X}_W/t^*J & \xrightarrow{t_!^J} & \mathcal{T}/J \end{array}$$

should be an isomorphism (see [6, B.3.1.1]). Here, the functor

$$\begin{aligned} x^* : \mathcal{T}/J &\longrightarrow \mathcal{T}/I \\ (\mathcal{L} \rightarrow J) &\longmapsto (\mathcal{L} \times_J I \rightarrow I) \end{aligned}$$

is the usual base change and one has

$$\begin{aligned} t_!^J : \bar{X}_W/t^*J &\longrightarrow \mathcal{T}/J \\ (\mathcal{F} \rightarrow t^*J) &\longrightarrow (t_!\mathcal{F} \rightarrow t_!t^*J \rightarrow J) \end{aligned}$$

where the map $t_!t^*J \rightarrow J$ is given by adjunction. Let $\mathcal{F} \rightarrow t^*J$ be an object of \bar{X}_W/t^*J , and denote it by \mathcal{F} . On the one hand, one has

$$t_!^I \circ (t^*x)^* \mathcal{F} = t_!(\mathcal{F} \times_{t^*J} t^*I)$$

and

$$x^* \circ t_!^J (\mathcal{F} \rightarrow t^*J) = t_!(\mathcal{F}) \times_J I$$

one the other. Hence, the natural transformation (3.5) is given by the canonical morphism from

$$t_!(\mathcal{F} \times_{t^*J} t^*I) = u_!j^*(\mathcal{F} \times_{t^*J} t^*I) = u_!(j^*\mathcal{F} \times_{j^*t^*J} j^*t^*I) = u_!(j^*\mathcal{F} \times_{u^*J} u^*I)$$

to

$$u_!j^*\mathcal{F} \times_{u_!u^*J} u_!u^*I \simeq u_!j^*\mathcal{F} \times_J I.$$

This morphism is an isomorphism because $u : B_{W_F} \rightarrow \mathcal{T}$ is connected and locally connected. Indeed, the adjunction map $u_!u^*I \rightarrow I$ is an isomorphism since u^* is fully faithful. Then,

$$u_!(j^*\mathcal{F} \times_{u^*J} u^*I) = (j^*\mathcal{F} \times_J I)/yW_F$$

is canonically isomorphic to

$$(j^*\mathcal{F}/yW_F) \times_J I$$

since inductive limits (in particular quotients of group actions) are universal in \mathcal{T} . For a down to earth argument proving the very last claim of this proof, one can use the fact that \mathcal{T} has enough Sets-valued points, and check that $(j^*\mathcal{F} \times_J I)/yW_F \rightarrow (j^*\mathcal{F}/yW_F) \times_J I$ induces isomorphisms on stalks. \square

Definition 3.11 An object \mathcal{F} of \bar{X}_W is said to be *constant over \mathcal{T}* if there is an isomorphism $\mathcal{F} \simeq t^*\mathcal{L}$, where \mathcal{L} is an object of \mathcal{T} .

Corollary 3.12 *If \mathcal{F} is a constant object over \mathcal{T} then the adjunction map*

$$\mathcal{F} \longrightarrow j_*j^*\mathcal{F}$$

is an isomorphism.

Proof This follows immediately from (3.3). Indeed, if $\mathcal{F} = t^*\mathcal{L}$ then

$$\mathcal{F}(Z_0, Z_v, f_v) = t^*\mathcal{L}(Z_0, Z_v, f_v) = u^*\mathcal{L}(Z_0) = j^*\mathcal{F}(Z_0) = j_*j^*\mathcal{F}(Z_0, Z_v, f_v).$$

for any object (Z_0, Z_v, f_v) of $T_{\bar{X}}$. \square

Definition 3.13 Let \mathcal{F} be an object of \bar{X}_W . The object of \mathcal{T}

$$t_!\mathcal{F} := (j^*\mathcal{F})/y(W_F)$$

is called the *space of connected components* of \mathcal{F} .

Definition 3.14 An object \mathcal{F} of \bar{X}_W is said to be *connected over \mathcal{T}* if its space of connected components $t_!\mathcal{F}$ is the final object of \mathcal{T} .

Consider for example a constant object $\mathcal{F} = t^*\mathcal{L}$ over \mathcal{T} . Then, the space of connected components of \mathcal{F} is

$$t_!\mathcal{F} = t_!t^*\mathcal{L} \simeq \mathcal{L}$$

since t^* is fully faithful. Therefore, a constant object $\mathcal{F} = t^*\mathcal{L}$ of \bar{X}_W over \mathcal{T} is connected over \mathcal{T} if and only if \mathcal{F} is the final object of \bar{X}_W .

Remark 3.15 Note that $t_!\mathcal{F}$ is not a topological space in general. However, this is a topological space when \mathcal{F} is representable by an object (Z_0, Z_v, f_v) such that $y(Z_0)/y(W_F) = y(Z_0/W_F)$. Our terminology is justified by the fact that any object of \mathcal{T} is topological in nature.

3.2 The morphism from the Weil-étale topos to the Artin–Verdier étale topos

Let \bar{X} be the Arakelov compactification of a number ring \mathcal{O}_F . The set \bar{X} is given with the Zariski topology. We recall below the definition of the Artin–Verdier étale site of \bar{X} . We refer to [11] for more details.

A *connected étale \bar{X} -scheme* is a map

$$\bar{U} = (U; U_\infty) \longrightarrow \bar{X} = (X; X_\infty),$$

where U is a connected étale X -scheme in the usual sense. The set U_∞ is a subset of $U(\mathbb{C})/\sim$, where $U(\mathbb{C})/\sim$ is the quotient of the set of complex valued points of U under the equivalence relation defined by complex conjugation. Moreover, U_∞/X_∞ is unramified in the sense that if $v \in X_\infty$ is real, then so is any point w of U_∞ lying over v . An *étale \bar{X} -scheme* is a finite sum of connected étale \bar{X} -schemes, called the *connected components* of \bar{X} . A morphism $\bar{U} \rightarrow \bar{V}$ in the category $Et_{\bar{X}}$ is a morphism of X -schemes $U \rightarrow V$ inducing a map $U_\infty \rightarrow V_\infty$ over X_∞ . The *Artin–Verdier étale site of \bar{X}* is defined by the category $Et_{\bar{X}}$ endowed with the topology \mathcal{J}_{et} generated by the pretopology for which the coverings are the surjective families.

Definition 3.16 The *Artin–Verdier étale topos* of \bar{X} is the category of sheaves of sets on the Artin–Verdier étale site:

$$\bar{X}_{et} := (\widetilde{Et_{\bar{X}}}, \mathcal{J}_{et}).$$

Let v be a closed point of \bar{X} . Data 3.1 gives an embedding $G_{F_v} \hookrightarrow G_F$; hence, we have an inertia subgroup $I_v \subset G_F$ at v . One can define the *strict henselization* of \bar{X} at v as the projective limit $\bar{X}_v^{sh} = \varprojlim \bar{U}$, where \bar{U} runs over the filtered system of étale neighborhoods in \bar{X} of a geometric point over v . We refer to [11, Section 6.2] for a precise definition. For v ultrametric, one has $\bar{X}_v^{sh} := Spec(\mathcal{O}_{\bar{X},v}^{sh})$ where the ring $\mathcal{O}_{\bar{X},v}^{sh}$ is the strict henselization of the local ring $\mathcal{O}_{X,v}$. The generic point of \bar{X}_v^{sh} is $Spec(\bar{F}^{I_v})$. For an archimedean valuation v , \bar{X}_v^{sh} can be formally defined as the pair $(Spec(\bar{F}^{I_v}), v) \rightarrow (X, X_\infty)$. Hence for any closed point v of \bar{X} , Data 3.1 gives a *specialization map over \bar{X}*

$$Spec(\bar{F}) \rightarrow Spec(\bar{F}^{I_v}) \hookrightarrow \bar{X}_v^{sh}. \tag{3.6}$$

Proposition 3.17 *There exists a morphism of left exact sites*

$$\begin{aligned} \gamma^* : (Et_{\bar{X}}; \mathcal{J}_{et}) &\longrightarrow (T_{\bar{X}}; \mathcal{J}_{Is}) \\ \bar{U} &\longmapsto (U_0, U_v, f_v) \end{aligned}$$

The functor γ^* is fully faithful, and the essential image of γ^* consists in objects (U_0, U_v, f_v) where U_0 is a finite W_F -set.

This result is a reformation of [10, Proposition 4.61] and [10, Proposition 4.62]. Let us fix some notations. For any point $v \in \bar{X}$, we define the Galois group of the “residue field at v ” as follows:

$$G_{k(v)} := G_{F_v}/I_v$$

while the Weil group of the residue field at v is defined as $W_{k(v)} := W_{F_v}/W_{F_v}^1$. Note that we have a morphism $\alpha_v : W_{k(v)} \rightarrow G_{k(v)}$ compatible with the Weil map $\theta_v : W_{F_v} \rightarrow W_F$ for any $v \in \bar{X}$. Note also that for an archimedean valuation v , the group $G_{k(v)}$ is trivial and $W_{k(v)}$ is isomorphic to \mathbb{R}_+^\times as a topological group.

Proof For any étale \bar{X} -scheme \bar{U} , we define an object $\gamma^*(\bar{U}) = (U_0, U_v, f_v)$ of $T_{\bar{X}}$ as follows. An algebraic closure \bar{F}/F has been chosen in Data 3.1. The generic point $\bar{U} \times_{\bar{X}} \text{Spec } F$ is the spectrum of a finite étale F -algebra. The Grothendieck–Galois theory shows that this étale F -algebra is uniquely determined by the finite G_F -set

$$U_0 := \text{Hom}_{\text{Spec } F}(\text{Spec } \bar{F}, \bar{U} \times_{\bar{X}} \text{Spec } F) = \text{Hom}_{\bar{X}}(\text{Spec } \bar{F}, \bar{U}).$$

Let v be an ultrametric place of F . The maximal unramified sub-extension of the algebraic closure \bar{F}_v/F_v chosen in Data 3.1 yields an algebraic closure of the residue field $\overline{k(v)}/k(v)$. The scheme $\bar{U} \times_{\bar{X}} \text{Spec } k(v)$ is the spectrum of an étale $k(v)$ -algebra, corresponding to the finite $G_{k(v)}$ -set

$$U_v := \text{Hom}_{\text{Spec } k(v)}(\text{Spec } \overline{k(v)}, \bar{U} \times_{\bar{X}} \text{Spec } k(v)) = \text{Hom}_{\bar{X}}(\text{Spec } \overline{k(v)}, \bar{U})$$

Let v be an ultrametric place of F . Here, we define the set

$$U_v := \text{Hom}_{\bar{X}}((\emptyset, v), \bar{U}) = v \times_{X_\infty} U_\infty$$

For any closed point v of \bar{X} , we have $U_v = \text{Hom}_{\bar{X}}(\bar{X}_v^{sh}, \bar{U})$; hence, the specialization map (3.6) gives a G_{F_v} -equivariant map

$$f_v : U_v \longrightarrow U_0.$$

This map is bijective for almost all valuations and injective for all valuations. For any place v of F , the set U_v is viewed as a $W_{k(v)}$ -topological space via the morphism $W_{k(v)} \rightarrow G_{k(v)}$. Respectively, U_0 is viewed as a W_F -topological space via $W_F \rightarrow G_F$. Then, the map f_v defined above is W_{F_v} -equivariant. We obtain a functor

$$\gamma^* : \text{Et}_{\bar{X}} \longrightarrow T_{\bar{X}}.$$

Note that if \bar{U} is the a finite sum of connected étale \bar{X} -schemes $\bar{U} = \coprod \bar{U}_i$, then we have

$$\gamma^*(\bar{U}) = \coprod \gamma^*(\bar{U}_i)$$

where the sum on the right-hand side is understood in $T_{\bar{X}}$. The functor γ^* is easily seen to be left exact (i.e. it preserves the final object and fiber products) and continuous (i.e. it preserves covering families). Hence, we do have a morphism of left exact sites.

Let $\mathcal{U} = (U_0, U_v, f_v)$ be an object of $T_{\bar{X}}$ such that U_0 is a finite W_F -set. Writing U_0 as the sum of its W_F -orbits, we can decompose $\mathcal{U} = \coprod_{i \in I} \mathcal{U}_i$ as a sum in $T_{\bar{X}}$. The action of W_F on U_0 factors through $W_F/W_F^0 = G_F$, where W_F^0 is the connected of 1

in W_F , since U_0 is finite. Hence, we can see U_0 as finite G_F -sets. By Galois theory, U_0 corresponds to an essentially unique étale F -algebra $A = \prod_{i \in I} K_i$. Then, for any $i \in I$ one has a finite set S_i of places of K_i and an isomorphism in $T_{\bar{X}}$:

$$\mathcal{U}_i \simeq \gamma^*(\overline{\text{Spec}(\mathcal{O}_{K_i})} - S_i)$$

This shows that the essential image of γ^* consists in objects (U_0, U_v, f_v) such that U_0 is a finite W_F -set.

Let \bar{U} and \bar{U}' be two objects of $Et_{\bar{X}}$. We set $\gamma^*(\bar{U}) = (U_0, U_v, f_v)$ and $\gamma^*(\bar{U}') = (U'_0, U'_v, f'_v)$. By functoriality, we have a map

$$\text{Hom}_{\bar{X}}(\bar{U}, \bar{U}') \rightarrow \text{Hom}_{T_{\bar{X}}}((U_0, U_v, f_v), (U'_0, U'_v, f'_v)). \quad (3.7)$$

We define the inverse map as follows. A morphism $\phi : (U_0, U_v, f_v) \rightarrow (U'_0, U'_v, f'_v)$ is given by a map of finite G_F -sets $\phi_0 : U_0 \rightarrow U'_0$. This map gives a uniquely determined morphism of F -algebras $A' \rightarrow A$, where $\text{Spec}(A) := \bar{U} \times_{\bar{X}} \text{Spec}(F)$ and $\text{Spec}(A') := \bar{U}' \times_{\bar{X}} \text{Spec}(F)$, again by Galois theory. The morphism $A' \rightarrow A$ induces a morphism of étale \bar{X} -schemes $\tilde{\phi} : \bar{U} \rightarrow \bar{U}'$ precisely because ϕ_0 induces a map $\phi_v : U_v \rightarrow U'_v$ for any closed point v of \bar{X} . Then $\phi \mapsto \tilde{\phi}$ is the inverse isomorphism to (3.7). This shows that γ^* is fully faithful. \square

The next corollary follows immediately from the fact that a morphism of left exact sites induces a morphism of topoi.

Corollary 3.18 *There is a morphism of topoi*

$$\gamma : \bar{X}_W \longrightarrow \bar{X}_{et}.$$

Remark 3.19 Let \mathcal{F} be an object of \bar{X}_W represented by an étale \bar{X} -scheme \bar{U} . In other words, we assume that

$$\mathcal{F} = \gamma^*y(\bar{U}) = y(\gamma^*\bar{U}) = y(U_0, U_v, f_v)$$

where U_0 is a finite G_F -set. The space of connected components

$$t_1\mathcal{F} := (j^*\mathcal{F})/y(W_F) = U_0/G_F$$

is the object of \mathcal{T} represented by the finite set U_0/G_F , which is the set of connected components of \bar{U} in the usual sense.

3.3 Structure of \bar{X}_W at the closed points

Let v be a place of F . We consider the Weil group $W_{k(v)}$ and the Galois group $G_{k(v)}$ of the residue field $k(v)$ at $v \in \bar{X}$. Recall that for v archimedean, one has $W_{k(v)} \simeq \mathbb{R}$ and

$G_{k(v)} = \{1\}$. We consider the big classifying topos $B_{W_{k(v)}}$, i.e. the category of $y(W_{k(v)})$ -objects in \mathcal{T} , and the small classifying topos $B_{G_{k(v)}}^{sm}$, which is defined as the category of continuous $G_{k(v)}$ -sets. The category of $W_{k(v)}$ -topological spaces $B_{Top}W_{k(v)}$ is endowed with the local section topology \mathcal{J}_{ls} . Then, $(B_{Top}W_{k(v)}, \mathcal{J}_{ls})$ is a site for the classifying topos $B_{W_{k(v)}}$. Respectively, let $B_{fSets}G_{k(v)}$ be the category of finite $G_{k(v)}$ -sets endowed with the canonical topology \mathcal{J}_{can} . The site $(B_{fSets}G_{k(v)}, \mathcal{J}_{can})$ is a site for the small classifying topos $B_{G_{k(v)}}^{sm}$.

For any place v of F , we have a morphism of left exact sites

$$\begin{aligned} i_v^* : (T_{\bar{X}}, \mathcal{J}_{ls}) &\longrightarrow (B_{Top}W_{k(v)}, \mathcal{J}_{ls}) \\ (Z_0, Z_v, f_v) &\longmapsto Z_v \end{aligned}$$

hence a morphism of topoi

$$i_v : B_{W_{k(v)}} \longrightarrow \bar{X}_W.$$

Assume that v is ultrametric. The morphism of schemes $Spec k(v) \rightarrow \bar{X}$ induces a morphism of topoi

$$u_v : B_{G_{k(v)}}^{sm} \longrightarrow \bar{X}_{et}$$

since the étale topos of $Spec k(v)$ is equivalent to the category $B_{G_{k(v)}}^{sm}$ of continuous $G_{k(v)}$ -sets. This equivalence is induced by the choice of an algebraic closure of $k(v)$ given in Data 3.1. For v archimedean, we still have a morphism

$$u_v : B_{G_{k(v)}}^{sm} = \underline{Sets} = Sh(v) \longrightarrow \bar{X}_{et}.$$

The category of finite $G_{k(v)}$ -sets endowed with the canonical topology is a site for the small classifying topos $B_{G_{k(v)}}^{sm}$. We have a commutative diagram of left exact sites

$$\begin{array}{ccc} (B_{Top}W_{k(v)}, \mathcal{J}_{ls}) & \xleftarrow{\alpha_v^*} & (B_{fSets}G_{k(v)}, \mathcal{J}_{ls}) \\ i_v^* \uparrow & & \uparrow u_v^* \\ (T_{\bar{X}}, \mathcal{J}_{ls}) & \xleftarrow{\gamma^*} & (Et_{\bar{X}}, \mathcal{J}_{ls}) \end{array}$$

where $u_v^*(\bar{U})$ is the finite $G_{k(v)}$ -set

$$U_v := Hom_{\bar{X}}(Spec \overline{k(v)}, \bar{U}).$$

The diagram of sites above induces the commutative of topoi of the following result, which is proven in [4].

Theorem 3.20 *For any closed point v of \bar{X} , the following diagram is a pull-back of topoi.*

$$\begin{array}{ccc}
 B_{W_{k(v)}} & \xrightarrow{\alpha_v} & B_{G_{k(v)}}^{sm} \\
 i_v \downarrow & & u_v \downarrow \\
 \bar{X}_W & \xrightarrow{\gamma} & \bar{X}_{et}
 \end{array}$$

Corollary 3.21 *For any closed point v of \bar{X} , the morphism i_v is a closed embedding.*

Proof It is well known that the morphism of étale topoi

$$u_v : B_{G_{k(v)}}^{sm} \longrightarrow \bar{X}_{et}$$

is a closed embedding. The result then follows from the fact that closed embeddings are stable under pull-backs. Indeed, the image of u_v is a closed subtopos $Im(u_v)$ of \bar{X}_{et} . But the inverse image of $Im(u_v)$ under γ is precisely the image of i_v , as it follows from the previous theorem. Hence, $Im(i_v)$ is a closed subtopos of \bar{X}_W , and i_v induces an equivalence $B_{W_{k(v)}} \simeq Im(i_v)$. \square

3.4 The Weil-étale topoi of an étale \bar{X} -scheme

Remark 3.22 In this section, we define the Weil-étale topoi \bar{U}_W for any étale \bar{X} -scheme \bar{U} . Such a definition must be functorial. According to Proposition 3.28 below, there are two possible definitions for \bar{U}_W . If one defines \bar{U}_W as in Definition 3.4 for any \bar{U} étale over \bar{X} , then $\bar{U} \mapsto \bar{U}_W$ is not functorial. In order to get functoriality, we define \bar{U}_W as a slice topos (see Definition 3.23 and Proposition 3.24 below). The fact that \bar{U}_W is equivalent to $(\widetilde{T_{\bar{U}}}, \widetilde{\mathcal{I}_{ls}})$ will be used as a technical tool in the remaining part of this paper.

Definition 3.23 Let \bar{U} be an étale \bar{X} -scheme. We define the *Weil-étale topoi* of \bar{U} as the slice topos

$$\bar{U}_W := \bar{X}_W / \gamma^*(\bar{U}).$$

Proposition 3.24 *One has a pseudo-functor*

$$\begin{array}{ccc}
 Et_{\bar{X}} & \longrightarrow & \mathfrak{Top} \\
 \bar{U} & \longmapsto & \bar{U}_W
 \end{array}$$

where \mathfrak{Top} is the 2-category of topoi.

Proof The assignment $\bar{U} \mapsto \bar{U}_W$ is obtained by composing the Yoneda embedding, the functor γ^* , and the pseudo-functor (see [5, IV.5.6])

$$\begin{array}{ccc}
 \bar{X}_W & \longrightarrow & \mathfrak{Top} \\
 \mathcal{F} & \longmapsto & \bar{X}_W / \mathcal{F}
 \end{array}$$

The result follows. \square

Proposition 3.25 *We have a pull-back of topoi*

$$\begin{array}{ccc} \bar{U}_W & \xrightarrow{\gamma_{\bar{U}}} & \bar{U}_{et} \\ \downarrow & & \downarrow \\ \bar{X}_W & \xrightarrow{\gamma} & \bar{X}_{et} \end{array}$$

In other words, one has an equivalence $\bar{U}_W \simeq \bar{X}_W \times_{\bar{X}_{et}} \bar{U}_{et}$, where the fiber product is defined in the 2-category of topoi.

