

A REMARK ON GAMMA FACTORS AND LOCAL WEIL-GROUPS

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Let F be a number field and let \mathfrak{p} be an archimedean prime, so that $F_{\mathfrak{p}} = \mathbb{R}$ or $F_{\mathfrak{p}} = \mathbb{C}$. We denote by $W_{F_{\mathfrak{p}}}$ the corresponding local Weil group, and by $B_{W_{F_{\mathfrak{p}}}}$ its classifying space. The aim of this short note is to show the following observation, which is based on Deninger's description of local factors of zeta-functions in terms of zeta-regularized determinants. The cohomology with (discrete) real coefficients $H^*(B_{W_{F_{\mathfrak{p}}}}, \mathbb{R})$ of the local Weil-group $W_{F_{\mathfrak{p}}}$ is naturally endowed with a Hodge structure inducing an \mathbb{R} -action whose infinitesimal generator we denote by Θ . Then the zeta-regularized determinant of Θ acting on the homology $H_*(B_{W_{F_{\mathfrak{p}}}}, \mathbb{R})$ gives the corresponding Gamma-factor $\Gamma_{F_{\mathfrak{p}}}(s)$. This is simply due to the fact that $B_{W_{\mathbb{C}}}$ is the infinite projective space $\mathbf{P}^{\infty}(\mathbb{C})$, whereas $B_{W_{\mathbb{R}}}$ can be seen as the quotient of $\mathbf{P}^{\infty}(\mathbb{C})$ by complex conjugation.

1. HODGE-TATE STRUCTURES

We say that a semi-simple real Hodge structure $(V, V_{\mathbb{C}} = \bigoplus_{p,q} V^{p,q})$ is a Hodge-Tate structure if $V^{p,q} = 0$ whenever $p \neq q$. A semi-simple real Hodge structure on V can be seen as a representation $h : \mathbb{S} \rightarrow \mathbf{GL}(V)$, where $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{S}^1$ is Deligne's torus, such that $h(z) \cdot v = z^{-p} \bar{z}^{-q} \cdot v$ for $v \in V^{p,q}$ and $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^*$. We consider the exact sequence

$$1 \rightarrow \mathbb{S}^1 \rightarrow \mathbb{C}^{\times} \xrightarrow{N} \mathbb{R}_{>0}^{\times} \rightarrow 1$$

where $N(z) = z\bar{z}$ is the norm map. If (V, h) is a Hodge-Tate structure then the \mathbb{C}^{\times} -action on V factors through $\mathbb{R}_{>0}^{\times}$, so that $r \in \mathbb{R}_{>0}^{\times}$ acts as $v \mapsto r^{-p} \cdot v$ for $v \in V^{pp} := V \cap V^{p,p}$. Composing with the exponential map $\mathbb{R} \rightarrow \mathbb{R}_{>0}^{\times}$, we obtain an action

$$\begin{aligned} \phi : \mathbb{R} \times V &\longrightarrow V \\ (t, v) &\longmapsto \phi^t(v) \end{aligned}$$

such that $\phi^t(v) = \exp(-p \cdot t) \cdot v$ if $v \in V^{pp}$. We define an operator Θ on V by the formula

$$\Theta := \lim_{t \rightarrow 0} \frac{(\phi^t)^{-1} - \mathrm{Id}}{t}$$

Then Θ may be seen as an archimedean analogue of the *geometric* Frobenius (by geometric I mean $(\phi^t)^{-1}$ is involved rather than ϕ^t). Notice that $\Theta = -\mathrm{Id}$ on the Hodge-Tate structure $\mathbb{R}(1)$. In the next section Hodge-Tate structures are often seen as real vector spaces endowed with the operator Θ defined above. For example, $\mathbb{R}(n)$ denotes either the Hodge-Tate structure or the Θ -equivariant vector-space \mathbb{R} on which Θ acts as $-n \cdot \mathrm{Id}$.