Proof One has a canonical equivalence $\bar{X}_{et}/\bar{U} \simeq \bar{U}_{et}$, as it follows from (see [5, III Prop 5.4])

$$\bar{X}_{et}/y\bar{U} := (\widetilde{Et_{\bar{X}}}, \mathcal{J}_{et})/\bar{U} \simeq (\widetilde{Et_{\bar{X}}/\bar{U}}, \mathcal{J}_{ind}) = (\widetilde{Et_{\bar{U}}}, \mathcal{J}_{et}) =: \bar{U}_{et}.$$

We write below $\gamma^*\bar{U}$ (respectively \bar{U}) for the object $y(\gamma^*\bar{U}) = \gamma^*(y\bar{U})$ (respectively $y\bar{U}$) of the topoi \bar{X}_W (respectively of \bar{X}_{et}). By ([5, IV Prop 5.11]), the following commutative diagram

$$\begin{array}{ccc} \bar{X}_W/\gamma^*\bar{U} & \xrightarrow{\gamma_{\bar{U}}} & \bar{X}_{et}/\bar{U} \\ \downarrow & & \downarrow \\ \bar{X}_W & \xrightarrow{\gamma} & \bar{X}_{et} \end{array}$$

is a pull-back, i.e. 2-cartesian in the terminology of [5], where the vertical arrows are the localization morphisms. The result then follows from the definition $\bar{U}_W := \bar{X}_W/\gamma^*\bar{U}$. \square

For any étale \bar{X} -scheme \bar{U} , a site for the topoi \bar{U}_W is given by the category $T_{\bar{X}}/\gamma^*\bar{U}$ endowed with the topology induced by the local sections topology via the forgetful functor $T_{\bar{X}}/\gamma^*\bar{U} \rightarrow T_{\bar{X}}$. We want to define a site for \bar{U}_W analogous to $T_{\bar{X}}$. Let \bar{U} be a connected étale \bar{X} -scheme. Again, we need to make non-canonical choices.

Data 3.26 1. We choose a geometric point $q_{\bar{U}} : Spec \bar{F} \rightarrow \bar{U}$ over the geometric point $Spec \bar{F} \rightarrow \bar{X}$ chosen in Sect. 3.1 (1). In other words, the following triangle

$$\begin{array}{ccc} & & \bar{U} \\ & \nearrow q_{\bar{U}} & \downarrow \\ Spec \bar{F} & \longrightarrow & \bar{X} \end{array}$$

is required to be commutative. The geometric point $q_{\bar{U}}$ yields a sub-extension $\bar{F}/K/F$, where K is the function field of \bar{U} .

2. For any closed point u of \bar{U} over $v \in \bar{X}$, we choose an embedding $K_u \rightarrow \bar{F}_v$ such that the following diagram commutes.

$$\begin{array}{ccccc} F_v & \longrightarrow & K_u & \longrightarrow & \bar{F}_v \\ \uparrow & & \uparrow & & \uparrow \\ F & \longrightarrow & K & \longrightarrow & \bar{F} \end{array}$$

Then, the Weil group of \bar{F}/K is defined by

$$W_K := \varphi^{-1} G_K$$

where $\varphi : W_F \rightarrow G_F$ is the map chosen in 3.1(2). For any closed point u of \bar{U} over $v \in \bar{X}$, the Weil group of \bar{F}_v/K_u is defined by

$$W_{K_u} := \varphi_v^{-1} G_{K_u}$$

where $\varphi_v : W_{F_v} \rightarrow G_{F_v}$ is the map chosen in 3.1(4). Finally, the Weil map $\theta_v : W_{F_v} \rightarrow W_F$ of Data 3.1(6) induces a Weil map

$$\theta_u : W_{K_u} \rightarrow W_K.$$

Definition 3.27 Let \bar{U} be a connected étale \bar{X} -scheme endowed with the data 3.26. We consider the category $T_{\bar{U}}$ of objects $(Z_0, Z_u, f_u)_{u \in \bar{U}}$ defined as follows. The space Z_0 is locally compact and given with a continuous action of W_K . For any point u of \bar{U} , Z_u is a locally compact topological space endowed with a continuous action of $W_{K(u)}$. The map $f_u : Z_u \rightarrow Z_0$ is continuous and W_{K_u} -equivariant.

The action of W_K on Z_0 factors through $W_{L/K}$ for a finite Galois sub-extension $\bar{F}/L/K$. The map f_u is an *homeomorphism for almost all points u of \bar{U} and a continuous injective map for all points of \bar{U}* . An arrow $\phi : (Z_0, Z_u, f_u) \rightarrow (Z'_0, Z'_u, f'_u)$ in the category $T_{\bar{U}}$ is a W_K -equivariant continuous map $\phi : Z_0 \rightarrow Z'_0$ inducing a continuous map $Z_u \rightarrow Z'_u$ for any $u \in \bar{U}$. The category $T_{\bar{U}}$ is endowed with the local section topology \mathcal{J}_{I_S} .

The argument of the proof of Proposition 3.17 gives a morphism of topoi

$$(T_{\bar{U}}, \widetilde{\mathcal{J}}_{I_S}) \longrightarrow \bar{U}_{et}.$$

Moreover, the choices 3.26 above define a morphism of topoi

$$(T_{\bar{U}}, \widetilde{\mathcal{J}}_{I_S}) \longrightarrow (T_{\bar{X}}, \widetilde{\mathcal{J}}_{I_S}) =: \bar{X}_W.$$

Indeed, we have a morphism of left exact sites

$$\begin{array}{ccc} (T_{\bar{X}}, \mathcal{J}_{I_S}) & \longrightarrow & (T_{\bar{U}}, \mathcal{J}_{I_S}) \\ (Z_0, Z_v, f_v)_{v \in \bar{X}} & \longmapsto & (Z_0, Z_u, f_u)_{u \in \bar{U}} \end{array}$$

defined as follows. The space Z_0 on the right-hand side is given with the action of W_K induced by the morphism $W_K \hookrightarrow W_F$. For any closed point u of \bar{U} lying above $v \in \bar{X}$, the space Z_u is Z_v endowed with the action of $W_{k(u)}$ induced by the morphism $W_{k(u)} \hookrightarrow W_{k(v)}$, which in turn induced by the morphism $W_{K_u} \hookrightarrow W_{F_v}$.

We obtain a commutative square

$$\begin{array}{ccc} (\widetilde{T_{\bar{U}}}, \widetilde{\mathcal{I}_{ls}}) & \longrightarrow & \bar{U}_{et} \\ \downarrow & & \downarrow \\ \bar{X}_W & \xrightarrow{\gamma} & \bar{X}_{et} \end{array}$$

since the corresponding diagram of sites is commutative. By the universal property of fiber products in the 2-category of topoi, this commutative square gives rise to an essentially unique morphism

$$(\widetilde{T_{\bar{U}}}, \widetilde{\mathcal{I}_{ls}}) \longrightarrow \bar{X}_W \times_{\bar{X}_{et}} \bar{U}_{et} \simeq \bar{U}_W.$$

Proposition 3.28 *Let \bar{U} be a connected étale \bar{X} -scheme endowed with the Data 3.26. Then, the morphism defined above*

$$(T_{\bar{U}}, \mathcal{I}_{ls}) \longrightarrow \bar{U}_W.$$

is an equivalence.

Proof Recall that $\gamma^*\bar{U} = (U_0, U_v, h_v)$, where $U_0 := \text{Hom}_{\bar{X}}(\text{Spec } \bar{F}, \bar{U})$ as a W_F -set. The sub-extension $\bar{F}/K/F$ given by the point $q_{\bar{U}}$ yields an isomorphism of W_F -sets

$$U_0 := \text{Hom}_F(K, \bar{F}) \simeq G_F/G_K \simeq W_F/W_K,$$

sending $q_{\bar{U}} \in U_0$ to the distinguished element of W_F/W_K . This gives an isomorphism of categories

$$B_{\text{Top}} W_F/U_0 \simeq B_{\text{Top}} W_F/(W_F/W_K).$$

Hence, the functor

$$\begin{aligned} \Psi_0 : B_{\text{Top}} W_F/U_0 &\longrightarrow B_{\text{Top}} W_K \\ \phi_0 : Z_0 \rightarrow U_0 &\longmapsto Z_{u_0} := \phi_0^{-1}(q_{\bar{U}}) \end{aligned}$$

is an equivalence of categories. Let $\phi : (Z_0, Z_v, f_v) \rightarrow (U_0, U_v, h_v)$ be an object of the slice category $T_{\bar{X}}/\gamma^*\bar{U}$ and let $u \in \bar{U}$ be a closed point lying above $v \in \bar{X}$. The action of W_{K_u} on

$$f_v^{-1}(Z_{u_0}) \hookrightarrow Z_{u_0} := \phi_0^{-1}(q_{\bar{U}})$$

via the map $W_{K_u} \rightarrow W_K$ is unramified, in the sense that $W_{K_u}^1$ acts trivially on $f_v^{-1}(Z_{u_0})$. Then, we see $f_v^{-1}(Z_{u_0})$ as a $W_{k(u)}$ -space, where $f_v^{-1}(Z_{u_0})$ is given with the topology induced by the inclusion $f_v^{-1}(Z_{u_0}) \subseteq Z_v$. We define Z_u to be the space

$$Z_u := f_v^{-1}(Z_{u_0})$$

endowed with its $W_{k(u)}$ -action. Finally, the W_{F_v} -equivariant map $f_v : Z_v \rightarrow Z_0$ induces a W_{K_u} -equivariant map $g_u : Z_u \rightarrow Z_{u_0}$, which is injective and continuous everywhere and an homeomorphism almost everywhere. Then, the equivalence Ψ_0 induces a functor

$$\begin{aligned} \Psi : T_{\bar{X}}/\gamma^*\bar{U} &\longrightarrow T_{\bar{U}} \\ (Z_0, Z_v, f_v) \rightarrow (U_0, U_v, h_v) &\longmapsto (Z_{u_0}, Z_u, g_u) \end{aligned}$$

which is an equivalence as well. Moreover, the topology induced on $B_{Top}W_K$ by the local section topology on $B_{Top}W_F$ via the functor (forget the map to U_0)

$$B_{Top}W_K \simeq B_{Top}W_F/U_0 \rightarrow B_{Top}W_F$$

is still the local section topology on $B_{Top}W_K$. The same is true for any place v of F , and we obtain an equivalence of sites:

$$(T_{\bar{X}}/\gamma^*\bar{U}, \mathcal{J}_{Is}) \longrightarrow (T_{\bar{U}}, \mathcal{J}_{Is})$$

Therefore, the induced morphism of topoi

$$(\widetilde{T_{\bar{U}}}, \widetilde{\mathcal{J}_{Is}}) \longrightarrow (\widetilde{T_{\bar{X}}/\gamma^*\bar{U}}, \widetilde{\mathcal{J}_{Is}}) \simeq (\widetilde{T_{\bar{X}}}, \widetilde{\mathcal{J}_{Is}})/\gamma^*y(\bar{U}) =: \bar{U}_W$$

is an equivalence (see [5, III Prop. 5.4] for the last equivalence). □

3.5 The Weil-étale topos of a function field

In this section, we show that our definition of the (small) Weil-étale topos of a function field coincides with the definition given by Lichtenbaum in [7]. More precisely, let Y be an open subscheme of a smooth projective curve over a finite field k . The most natural definition for the Weil-étale topos is given by the category $\mathcal{S}_{et}(W_k, \bar{Y})$ of W_k -equivariant étale sheaves on the geometric curve $\bar{Y} = Y \otimes_k \bar{k}$. On the other hand, Definition 3.3 yields a left exact category T_Y^{sm} endowed with the local section topology \mathcal{J}_{Is} , where we replace Top by \underline{Sets} . We define below an equivalence

$$(\widetilde{T_Y^{sm}}, \widetilde{\mathcal{J}_{Is}}) \simeq \mathcal{S}_{et}(W_k, \bar{Y}).$$

In other words, we show that the artificial definition of the (small) Weil-étale topos coincides with the natural one in the case of a function field. This justifies the term “Weil-étale topos” for the topos defined in this paper.

Data 3.29 Let Y be an open subscheme of a geometrically connected smooth projective curve over a finite field k with function field K .

1. We choose a separable algebraic closure \bar{K}/K .
2. For closed point y of Y , we choose a separable algebraic closure \bar{K}_v/K_v and a K -embedding $\bar{K} \rightarrow \bar{K}_v$.

We have a natural map $G_K \rightarrow G_k$, and the global Weil group W_K is defined as the fiber product topological group $W_K := G_K \times_{G_k} W_k$. For any closed point v of Y , one has $G_{k(v)} = G_{K_v}/I_{K_v}$, and $W_{K_v} := G_{K_v} \times_{G_{k(v)}} W_{k(v)}$. There exists a unique Weil map $W_{K_v} \rightarrow W_K$ such that the following diagram is commutative

$$\begin{array}{ccc} W_{K_v} & \longrightarrow & G_{K_v} \\ \downarrow & & \downarrow \\ W_K & \longrightarrow & G_K \end{array}$$

Definition 3.30 Let T_Y^{sm} be the category of objects (Z_0, Z_v, f_v) defined as follows. The set Z_0 is endowed with a continuous W_K -action. For any closed point v of Y , Z_v is a set endowed with a continuous $W_{k(v)}$ -action. The map $f_v : Z_v \rightarrow Z_0$ is W_{K_v} -equivariant, when Z_v and Z_0 are seen as W_{K_v} -spaces via the maps $W_{K_v} \rightarrow W_K$ and $q_v : W_{K_v} \rightarrow W_{k(v)}$. We require the following facts:

- The map f_v is *bijective for almost all closed points and injective for all closed points v of Y* .
- The action of W_K on Z_0 factors through $W_{L/K}$, for some finite Galois subextension $\bar{K}/L/K$.

A *morphism*

$$\phi : (Z_0, Z_v, f_v) \longrightarrow (Z'_0, Z'_v, f'_v)$$

in the category T_Y^{sm} is a W_K -equivariant map $\phi : Z_0 \rightarrow Z'_0$ inducing a W_{K_v} -equivariant map $\phi_v : Z_v \rightarrow Z'_v$ for all closed points v of Y .

The category T_Y^{sm} is endowed with the local section topology \mathcal{J}_{I_s} , i.e. the topology generated by the pretopology for which a family

$$\{\varphi_i : (Z_{i,0}, Z_{i,v}, f_{i,v}) \rightarrow (Z_0, Z_v, f_v), i \in I\}$$

is a covering family if the map $\coprod_{i \in I} Z_{i,v} \rightarrow Z_v$ is surjective, for any point v of Y .

Definition 3.31 We define the *small Weil-étale topos* Y_W^{sm} as the topos of sheaves on the site $(T_Y^{sm}, \mathcal{J}_{I_s})$.

Let $K\bar{k}$ be the function field of the geometric curve \bar{Y} . We have the sub-extension $\bar{K}/K\bar{k}/K$ and we set $G_{K\bar{k}} := G(\bar{K}/K\bar{k})$. For any closed point $y \in Y$, we denote by I_y the Galois group of the completion of $K\bar{k}$ at y . We choose maps $I_y \hookrightarrow G_{K\bar{k}}$ compatible with *Data 3.29*.

Definition 3.32 Let $T_{\bar{Y}}^{sm}$ be the category of objects (Z_0, Z_y, f_y) , where y runs over the closed points of \bar{Y} , defined as follows. The set Z_0 is endowed with a continuous $G_{K\bar{k}}$ -action. For any closed point y of \bar{Y} , Z_y is a set endowed with a I_y -equivariant a map $f_y : Z_y \rightarrow Z_0$, where I_y acts trivially on Z_y and I_y acts on Z_0 via the map $I_y \rightarrow G_{K\bar{k}}$. Moreover, we assume that

- The map f_y is bijective for almost all closed points y of \bar{Y} and injective for all closed points y of \bar{Y} .
- The action of $G_{K\bar{k}}$ on Z_0 factors through $G(L/K\bar{k})$, for some finite Galois sub-extension $\bar{K}/L/K\bar{k}$.

The morphisms in the category $T_{\bar{Y}}^{sm}$ are defined as above. The local section topology \mathcal{J}_{ls} on the category $T_{\bar{Y}}^{sm}$ is generated by the pretopology of surjective families as above.

We consider below the category $Et_{\bar{Y}}$ of finitely presented étale \bar{Y} -schemes. The site $(Et_{\bar{Y}}, \mathcal{J}_{et})$ is called the *restricted étale site*. Since \bar{Y} is quasi-compact and quasi-separated, the restricted étale site $(Et_{\bar{Y}}, \mathcal{J}_{et})$ is a site for the étale topos of \bar{Y} , i.e. we have

$$\bar{Y}_{et} = (\widetilde{Et_{\bar{Y}}, \mathcal{J}_{et}}).$$

Proposition 3.33 *There is an equivalence*

$$(\widetilde{T_{\bar{Y}}^{sm}, \mathcal{J}_{ls}}) \simeq \bar{Y}_{et}.$$

Proof The arguments of Proposition 3.17 can be generalized to this context. This yields a natural functor $Et_{\bar{Y}} \rightarrow T_{\bar{Y}}^{sm}$. This functor is not essentially surjective because an object (Z_0, Z_y, f_y) of $T_{\bar{Y}}^{sm}$ can have an infinite number of connected components (i.e. $Z_0/G_{K\bar{k}}$ is infinite), while a finitely presented étale \bar{Y} -scheme has finitely many connected components. However, the previous functor is fully faithful, $Et_{\bar{Y}}$ is a topologically generating family of the site $(T_{\bar{Y}}^{sm}, \mathcal{J}_{ls})$, and the étale topology on $Et_{\bar{Y}}$ is induced by the local section topology on $T_{\bar{Y}}^{sm}$. Hence, the result follows from ([5, IV Corollary 1.2.1]). \square

We recall below some basic facts concerning truncated simplicial topoi. We refer to ([10, Chapter 10 Sect. 1.2]) for more details and references. A truncated simplicial topos \mathcal{S}_\bullet is given by the usual diagram

$$\mathcal{S}_2 \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} \mathcal{S}_1 \begin{array}{c} \rightrightarrows \\ \rightrightarrows \\ \rightrightarrows \end{array} \mathcal{S}_0$$

Given such truncated simplicial topos \mathcal{S}_\bullet , we define the category $Desc(\mathcal{S}_\bullet)$ of objects of \mathcal{S}_0 endowed with a descent data. One can prove that $Desc(\mathcal{S}_\bullet)$ is always a topos. More precisely, $Desc(\mathcal{S}_\bullet)$ is the inductive limit of the diagram \mathcal{S}_\bullet in the 2-category of topoi.

The most simple non-trivial example is the following. Let \mathcal{S} be a topos and let U be an object of \mathcal{S} . We consider the truncated simplicial topos

$$(\mathcal{S}, U)_\bullet : \quad \mathcal{S}/(U \times U \times U) \rightrightarrows \mathcal{S}/(U \times U) \rightleftarrows \mathcal{S}/U$$

where these morphisms of topoi are induced by the projections maps (of the form $U \times U \times U \rightarrow U \times U$ and $U \times U \rightarrow U$) and by the diagonal map $U \rightarrow U \times U$. It is well known that if U covers the final object of \mathcal{S} , then the natural morphism

$$Desc(\mathcal{S}, U)_\bullet \longrightarrow \mathcal{S}$$

is an equivalence. In other words, $\mathcal{S}/U \rightarrow \mathcal{S}$ is an effective descent morphism for any U covering the final object of \mathcal{S} .

We will also use the following example. Let G be a discrete group acting on a scheme \bar{Y} . The truncated simplicial scheme

$$G \times G \times \bar{Y} \rightrightarrows G \times \bar{Y} \rightleftarrows \bar{Y}$$

defined by the action of the group G on \bar{Y} induces a truncated simplicial topos:

$$(G, \bar{Y}_{et})_\bullet : \quad (G \times G \times \bar{Y})_{et} \rightrightarrows (G \times \bar{Y})_{et} \rightleftarrows \bar{Y}_{et}$$

The descent topos of this truncated simplicial topos is precisely the category of G -equivariant étale sheaves on \bar{Y} :

$$\mathcal{S}_{et}(G, \bar{Y}) := Desc((G, \bar{Y}_{et})_\bullet).$$

Theorem 3.34 *There is an equivalence*

$$(T_Y^{sm}, \widetilde{\mathcal{I}}_{ls}) \simeq \mathcal{S}_{et}(W_k, \bar{Y}).$$

Proof First, there is a canonical morphism of topoi

$$f : (T_Y^{sm}, \widetilde{\mathcal{I}}_{ls}) \longrightarrow B_{W_k}^{sm}$$

induced by the morphism f^* of left exact sites defined as follows. The functor f^* sends a W_k -set Z to the object (Z, Z, Id_Z) of T_Y^{sm} , where W_K (respectively $W_{k(v)}$) acts on Z via the map $W_K \rightarrow W_k$ (respectively via $W_{k(v)} \rightarrow W_k$). Let EW_k be the object of $B_{W_k}^{sm}$ defined by the action of W_k on itself by multiplications. One has $f^*(EW_k) = y(\bar{E}W_k, EW_k, Id)$. Adapting the proof of Proposition 3.28 to this context, we obtain the following equivalences:

$$(T_Y^{sm}, \widetilde{\mathcal{I}}_{ls})/f^*(EW_k) \simeq (T_Y^{sm}/f^*EW_k, \widetilde{\mathcal{I}}_{ls}) \simeq (T_Y^{sm}, \widetilde{\mathcal{I}}_{ls}).$$

By Proposition 3.33, we have

$$Y_W^{sm}/f^*(EW_k) = (T_Y^{sm}, \widetilde{\mathcal{J}}_{ls})/f^*(EW_k) \simeq (T_{\overline{Y}}^{sm}, \widetilde{\mathcal{J}}_{ls}) \simeq (Et_{\overline{Y}}, \widetilde{\mathcal{J}}_{et}) = \overline{Y}_{et} \quad (3.8)$$

The morphism from f^*EW_k to the final object of $(T_Y^{sm}, \widetilde{\mathcal{J}}_{ls})$ is certainly an epimorphism, i.e. a covering morphism. We consider the truncated simplicial topos \mathcal{S}_\bullet^1 obtained by localization:

$$Y_W^{sm}/(f^*EW_k \times f^*EW_k \times f^*EW_k) \rightrightarrows Y_W^{sm}/(f^*EW_k \times f^*EW_k) \rightleftarrows Y_W^{sm}/f^*EW_k$$

The descent topos of this truncated simplicial topos \mathcal{S}_\bullet^1 is canonically equivalent to Y_W^{sm} since f^*EW_k covers the final object:

$$Y_W^{sm} \simeq Desc(\mathcal{S}_\bullet^1)$$

By (3.8), the truncated simplicial topos \mathcal{S}_\bullet^1 is equivalent to

$$\mathcal{S}_\bullet^2 : \quad \overline{Y}_{et}/(g^*W_k \times g^*W_k) \rightrightarrows \overline{Y}_{et}/g^*W_k \rightleftarrows \overline{Y}_{et}$$

where $g : \overline{Y}_{et} \rightarrow \underline{SetS}$ is the unique morphism. One has $g^*W_k \times g^*W_k = g^*(W_k \times W_k)$ and

$$g^*W_k = g^* \left(\coprod_{W_k} \{*\} \right) = \coprod_{W_k} g^* \{*\} = \coprod_{W_k} y(\overline{Y}) = y \left(\coprod_{W_k} \overline{Y} \right) = y(W_k \times \overline{Y})$$

where $\{*\}$ and $y(\overline{Y})$ are the final objects of \underline{SetS} and \overline{Y}_{et} respectively, since g^* commutes with finite projective limits and arbitrary inductive limits. We obtain

$$\overline{Y}_{et}/g^*W_k = \overline{Y}_{et}/y(W_k \times \overline{Y}) = (W_k \times \overline{Y})_{et}$$

since the projection $W_k \times \overline{Y} \rightarrow \overline{Y}$ is an étale morphism of schemes, and

$$\overline{Y}_{et}/(g^*W_k \times g^*W_k) = \overline{Y}_{et}/y(W_k \times W_k \times \overline{Y}) = (W_k \times W_k \times \overline{Y})_{et}$$

The truncated simplicial topos \mathcal{S}_\bullet^2 is equivalent to

$$\mathcal{S}_\bullet^3 : \quad (W_k \times W_k \times \overline{Y})_{et} \rightrightarrows (W_k \times \overline{Y})_{et} \rightleftarrows \overline{Y}_{et}$$

where the maps of this simplicial topos are given by the group structure of W_k and its action on \overline{Y} . Hence, we have equivalences of truncated simplicial topoi:

$$\mathcal{S}_\bullet^1 \simeq \mathcal{S}_\bullet^2 \simeq \mathcal{S}_\bullet^3$$

inducing equivalences between the associated descent topoi:

$$Y_W^{sm} \simeq \text{Desc}(\mathcal{S}_\bullet^1) \simeq \text{Desc}(\mathcal{S}_\bullet^2) \simeq \text{Desc}(\mathcal{S}_\bullet^3) \simeq \mathcal{S}_{et}(W_k, \bar{Y}).$$

□

Recall that Lichtenbaum has defined in [7] the Weil-étale cohomology as follows:

$$H_W^n(Y, \mathcal{A}) := R^n(\Gamma_{\bar{Y}}^{W_k})\mathcal{A}$$

Here, \mathcal{A} is a W_k -equivariant abelian étale sheaf on \bar{Y} and $\Gamma_{\bar{Y}}^{W_k}$ is the functor of W_k -invariant global sections on \bar{Y} . This cohomology is precisely the cohomology of the Weil-étale topos $\mathcal{S}_{et}(W_k, \bar{Y})$. Indeed, the latter is defined as $R^n(\alpha_*)\mathcal{A}$, where $\alpha : \mathcal{S}_{et}(W_k, \bar{Y}) \rightarrow \underline{\text{Sets}}$ is the unique map from the Weil-étale topos to the final topos. But we have canonically $\alpha_* = \Gamma_{\bar{Y}}^{W_k}$.

4 The Weil-étale fundamental group

4.1 Local sections

For W a locally compact topological group and I a closed subgroup of W , it is not known in general that the continuous projection $W \rightarrow W/I$ admits local sections. The result below, due to Mostert, shows that local sections do exist when W/I is finite dimensional. We denote below by $\text{dim}(X)$ the covering dimension of the space X in the sense of Lebesgue. Note that for any locally compact space X , we have the inequality

$$\text{cd}(X) \leq \text{dim}(X)$$

where $\text{cd}(X)$ denotes the cohomological dimension that is used in [13].

Theorem 4.1 *Let W be a locally compact topological group and let $I \subseteq W$ be a closed subgroup such that W/I is finite dimensional. Then, the continuous projection $W \rightarrow W/I$ has local sections. If W/I is 0-dimensional, then the projection $W \rightarrow W/I$ has a global continuous section.*

Proof This is [13, Theorem 8]. □

Corollary 4.2 *Let W be a finite dimensional locally compact topological group and let $I \subseteq W$ be a closed subgroup. Then the continuous projection $W \rightarrow W/I$ has local sections.*

Proof By [13, Corollary 2], if W is finite dimensional then so is W/I , and the result follows from the previous theorem. □

Lemma 4.3 *Let L/K be a finite Galois extension of number fields. The idèle class group C_L and relative Weil group $W_{L/K}$ both have finite dimension.*

Proof Let us first note that C_L is an open (and closed) subgroup of $W_{L/K}$, hence

$$\dim(W_{L/K}) = \dim(C_L).$$

Global class field theory provides us with an exact sequence of topological groups

$$1 \rightarrow C_L^0 \rightarrow C_L \rightarrow G_L^{ab} \rightarrow 1,$$

where C_L^0 is the connected component of C_L . We mean by the term exact sequence that C_L^0 is a closed normal subgroup of C_L endowed with the induced topology and that G_L^{ab} becomes isomorphic to the group C_L/C_L^0 endowed with the quotient topology. The space G_L^{ab} is profinite hence compact and totally disconnected. Hence, $\dim(G_L^{ab}) = 0$. By Theorem 4.1, the continuous map $C_L \rightarrow G_L^{ab}$ has a global continuous section. We obtain an homeomorphism $C_L \simeq C_L^0 \times G_L^{ab}$ (which is not a group morphism). We obtain

$$\dim(C_L) \leq \dim(C_L^0) + \dim(G_L^{ab}) = \dim(C_L^0).$$

But the connected component C_L^0 is the product of \mathbb{R} with $r_1(L) + r_2(L) - 1$ solenoids and $r_2(L)$ circles. Recall that a solenoid is a filtered projective limit of circles:

$$\mathbb{V} := \varprojlim \mathbb{S}^1$$

hence \mathbb{V} is of dimension 1. We obtain

$$\dim(W_{L/K}) = \dim(C_L) \leq \dim(C_L^0) \leq r_1(L) + 2r_2(L) = [L : \mathbb{Q}]$$

□

Corollary 4.4 *Let L/K be a finite Galois extension of number fields. The map $W_K \rightarrow W_{L/K}$ has local sections. The relative Weil group $W_{L/K,S}$ has finite dimension, and $W_K \rightarrow W_{L/K,S}$ has local sections. For two Galois extensions $L'/L/K$, the map $W_{L'/K,S} \rightarrow W_{L/K,S}$ has local sections.*

Proof The group $W_{L/K,S}$ is the quotient of $W_{L/K}$ by a closed subgroup. Hence, $W_{L/K,S}$ has finite dimension by Lemma 4.3 and [13, Corollary 2]. The maps $W_K \rightarrow W_{L/K}$, $W_K \rightarrow W_{L/K,S}$, and $W_{L'/K,S} \rightarrow W_{L/K,S}$ are all quotient maps of locally compact groups by closed subgroups with finite dimensional targets. Those results follow from Theorem 4.1. □

4.2 Weil groups

Again we consider a connected étale \bar{X} -scheme \bar{U} endowed with the Data 3.26. Thus, we have a geometric point $q_{\bar{U}} : \text{Spec } \bar{F} \rightarrow \bar{U}$ over $\text{Spec } \bar{F} \rightarrow \bar{X}$, i.e. a sub-extension $\bar{F}/K/F$, where K is the function field of \bar{U} . The Weil group of K is $W_K := \varphi^{-1}G_K$.

If u is a closed point of \bar{U} lying over $v \in \bar{X}$, then the Weil map $\theta_v : W_{F_v} \rightarrow W_F$ of Data 3.1(6) induces a Weil map $\theta_u : W_{K_u} \rightarrow W_K$. We denote by $W_{K_u}^1 \subset W_{K_u}$ the maximal compact subgroup, which should be thought of as the inertia subgroup.

Definition 4.5 We define the Weil group $W(\bar{U}, q_{\bar{U}})$ of the pair $(\bar{U}, q_{\bar{U}})$ as the quotient

$$W(\bar{U}, q_{\bar{U}}) := W_K / N_{\bar{U}}$$

where $N_{\bar{U}}$ is the closure of the normal subgroup in W_K generated by the images of the maps

$$W_{K_u}^1 \hookrightarrow W_{K_u} \rightarrow W_K$$

where u runs through the closed points of \bar{U} .

We will show below that this group $W(\bar{U}, q_{\bar{U}})$ is the limit in the category of topological groups of a projective system of topological groups, i.e. of a topological pro-group. We can either consider this topological pro-group or we can consider its limit as a topological group. It turns out that a topological pro-group contains more information than its limit computed in the category of topological groups. For example, there exist non-trivial strict pro-groups whose limit, computed in the category of topological groups, is the trivial group. Topos theory provides a natural framework to overcome this kind of pathologies.

Let $(\bar{U}, q_{\bar{U}})$ be as above and let $\bar{F}/L/K$ be a finite Galois subextension, where the algebraic closure \bar{F}/K is given by the geometric point $q_{\bar{U}}$.

Definition 4.6 We consider the topological group $W(\bar{U}, L)$ defined as the quotient

$$W(\bar{U}, L) := W_{L/K} / N_{\bar{U}, L},$$

where $N_{\bar{U}, L}$ is the closure of the normal subgroup in $W_{L/K}$ generated by the images of the maps

$$W_{K_u}^1 \hookrightarrow W_{K_u} \rightarrow W_{L/K}$$

where u runs through the closed points of \bar{U} .

Definition 4.7 We denote by $\underline{W}(\bar{U}, q_{\bar{U}})$ the strict topological pro-group

$$\underline{W}(\bar{U}, q_{\bar{U}}) := \{W(\bar{U}, L) ; \bar{F}/L/K \text{ finite Galois}\}$$

indexed over the system of finite Galois subextension of \bar{F}/K .

Recall that the term “strict” means that the transition maps

$$W(\bar{U}, L') \longrightarrow W(\bar{U}, L)$$

have local sections. This follows from Theorem 4.1, Lemma 4.3 and from the fact that $W(\bar{U}, L)$ is the quotient of $W_{L/K}$ by a compact subgroup.

Proposition 4.8 *The canonical morphism of topological groups*

$$\alpha : W(\bar{U}, q_{\bar{U}}) \longrightarrow \varprojlim W(\bar{U}, L)$$

is an isomorphism, where the right-hand side is the projective limit computed in the category of topological groups. Moreover, the map

$$W(\bar{U}, q_{\bar{U}}) \longrightarrow W(\bar{U}, L)$$

has local sections.

Proof Consider the product decompositions

$$W(\bar{U}, q_{\bar{U}}) \simeq W^1(\bar{U}, q_{\bar{U}}) \times \mathbb{R} \text{ and } W(\bar{U}, L) \simeq W^1(\bar{U}, L) \times \mathbb{R}$$

where $W^1(-)$ is the maximal compact subgroup of $W(-)$. We have

$$\varprojlim W(\bar{U}, L) \simeq \varprojlim W^1(\bar{U}, L) \times \mathbb{R}$$

since projective limits commute between themselves (in particular with products). Hence, it is enough to show that

$$\alpha^1 : W^1(\bar{U}, q_{\bar{U}}) \longrightarrow \varprojlim W^1(\bar{U}, L)$$

is an isomorphism of topological groups. For any L/K finite Galois, we have an exact sequence of topological groups:

$$1 \rightarrow N_{\bar{U},L} \rightarrow W_{L/K}^1 \rightarrow W^1(\bar{U}, L) \rightarrow 1$$

Passing to the limit we obtain an exact sequence (since projective limits are left exact)

$$1 \rightarrow \varprojlim N_{\bar{U},L} \rightarrow \varprojlim W_{L/K}^1 \rightarrow \varprojlim W^1(\bar{U}, L) \tag{4.1}$$

By definition of $N_{\bar{U}}$ and $N_{\bar{U},L}$, the inclusion $N_{\bar{U}} \hookrightarrow W_K^1$ factors through $\varprojlim N_{\bar{U},L}$. We obtain an injective continuous map

$$n : N_{\bar{U}} \hookrightarrow \varprojlim N_{\bar{U},L}$$

which has dense image, since all the maps $N_{\bar{U}} \hookrightarrow N_{\bar{U},L}$ are surjective. This morphism n is an isomorphism of topological groups because $N_{\bar{U}}$ is compact. The exact sequence (4.1) then shows that α^1 is injective. On the other hand, the map

$$W^1(\bar{U}, q_{\bar{U}}) \longrightarrow W^1(\bar{U}, L)$$

is surjective for any L , hence α^1 has dense image. Therefore, α^1 is surjective and bicontinuous, since $W^1(\bar{U}, q_{\bar{U}})$ is compact. Finally, the map

$$W(\bar{U}, q_{\bar{U}}) \longrightarrow W(\bar{U}, L)$$

has local sections by Theorem 4.1. □

Let \bar{V} be another connected étale \bar{X} -scheme. The generic point of \bar{V} is the spectrum of a number field L and we denote by S the finite set of places of L not corresponding to a point of \bar{V} . The S -idèle class group $C_{L,S}$ of L is defined by the following exact sequence of topological groups

$$0 \rightarrow \prod_{w \in \bar{V}} \mathcal{O}_{L_w}^\times \rightarrow C_L \rightarrow C_{L,S} \rightarrow 0 \tag{4.2}$$

where $\prod_{w \in \bar{V}} \mathcal{O}_{L_w}^\times$ is the product of the local units $\mathcal{O}_{L_w}^\times := \text{Ker}(L_w^* \rightarrow \mathbb{R}_{>0})$ indexed by the sets of places of L corresponding to a point of \bar{V} .

Definition 4.9 For any connected étale \bar{X} -scheme \bar{V} with function field L , we define the *formation module* $C_{\bar{V}}$ of \bar{V} as the S -idèle class group of L

$$C_{\bar{V}} := C_{L,S}$$

where S is the set of places of L not corresponding to a point of \bar{V} .

The geometric point $q_{\bar{U}} : \text{Spec } \bar{F} \rightarrow \bar{U}$ gives a point of the étale topos

$$q_{\bar{U}} : \underline{\text{Sets}} \longrightarrow \bar{U}_{\text{ét}}$$

and the étale fundamental group $\pi_1(\bar{U}_{\text{ét}}, q_{\bar{U}})$ is well defined as a profinite group. This group is the Galois group of the maximal sub-extension of \bar{F}/K unramified at any place of K corresponding to of \bar{U} (regardless if such a place is ultrametric or archimedean). More geometrically, we consider the filtered set of pointed Galois étale cover $\{(\bar{V}, q_{\bar{V}}) \rightarrow (\bar{U}, q_{\bar{U}})\}$ to define the étale fundamental group

$$\pi_1(\bar{U}_{\text{ét}}, q_{\bar{U}}) := \varprojlim_{(\bar{V}, q_{\bar{V}})} \text{Gal}(\bar{V}/\bar{U})$$

The pair

$$(\pi_1(\bar{U}_{\text{ét}}, q_{\bar{U}}), \varinjlim_{(\bar{V}, q_{\bar{V}})} C_{\bar{V}}) \tag{4.3}$$

is a (topological) class formation (see [14, Proposition 8.3.8] and [14, Theorem 8.3.12]). This follows from the fact that if L/K is a Galois extension unramified over \bar{U} , then the $G_{L/K}$ -module $\prod_{w \in \bar{V}} \mathcal{O}_{L_w}^\times$ in (4.2) is cohomologically trivial. Therefore, one can consider the Weil group associated to this class formation (see [15]). More

precisely, one has a compatible system of fundamental class leading to a projective system of extensions

$$1 \rightarrow C_{\bar{V}} \rightarrow W_{\bar{V}/\bar{U}} \rightarrow \text{Gal}(\bar{V}/\bar{U}) \rightarrow 1.$$

This projective system is indexed by the filtered set of pointed Galois cover of $(\bar{U}, q_{\bar{U}})$.

Definition 4.10 The Weil group of the class formation (4.3) is the projective limit

$$W_{\bar{U}, q_{\bar{U}}} := \varprojlim_{(\bar{V}, q_{\bar{V}})} W_{\bar{V}/\bar{U}}$$

computed in the category of topological groups.

We have a canonical map $W(\bar{U}, p_{\bar{U}}) \rightarrow \varprojlim_{(\bar{V}, q_{\bar{V}})} W_{\bar{V}/\bar{U}} =: W_{\bar{U}, q_{\bar{U}}}$.

If W is an Hausdorff topological group, we denote W^c the closure of the commutator subgroup of W , and by $W^{ab} = W/W^c$ the maximal Hausdorff abelian quotient of W .

Lemma 4.11 We have topological isomorphisms

$$W(\bar{U}, q_{\bar{U}})^{ab} \simeq C_{\bar{U}} \text{ and } W(\bar{U}, L)^{ab} \simeq C_{\bar{U}}$$

for any finite Galois extension L/K .

Proof Recall that K is the number field of \bar{U} and that L/K is a finite Galois extension. We have $W_{L/K}^{ab} \simeq C_K$. The morphism $W_{L/K}^1 \rightarrow W^1(\bar{U}, L)$ is surjective and closed. Hence, $W(\bar{U}, L)^c$ is the image of $W_{L/K}^c$. On the other hand, the image of $N_{\bar{U}, L} \subset W_{L/K}$ in C_K is $\prod_{v \in \bar{U}} \mathcal{O}_{K_v}^\times$. Since quotients commute between themselves, we have

$$W(\bar{U}, L)^{ab} = W_{L/K}^{ab} / \prod_{v \in \bar{U}} \mathcal{O}_{K_v}^\times =: C_{\bar{U}}.$$

The proof concerning $W(\bar{U}, q_{\bar{U}})$ is similar. □

Corollary 4.12 The topological pro-group

$$\underline{W}(\bar{U}, q_{\bar{U}})^{ab} := \{W(\bar{U}, L)^{ab}; L/K \text{ finite Galois}\}$$

is constant and can therefore be identified with a usual topological group. One has an isomorphism of topological groups

$$\underline{W}(\bar{U}, q_{\bar{U}})^{ab} \simeq C_{\bar{U}}.$$

4.3 Normal subgroups

Let \mathcal{G} be a group object in a topos \mathcal{S} . A *subgroup* of \mathcal{G} is an equivalence class of monomorphisms of group objects $\mathcal{H} \hookrightarrow \mathcal{G}$. A *quotient* of \mathcal{G} , the dual notion, is an equivalence class of epimorphisms of group objects $\mathcal{G} \twoheadrightarrow \mathcal{Q}$.

Such a subgroup is said to be *normal* if the conjugation action of \mathcal{G} on itself induces an action on \mathcal{H} . In other words, \mathcal{H} is normal if there exists a commutative diagram

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{G} & \xrightarrow{c} & \mathcal{G} \\ \uparrow & & \uparrow \\ \mathcal{G} \times \mathcal{H} & \longrightarrow & \mathcal{H} \end{array}$$

where c is the conjugation action on \mathcal{G} (which can be defined on sections, or more directly as the conjugation action on a group object in any category). If such an induced action of \mathcal{G} on \mathcal{H} does exist, then it is unique since $\mathcal{H} \hookrightarrow \mathcal{G}$ is mono.

Let $\mathcal{H} \hookrightarrow \mathcal{G}$ be a subgroup. Consider the quotient \mathcal{G}/\mathcal{H} in \mathcal{T} of the equivalence relation

$$\mathcal{H} \times \mathcal{G} \rightrightarrows \mathcal{G},$$

where the arrows are given by projection and multiplication. Then \mathcal{G}/\mathcal{H} has a group structure compatible with the group structure on \mathcal{G} (i.e. the map $\mathcal{G} \rightarrow \mathcal{G}/\mathcal{H}$ is a group morphism) if and only if \mathcal{H} is normal in \mathcal{G} .

In particular, let $f : \mathcal{G}' \rightarrow \mathcal{G}$ be a morphism of group objects. The kernel of f is defined as

$$Ker(f) := \mathcal{G}' \times_{\mathcal{G}} *$$

where $* \rightarrow \mathcal{G}$ is the unit section. Then, $Ker(f)$ is a normal subgroup of \mathcal{G}' .