For example, let G be a complex linear algebraic group, and let B_G be its classifying space. Then $H^{2n+1}(B_G, \mathbb{R}) = 0$ and $H^{2n}(B_G, \mathbb{R})$ carries a Hodge-Tate structure of weight $-2n$ (see [1] Théorème 9.1.1).

2. GAMMA FACTORS AND LOCAL WEIL-GROUPS

We consider the Gamma factors

$$\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s) \text{ and } \Gamma_{\mathbb{R}}(s) = 2^{-1/2}\pi^{-s/2}\Gamma(s/2).$$

For \mathfrak{p} an archimedean prime of a number field F , we set $\zeta_{\mathfrak{p}}(s) := \Gamma_{F_{\mathfrak{p}}}(s)$. We consider the archimedean local Weil-group $W_{F_{\mathfrak{p}}}$. Recall that $W_{\mathbb{C}} := \mathbb{C}^{\times} = \mathbb{G}_m(\mathbb{C})$, and that $W_{\mathbb{R}}$ is given by an extension

$$(1) \quad 1 \longrightarrow \mathbb{C}^{\times} \longrightarrow W_{\mathbb{R}} \longrightarrow \text{Gal}(\mathbb{C}/\mathbb{R}) \longrightarrow 1,$$

inducing the $\text{Gal}(\mathbb{C}/\mathbb{R})$ -action on \mathbb{C}^{\times} given by complex conjugation, and whose class in

$$H^2(\text{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^{\times}) \simeq \widehat{H}^0(\text{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^{\times}) \simeq \mathbb{R}^{\times}/\mathbb{R}^{\times 2} \simeq \{\pm 1\}$$

is the non-trivial class.

Theorem 2.1. *For any archimedean prime $\mathfrak{p} \mid \infty$ of the number field F , we have*

$$\zeta_{\mathfrak{p}}(s) = \det_{\infty} \left(\frac{1}{2\pi} (\Theta - s \cdot \text{Id}) \mid H_*(B_{W_{F_{\mathfrak{p}}}}, \mathbb{R}) \right)^{-1}$$

where Θ is induced by the natural Hodge-Tate structure on $H^n(B_{W_{F_{\mathfrak{p}}}}, \mathbb{R})$.

Proof. Assume first that \mathfrak{p} is a complex prime. The \mathbb{G}_m -principal homogeneous space

$$\left(\mathbf{A}_{\mathbb{C}}^{N+1} - \{0\} \right) \rightarrow \mathbf{P}_{\mathbb{C}}^N$$

induces a map

$$H^n(B_{W_{\mathbb{C}}}, \mathbb{R}) \xrightarrow{\sim} H^n(\mathbf{P}^N(\mathbb{C}), \mathbb{R})$$

which is an isomorphism of Hodge structures for any $n \leq 2N$. We obtain Θ -equivariant isomorphisms

$$H^*(B_{W_{\mathbb{C}}}, \mathbb{R}) \simeq H^*(\mathbf{P}^{\infty}(\mathbb{C}), \mathbb{R}) \simeq \prod_{n \geq 0} \mathbb{R}(-n)[-2n]$$

and

$$H_*(B_{W_{\mathbb{C}}^1}, \mathbb{R}) \simeq H_*(\mathbf{P}^{\infty}(\mathbb{C}), \mathbb{R}) \simeq \bigoplus_{n \geq 0} \mathbb{R}(n)[2n]$$

where \mathbf{P}^{∞} denotes the infinite projective space. The spectrum of Θ acting on $H_*(B_{W_{\mathbb{C}}}, \mathbb{R})$ is therefore $\{0, -1, -2, \dots\}$. Assume now that \mathfrak{p} is real. The extension (1) induces the natural action of $\text{Gal}(\mathbb{C}/\mathbb{R})$ on \mathbb{C}^{\times} . It gives an (anti-holomorphic) action on the simplicial space $B_{W_{\mathbb{C}}}$, hence on its cohomology $H^*(B_{W_{\mathbb{C}}}, \mathbb{R})$. The Hirsch-Serre spectral sequence for the extension (1) reads as follows:

$$E_2^{i,j} = H^i(\text{Gal}(\mathbb{C}/\mathbb{R}), H^j(B_{W_{\mathbb{C}}}, \mathbb{R})) \implies H^{i+j}(B_{W_{\mathbb{R}}}, \mathbb{R}).$$