4.3.1 Normal subgroup generated by a subgroup

Let $i : \mathcal{H} \hookrightarrow \mathcal{G}$ be a subgroup. Consider the category of triangles

$$\begin{array}{ccc} & & N \\ & \nearrow & \downarrow \\ \mathcal{H} & \xrightarrow{i} & \mathcal{G} \end{array}$$

where the maps are all monomorphisms of groups and N is normal in \mathcal{G} . This category is not empty since it contains $Id_{\mathcal{G}}$ as the final object. The *normal subgroup generated by \mathcal{H} in \mathcal{G}* is the projective limit in \mathcal{T}

$$\mathcal{N}(\mathcal{H}) := \varprojlim N,$$

More precisely, $\mathcal{N}(\mathcal{H})$ is the projective limit of the functor that sends a triangle as above to N . We check immediately that $\mathcal{N}(\mathcal{H})$ is the smallest normal subgroup of \mathcal{G} containing \mathcal{H} .

4.3.2 Subgroup generated by a family of subgroups

Let $\{\mathcal{H}_j \hookrightarrow \mathcal{G}, j \in J\}$ be a family of subgroup of \mathcal{G} . We define analogously the subgroup

$$\langle \mathcal{H}_j, j \in J \rangle \hookrightarrow \mathcal{G}$$

generated by the \mathcal{H}_j 's in \mathcal{G} . In what follows, we denote by

$$\mathcal{H}_1 \vee \mathcal{H}_2$$

the subgroup generated by two subgroups \mathcal{H}_1 and \mathcal{H}_2 in \mathcal{G} .

4.3.3 Special cases

We consider now subgroups of representable group objects in \mathcal{T} . Let G be a (locally compact) topological group. A *topological subgroup* of G is a subgroup $H \subseteq G$ endowed with the induced topology. A *topological quotient* of G is a quotient G/H endowed with the quotient topology, where H is a normal subgroup.

Lemma 4.13 *Let $y(G)$ be a group of \mathcal{T} representable by a topological group G . The following are equivalent.*

1. G is discrete.
2. Any subgroup of yG is representable by a topological subgroup of G .
3. Any quotient of yG is representable by a topological quotient of G .

Proof By ([10, Lemma 10.29]) the unique morphism

$$e_{\mathcal{T}} : \mathcal{T} \longrightarrow \underline{Sets}$$

is *hyperconnected*. This means that for any set I , the (ordered) set $Sub_{\underline{Sets}}(I)$ of subobjects of I in \underline{Sets} is in 1 – 1 correspondence with the set $Sub_{\mathcal{T}}(e_{\mathcal{T}}^*I)$ of subobjects of $e_{\mathcal{T}}^*I$ in \mathcal{T} . Note that $Sub_{\underline{Sets}}(I)$ is just the family of subsets of I , and that $e_{\mathcal{T}}^*I = y(I)$ is the sheaf of \mathcal{T} represented by the discrete topological space I . Thus we have (1) \Rightarrow (2). We have also (1) \Rightarrow (3) for the same reason. Let us write a more direct proof of this fact using (1) \Rightarrow (2). Let G be a discrete group. If

$$f : yG \twoheadrightarrow \mathcal{Q}$$

is a quotient in \mathcal{T} , then the kernel

$$Ker(f) := yG \times_{\mathcal{Q}} * \hookrightarrow yG$$

is a subobject of yG in \mathcal{T} . Therefore, $Ker(f) = y(K)$ is representable by a usual subgroup $K \subseteq G$, and we have

$$\mathcal{Q} = y(G)/y(K) = y(G/K)$$

since the map $G \rightarrow G/K$ has (obviously) local sections (see [3, Lemma 4]).

We claim that (2) \Rightarrow (1) and (3) \Rightarrow (1). Let G be a non-discrete topological group. We denote by G^δ the abstract group G endowed with the discrete topology. The injective continuous map $G^\delta \rightarrow G$ yields a monomorphism in \mathcal{T} :

$$yG^\delta \hookrightarrow yG.$$

This map is not an isomorphism. Indeed, the Yoneda functor is fully faithful and the identity map $G \rightarrow G^\delta$ is not continuous. Hence, yG^δ is a proper subgroup of yG . But yG^δ is not representable by a topological subgroup of G , since the induced morphism on global sections

$$e_{\mathcal{T},*}(yG^\delta) := Hom_{Top}(*, G^\delta) = G^\delta \longrightarrow e_{\mathcal{T},*}(yG) := Hom_{Top}(*, G) = G^\delta$$

is an isomorphism. Similarly, the quotient

$$yG/yG^\delta$$

is not representable by a quotient of G , since the kernel of the map $yG \rightarrow yG/yG^\delta$ is not representable by a topological subgroup of G (this is yG^δ). \square

Lemma 4.14 *Let W be a locally compact finite dimensional topological group and let N_1 and N_2 be two normal compact subgroups of W . Let $N_1 \vee N_2$ be the normal topological subgroup of W generated by N_1 and N_2 , endowed with the induced topology. Then, $N_1 \vee N_2$ is compact, and the canonical map*

$$yN_1 \vee yN_2 \longrightarrow y(N_1 \vee N_2)$$

is an isomorphism of subgroups of yW . Moreover, one has

$$yW/(yN_1 \vee yN_2) \simeq y(W/N_1 \vee N_2).$$

Proof The subgroup $y(N_1 \vee N_2) \hookrightarrow y(W)$ contains both yN_1 and yN_2 . Hence, it contains $(yN_1 \vee yN_2)$ as well, i.e. one has

$$(yN_1 \vee yN_2) \hookrightarrow y(N_1 \vee N_2) \hookrightarrow yW$$

We show below that the inverse inclusion holds.

Any element of $N_1 \vee N_2$ is of the form xy for $x \in N_1$ and $y \in N_2$. We have a continuous map

$$N_1 \times N_2 \rightarrow W \times W \rightarrow W,$$

where the second map is the multiplication. The image of this map is precisely $N_1 \vee N_2$; hence, we obtain a surjective continuous map

$$N_1 \times N_2 \longrightarrow N_1 \vee N_2.$$

This shows that $N_1 \vee N_2$ is compact. Note that this map is not a morphism of groups in general. This map induces a bijective continuous map

$$N_1 \times N_2 / (N_1 \cap N_2) \longrightarrow N_1 \vee N_2 \tag{4.4}$$

where the group $(N_1 \cap N_2)$ acts on the space $N_1 \times N_2$ by

$$\sigma(x, y) = (x\sigma^{-1}, \sigma y)$$

for any $\sigma \in (N_1 \cap N_2)$ and $(x, y) \in N_1 \times N_2$. The map (4.4) is also closed since $N_1 \times N_2 / (N_1 \cap N_2)$ is compact, hence we get an homeomorphism

$$N_1 \times N_2 / (N_1 \cap N_2) \simeq N_1 \vee N_2.$$

The map $N_1 \times N_2 \rightarrow N_1 \times N_2 / (N_1 \cap N_2)$ is a local section cover by Corollary 4.2. Indeed, $(N_1 \cap N_2)$ is a closed subgroup of the compact group $N_1 \times N_2$, and $N_1 \times N_2$ is finite dimensional since N_1 and N_2 are two compact subgroups of W which is finite dimensional (see [13, Corollary 2]). Hence, the map

$$y(N_1 \times N_2) \rightarrow y(N_1 \vee N_2)$$

is an epimorphism in \mathcal{T} (again, this is not a morphism of groups in general).

It follows that $y(N_1 \vee N_2)$ is the image of the map

$$y(N_1 \times N_2) \rightarrow y(W \times W) \rightarrow yW,$$

where the second map is the multiplication. In other words, one has the epi-mono factorization

$$y(N_1 \times N_2) \rightarrow y(N_1 \vee N_2) \hookrightarrow yW.$$

But the image of $y(N_1 \times N_2)$ in $y(W)$ is contained in $(yN_1 \vee yN_2)$ (check this on sections), hence we have

$$y(N_1 \vee N_2) \hookrightarrow (yN_1 \vee yN_2) \hookrightarrow yW$$

We obtain $y(N_1 \vee N_2) = (yN_1 \vee yN_2)$ (recall that the set of subgroups of yW has is an ordered set). In particular, one has

$$yW / (y(N_1) \vee y(N_2)) = yW / y(N_1 \vee N_2) = y(W / N_1 \vee N_2)$$

where the last equality follows from the fact that $W \rightarrow W/N_1 \vee N_2$ has local sections, since $N_1 \vee N_2$ is a compact subgroup of the locally compact and finite dimensional group W (see Corollary 4.2). \square

Remark 4.15 The previous result generalizes immediately to the case of a finite number of compact normal topological subgroups $\{N_j \subset W, 1 \leq j \leq n\}$.

Let L/K be a finite Galois extension of number fields (inside the fixed algebraic closure \bar{F}/K) and let

$$1 \rightarrow C_L \rightarrow W_{L/K} \rightarrow G_{L/K} \rightarrow 1$$

be the associated relative Weil group. Let v be a place of K and let $\tilde{W}_{K_v}^1$ be the image of the composite morphism

$$W_{K_v}^1 \hookrightarrow W_{K_v} \hookrightarrow W_K \twoheadrightarrow W_{L/K}$$

endowed with the induced topology. We consider the topological normal subgroup $N(\tilde{W}_{K_v}^1)$ of $W_{L/K}$ generated by $\tilde{W}_{K_v}^1$. We consider also the normal subgroup $\mathcal{N}(y\tilde{W}_{K_v}^1)$ of $yW_{L/K}$ generated by $y\tilde{W}_{K_v}^1$.

Lemma 4.16 *With the notations above, the group $N(\tilde{W}_{K_v}^1)$ is a compact subgroup of $W_{L/K}$.*

Proof Let $\bar{F}/L/K$ be a finite Galois sub-extension. The image of $W_{K_v}^1$ in W_K/F is topologically isomorphic to W_{L_w/K_v}^1 , i.e. one has

$$\tilde{W}_{K_w}^1 \simeq W_{L_w/K_v}^1.$$

Here, W_{L_w/K_v}^1 is the maximal compact subgroup of W_{L_w/K_v} , which is in turn given by the group extension

$$1 \rightarrow L_w^\times \rightarrow W_{L_w/K_v} \rightarrow G(L_w/K_v) \rightarrow 1$$

where w is a place of L lying above v . More precisely, W_{L_w/K_v}^1 is given by the following extension

$$1 \rightarrow \mathcal{O}_{L_w}^\times \rightarrow W_{L_w/K_v}^1 \rightarrow I(L_w/K_v) \rightarrow 1$$

where $I(L_w/K_v)$ is the usual inertia subgroup of $G(L_w/K_v)$. The map $W_{L_w/K_v}^1 \rightarrow W_{L/K}$ sits in the (injective) morphism of group extensions

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{O}_{L_w}^\times & \longrightarrow & W_{L_w/K_v}^1 & \longrightarrow & I(L_w/K_v) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & C_L & \longrightarrow & W_{L/K} & \xrightarrow{\varphi} & G(L/K) \longrightarrow 1 \end{array}$$

We thus have $W_{L_w/K_v}^1 \cap C_L = \mathcal{O}_{L_w}^\times$ where the intersection makes sense inside $W_{L/K}$. The conjugation action of $W_{L/K}$ on C_L corresponds to the Galois action. Hence for any $\eta \in W_{L/K}$, we set $\sigma = \varphi(\eta)$ and we have

$$\eta \mathcal{O}_{L_w}^\times \eta^{-1} = \mathcal{O}_{L_{\sigma(w)}}^\times \subset C_L.$$

We denote by $N_v := N(\widetilde{W}_{K_v}^1)$ the normal subgroup generated by W_{L_w/K_v}^1 in $W_{L/K}$. We obtain $\prod_{w|v} \mathcal{O}_{L_w}^\times \subset N_v \cap C_L$ and a quotient map (hence surjective, continuous, and open)

$$C_L^v := C_L / \prod_{w|v} \mathcal{O}_{L_w}^\times \longrightarrow C_L / (N_v \cap C_L).$$

On the other hand, we have

$$\varphi(N_v) = G(L/K^{un}) \subset G(L/K)$$

where K^{un}/K is the maximal subextension of L/K unramified above v , since $\varphi(N_v)$ is the normal subgroup of $G(L/K)$ generated by $I(L_w/K_v)$. We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N_v \cap C_L & \longrightarrow & N_v & \longrightarrow & G(L/K^{un}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & C_L & \longrightarrow & W_{L/K} & \xrightarrow{\varphi} & G(L/K) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & C_L/C_L \cap N_v & \longrightarrow & W_{L/K}/N_v & \longrightarrow & G(K^{un}/K) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & C_{K^{un}}^v & \longrightarrow & W_{K^{un}/K}^v & \longrightarrow & G(K^{un}/K) & \longrightarrow & 1 \end{array}$$

In the diagram above, $W_{K^{un}/K}^v$ is the extension of $G(K^{un}/K)$ by $C_{K^{un}}^v := C_{K^{un}} / \prod_{w|v} \mathcal{O}_{K_w^{un}}^\times$ corresponding to the fundamental class (note that $\prod_{w|v} \mathcal{O}_{K_w^{un}}^\times$ is a cohomologically trivial $G(K^{un}/K)$ -module since K^{un}/K is unramified at v). It can be seen from the diagram above that $C_L/C_L \cap N_v$ is a $G(L/K^{un})$ -invariant quotient of C_L . To reach the same conclusion, one can also observe that the group $C_L \cap N_v$ contains the group generated by the family

$$\{\alpha\sigma(\alpha)^{-1} = \alpha\eta\alpha^{-1}\eta^{-1}, \alpha \in C_L, \sigma \in G(L/K^{un}), \eta \in N_v, \sigma := \varphi(\eta)\}$$

since $\alpha\sigma(\alpha)^{-1} \in C_L$ and $\alpha\eta\alpha^{-1}\eta^{-1} \in N_v$ for any $\alpha \in C_L$ and $\eta \in N_v$. Let $H_0(G(L/K^{un}), C_L^v)$ be the maximal $G(L/K^{un})$ -invariant quotient of C_L^v , endowed

with the quotient topology. We obtain a continuous, surjective, open map

$$H_0(G(L/K^{un}), C_L^v) \longrightarrow C_L/C_L \cap N_v.$$

Considering the norm map, we obtain a commutative triangle

$$\begin{array}{ccc} H_0(G(L/K^{un}), C_L^v) & \longrightarrow & C_L/C_L \cap N_v \\ & \searrow \text{N} & \downarrow \\ & & H^0(G(L/K^{un}), C_L^v) \end{array}$$

More precisely, the norm map N can be decomposed as follows:

$$N : H_0(G(L/K^{un}), C_L^v) \twoheadrightarrow C_L/C_L \cap N_v \twoheadrightarrow C_{K^{un}}^v \hookrightarrow H^0(G(L/K^{un}), C_L^v). \quad (4.5)$$

The kernel and cokernel of the norm map N are given by the following exact sequence

$$\begin{aligned} 0 \rightarrow \widehat{H}^{-1}(G(L/K^{un}), C_L^v) &\rightarrow H_0(G(L/K^{un}), C_L^v) \\ &\rightarrow H^0(G(L/K^{un}), C_L^v) \rightarrow \widehat{H}^0(G(L/K^{un}), C_L^v) \rightarrow 0 \end{aligned}$$

It follows from class field theory that $\widehat{H}^{-1}(G(L/K^{un}), C_L^v)$ and $\widehat{H}^0(G(L/K^{un}), C_L^v)$ are both finite. In particular, the continuous, open and surjective map

$$C_L/C_L \cap N_v \longrightarrow C_{K^{un}}^v$$

has finite kernel. It is a finite étale Galois cover (in the topological sense), hence a local homeomorphism. Hence, $C_L/C_L \cap N_v$ is Hausdorff, i.e. $C_L \cap N_v$ is closed in C_L . But N_v is contained in $W_{L/K}^1$, hence $C_L \cap N_v$ is a closed subgroup of C_L^1 , where C_L^1 denotes the maximal compact subgroup of C_L . Therefore, $C_L \cap N_v$ is compact, and N_v is an extension of the finite group $G(L/K^{un})$ by $C_L \cap N_v$. Hence, N_v is compact as well. \square

Lemma 4.17 *We keep the notations of Lemma 4.16. One has the equality*

$$\mathcal{N}(y\widetilde{W}_{K_v}^1) = yN(\widetilde{W}_{K_v}^1).$$

of subgroups of $y(W_{L/K})$ in \mathcal{T} .

Proof Following the notations of the previous proof, we set

$$N_v := N(\widetilde{W}_{K_v}^1) \quad \text{and} \quad \mathcal{N}_v := \mathcal{N}(y\widetilde{W}_{K_v}^1).$$

We have the following morphism of exact sequences of group objects in \mathcal{T} , where the vertical maps are all monomorphisms.

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & \mathcal{N}_v \times_{yW_{L/K}} yC_L & \longrightarrow & \mathcal{N}_v & \longrightarrow & yG(L/K^{un}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & yC_L & \longrightarrow & yW_{L/K} & \xrightarrow{\varphi} & yG(L/K) & \longrightarrow & 1
 \end{array}$$

The subgroup $\mathcal{N}_v \times_{yW_{L/K}} yC_L$ contains

$$yW_{L_w/K_v}^1 \times_{yW_{L/K}} yC_L = y(W_{L_w/K_v}^1 \cap C_L) = y(\mathcal{O}_{L_w}^\times) \tag{4.6}$$

since the Yoneda functor commutes with fiber products. Hence, $\mathcal{N}_v \times_{yW_{L/K}} yC_L$ contains the conjugates in $yW_{L/K}$ of the subgroup (4.6):

$$\eta(y\mathcal{O}_{L_w}^\times)\eta^{-1} = y(\eta\mathcal{O}_{L_w}^\times\eta^{-1}) = y\mathcal{O}_{L_{\sigma(w)}}^\times$$

for any $\eta \in W_{L/K}$ with $\sigma = \phi(\eta)$. Thus, $\mathcal{N}_v \times_{yW_{L/K}} yC_L$ contains the subgroup of $yW_{L/K}$ generated by those subgroups:

$$\langle y\mathcal{O}_{L_{\sigma(w)}}^\times, \sigma \in G(L/K) \rangle = y(\langle \mathcal{O}_{L_{\sigma(w)}}^\times, \sigma \in G(L/K) \rangle) = y\left(\prod_{w|v} \mathcal{O}_{L_w}^\times\right),$$

where the first identity follows from Lemma 4.14. Let $\sigma \in G(L/K^{un})$, and consider the topological subgroup of C_L given by

$$(Id - \sigma)(C_L) := \{\alpha\sigma(\alpha)^{-1}, \alpha \in C_L\}.$$

Then $(Id - \sigma)(C_L)$ is compact, since it is the image of the continuous morphism

$$\begin{array}{ccc}
 C_L^1 & \longrightarrow & C_L \\
 \alpha & \longmapsto & \alpha\sigma(\alpha)^{-1}
 \end{array}$$

where C_L^1 is the maximal compact subgroup of C_L . Using this fact and an argument similar to the proof of Lemma 4.14, we see that we have the inclusion

$$y((Id - \sigma)(C_L)) \hookrightarrow \mathcal{N}_v \times_{yW_{L/K}} yC_L$$

of subgroups of $yW_{L/K}$. Therefore, $\mathcal{N}_v \times_{yW_{L/K}} yC_L$ contains the subgroup of $yW_{L/K}$ generated by all the subgroups considered above:

$$\langle y\left(\prod_{w|v} \mathcal{O}_{L_w}^\times\right); y(Id - \sigma)(C_L) \forall \sigma \in G(L/K^{un}) \rangle \hookrightarrow \mathcal{N}_v \times_{yW_{L/K}} yC_L.$$

We have above a finite number of compact subgroups of $W_{L/K}$. By Lemma 4.14, we obtain

$$y(\Theta) := y \left(\left\langle \prod_{w|v} \mathcal{O}_{L_w}^\times ; (Id - \sigma)(C_L) \forall \sigma \in G(L/K^{un}) \right\rangle \right) \hookrightarrow \mathcal{N}_v \times_{yW_{L/K}} yC_L.$$

where $\Theta := \left\langle \prod_{w|v} \mathcal{O}_{L_w}^\times ; (Id - \sigma)(C_L) \forall \sigma \in G(L/K^{un}) \right\rangle$ is a topological subgroup of C_L . Note that we have

$$C_L / \Theta = H_0(G(L/K^{un}), C_L^v).$$

The proof of the previous lemma shows that Θ is a subgroup of finite index in $N_v \cap C_L$, since the norm map N has finite kernel (see (4.5)). More precisely, we have the following exact sequence of topological groups

$$1 \rightarrow H' \rightarrow C_L / \Theta = H_0(G(L/K^{un}), C_L^v) \rightarrow C_L / N_v \cap C_L \rightarrow 1$$

where H' is a finite subgroup of $\widehat{H}^{-1}(G(L/K^{un}), C_L^v)$. In particular, Θ is open in $N_v \cap C_L$. We have monomorphisms

$$y\Theta \hookrightarrow \mathcal{N}_v \times_{yW_{L/K}} yC_L \hookrightarrow y(N_v \cap C_L).$$

This implies that $\mathcal{N}_v \times_{yW_{L/K}} yC_L$ is representable by a topological group, as it follows from Lemma 4.13.

Now the exact sequence

$$1 \rightarrow \mathcal{N}_v \times_{yW_{L/K}} yC_L \rightarrow \mathcal{N}_v \rightarrow yG(L/K^{un}) \rightarrow 1$$

and the fact that the Yoneda functor $y : \mathcal{T}op \rightarrow \mathcal{T}$ commutes with (disjoint) sums (of topological spaces) show that \mathcal{N}_v is itself representable. Hence, \mathcal{N}_v is representable by a topological group N'_v , and we have continuous injective morphisms of topological groups

$$\Theta \hookrightarrow N'_v \hookrightarrow N_v$$

since the maps $y\Theta \hookrightarrow$ and $yN'_v \hookrightarrow yN_v$ are both monomorphisms in \mathcal{T} . But Θ is open in N_v , hence N'_v is a topological subgroup of N_v , i.e. $N'_v \subseteq N_v$ is endowed with the induced topology.