It degenerates and gives

$$H^n(B_{W_{\mathbb{R}}}, \mathbb{R}) = H^0(\text{Gal}(\mathbb{C}/\mathbb{R}), H^n(B_{W_{\mathbb{C}}}, \mathbb{R}))$$

which provides $H^n(B_{W_{\mathbb{R}}}, \mathbb{R})$ with an Hodge-Tate structure for any $n \in \mathbb{Z}$. The action of $\text{Gal}(\mathbb{C}/\mathbb{R})$ on $H^*(B_{W_{\mathbb{C}}}, \mathbb{R})$ is the obvious one. Indeed, the action

$$\mathbb{C}^{\times} \times (\mathbf{A}^{N+1}(\mathbb{C}) - \{0\}) \longrightarrow (\mathbf{A}^{N+1}(\mathbb{C}) - \{0\})$$

as well as the projection map $(\mathbf{A}^{N+1}(\mathbb{C}) - \{0\}) \rightarrow \mathbf{P}^N(\mathbb{C})$ are compatible with complex conjugation. It follows that the map

$$H^n(B_{W_{\mathbb{C}}}, \mathbb{R}) \xrightarrow{\sim} H^n(\mathbf{P}^N(\mathbb{C}), \mathbb{R})$$

is $\text{Gal}(\mathbb{C}/\mathbb{R})$ -equivariant, where the $\text{Gal}(\mathbb{C}/\mathbb{R})$ -action on the left hand side is induced by complex conjugation on $W_{\mathbb{C}} = \mathbb{C}^*$, and the $\text{Gal}(\mathbb{C}/\mathbb{R})$ -action on the right hand side is induced by complex conjugation on $\mathbf{P}^N(\mathbb{C})$. Hence the non-trivial element $c \in \text{Gal}(\mathbb{C}/\mathbb{R})$ acts on $H^{2n}(\mathbf{P}^N(\mathbb{C}), \mathbb{R}) = \mathbb{R}(-n)$ as Id (resp. $-\text{Id}$) if n is even (resp. odd). We get

$$H^*(B_{W_{\mathbb{R}}}, \mathbb{R}) = \prod_{n \geq 0} \mathbb{R}(-2n)[-4n]$$

and

$$H_*(B_{W_{\mathbb{R}}}, \mathbb{R}) = \bigoplus_{n \geq 0} \mathbb{R}(2n)[4n]$$

so that the spectrum of Θ acting on $H_*(B_{W_{\mathbb{R}}}, \mathbb{R})$ is $\{0, -2, -4, \dots\}$. The result then follows from (see for example [2] Proposition 3.1)

$$\Gamma_{\mathbb{C}}(s) = \det_{\infty} \left(\frac{1}{2\pi} (\Theta - s \cdot \text{Id}) \mid \bigoplus_{n \geq 0} \mathbb{R}(n) \right)^{-1}$$

and

$$\Gamma_{\mathbb{R}}(s) = \det_{\infty} \left(\frac{1}{2\pi} (\Theta - s \cdot \text{Id}) \mid \bigoplus_{n \geq 0} \mathbb{R}(2n) \right)^{-1}.$$

□

REFERENCES

- [1] Deligne, P. : *Théorie de Hodge. III*. Inst. Hautes Études Sci. Publ. Math. No. 44 (1974), 5–77.
- [2] Deninger, C. : *Arithmetic Geometry and Analysis on Foliated Spaces*. arXiv:math/0505354