Moreover, $\mathcal{N}_v = yN'_v$ is normal in $yW_{L/K}$, hence so is N'_v in $W_{L/K}$ (since Yoneda is fully faithful). Finally N'_v must contain $\widetilde{W}_{K_v}^1$ and we get $N'_v = N(\widetilde{W}_{K_v}^1) = N_v$ hence

$$\mathcal{N}_v = yN'_v = yN(\widetilde{W}_{K_v}^1).$$

□

4.4 A generating family for the Weil-étale topos

Let \bar{U} be a connected étale \bar{X} -scheme endowed with Data 3.26. We denote by K the function field of \bar{U} . In this section, we define a simple topologically generating family for the site $(T_{\bar{U}}, \mathcal{J}_{Is})$ (hence a generating family for the topos \bar{U}_W). This has already been used to show that \bar{U}_W is connected and locally connected over \mathcal{T} , and this will be necessary to compute the fundamental group of \bar{U}_W .

Let us fix a finite Galois sub-extension $\bar{F}/L/K$, an open subset V of \bar{U} , a point u of \bar{U} and a locally compact topological space T . In this section, we denote by N the *closed normal subgroup* of $W_{L/K}$ generated by the subgroups $\tilde{W}_{K_v}^1 \subseteq W_{L/K}$ for any closed point $v \in V$. Let $(N, \tilde{W}_{K_u}^1)$ be the subgroup of $W_{L/K}$ generated by N and $\tilde{W}_{K_u}^1$. This subgroup is compact hence closed. We define an object of $T_{\bar{U}}$

$$\mathcal{G}_{L,V,u,T} := (G_0 \times T, G_v \times T, g_v)$$

as follows. If u is not in V , we consider

$$G_0 = W_{L/K}/(N, \tilde{W}_{K_u}^1)$$

as a W_K -space and

$$G_v = W_{L/K}/(N, \tilde{W}_{K_u}^1)$$

as a $W_{k(v)}$ -space for any point v of $V \subseteq \bar{U}$. Then we set $G_u = W_{k(u)}$ on which $W_{k(u)}$ acts by multiplication, and $G_v = \emptyset$ anywhere else. The group W_K (respectively $W_{k(v)}$) acts on the first factor of $G_0 \times T$ (respectively of $G_v \times T$). The map

$$g_v : G_v \times T \longrightarrow G_0 \times T$$

is the identity for any point v of $V \subseteq \bar{U}$, and the continuous map

$$g_u : W_{k(u)} \times T = W_{K_u}/W_{K_u}^1 \times T \longrightarrow W_{L/K}/(N, \tilde{W}_{K_u}^1) \times T$$

is induced by the Weil map $W_{K_u} \rightarrow W_K$.

If $u \in V$ we define $\mathcal{G}_{L,V,u,T}$ as above except that we set

$$G_u = W_{L/K}/(N, \tilde{W}_{K_u}^1) = W_{L/K}/N.$$

Notation 4.18 We denote by $\mathcal{G}_{L,V,u,T}$ the object of $T_{\bar{U}}$ defined above. If $T = *$ is the one point space, then we set $\mathcal{G}_{L,V,u} := \mathcal{G}_{L,V,u,*}$.

For any space T of *Top*, one has a product decomposition in $T_{\bar{U}}$:

$$\mathcal{G}_{L,V,u,T} = \mathcal{G}_{L,V,u} \times t^*T$$

where $t^*T = (T, T, Id_T)$ is the constant object of $T_{\bar{U}}$ associated to the space T .

Definition 4.19 Let $\mathbb{G}_{\bar{U}}$ be the full subcategory of $T_{\bar{U}}$ consisting in objects of the form $\mathcal{G}_{L,V,u,T}$, where $\bar{F}/L/K$ is a finite Galois sub-extension, V is an open subset of \bar{U} , u is a point of \bar{U} , and T is a locally compact topological space. The category $\mathbb{G}_{\bar{U}}$ is endowed with the local section topology \mathcal{J}_{ls} .

Theorem 4.20 *The canonical morphism*

$$\bar{U}_W \longrightarrow (\widetilde{\mathbb{G}_{\bar{U}}}, \widetilde{\mathcal{J}_{ls}})$$

is an equivalence.

Proof We have a composition of fully faithful functors

$$\mathbb{G}_{\bar{U}} \rightarrow T_{\bar{U}} \rightarrow \bar{U}_W$$

where the second functor is the Yoneda embedding. The local section topology on $\mathbb{G}_{\bar{U}}$ is the topology induced by the local section topology on $T_{\bar{U}}$ via the inclusion $\mathbb{G}_{\bar{U}} \rightarrow T_{\bar{U}}$. Hence, \mathcal{J}_{ls} on $\mathbb{G}_{\bar{U}}$ is the topology induced by the canonical topology of \bar{U}_W via the composite functor defined above. But the Yoneda embedding takes a topologically generating family of a site to a generating family of the corresponding topos. Hence, it remains to show that $\mathbb{G}_{\bar{U}}$ is a topologically generating family for the site $(T_{\bar{U}}, \mathcal{J}_{ls})$. In other words, we need to prove that any object of $T_{\bar{U}}$ admits a local section cover by objects of $\mathbb{G}_{\bar{U}}$.

Let (Z_0, Z_v, f_v) be an object of $T_{\bar{U}}$. The action of W_K on Z_0 factors through $W_{L/K}$, for a finite Galois extension L/K . Since the group $W_{L/K}$ is locally compact, its action on the space Z_0 yields a continuous morphism of topological groups

$$\rho : W_{L/K} \longrightarrow \underline{Aut}_{Top}(Z_0)$$

where the group $\underline{Aut}_{Top}(Z_0)$, of homeomorphisms of Z_0 , is endowed with the compact-open topology. The space Z_0 is Hausdorff hence so is the topological group $\underline{Aut}_{Top}(Z_0)$. It follows that the kernel of ρ is a closed normal subgroup of $W_{L/K}$:

$$Ker(\rho) \subseteq W_{K/F}.$$

Let V be the open set of points of \bar{U} such that f_v is an homeomorphism. Take the generic point $u = u_0$ of \bar{U} and $T = Z_0$ as a topological space. Let N be the closed normal subgroup of $W_{L/K}$ generated by the subgroups $\tilde{W}_{K_v}^1 \subseteq W_{L/K}$ for any closed point $v \in V$. The action of W_K on Z_0 factors through $W_{L/K}/N$, since the kernel of the continuous morphism ρ is closed in $W_{L/K}$. Hence ρ induces a continuous morphism

$$W_{L/K}/N \longrightarrow \underline{Aut}_{Top}(Z_0).$$

Such an action is given by a continuous map

$$G_0 \times T := W_{L/K}/N \times Z_0 \longrightarrow Z_0$$

which is W_K -equivariant. This map has an obvious global continuous section. We obtain a morphism in $T_{\bar{U}}$

$$\mathcal{G}_{L,V,u_0,Z_0} \longrightarrow (Z_0, Z_v, f_v)$$

which is a global section cover over any point $v \in V$.

Let $u \in \bar{U} - V$. Here, we consider

$$\mathcal{G}_{L,v,u,Z_u} = (G_0 \times Z_u, G_v \times Z_u, g_v),$$

with $G_v = G_0 = W_{L/K}/(N, \tilde{W}_{K_u}^1)$ for any $v \in V$. The second projection gives a $W_{k(u)}$ -equivariant continuous map

$$\phi_u : G_u \times Z_u := W_{k(u)} \times Z_u \longrightarrow Z_u$$

which has a global continuous section. Then there exists a unique morphism in $T_{\bar{U}}$

$$\phi : \mathcal{G}_{L,v,u,Z_u} \longrightarrow (Z_0, Z_v, f_v)$$

inducing ϕ_u at the point $u \in \bar{U}$. Indeed, the given $W_{k(u)}$ -equivariant continuous map $f_u : Z_u \rightarrow Z_0$ provide us with a W_K -equivariant map

$$\phi_u : G_0 \times Z_u := W_{L/K}/(N, \tilde{W}_{K_u}^1) \times Z_u \longrightarrow Z_0$$

For any point v of V , the same map $\phi_v := \phi_u$ is also $W_{k(v)}$ -equivariant and continuous:

$$\phi_v : G_v \times Z_u := W_{L/K}/(N, \tilde{W}_{K_u}^1) \times Z_u \longrightarrow Z_0 \simeq Z_v.$$

We have obtained a local section cover of $\mathcal{Z} := (Z_0, Z_v, f_v)$ by objects of $\mathbb{G}_{\bar{U}}$:

$$\{\mathcal{G}_{L,v,u_0,Z_0} \rightarrow \mathcal{Z}, \mathcal{G}_{L,v,u,Z_u} \rightarrow \mathcal{Z} \text{ for } u \in \bar{U} - V\}$$

Hence, (the essential image of) $\mathbb{G}_{\bar{U}}$ is a generating full subcategory of \bar{U}_W endowed with the topology induced by the canonical topology. The result then follows from ([5, IV Corollary 1.2.1]). \square

Corollary 4.21 *Consider the full subcategory $\mathbb{C}_{\bar{U}}$ of $T_{\bar{U}}$ consisting in objects (Z_0, Z_v, f_v) such that the canonical morphism in \mathcal{T}*

$$yZ_0/yW_K \longrightarrow y(Z_0/W_K)$$

is an isomorphism with Z_0/W_K locally compact. Then, $\mathbb{C}_{\bar{U}}$ is a topologically generating family of $T_{\bar{U}}$.

Proof It is enough to show that

$$\mathcal{G}_{L,V,u,T} = (G_0 \times T, G_v \times T, g_v)$$

satisfies those properties. The map

$$W_{L/K} \rightarrow W_{L/K}/(N, \tilde{W}_{K_u}^1)$$

admits local sections by Corollary 4.2, since $W_{L/K}$ is locally compact and finite dimensional, and $(N, \tilde{W}_{K_u}^1)$ is compact hence closed. The map $W_K \rightarrow W_{L/K}$ admits local sections by Corollary 4.4. We obtain an epimorphism in \mathcal{T}

$$yW_K \rightarrow y(W_{L/K}/(N, \tilde{W}_{K_u}^1)).$$

Hence, the quotient of the action of yW_K on $y(W_{L/K}/(N, \tilde{W}_{K_u}^1))$ is the final object of \mathcal{T} . Thus, the quotient of

$$y(G_0 \times T) := y(W_{L/K}/(N, \tilde{W}_{K_u}^1) \times T) = y(W_{L/K}/(N, \tilde{W}_{K_u}^1)) \times yT$$

under the action of yW_K is yT , since inductive limits (in particular quotients of group actions) are universal in \mathcal{T} .

On the other hand, the quotient of the topological space

$$G_0 \times T := W_{L/K}/(N, \tilde{W}_{K_u}^1) \times T$$

by the action of the topological group W_K is the locally compact space T . □

Remark 4.22 The space of connected components of $\mathcal{G}_{L,V,u,T}$ is

$$t!\mathcal{G}_{L,V,u,T} = T.$$

4.5 The category $SLC_{\mathcal{T}}(\bar{U}_W)$ of sums of locally constant sheaves

Let \bar{U} be a connected étale \bar{X} -scheme endowed with Data 3.26. In this section, we denote by $t : \bar{U}_W \rightarrow \mathcal{T}$ the canonical map. This morphism t is connected and locally connected (see Theorem 4.27 (i)).

4.5.1 Locally constant sheaves

Recall that an object \mathcal{L} of \bar{U}_W is said to be *locally constant over \mathcal{T}* if there exists a covering morphism $\mathcal{F} \rightarrow 1$ of the final object of \bar{U}_W , an object S of \mathcal{T} and an isomorphism over \mathcal{F}

$$\mathcal{L} \times \mathcal{F} \simeq t^*S \times \mathcal{F}.$$

Definition 4.23 An object \mathcal{L} of \bar{U}_W is said to be *locally component-wise constant over \mathcal{T}* if there exists an epimorphism $\mathcal{F} \rightarrow 1$ where 1 denotes the final object of \bar{U}_W , an object $S \rightarrow t_! \mathcal{F}$ of $\mathcal{T}/t_! \mathcal{F}$ and an isomorphism over \mathcal{F}

$$\mathcal{L} \times \mathcal{F} \simeq t^* S \times_{t^* t_! \mathcal{F}} \mathcal{F}.$$

Proposition 4.24 *An object \mathcal{L} of \bar{U}_W is locally component-wise constant if and only if \mathcal{L} is locally constant.*

Proof Any locally constant object is locally component-wise constant. Indeed, if \mathcal{L} is locally constant, then one has

$$\mathcal{L} \times \mathcal{F} \simeq t^* S \times \mathcal{F} = t^* S \times t^* t_! \mathcal{F} \times_{t^* t_! \mathcal{F}} \mathcal{F} = t^*(S \times t_! \mathcal{F}) \times_{t^* t_! \mathcal{F}} \mathcal{F}.$$

The converse is also true. Let \mathcal{L} be a locally component-wise constant object. There exist \mathcal{F} covering the final object, $S \rightarrow t_! \mathcal{F}$ and an isomorphism over \mathcal{F}

$$\mathcal{L} \times \mathcal{F} \simeq t^* S \times_{t^* t_! \mathcal{F}} \mathcal{F}.$$

By Theorem 4.20, there exists an epimorphic family $\{\mathcal{F}_i \rightarrow \mathcal{F}, i \in I\}$ where \mathcal{F}_i is represented by an object $\mathcal{G}_{L_i, V_i, u_i, T_i}$ of $\mathbb{G}_{\bar{U}}$. Choosing a point of T_i for any element i of the set I , we obtain a map

$$\mathcal{G} := \coprod_{i \in I} y \mathcal{G}_{L_i, V_i, u_i, *} \rightarrow \coprod_{i \in I} y \mathcal{G}_{L_i, V_i, u_i, T_i} \rightarrow \mathcal{F} \rightarrow 1$$

which is a cover of the final object of \bar{U}_W . Then we have

$$\begin{aligned} \mathcal{L} \times \mathcal{G} &= \mathcal{L} \times \mathcal{F} \times_{\mathcal{F}} \mathcal{G} \simeq t^* S \times_{t^* t_! \mathcal{F}} \mathcal{F} \times_{\mathcal{F}} \mathcal{G} = t^* S \times_{t^* t_! \mathcal{F}} \mathcal{G} \\ &= t^*(S \times_{t_! \mathcal{F}} t_! \mathcal{G}) \times_{t^* t_! \mathcal{G}} \mathcal{G}. \end{aligned}$$

Hence, one can assume that $\mathcal{F} = \mathcal{G}$. Note that $t_! \mathcal{G}$ is the object of \mathcal{T} represented by the discrete set I , so that $S \rightarrow t_! \mathcal{G} = I$ can be seen as a family of objects S_i of \mathcal{T} , indexed by the set I . We set $\mathcal{G}_i := \mathcal{G}_{L_i, V_i, u_i, *}$ and we have $\mathcal{L} \times \mathcal{G}_i \simeq S_i \times \mathcal{G}_i$. For any $i, j \in I$ we consider an object $\mathcal{K} = \mathcal{G}_{L, V, u, *}$ of $\mathbb{G}_{\bar{U}}$ endowed with a map $\mathcal{K} \rightarrow \mathcal{G}_i \times \mathcal{G}_j$. Then we have an isomorphism in the slice topos \bar{U}_W/\mathcal{K}

$$S_i \times \mathcal{K} \simeq \mathcal{L} \times \mathcal{K} \simeq S_j \times \mathcal{K}. \tag{4.7}$$

But \mathcal{K} is connected over \mathcal{T} (i.e. $t_! \mathcal{K}$ is the final object of \mathcal{T}) and it follows that $\bar{U}_W/\mathcal{K} \rightarrow \mathcal{T}$ is connected, so that there exists a (unique) isomorphism $S_i \simeq S_j$ in \mathcal{T} inducing (4.7). We obtain an isomorphism $S \simeq \coprod_I S_i \simeq S_{i_0} \times I$ over I and one has

$$\mathcal{L} \times \mathcal{G} = S \times_{t_! \mathcal{G}} \mathcal{G} = S \times_I \mathcal{G} \simeq S_{i_0} \times \mathcal{G}$$

where i_0 is some fixed element of I . Hence, \mathcal{L} is locally constant. □

The category of “sums” of locally constant objects can be defined as follows (see [1], and [2, section 2] for more details). For any \mathcal{F} covering the final object of \bar{U}_W , one defines the push-out topos

$$\begin{array}{ccc} \bar{U}_W/\mathcal{F} & \longrightarrow & \bar{U}_W \\ \downarrow & & \downarrow \sigma_{\mathcal{F}} \\ \mathcal{T}/t!\mathcal{F} & \longrightarrow & \mathbf{Spl}_{\mathcal{F}}(\bar{U}_W) \end{array}$$

By definition of the push-out topos, an object of $\mathbf{Spl}_{\mathcal{F}}(\bar{U}_W)$ is a triple (\mathcal{L}, S, χ) where \mathcal{L} is an object of \bar{U}_W , S an object of $\mathcal{T}/t!\mathcal{F}$ and χ is an isomorphism in \bar{U}_W/\mathcal{F}

$$\mathcal{L} \times \mathcal{F} \simeq t^*S \times_{t^*t!\mathcal{F}} \mathcal{F}.$$

The morphisms in the category $\mathbf{Spl}_{\mathcal{F}}(\bar{U}_W)$ are the obvious ones.

The inverse image functor

$$\begin{array}{ccc} \sigma_{\mathcal{F}}^* : \mathbf{Spl}_{\mathcal{F}}(\bar{U}_W) & \longrightarrow & \bar{U}_W \\ (\mathcal{L}, S, \chi) & \longmapsto & \mathcal{L} \end{array}$$

is fully faithful, and its essential image is precisely the full subcategory of \bar{U}_W consisting in locally component-wise constant objects split by \mathcal{F} .

Given two epimorphisms $\mathcal{F} \rightarrow e$ and $\mathcal{F}' \rightarrow e$ and any map $\mathcal{F}' \rightarrow \mathcal{F}$, we have a canonical morphism $\mathbf{Spl}_{\mathcal{F}'}(\bar{U}_W) \rightarrow \mathbf{Spl}_{\mathcal{F}}(\bar{U}_W)$ such that the triangle

$$\begin{array}{ccc} \bar{U}_W & \xrightarrow{\sigma_{\mathcal{F}'}} & \mathbf{Spl}_{\mathcal{F}'}(\bar{U}_W) \\ & \searrow \sigma_{\mathcal{F}} & \downarrow \rho_{\mathcal{F}', \mathcal{F}} \\ & & \mathbf{Spl}_{\mathcal{F}}(\bar{U}_W) \end{array}$$

is commutative. Hence, two different maps $f_1 : \mathcal{F}' \rightarrow \mathcal{F}$ and $f_2 : \mathcal{F}' \rightarrow \mathcal{F}$ yield two morphisms $\rho_{\mathcal{F}', \mathcal{F}}^1$ and $\rho_{\mathcal{F}', \mathcal{F}}^2$ which are isomorphic.

Definition 4.25 The topos $SLC_{\mathcal{T}}(\bar{U}_W)$ is defined as the projective limit topos

$$SLC_{\mathcal{T}}(\bar{U}_W) := \varprojlim \mathbf{Spl}_{\mathcal{F}}(\bar{U}_W)$$

where \mathcal{F} runs over a small cofinal system of coverings of the final object of \bar{U}_W .

The canonical morphism

$$\sigma : \bar{U}_W \longrightarrow SLC_{\mathcal{T}}(\bar{U}_W), \tag{4.8}$$

induced by the compatible maps $\sigma_{\mathcal{F}}$, is connected and locally connected (see [2, Theorem 2.2]) so that $SLC_{\mathcal{T}}(\bar{U}_W)$ can be seen as a full subcategory of \bar{U}_W , which we call the category of sums of locally constant objects.

4.5.2 The role of the fundamental group

The purpose of the fundamental group is to classify the category of sums of locally constant objects. The Weil-étale topos \bar{U}_W is connected and locally connected over \mathcal{T} (see Theorem 4.27 (i)). Consider a \mathcal{T} -point p of \bar{U}_W (see Theorem 4.27 (ii)), i.e. a section of the structure map

$$t : \bar{U}_W \longrightarrow \mathcal{T}.$$

Composing p and the morphism (4.8), we obtain a point

$$\tilde{p} : \mathcal{T} \longrightarrow \bar{U}_W \longrightarrow SLC_{\mathcal{T}}(\bar{U}_W)$$

of the topos $SLC_{\mathcal{T}}(\bar{U}_W)$ over \mathcal{T} . The theory of the fundamental group in the context of topos theory shows the following. We refer to [9] and [2, Sect. 1], or [2, Sect. 2] (and [1] for more details) for a different approach. There exists a “pro-discrete localic group” $\pi_1(\bar{U}_W, p)$ in \mathcal{T} well defined up to a canonical isomorphism and an equivalence

$$B_{\pi_1(\bar{U}_W, p)} \simeq SLC_{\mathcal{T}}(\bar{U}_W),$$

where $B_{\pi_1(\bar{U}_W, p)}$ is the classifying topos of $\pi_1(\bar{U}_W, p)$. Moreover, the equivalence above identifies the inverse image of the point $\tilde{p} : \mathcal{T} \rightarrow SLC_{\mathcal{T}}(\bar{U}_W)$ with the forgetful functor $B_{\pi_1(\bar{U}_W, p)} \rightarrow \mathcal{T}$. In our situation, the “pro-discrete localic group” $\pi_1(\bar{U}_W, p)$ is in fact (the “limit” of) a strict pro-group in \mathcal{T} , as it follows from Theorem 4.27. More precisely, $\pi_1(\bar{U}_W, p)$ is pro-represented by a strict locally compact topological pro-group in the sense of Definition 4.7, and $B_{\pi_1(\bar{U}_W, p)}$ is the classifying topos of $\pi_1(\bar{U}_W, p)$ in the sense of Definition 2.4.

4.6 Computation of the fundamental group

Recall that one has a morphism

$$j : B_{W_F} \longrightarrow \bar{X}_W.$$

Lemma 4.26 *If \mathcal{L} is a locally constant object of \bar{X}_W over \mathcal{T} , then the adjunction map*

$$\mathcal{L} \longrightarrow j_* j^* \mathcal{L}$$

is an isomorphism.

Proof Let \mathcal{L} be a locally constant object of \bar{X}_W over \mathcal{T} . There exist an object \mathcal{S} of \mathcal{T} , an epimorphism $\mathcal{F} \rightarrow e$ where e is the final object of \bar{X}_W , and an isomorphism $\mathcal{L} \times \mathcal{F} \simeq t^*\mathcal{S} \times \mathcal{F}$ over \mathcal{F} . Consider the morphism defined by base change of the adjunction map:

$$\mathcal{L} \times \mathcal{F} \longrightarrow j_*j^*\mathcal{L} \times \mathcal{F}. \quad (4.9)$$

For any object $\mathcal{U} \rightarrow \mathcal{F}$ of \bar{X}_W/\mathcal{F} one has (using several adjunctions):

$$\begin{aligned} \text{Hom}_{\bar{X}_W/\mathcal{F}}(\mathcal{U}, j_*j^*\mathcal{L} \times \mathcal{F}) &= \text{Hom}_{\bar{X}_W}(\mathcal{U}, j_*j^*\mathcal{L}) \\ &= \text{Hom}_{B_{W_F}}(j^*\mathcal{U}, j^*\mathcal{L}) \\ &= \text{Hom}_{B_{W_F}/j^*\mathcal{F}}(j^*\mathcal{U}, j^*(\mathcal{L} \times \mathcal{F})) \\ &\simeq \text{Hom}_{B_{W_F}/j^*\mathcal{F}}(j^*\mathcal{U}, j^*(t^*\mathcal{S} \times \mathcal{F})) \\ &= \text{Hom}_{B_{W_F}}(j^*\mathcal{U}, j^*t^*\mathcal{S}) \\ &= \text{Hom}_{\bar{X}_W}(\mathcal{U}, j_*j^*t^*\mathcal{S}) \\ &= \text{Hom}_{\bar{X}_W/\mathcal{F}}(\mathcal{U}, j_*j^*t^*\mathcal{S} \times \mathcal{F}) \end{aligned}$$

Hence, we have an isomorphism over \mathcal{F}

$$j_*j^*\mathcal{L} \times \mathcal{F} \simeq j_*j^*t^*\mathcal{S} \times \mathcal{F},$$

and a commutative diagram

$$\begin{array}{ccc} \mathcal{L} \times \mathcal{F} & \xrightarrow{\simeq} & t^*\mathcal{S} \times \mathcal{F} \\ \downarrow & & \simeq \downarrow \\ j_*j^*\mathcal{L} \times \mathcal{F} & \xrightarrow{\simeq} & j_*j^*t^*\mathcal{S} \times \mathcal{F} \end{array}$$

where the map $t^*\mathcal{S} \times \mathcal{F} \rightarrow j_*j^*t^*\mathcal{S} \times \mathcal{F}$ is an isomorphism by Corollary 3.12. This shows that the morphism (4.9) is an isomorphism. But $\mathcal{F} \rightarrow e$ is epimorphic, so that the base change functor $\bar{X}_W \rightarrow \bar{X}_W/\mathcal{F}$ is faithful, hence conservative. Therefore, the adjunction map

$$\mathcal{L} \longrightarrow j_*j^*\mathcal{L}$$

is an isomorphism. □

The following theorem is the main result of this paper. Data 3.1 gives a geometric point $q_{\bar{X}} : \text{Spec } \bar{F} \rightarrow \bar{X}$. Then, we defined a \mathcal{T} -point of \bar{X}_W (see Proposition 3.9):

$$p_{\bar{X}} : \mathcal{T} \longrightarrow \bar{X}_W.$$

Recall also that the Weil-étale topos of a connected étale \bar{X} -scheme \bar{U} is defined as the slice topos

$$\bar{U}_W := \bar{X}_W / \gamma^* \bar{U}.$$

We consider below the topological pro-group $\underline{W}(\bar{U}, q_{\bar{U}})$ introduced in Definition 4.7.

Theorem 4.27 *For any connected étale \bar{X} -scheme \bar{U} , one has*

- (i) *The topos \bar{U}_W is connected and locally connected over \mathcal{T} .*
- (ii) *A geometric point $q_{\bar{U}}$ of the scheme \bar{U} over $q_{\bar{X}}$ induces a \mathcal{T} -valued point $p_{\bar{U}}$ over $p_{\bar{X}}$ of the Weil-étale topos \bar{U}_W , and respectively.*
- (iii) *One has an isomorphism of topological pro-groups*

$$\pi_1(\bar{U}_W, p_{\bar{U}}) \simeq \underline{W}(\bar{U}, q_{\bar{U}}).$$

Proof (i) Composing the localization map $l_{\bar{U}} : \bar{X}_W / \gamma^* \bar{U} \rightarrow \bar{X}_W$ with t , we obtain the canonical morphism

$$t_{\bar{U}} : \bar{U}_W := \bar{X}_W / \gamma^* \bar{U} \longrightarrow \bar{X}_W \longrightarrow \mathcal{T}.$$

The morphism $l_{\bar{U}}$ is locally connected, since it is a localization map, i.e. a local homeomorphism (the left adjoint of $l_{\bar{U}}^*$ is $l_{\bar{U}!}(\mathcal{F} \rightarrow \gamma^* \bar{U}) := \mathcal{F}$). By [6, C3.3.2], the class of locally connected morphisms is closed under composition. Hence, $t_{\bar{U}}$ is locally connected, i.e. $t_{\bar{U}}^*$ has a \mathcal{T} -indexed left adjoint $t_{\bar{U}!}$. This functor is defined as follows $t_{\bar{U}!} = t_! \circ l_{\bar{U}!}$, so that we have

$$t_{\bar{U}!} = t_! \circ l_{\bar{U}!}(\mathcal{F} \rightarrow \gamma^* \bar{U}) = t_!(\mathcal{F}).$$

for any object $\mathcal{F} \rightarrow \gamma^* \bar{U}$ of the slice topos \bar{U}_W . Let $Id_{\gamma^* \bar{U}}$ be the final object of \bar{U}_W . Then

$$t_{\bar{U}!}(Id_{\gamma^* \bar{U}}) = t_!(\gamma^* \bar{U}) = \{*\}$$

is the final object of \mathcal{T} since \bar{U} is connected (see Remark 3.19). It follows from ([6, C3.3.3]) that $t_{\bar{U}} : \bar{U}_W \rightarrow \mathcal{T}$ is connected and locally connected.

One can also give the following easier—but less canonical—argument. By Proposition 3.28, $(T_{\bar{U}}, \mathcal{J}_{ls})$ is a site for the topos U_L . The proof of Proposition 3.10 is still valid by replacing $T_{\bar{X}}$ with $T_{\bar{U}}$ (without any other change). This shows that \bar{U}_W is connected and locally connected over \mathcal{T} .

- (ii) A geometric point $q_{\bar{U}} : Spec \bar{F} \rightarrow \bar{U}$ gives a point of the étale topos

$$q_{\bar{U}} : \underline{Sets} \longrightarrow \bar{U}_{et}$$

where $q_{\bar{U}}^*$ is the usual fiber functor. We obtain a morphism

$$p_{\bar{U}} = q_{\bar{U}} \times_{q_{\bar{X}}} p_{\bar{X}} : \mathcal{T} = \underline{Sets} \times_{\underline{Sets}} \mathcal{T} \longrightarrow \bar{U}_{et} \times_{\bar{X}_{et}} \bar{X}_W =: \bar{X}_W$$

defined over \mathcal{T} . One can recover the geometric point $q_{\bar{U}}$ from $p_{\bar{U}}$. Indeed, let $p_{\bar{U}} : \mathcal{T} \rightarrow \bar{U}_W$ be a \mathcal{T} -point of \bar{U}_W . Then, we have a point of the étale topos

$$\underline{Sets} \longrightarrow \mathcal{T} \longrightarrow \bar{U}_W \longrightarrow \bar{U}_{et}, \quad (4.10)$$

where the map $\underline{Sets} \rightarrow \mathcal{T}$ is the canonical one (see [5, IV4.10]). By ([5, VIII Theorem 7.9]), the category of points of the étale topos of a scheme is equivalent to the category of geometric points (algebraic and separable) and specialization maps. Then, the map (4.10) corresponds to the given geometric point $q_{\bar{U}}$ of \bar{U} . However, two distinct \mathcal{T} -points of \bar{U}_W over $p_{\bar{X}}$ can induce the same \underline{Sets} -valued point of \bar{U}_{et} , hence the same geometric point.

(iii) We make the choices listed in Data 3.26. Proposition 3.28 yields an equivalence

$$(\widetilde{T_{\bar{U}}, \mathcal{J}_{ls}}) \longrightarrow \bar{U}_W.$$

This equivalence provides us with the morphism

$$j : B_{W_K} \longrightarrow (\widetilde{T_{\bar{U}}, \mathcal{J}_{ls}}) \simeq \bar{U}_W$$

corresponding to the generic point of the connected étale \bar{X} -scheme \bar{U} . Then the \mathcal{T} -point $p_{\bar{U}}$ defined in (ii), using the geometric point $q_{\bar{U}}$ of \bar{U} given by Data 3.26, is isomorphic to the map defined over \mathcal{T} :

$$p := j \circ u : \mathcal{T} \longrightarrow B_{W_K} \longrightarrow (\widetilde{T_{\bar{U}}, \mathcal{J}_{ls}}) \simeq \bar{U}_W$$

where $\mathcal{T} \rightarrow B_{W_K}$ is the canonical \mathcal{T} -point of B_{W_K} (see Proposition 3.9). In order to ease the notations, we denote here by j and p the maps $j_{\bar{U}}$ and $p_{\bar{U}}$. Finally, we denote by $u : B_{W_K} \rightarrow \mathcal{T}$ the canonical map, i.e. the map induced by the morphism of groups $W_K \rightarrow 1$.

If \mathcal{L} is an object of \bar{U}_W , then $j^*\mathcal{L}$ is the object $p^*\mathcal{L}$ of \mathcal{T} endowed with an action of $y(W_K)$. In other words, $j^*\mathcal{L}$ comes with a morphism of groups in \mathcal{T} :

$$y(W_K) \longrightarrow \underline{Aut}_{\mathcal{T}}(p^*\mathcal{L}).$$

The following proof consists in two steps:

Step 1 We define a projective system of Galois torsors in the topos \bar{U}_W .

Let $\bar{F}/L/K$ be a finite Galois subextension given by a geometric point $q_{\bar{U}} : \text{Spec } \bar{F} \rightarrow \bar{U}$ over \bar{X} . Consider the topological group $W(\bar{U}, L)$ of Definition 4.6. The morphism of left exact sites

$$t_{\bar{U}}^* : (Top, \mathcal{J}_{op}) \longrightarrow (T_{\bar{U}}, \mathcal{J}_{ls}) \quad (4.11)$$

factors through the morphism

$$\begin{aligned} (B_{Top} W(\bar{U}, L), \mathcal{J}_{Is}) &\longrightarrow (T_{\bar{U}}, \mathcal{J}_{Is}) \\ Z &\longmapsto (Z, Z, Id_Z) \end{aligned} \tag{4.12}$$

where W_K acts on Z via the morphism $W_K \rightarrow W(\bar{U}, L)$. Respectively, $W_{k(u)}$ acts on Z via the morphism $W_{k(v)} = W_{K_u}/W_{K_u}^1 \rightarrow W(\bar{U}, L)$. We obtain a commutative diagram of topoi

$$\begin{array}{ccc} B_{W_K} & \longrightarrow & B_{W(\bar{U}, L)} \\ j_{\bar{U}} \downarrow & \nearrow \pi & \downarrow \\ \bar{U}_W & \xrightarrow{t_{\bar{U}}} & \mathcal{T} \end{array}$$

where the map $B_{W_K} \rightarrow B_{W(\bar{U}, L)}$ is induced by the surjection

$$W_K \longrightarrow W_K/N(\bar{U}, L) = W(\bar{U}, L).$$

The map $\pi : \bar{U}_W \rightarrow B_{W(\bar{U}, L)}$ corresponds to the torsor

$$\text{TORS}(\bar{U}, L) := \pi^* E W(\bar{U}, L)$$

where $E W(\bar{U}, L)$ is the universal torsor of $B_{W(\bar{U}, L)}$ given by $W(\bar{U}, L)$ acting on itself by multiplications. Note that $\text{TORS}(\bar{U}, L)$ is a torsor of group $W(\bar{U}, L)$, which is *connected* over \mathcal{T} . Indeed, its space of connected components

$$t_{\bar{U},!} \text{TORS}(\bar{U}, L) = yW(\bar{U}, L)/yW_K$$

is the final object of \mathcal{T} , since $yW_K \rightarrow yW(\bar{U}, L)$ is an epimorphism in \mathcal{T} . The last claim follows from the fact that $W_K \rightarrow W(\bar{U}, L)$ has local sections since $W(\bar{U}, L)$ is finite dimensional (see Theorem 4.1 and Lemma 4.3). The topological pro-group

$$\underline{W}(\bar{U}, q_{\bar{U}}) := \{W(\bar{U}, L), \text{ for } \bar{F}/L/K \text{ finite Galois}\}$$

yields a projective system of connected torsors

$$\{\text{TORS}(\bar{U}, L), \text{ for } \bar{F}/L/K \text{ finite Galois}\} \tag{4.13}$$

This projective system of torsors is given by compatible maps to classifying topoi. By the universal property of projective limits, the pro-torsor (4.13) corresponds to an essentially unique morphism

$$\bar{U}_W \longrightarrow \varprojlim B_{W(\bar{U}, L)} =: B_{\underline{W}(\bar{U}, q_{\bar{U}})}$$

into the classifying topos of the topological pro-group $\underline{W}(\bar{U}, q_{\bar{U}})$.

Step 2 The pro-torsor (4.13) is universal.

In other words, we have to show that any locally constant object \mathcal{L} of \bar{U}_W over \mathcal{T} is trivialized by a torsor of the form $\text{TORS}(\bar{U}, L)$. This is the technical part of the proof.

Step 2.1 Let \mathcal{L} be such a locally constant object. There exist an object \mathcal{S} of \mathcal{T} , an epimorphism $\mathcal{F} \rightarrow e$ where e is the final object of \bar{U}_W , and an isomorphism

$$\mathcal{L} \times \mathcal{F} \simeq t^* \mathcal{S} \times \mathcal{F} \tag{4.14}$$

over \mathcal{F} . Since the full-subcategory $\mathbb{G}_{\bar{U}}$ of \bar{U}_W defined in Sect. 4.4 is a generating subcategory (see Theorem 4.20), one can assume that \mathcal{F} is representable by a sum of objects in $\mathbb{G}_{\bar{U}}$:

$$\mathcal{F} = \coprod_{i \in I} \mathcal{F}_i = y \mathcal{G}_{L_i, V_i, u_i, T_i}.$$

For any index $i \in I$, a point of the topological space $T_i \neq \emptyset$ yields a morphism

$$\mathcal{G}_{L_i, V_i, u_i} := \mathcal{G}_{L_i, V_i, u_i, *} \longrightarrow \mathcal{G}_{L_i, V_i, u_i, T_i}$$

in the category $T_{\bar{U}}$, where $*$ denotes the one point space as usual.

Recall that $\mathcal{G}_{L_i, V_i, u_i}$ is defined as follows. Here $\bar{F}/L_i/K$ be a finite Galois sub-extension, V_i is an open subset of \bar{U} , u_i is a point of \bar{U} and T_i is a separated topological space. We denote by N_i the closed normal subgroup of $W_{L_i/K}$ generated by the subgroups $\tilde{W}_{K_v}^1 \subseteq W_{L_i/K}$ for any point $v \in V_i$. Let $(N_i, \tilde{W}_{K_{u_i}}^1)$ be the compact subgroup of $W_{L_i/K}$ generated by N_i and $\tilde{W}_{K_{u_i}}^1$. The object

$$\mathcal{G}_i := y \mathcal{G}_{L_i, V_i, u_i} = y(G_{i,0}, G_{i,u}, g_{i,u})$$

is then defined as follows. Assume that u_i is not in V_i . We consider

$$G_{i,0} = W_{L_i/K} / (N_i, \tilde{W}_{K_{u_i}}^1)$$

as a W_K -space and

$$G_{i,v} = W_{L_i/K} / (N_i, \tilde{W}_{K_{u_i}}^1)$$

as a $W_{k(v)}$ -space for any point v of $V_i \subseteq \bar{U}$. Then we set $G_{i,u_i} = W_{k(u_i)}$ on which $W_{k(u_i)}$ acts by multiplication, and $G_{i,u} = \emptyset$ anywhere else.

Note that the image of \mathcal{G}_i in the final object $y\bar{U}$ of \bar{U}_W , i.e. the support of the sheaf \mathcal{G}_i , is precisely the subobject of $y\bar{U}$ given by

$$V_i \cup \{u_i\} \hookrightarrow \bar{U}.$$

The family

$$\{\mathcal{G}_i \rightarrow y\bar{U}, i \in I\}$$

is an epimorphic family, i.e. a covering family of the final object of the topos \bar{U}_W for the canonical topology. Indeed, the corresponding family of $T_{\bar{U}}$ is a local section cover, as it follows from the facts that a map from a non-empty space to the one point space is a local section cover and that we have

$$\bigcup_{i \in I} (V_i \cup \{u_i\}) = \bar{U}. \tag{4.15}$$

Step 2.2 Applying the base change functor along the map (given by any point of T_i)

$$\mathcal{G}_i := y\mathcal{G}_{L_i, V_i, u_i} \longrightarrow y\mathcal{G}_{L_i, V_i, u_i, T_i} = \mathcal{F}_i$$

to the trivialization (4.14), we obtain an isomorphism over \mathcal{G}_i :

$$\mathcal{L} \times \mathcal{G}_i = (\mathcal{L} \times \mathcal{F}_i) \times_{\mathcal{F}_i} \mathcal{G}_i \simeq (t^*\mathcal{S} \times \mathcal{F}_i) \times_{\mathcal{F}_i} \mathcal{G}_i = t^*\mathcal{S} \times \mathcal{G}_i$$

where \mathcal{S} is an object of \mathcal{T} . Applying in turn the functor j^* , we get an isomorphism

$$j^*\mathcal{L} \times yG_{i,0} = j^*(\mathcal{L} \times \mathcal{G}_i) \simeq j^*(t^*\mathcal{S} \times \mathcal{G}_i) = j^*t^*\mathcal{S} \times j^*\mathcal{G}_i = u^*\mathcal{S} \times yG_{i,0} \tag{4.16}$$

over $y(G_{i,0}) = j^*\mathcal{G}_i$, i.e. an isomorphism in the topos $B_{W_K}/yG_{i,0}$.

Assume for a moment that the action of yW_K on $p^*\mathcal{L}$ (given by the object $j^*\mathcal{L}$ of B_{W_K}) factors through $W_{L_i/K}$. In other words, suppose that one has a commutative triangle

$$\begin{array}{ccc} yW_K & & \\ \downarrow & \searrow & \\ yW_{L_i/K} & \longrightarrow & \underline{Aut}_{\mathcal{T}}(p^*\mathcal{L}) \end{array}$$

Then $j^*\mathcal{L}$ is an object of the full subcategory

$$B_{W_{L_i/K}} \hookrightarrow B_{W_K}.$$

Recall that

$$G_{i,0} = W_{L_i/K}/(N, \tilde{W}_{K_{u_i}}^1).$$

But there is a canonical equivalence

$$\begin{aligned} B_{W_{L_i/K}/yG_{i,0}} &:= B_{W_{L_i/K}/y(W_{L_i/K}/(N_i, W_{\tilde{K}_{u_i}}^1))} \\ &= B_{W_{L_i/K}/(yW_{L_i/K}/y(N_i, \tilde{W}_{\tilde{K}_{u_i}}^1))} \\ &\simeq B_{(N_i, \tilde{W}_{\tilde{K}_{u_i}}^1)} \end{aligned}$$

where the second equality follows (by [3, Lemma 3]) from the fact that the projection

$$W_{L_i/K} \longrightarrow W_{L_i/K}/(N_i, \tilde{W}_{\tilde{K}_{u_i}}^1)$$

admits local sections, as it follows from Corollary 4.2 and Lemma 4.3. Let us make the equivalence above more explicit. The homogeneous space $G_{i,0}$ has a distinguished (non-equivariant) point $* \rightarrow G_{i,0}$, and we have

$$\begin{aligned} B_{W_{L_i/K}/yG_{i,0}} &\longrightarrow B_{(N_i, \tilde{W}_{\tilde{K}_{u_i}}^1)} \\ (\mathcal{X} \rightarrow yG_{i,0}) &\longmapsto \mathcal{X} \times_{yG_{i,0}} * \end{aligned}$$

Under this equivalence, the base change functor

$$\begin{aligned} B_{W_{L_i/K}} &\longrightarrow B_{W_{L_i/K}/yG_{i,0}} \\ \mathcal{F} &\longmapsto \mathcal{F} \times_{yG_{i,0}} * \end{aligned}$$

takes a $yW_{L_i/K}$ -object \mathcal{F} of \mathcal{T} to the same object of \mathcal{T} :

$$\mathcal{F} = \mathcal{F} \times_{yG_{i,0}} \times_{yG_{i,0}} *$$

endowed with the induced $y(N_i, \tilde{W}_{\tilde{K}_{u_i}}^1)$ -action. Therefore, (4.16) means that $y(N_i, \tilde{W}_{\tilde{K}_{u_i}}^1)$ acts trivially on $j^*\mathcal{L}$, i.e. $y(N_i, \tilde{W}_{\tilde{K}_{u_i}}^1)$ is in the kernel of the map

$$yW_{L_i/K} \longrightarrow \underline{\text{Aut}}_{\mathcal{T}}(p^*\mathcal{L}). \quad (4.17)$$

Hence, the action of $yW_{L_i/K}$ on $p^*\mathcal{L}$ factors through

$$yW_{L_i/K}/y(N_i, \tilde{W}_{\tilde{K}_{u_i}}^1) = y(W_{L_i/K}/(N_i, \tilde{W}_{\tilde{K}_{u_i}}^1)).$$

The same argument shows that the action of yW_K on $p^*\mathcal{L}$ factors through $yW_{L_i/K}$, i.e. that the commutative triangle considered above exists. Indeed, by (4.16) one has an isomorphism

$$\begin{aligned} j^*\mathcal{L} \times yW_{L_i/K} &= (j^*\mathcal{L} \times yG_{i,0}) \times_{yG_{i,0}} yW_{L_i/K} \\ &\simeq (u^*\mathcal{S} \times yG_{i,0}) \times_{yG_{i,0}} yW_{L_i/K} \\ &= u^*\mathcal{S} \times yW_{L_i/K} \end{aligned}$$

where we consider the object of B_{W_K}

$$yW_{L_i/K} = y(W_K/W_{L_i}^c) = yW_K/yW_{L_i}^c.$$

Note that $W_{L_i}^c$, which is the closure of the commutator subgroup of W_{L_i} , is compact in W_K . Then, the previous argument shows that the action of yW_K on $p^*\mathcal{L}$ factors through $yW_{L_i/K} = yW_K/yW_{L_i}^c$ (this last identification is valid by Theorem 4.1, Lemma 4.3 and [3, Lemma 3]). In summary, we have proven the following

Proposition 4.28 *The action $yW_K \rightarrow \underline{Aut}_{\mathcal{T}}(p^*\mathcal{L})$ induces a morphism*

$$\rho_i : yW_{L_i/K} \longrightarrow \underline{Aut}_{\mathcal{T}}(p^*\mathcal{L})$$

for any $i \in I$, and we have

$$y(N_i, \widetilde{W}_{K_{u_i}}^1) \hookrightarrow \text{Ker}(\rho_i).$$

Step 2.3 Choose an object \mathcal{G}_{i_0} (i.e. an index $i_0 \in I$). We have a morphism

$$\rho_{i_0} : yW_{L_{i_0}/K} \longrightarrow \underline{Aut}_{\mathcal{T}}(p^*\mathcal{L}).$$

Let N_{i_0} be the closed normal subgroup of $W_{L_{i_0}/K}$ defined above. We have an open subset $V_{i_0} \subseteq \bar{U}$ such that $\widetilde{W}_{K_v}^1 \subseteq N_{i_0}$ for any $v \in V_{i_0}$. We claim that for any $u \in \bar{U} - V_{i_0}$, the subgroup $y\widetilde{W}_{K_u}^1 \subset yW_{L_{i_0}/K}$ is in the kernel of ρ_{i_0} , i.e. one has

$$y\widetilde{W}_{K_u}^1 \hookrightarrow \text{Ker}(\rho_{i_0}); \text{ for any } u \in \bar{U} - V_{i_0}. \quad (4.18)$$

Let $u \in \bar{U} - V_{i_0}$. By (4.15), there exists an index $i \in I$ such that the support of \mathcal{G}_i contains u , i.e. one has:

$$u \in V_i \cup \{u_i\}.$$

Proposition 4.28 then shows that the subgroup $y\widetilde{W}_{K_u}^1 \subset yW_{L_i/K}$ is in the kernel of the morphism

$$\rho_i : yW_{L_i/K} \longrightarrow \underline{Aut}_{\mathcal{T}}(p^*\mathcal{L}).$$

Hence the subgroup $yW_{K_u}^1 \subset yW_K$ is in the kernel of the morphism

$$\rho : yW_K \longrightarrow \underline{Aut}_{\mathcal{T}}(p^*\mathcal{L}).$$

It follows that the image of $yW_{K_u}^1$ in $yW_{L_{i_0}/K}$ is in the kernel of ρ_{i_0} . But the continuous surjection

$$W_{K_u}^1 \twoheadrightarrow \widetilde{W}_{K_u}^1 \subset W_{L_{i_0}/K}$$

admits local sections (using Theorem 4.1 and the fact that $\tilde{W}_{K_u}^1$ is finite dimensional), hence induces an epimorphism in \mathcal{T} . Thus the image of $yW_{K_u}^1$ in $yW_{L_{i_0}/K}$ is $y\tilde{W}_{K_u}^1 \subset y(W_{L_{i_0}/K})$, which is therefore in the kernel of ρ_{i_0} . We have proven (4.18).

Let $N(\tilde{W}_{K_u}^1)$ be the normal topological subgroup of $W_{L_{i_0}/K}$ generated by the subgroup $\tilde{W}_{K_u}^1$. By Lemma 4.17, $N(\tilde{W}_{K_u}^1)$ is compact and we have

$$\mathcal{N}(y\tilde{W}_{K_u}^1) = yN(\tilde{W}_{K_u}^1),$$

where $\mathcal{N}(y\tilde{W}_{K_u}^1)$ is the normal subgroup of $yW_{L_{i_0}/K}$ generated by $y\tilde{W}_{K_u}^1$. We obtain

$$yN(\tilde{W}_{K_u}^1) = \mathcal{N}(y\tilde{W}_{K_u}^1) \hookrightarrow \text{Ker}(\rho_{i_0}).$$

Therefore, the subgroup of $yW_{L_{i_0}/K}$ generated by yN_{i_0} and $yN(\tilde{W}_{K_u}^1)$, for any $u \in \bar{U} - V_{i_0}$, is contained in $\text{Ker}(\rho_{i_0})$:

$$\langle yN_{i_0}; yN(\tilde{W}_{K_u}^1); \text{ for any } u \in \bar{U} - V_{i_0} \rangle \hookrightarrow \text{Ker}(\rho_{i_0})$$

The topological subgroup of $W_{L_{i_0}/K}$, generated by N_{i_0} and $N(\tilde{W}_{K_u}^1)$ for any $u \in \bar{U} - V_{i_0}$, is normal and compact. Hence, this subgroup is precisely $N(\bar{U}, L_{i_0})$, which is the closed normal subgroup of $W_{L_{i_0}/K}$ generated by all the subgroups $\tilde{W}_{K_u}^1 \subset W_{L_{i_0}/K}$ for any $u \in \bar{U}$ (see Sect. 4.2).

Lemma 4.14 then shows that

$$yN(\bar{U}, L_{i_0}) \hookrightarrow \text{Ker}(\rho_{i_0})$$

and that the morphism ρ_{i_0} induces a morphism from

$$yW_{L_{i_0}/K}/yN(\bar{U}, L_{i_0}) = y(W_{L_{i_0}/K}/N(\bar{U}, L_{i_0})) = yW(\bar{U}, L_{i_0})$$

to the automorphism group

$$\underline{\text{Aut}}_{\mathcal{T}}(p^*\mathcal{L}).$$

Therefore, we have an isomorphism

$$j^*\mathcal{L} \times yW(\bar{U}, L_{i_0}) \simeq u^*\mathcal{S} \times yW(\bar{U}, L_{i_0})$$

in the topos $B_{W_K}/yW(\bar{U}, L_{i_0})$, where \mathcal{S} is an object of \mathcal{T} . In other words, we have

$$j^*\mathcal{L} \times j^*\text{Tors}(\bar{U}, L_{i_0}) \simeq u^*\mathcal{S} \times j^*\text{Tors}(\bar{U}, L_{i_0})$$

since

$$j^*\text{Tors}(\bar{U}, L_{i_0}) = yW(\bar{U}, L_{i_0})$$

where $\text{TORS}(\bar{U}, L_{i_0})$ is the torsor corresponding to the morphism

$$\pi : \bar{U}_W \longrightarrow B_{W(\bar{U}, L_{i_0})}$$

defined in Step 1.

Step 2.4 The torsor $\text{TORS}(\bar{U}, L_{i_0})$ is locally constant over \mathcal{T} , since any torsor is trivialized by itself, hence Lemma 4.26 applies. We obtain an isomorphism over $\text{TORS}(\bar{U}, L_{i_0})$:

$$\begin{aligned} \mathcal{L} \times \text{TORS}(\bar{U}, L_{i_0}) &\simeq j_* j^* (\mathcal{L} \times \text{TORS}(\bar{U}, L_{i_0})) \\ &\simeq j_* j^* \mathcal{L} \times j_* j^* \text{TORS}(\bar{U}, L_{i_0}) \\ &\simeq j_* (j^* \mathcal{L} \times j^* \text{TORS}(\bar{U}, L_{i_0})) \\ &\simeq j_* (u^* \mathcal{S} \times j^* \text{TORS}(\bar{U}, L_{i_0})) \\ &\simeq j_* u^* \mathcal{S} \times j_* j^* \text{TORS}(\bar{U}, L_{i_0}) \\ &\simeq j_* j^* t^* \mathcal{S} \times j_* j^* \text{TORS}(\bar{U}, L_{i_0}) \\ &\simeq t^* \mathcal{S} \times \text{TORS}(\bar{U}, L_{i_0}). \end{aligned}$$

This shows that any locally constant object \mathcal{L} of \bar{U}_W over \mathcal{T} is trivialized by a torsor of the form $\text{TORS}(\bar{U}, L)$. Hence, the pro-torsor (4.13)

$$\{\text{TORS}(\bar{U}, L), \text{ for } \bar{F}/L/K \text{ finite Galois}\}$$

is universal. The pro-group object of \mathcal{T} defined by this pro-torsor is the projective system of its Galois groups:

$$y \underline{W}(\bar{U}, q_{\bar{U}}) := \{yW(\bar{U}, L), \text{ for } \bar{F}/L/K \text{ finite Galois}\}.$$

Equivalently, this pro-group object of \mathcal{T} is obtain by applying the fiber functor $p_{\bar{U}}^*$:

$$y \underline{W}(\bar{U}, q_{\bar{U}}) := p_{\bar{U}}^* \{\text{TORS}(\bar{U}, L), \text{ for } \bar{F}/L/K \text{ finite Galois}\}.$$

This yields an isomorphism of pro-group objects in \mathcal{T}

$$\pi_1(\bar{U}_W, p_{\bar{U}}) \simeq y \underline{W}(\bar{U}, q_{\bar{U}}).$$

Hence, $\pi_1(\bar{U}_W, p_{\bar{U}})$ is a topological pro-group canonically isomorphic to $\underline{W}(\bar{U}, q_{\bar{U}})$. \square

5 Weil-étale Cohomology with coefficients in $\tilde{\mathbb{R}}$

Let \bar{X} be a connected étale \bar{X} -scheme. For any topos $t : \mathcal{E} \rightarrow \mathcal{T}$ defined over \mathcal{T} , we set $\tilde{\mathbb{R}} := t^*(y\mathbb{R})$, where $y\mathbb{R}$ is the sheaf of \mathcal{T} represented by the standard topological group \mathbb{R} .

Lemma 5.1 *Let $j : B_{W_K} \rightarrow \bar{U}_W$ be the canonical map. We have $j_*\tilde{\mathbb{R}} = \tilde{\mathbb{R}}$ and $R^n j_*\tilde{\mathbb{R}} = 0$ for any $n \geq 1$.*

Proof The identification $j_*\tilde{\mathbb{R}} = \tilde{\mathbb{R}}$ follows immediately from

$$\text{Hom}_{T_{\bar{U}}}((Z_0, Z_u, f_u), (\mathbb{R}, \mathbb{R}, Id)) = \text{Hom}_{B_{T_{op}W_K}}(Z_0, \mathbb{R}).$$

where (Z_0, Z_u, f_u) is any object of $T_{\bar{U}}$. By Theorem 4.20, the site $(\mathbb{G}_{\bar{U}}, \mathcal{J}_{ls})$ is a site for \bar{U}_W . Then $R^n j_*\tilde{\mathbb{R}}$ is the sheaf on $(\mathbb{G}_{\bar{U}}, \mathcal{J}_{ls})$ associated to the presheaf

$$\begin{aligned} P^n j_*\tilde{\mathbb{R}} : \mathbb{G}_{\bar{U}} &\longrightarrow Ab \\ \mathcal{G}_{L,V,u,T} &\longrightarrow H^n(B_{W_K}/(j^*\mathcal{G}_{L,V,u,T}), \tilde{\mathbb{R}}) \end{aligned}$$

for any $n \geq 1$. Recall that one has

$$j^*\mathcal{G}_{L,V,u,T} = W_{L/K}/(N, \tilde{W}_{K_u}^1) \times T$$

where N is the closure of the normal subgroup of $W_{L/K}$ generated by the images of the maps $W_{K_v}^1 \rightarrow W_{L/K}$ where v runs over the closed points of $V \subset \bar{U}$ (see Sect. 4.4). One can write $W_{L/K}/(N, \tilde{W}_{K_u}^1) = W_K/\Lambda$ where Λ is a compact subgroup of W_K . The map $W_K \rightarrow W_K/\Lambda$ has local sections as it follows from Theorem 4.1 and from the fact that $W_K/\Lambda = W_{L/K}/(N, \tilde{W}_{K_u}^1)$ is finite dimensional. We obtain $yW_K/y\Lambda = y(W_K/\Lambda)$, and the following identifications:

$$\begin{aligned} B_{W_K}/(j^*\mathcal{G}_{L,V,u,T}) &= B_{W_K}/y(W_{L/K}/\Lambda \times T) \\ &= B_{W_K}/(yW_{L/K}/y\Lambda \times yT) \\ &= (B_{W_K}/(yW_{L/K}/y\Lambda))/yT \\ &= B_{\Lambda}/T \end{aligned}$$

Therefore, for any $n \geq 1$, one has

$$P^n j_*\tilde{\mathbb{R}}(\mathcal{G}_{L,V,u,T}) = H^n(B_{\Lambda}/T, \tilde{\mathbb{R}}).$$

Consider the pull-back square

$$\begin{array}{ccc} \mathcal{T}/T & \xrightarrow{a} & \mathcal{T} \\ \downarrow b & & \downarrow d \\ B_{\Lambda}/T & \xrightarrow{c} & B_{\Lambda} \end{array}$$

This pull-back square is obtained by localization since $B_{\Lambda}/E\Lambda = \mathcal{T}$ and $(B_{\Lambda}/T)/(E\Lambda \times T) = \mathcal{T}/T$. One checks immediately that such a pull-back satisfies the Beck–Chevalley condition $d^*c_* \simeq a_*b^*$ (this is a special case of the Beck–Chevalley condition for locally connected morphisms). But b^* is a localization functor, hence it

preserves injective abelian objects. We obtain

$$d^* R^m(c_*) \simeq R^m(a_*)b^*. \tag{5.1}$$

The sheaf $R^m(a_*)(\widetilde{\mathbb{R}})$ is the sheaf associated to the presheaf

$$\begin{aligned} P^m(a_*)(\widetilde{\mathbb{R}}) : Top &\longrightarrow Ab \\ T' &\longmapsto H^m(\mathcal{T}/(T \times T'), \widetilde{\mathbb{R}}) = H^m(Sh(T \times T'), C^0(\mathbb{R})) \end{aligned}$$

where $C^0(\mathbb{R})$ denotes the sheaf of germs of continuous real-valued functions on the locally compact space $T \times T'$, and $Sh(T \times T')$ is the topos of sheaves (i.e. of étalé spaces) on $T \times T'$. The isomorphism $H^m(\mathcal{T}/(T \times T'), \widetilde{\mathbb{R}}) = H^m(Sh(T \times T'), C^0(\mathbb{R}))$ follows from the fact that the big topos $\mathcal{T}/(T \times T')$ of the space $T \times T'$ is cohomologically equivalent to $Sh(T \times T')$ (see [5, IV 4.10]). But $T \times T'$ is locally compact hence paracompact, so that the sheaf $C^0(\mathbb{R})$ is “fin” on $T \times T'$ hence acyclic for the global sections functor. We obtain $P^m(a_*)(\widetilde{\mathbb{R}}) = 0$ for any $m \geq 1$, so $R^m(a_*)(\widetilde{\mathbb{R}}) = 0$ for any $m \geq 1$. Then it follows from (5.1) that $R^m(c_*)(\widetilde{\mathbb{R}}) = 0$ for any $m \geq 1$, since d^* is faithful. Moreover, by (5.1) with $m = 0$, the sheaf $c_*(\widetilde{\mathbb{R}})$ can be identified with $a_*(\widetilde{\mathbb{R}})$ with trivial $y\Delta$ -action, which is in turn represented by the space $\underline{Hom}_{Top}(T, \mathbb{R})$ on which Δ acts trivially. Hence, the Leray spectral sequence

$$H^n(B_\Delta, R^m(c_*)(\widetilde{\mathbb{R}})) \Rightarrow H^{n+m}(B_\Delta/T, \widetilde{\mathbb{R}})$$

degenerates and yields

$$H^n(B_\Delta/T, \widetilde{\mathbb{R}}) \simeq H^n(B_\Delta, \underline{Hom}_{Top}(T, \mathbb{R}))$$

By ([3, Corollary 8]), we have $H^n(B_\Delta, \underline{Hom}_{Top}(T, \mathbb{R})) = 0$ for any $n \geq 1$, since Δ is compact and $\underline{Hom}_{Top}(T, \mathbb{R})$ is a locally convex, Hausdorff and quasi-complete real vector space. We have shown the following:

$$P^n j_* \widetilde{\mathbb{R}}(\mathcal{G}_{L,V,u,T}) = H^n(B_\Delta/T, \widetilde{\mathbb{R}}) = 0$$

for any $n \geq 1$ and any object $\mathcal{G}_{L,V,u,T}$ of $\mathbb{G}_{\bar{J}}$. Hence $R^n j_* \widetilde{\mathbb{R}} = 0$ for any $n \geq 1$. \square

Proposition 5.2 *We have $H^n(\bar{U}_W, \widetilde{\mathbb{R}}) = \mathbb{R}$ for $n = 0, 1$ and $H^n(\bar{U}_W, \widetilde{\mathbb{R}}) = 0$ for $n \geq 2$.*

Proof We use the spectral sequence associated with the morphism $j : B_{W_K} \rightarrow \bar{U}_W$ and obtain $H^n(\bar{U}_W, \widetilde{\mathbb{R}}) \simeq H^n(B_{W_K}, \widetilde{\mathbb{R}})$, thanks to the previous Lemma. The latter group can be computed using the product decomposition $W_K = W_K^1 \times \mathbb{R}$ and the fact that W_K^1 is compact (see [3]). \square

6 Consequences of the main result

6.1 Direct consequences

In this section, \bar{U} denotes a connected étale \bar{X} -scheme with function field K . We consider the classifying topos of the topological pro-group $\underline{W}(\bar{U}, q_{\bar{U}})$, which is defined as the projective limit:

$$B_{\underline{W}(\bar{U}, q_{\bar{U}})} := \varprojlim B_{W(\bar{U}, L)}$$

Recall from Sect. 4.5 the definition of the category $SLC_{\mathcal{T}}(\bar{U}_W)$ of sums of locally constant objects over \mathcal{T} . The following result, which is an immediate consequence -in fact a rewriting- of the previous theorem, gives an explicit description of the category of sums of locally constant objects.

Corollary 6.1 *There is an equivalence defined over \mathcal{T} and compatible with the point $p_{\bar{U}}$:*

$$SLC_{\mathcal{T}}(\bar{U}_W) \simeq B_{\underline{W}(\bar{U}, q_{\bar{U}})}$$

This equivalence is canonically induced by Data 3.26.

Corollary 6.2 *The fundamental group $\pi_1(\bar{U}_W, p_{\bar{U}})$ is pro-representable by a locally compact strict pro-group indexed over a filtered poset.*

If \mathcal{G} is a group object of \mathcal{T} , then we consider the internal Hom group object

$$\mathcal{G}^D := \underline{Hom}_{\mathcal{T}}(\mathcal{G}, y\mathbb{S}^1).$$

For a locally compact topological group G , one can show that

$$(yG)^{DD} \simeq y(G^{ab})$$

is represented by the maximal Hausdorff abelian quotient G^{ab} of G (see [12]). Let $\underline{\mathcal{G}}$ be a pro-group object of \mathcal{T} given by a covariant functor $\underline{\mathcal{G}} : I \rightarrow Gr(\mathcal{T})$, where $Gr(\mathcal{T})$ denotes the category of groups in \mathcal{T} , and I is a small filtered category. We consider the pro-abelian group object $\underline{\mathcal{G}}^{DD}$ of \mathcal{T} defined as the composite functor

$$(-)^{DD} \circ \underline{\mathcal{G}} : I \longrightarrow Gr(\mathcal{T}) \longrightarrow Ab(\mathcal{T}).$$

Recall from Definition 4.9 the definition of the abelian topological group $C_{\bar{U}}$.

Corollary 6.3 *The pro-group object $\pi_1(\bar{U}_W, p_{\bar{U}})^{DD}$ of \mathcal{T} is essentially constant, hence can be identified with an actual topological group. Then, we have a canonical isomorphism of topological groups*

$$r_{\bar{U}} : C_{\bar{U}} \simeq \pi_1(\bar{U}_W, p_{\bar{U}})^{DD}.$$

Proof The pro-group object $\pi_1(\bar{U}_W, p_{\bar{U}})^{DD}$ is the projective system of abelian objects given by the groups $(yW(\bar{U}, L))^{DD}$ for $\bar{K}/L/K$ finite and Galois. But one has

$$(yW(\bar{U}, L))^{DD} = y(W(\bar{U}, L)^{ab}) = yC_{\bar{U}}.$$

for any $\bar{K}/L/K$. The second equality has been proved in Sect. 4.2. □

We simply denote by $t : \bar{U}_W \rightarrow \mathcal{T}$ the canonical map. Since the Weil-étale topos \bar{U}_W is defined over the base topos \mathcal{T} , the cohomology groups of \bar{U}_W have a topological structure. To make this precise, we introduce the following notion.

Definition 6.4 The \mathcal{T} -cohomology of \bar{U}_W with coefficients in \mathcal{A} is defined as

$$H_{\mathcal{T}}^n(\bar{U}_W, \mathcal{A}) := R^n(t_*)(\mathcal{A})$$

Corollary 6.5 For any abelian object \mathcal{A} of \mathcal{T} , one has

$$H_{\mathcal{T}}^0(\bar{U}_W, t^*\mathcal{A}) = \mathcal{A} \quad \text{and} \quad H^0(\bar{U}_W, t^*\mathcal{A}) = \mathcal{A}(*)$$

where $\mathcal{A}(*)$ denotes the group of global sections of the abelian object \mathcal{A} of \mathcal{T} .

Proof We have

$$H_{\mathcal{T}}^0(\bar{U}_W, t^*\mathcal{A}) := t_*t^*\mathcal{A} = \mathcal{A}$$

since $t : \bar{U}_W \rightarrow \mathcal{T}$ is connected, i.e. t^* is fully faithful. Let $e_{\mathcal{T}}$ be the unique map $e_{\mathcal{T}} : \mathcal{T} \rightarrow \underline{\text{Sets}}$. We have

$$H^0(\bar{U}_W, t^*\mathcal{A}) := (e_{\mathcal{T}*} \circ t_*)t^*\mathcal{A} = e_{\mathcal{T}*}\mathcal{A} = \mathcal{A}(*).$$

□

Corollary 6.6 For any abelian locally compact topological group A , one has

$$H_{\mathcal{T}}^1(\bar{U}_W, t^*A) = \underline{\text{Hom}}_{\mathcal{T}op}(C_{\bar{U}}, A) \quad \text{and} \quad H^1(\bar{U}_W, t^*A) = \text{Hom}_{cont}(C_{\bar{U}}, A).$$

Proof Let A be an abelian locally compact group. One has

$$\begin{aligned} H_{\mathcal{T}}^1(\bar{U}_W, t^*yA) &= \underline{\text{Hom}}_{\mathcal{T}}(\pi_1(\bar{U}_W, p_{\bar{U}}), yA) \\ &= \varinjlim \underline{\text{Hom}}_{\mathcal{T}}(yW(L, \bar{U}), yA) \\ &= \varinjlim y(\underline{\text{Hom}}_{\mathcal{T}op}(W(L, \bar{U}), A)) \\ &= \varinjlim y(\underline{\text{Hom}}_{\mathcal{T}op}(W(L, \bar{U})^{ab}, A)) \\ &= y(\underline{\text{Hom}}_{\mathcal{T}op}(C_{\bar{U}}, A)). \end{aligned}$$

Here $\underline{\text{Hom}}_{\mathcal{T}op}(C_{\bar{U}}, A)$ is the group of continuous morphisms from $C_{\bar{U}}$ to A , endowed with the compact-open topology.

Consider the unique map $e_{\mathcal{T}} : \mathcal{T} \rightarrow \underline{Sets}$. This map has a canonical section $s_{\mathcal{T}}$ such that $e_{\mathcal{T}*} \simeq s_{\mathcal{T}}^*$. Hence, the direct image functor $e_{\mathcal{T}*} : \mathcal{T} \rightarrow \underline{Sets}$ commutes with arbitrary inductive limits (see [5, IV.4.10]). Then the first cohomology group

$$\begin{aligned} H^1(\bar{U}_W, t^*A) &= e_{\mathcal{T}*} H^1_{\mathcal{T}}(\bar{U}_W, t^*A) \\ &= e_{\mathcal{T}*} \underline{Hom}_{Top}(C_{\bar{U}}, A) \\ &= Hom_{cont}(C_{\bar{U}}, A) \end{aligned}$$

is the discrete group of continuous morphisms from $C_{\bar{U}}$ to A . □

Corollary 6.7 *There is a fundamental class*

$$\theta_{\bar{U}} \in H^1(\bar{U}_W, \tilde{\mathbb{R}}) = Hom_{cont}(C_{\bar{U}}, \mathbb{R})$$

given by the canonical continuous morphism $\theta_{\bar{U}} : C_{\bar{U}} \rightarrow \mathbb{R}$.

Remark 6.8 Recall that $C_{\overline{Spec \mathbb{Z}}} = Pic(\overline{Spec \mathbb{Z}}) = \mathbb{R}_+^{\times}$. For any \bar{U} , the fundamental class $\theta_{\bar{U}}$ is the pull-back of the logarithm morphism:

$$\theta_{\overline{Spec \mathbb{Z}}} := \log \in H^1(\overline{Spec \mathbb{Z}_W}, \tilde{\mathbb{R}}) = Hom_{cont}(\mathbb{R}_+^{\times}, \mathbb{R})$$

along the map $\bar{U} \rightarrow \overline{Spec \mathbb{Z}}$.

The maximal compact subgroup of $C_{\bar{U}}$, i.e. the kernel of the absolute value map $C_{\bar{U}} \rightarrow \mathbb{R}_+^{\times}$, is denoted by $C_{\bar{U}}^1$. The Pontryagin dual $(C_{\bar{U}}^1)^D$ is a discrete abelian group.

Proposition 6.9 *For any connected étale \bar{X} -scheme \bar{U} , we have canonically*

$$\begin{aligned} H^n(\bar{U}_W, \mathbb{Z}) &= \mathbb{Z} \text{ for } n = 0 \\ &= 0 \text{ for } n = 1 \\ &= (C_{\bar{U}}^1)^D \text{ for } n = 2. \end{aligned}$$

Proof The result for $n = 0$ follows from Corollary 6.5. By Corollary 6.6, we have

$$H^1(\bar{U}_W, \mathbb{Z}) = Hom_c(C_{\bar{U}}, \mathbb{Z}) = 0.$$

Moreover, we have an isomorphism

$$H^1(\bar{U}_W, \tilde{\mathbb{S}}^1) = Hom_c(C_{\bar{U}}, \mathbb{S}^1) = C_{\bar{U}}^D.$$

The exact sequence of topological groups $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{S}^1 \rightarrow 0$ induces an exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \tilde{\mathbb{R}} \rightarrow \tilde{\mathbb{S}}^1 \rightarrow 0$ of abelian sheaves on \bar{U}_W , where $\tilde{\mathbb{R}} := t^*(y\mathbb{R})$ and $\tilde{\mathbb{S}}^1 := t^*(y\mathbb{S}^1)$. The induced long exact sequence

$$0 = H^1(\bar{U}_W, \mathbb{Z}) \rightarrow H^1(\bar{U}_W, \tilde{\mathbb{R}}) \rightarrow H^1(\bar{U}_W, \tilde{\mathbb{S}}^1) \rightarrow H^2(\bar{U}_W, \mathbb{Z}) \rightarrow H^2(\bar{U}_W, \tilde{\mathbb{R}}) = 0$$

is canonically identified with

$$0 \rightarrow \text{Hom}_c(C_{\bar{U}}, \mathbb{R}) \rightarrow \text{Hom}_c(C_{\bar{U}}, \mathbb{S}^1) \rightarrow H^2(\bar{U}_W, \mathbb{Z}) \rightarrow 0$$

and we obtain $H^2(\bar{U}_W, \mathbb{Z}) = (C_{\bar{U}}^1)^D$. □

6.2 The Weil-étale topos and the axioms for the conjectural Lichtenbaum topos

Lichtenbaum conjectured in [8] the existence of a Grothendieck topology for an arithmetic scheme X such that the Euler characteristic of the cohomology groups of the constant sheaf \mathbb{Z} with compact support at infinity gives, up to sign, the leading term of the zeta-function $\zeta_X(s)$ at $s = 0$. We call the category of sheaves on this conjectural site the *conjectural Lichtenbaum topos*, which we denote by \bar{X}_L . In [12, Sect. 5.2], we gave a list of axioms that should be satisfied by the conjectural topos \bar{X}_L , in the case where $X = \text{Spec}(\mathcal{O}_F)$. We refer to them as Axioms (1)–(9). We also showed in [12] that any topos satisfying these axioms gives rise to complexes of étale sheaves computing the expected Lichtenbaum cohomology. The main motivation for the present work is to provide an example of a topos (the Weil-étale topos) satisfying Axioms (1)–(9). This shows that that Axioms (1)–(9) are consistent, and this gives a natural computation of the base change from the Weil-étale cohomology to the étale cohomology (see Corollary 6.13 below). Axioms (1)–(9) are recalled in the proof of Theorem 6.12.

The morphism $\gamma : \bar{U}_W \rightarrow \bar{U}_{et}$ induces a morphism $\varphi_{\bar{U}}$ of fundamental pro-groups. Applying the functor $(-)^{DD}$, we obtain a morphism $\varphi_{\bar{U}}^{DD}$ of abelian fundamental pro-groups.

Corollary 6.10 *The composite morphism*

$$\varphi_{\bar{U}}^{DD} \circ r_{\bar{U}} : C_{\bar{U}} \simeq \pi_1(\bar{U}_W)^{ab} \longrightarrow \pi_1(\bar{U}_{et})^{ab}$$

is the reciprocity law of class field theory.

Proof The fundamental group

$$\pi_1(\bar{U}_W, p_{\bar{U}}) = \underline{W}(\bar{U}, q_{\bar{U}}) := \{W(\bar{U}, L), \text{ for } \bar{K}/L/K \text{ finite Galois}\}$$

can be seen as the automorphism group of the pro-torsor

$$\{\text{TORS}(\bar{U}, L) := \pi^*EW(\bar{U}, L), \text{ for } \bar{F}/L/K \text{ finite Galois}\}$$

in \bar{X}_W . Consider the morphism of fundamental groups induced by γ :

$$\varphi_{\bar{U}} : \pi_1(\bar{U}_W, p_{\bar{U}}) \longrightarrow \pi_1(\bar{U}_{et}, q_{\bar{U}})$$

It follows from the definition of γ , in terms of morphism of left exact sites (see Proposition 3.17), that $\varphi_{\bar{U}}$ is the morphism of topological pro-groups from

$$\{W(\bar{U}, L), \bar{K}/L/K \text{ finite Galois}\}$$

to

$$\{G(L'/K), \bar{K}/L'/K \text{ finite Galois unramified over } \bar{U}\}$$

Here the morphism $\varphi_{\bar{U}}$ is given by the compatible family of morphisms $W(\bar{U}, L) \rightarrow G(L^{un}/K)$, where L^{un} is the maximal sub-extension of L/K unramified over \bar{U} . Indeed, the previous statement follows from the fact that the following square is commutative, where $K_{\bar{U}}/K$ is the maximal sub-extension of \bar{K}/K unramified over \bar{U} :

$$\begin{array}{ccc} \bar{U}_W & \xrightarrow{\gamma} & \bar{U}_{et} \\ \downarrow & & \downarrow \\ B_{\underline{W}(\bar{U}, q_{\bar{U}})} & \longrightarrow & B_{G(K_{\bar{U}}/K)}^{sm} \end{array}$$

The commutativity of this square in turn follows from the description of these morphisms in terms of morphisms of sites, which is given in Proposition 3.17 and (4.12).

Hence, the morphism $\varphi_{\bar{U}}^{DD} \circ r_{\bar{U}}$ is given by the family of compatible morphisms

$$C_{\bar{U}} \simeq W(\bar{U}, L)^{ab} \rightarrow G(L^{un}/K)^{ab}$$

indexed over the finite Galois sub-extensions $\bar{K}/L/K$. Let us fix such a sub-extension L/K . We consider the usual relative Weil group $W_{L/K}$, which is given with maps $W_{L/K} \rightarrow G_{L/K}$ and $C_K \simeq W_{L/K}^{ab}$, where C_K is the idèle class group of K . The corollary now follows from the commutative diagram

$$\begin{array}{ccccc} C_K & \xrightarrow{\simeq} & W_{L/K}^{ab} & \longrightarrow & G(L/K)^{ab} \\ \downarrow & & \downarrow & & \downarrow \\ C_{\bar{U}} & \xrightarrow{\simeq} & W(\bar{U}, L)^{ab} & \longrightarrow & G(L^{un}/K)^{ab} \end{array}$$

since the first row is the reciprocity map of class field theory. □

Remark 6.11 Let $K_{\bar{U}}/K$ be the maximal sub-extension of \bar{K}/K unramified over \bar{U} . The map

$$\varprojlim \varphi_{\bar{U}} : \varprojlim \pi_1(\bar{U}_W, p_{\bar{U}}) = W(\bar{U}, q_{\bar{U}}) \longrightarrow \varprojlim \pi_1(\bar{U}_{et}, q_{\bar{U}}) = G(K_{\bar{U}}/K)$$

sits in the following commutative square

$$\begin{array}{ccc} W_K & \longrightarrow & G_K \\ \downarrow & & \downarrow \\ W(\bar{U}, q_{\bar{U}}) & \longrightarrow & G(K_{\bar{U}}/K) \end{array}$$

Theorem 6.12 *The Weil-étale topos \bar{X}_W satisfies Axioms (1)–(9) of [12, Sect. 5.2].*

Proof Recall from [12, Sect. 5.2] the following expected properties of the conjectural Lichtenbaum topos.

1. There is a morphism $\gamma : \bar{X}_W \rightarrow \bar{X}_{et}$.
2. The topos \bar{X}_W is defined over \mathcal{T} . The structure map $\bar{X}_W \rightarrow \mathcal{T}$ is connected locally connected and \bar{X}_W has a \mathcal{T} -point p . For any connected étale \bar{X} -scheme \bar{U} , the object $\gamma^*\bar{U}$ of \bar{X}_W is connected over \mathcal{T} .
3. There is a canonical isomorphism $r_{\bar{U}} : C_{\bar{U}} \simeq \pi_1(\bar{U}_W)^{ab}$ such that the composition

$$C_{\bar{U}} \simeq \pi_1(\bar{U}_W)^{ab} \rightarrow \pi_1(\bar{U}_{et})^{ab}$$

is the reciprocity law of class field theory, where the second morphism is induced by γ .

4. The isomorphism $r_{\bar{U}}$ is covariantly functorial for any map $\bar{V} \rightarrow \bar{U}$ of connected étale \bar{X} -schemes.
5. For any Galois étale cover $\bar{V} \rightarrow \bar{U}$ of étale \bar{X} -schemes, the conjugation action on $\pi_1(\bar{V}_W)^{ab}$ corresponds to the Galois action on $C_{\bar{V}}$.
6. The isomorphism $r_{\bar{U}}$ is contravariantly functorial for an étale cover.
7. For any closed point v of \bar{X} , one has a pull-back of topoi:

$$\begin{array}{ccc} B_{W_{k(v)}} & \xrightarrow{\alpha_v} & B_{G_{k(v)}}^{sm} \\ i_v \downarrow & & u_v \downarrow \\ \bar{X}_W & \xrightarrow{\gamma} & \bar{X}_{et} \end{array}$$

8. For any closed point w of a connected étale \bar{X} -scheme \bar{U} , the composition

$$B_{W_{k(w)}} \longrightarrow \bar{U}_W \longrightarrow B_{C_{\bar{U}}}$$

is the morphism of classifying topoi induced by the canonical morphism of topological groups $W_{k(w)} \rightarrow C_{\bar{U}}$.

9. For any étale \bar{X} -scheme \bar{U} , one has $H^n(\bar{U}_W, \tilde{\mathbb{R}}) = 0$ for any $n \geq 2$.

Indeed, Axiom (1) is given by Corollary 3.18 and Axiom (2) is given by Theorem 4.27 (i) and (ii). Axiom (3) is given by Corollary 6.3 and Corollary 6.10. The fundamental group is functorial by Proposition 3.24. Then Axioms (4)–(6) follow from the usual functorial properties of the Weil group (see Remark 6.11). Axiom (7) is given by Theorem 3.20. Axiom (8) follows immediately from the description of the morphisms

$$\bar{U}_W \rightarrow B_{W(\bar{U}, L)} \rightarrow B_{C_{\bar{U}}} \quad \text{and} \quad i_v : B_{W_{k(v)}} \rightarrow \bar{U}_W$$

in terms of morphisms of left exact sites (see (4.12) and Theorem 3.20 respectively). Finally, Axiom (9) is given by Proposition 5.2. \square

We denote by $\varphi : X_W \rightarrow \bar{X}_W$ the natural open embedding, and by $H_c^n(X_W, \mathcal{A}) := H^n(\bar{X}_W, \varphi_! \mathcal{A})$ the cohomology with compact support with coefficients in the abelian sheaf \mathcal{A} .

Corollary 6.13 (Lichtenbaum's formalism) *Assume that F is totally imaginary. We denote by $\tau_{\leq 2} R\gamma_*$ the truncated functor of the total derived functor $R\gamma_*$. Then one has:*

- $\mathbb{H}^n(\bar{X}_{et}, \tau_{\leq 2} R\gamma_*(\varphi_! \mathbb{Z}))$ is finitely generated and zero for $n \geq 4$.
- The canonical map

$$\mathbb{H}^n(\bar{X}_{et}, \tau_{\leq 2} R\gamma_*(\varphi_! \mathbb{Z})) \otimes \mathbb{R} \longrightarrow H_c^n(X_W, \tilde{\mathbb{R}})$$

is an isomorphism for any $n \geq 0$.

- There exists a fundamental class $\theta \in H^1(\bar{X}_W, \tilde{\mathbb{R}})$. The complex of finite dimensional vector spaces

$$\dots \rightarrow H_c^{n-1}(X_W, \tilde{\mathbb{R}}) \rightarrow H_c^n(X_W, \tilde{\mathbb{R}}) \rightarrow H_c^{n+1}(X_W, \tilde{\mathbb{R}}) \rightarrow \dots$$

defined by cup product with θ is acyclic.

- The vanishing order of the Dedekind zeta function $\zeta_F(s)$ at $s = 0$ is given by

$$\text{ord}_{s=0} \zeta_F(s) = \sum_{n \geq 0} (-1)^n n \dim_{\mathbb{R}} H_c^n(X_W, \tilde{\mathbb{R}})$$

- The leading term coefficient $\zeta_F^*(s)$ at $s = 0$ is given by the Lichtenbaum Euler characteristic:

$$\zeta_F^*(s) = \pm \prod_{n \geq 0} |\mathbb{H}^n(\bar{X}_{et}, \tau_{\leq 2} R\gamma_*(\varphi_! \mathbb{Z}))_{\text{tors}}|^{(-1)^n} / \det(H_c^n(X_L, \tilde{\mathbb{R}}), \theta, B^*)$$

where B^n is a basis of $\mathbb{H}^n(\bar{X}_{et}, \tau_{\leq 2} R\gamma_*(\varphi_! \mathbb{Z})) / \text{tors}$.

Proof By [12, Theorem 6.3], this follows from Theorem 6.12. □

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References

1. Bunge, M.: Classifying topoi and fundamental localic groupoids. Category theory 1991 (Montreal, PQ, 1991). In: CMS Conference Proceedings, vol. 13, pp. 75–96. American Mathematical Society, Providence (1992)
2. Bunge, M., Moerdijk, I.: On the construction of the Grothendieck fundamental group of a topos by paths. J. Pure Appl. Algebra **116**(1-3), 99–113 (1997)
3. Flach, M.: Cohomology of topological groups with applications to the Weil group. Compos. Math. **144**(3), 633–656 (2008)
4. Flach, M., Morin, B.: On the Weil-étale topos of regular arithmetic schemes. Preprint (2010)

5. Grothendieck, A., Artin, M., Verdier, J.L.: Théorie des Topos et cohomologie étale des schémas (SGA4). In: Lectures Notes in Mathematics, vols. 269, 270, 305. Springer (1972)
6. Johnstone, P.T.: Sketches of an elephant: a topos theory compendium. In: Vol. 1, 2. Oxford Logic Guides, 43. The Clarendon Press, Oxford University Press, New York (2002)
7. Lichtenbaum, S.: The Weil-étale topology on schemes over finite fields. *Compos. Math.* **141**(3), 689–702 (2005)
8. Lichtenbaum, S.: The Weil-étale topology for number rings. *Ann. Math.* **170**(2), 657–683 (2009)
9. Moerdijk, I.: Prodiscrete groups and Galois toposes. *Nederl. Akad. Wetensch. Indag. Math.* **51**(2), 219–234 (1989)
10. Morin, B.: Sur le topos Weil-étale d'un corps de nombres. Thesis (2008)
11. Morin, B.: On the Weil-étale cohomology of number fields. *Trans. Am. Math. Soc.* (to appear)
12. Morin, B.: The Weil-étale fundamental group of a number field I. *Kyushu J. Math.* **65** (2011, to appear)
13. Mostert, P.S.: Sections in principal fibre spaces. *Duke Math. J.* **23**, 57–71 (1956)
14. Neukirch, J., Schmidt, A., Wingberg, K.: Cohomology of Number Fields. 2nd edn. Grundlehren der Mathematischen Wissenschaften, 323. Springer, Berlin (2008)
15. Tate, J.: Number theoretic background. Automorphic forms, representations and L -functions. In: Proceedings of Symposium on Pure Mathematics. Oregon State University, Corvallis, 1977, Part 2, 3–26, Proceedings of Symposium on Pure Mathematics, vol. XXXIII. American Mathematical Society, Providence (1979)