A REMARK ON GAMMA FACTORS AND LOCAL WEIL-GROUPS

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Let F be a number field and let $\mathfrak{p}$ be an archimedean prime, so that $F_{\mathfrak{p}} = \mathbb{R}$ or $F_{\mathfrak{p}} = \mathbb{C}$. We denote by $W_{F_{\mathfrak{p}}}$ the corresponding local Weil group, and by $B_{W_{F_{\mathfrak{p}}}}$ its classifying space. The aim of this short note is to show the following observation, which is based on Deninger's descrition of local factors of zeta-functions in terms of zeta-regularized determinants. The cohomology with (discrete) real coefficients $H^*(B_{W_{F_{\mathfrak{p}}}}, \mathbb{R})$ of the local Weil-group $W_{F_{\mathfrak{p}}}$ is naturally endowed with a Hodge structure inducing an $\mathbb{R}$-action whose infinitesimal generator we denote by Θ. Then the zeta-regularized determinant of Θ acting on the homology $H_*(B_{W_{F_{\mathfrak{p}}}}, \mathbb{R})$ gives the corresponding Gamma-factor $\Gamma_{F_{\mathfrak{p}}}(s)$. This is simply due to the fact that $B_{W_{\mathbb{C}}}$ is the infinite projective space $\mathbf{P}^{\infty}(\mathbb{C})$, whereas $B_{W_{\mathbb{R}}}$ can be seen as the quotient of $\mathbf{P}^{\infty}(\mathbb{C})$ by complex conjugation.

1. Hodge-Tate structures

We say that a semi-simple real Hodge structure $(V, V_{\mathbb{C}} = \bigoplus_{p,q} V^{p,q})$ is a Hodge-Tate structure if $V^{p,q} = 0$ whenever $p \neq q$. A semi-simple real Hodge structure on V can be seen as a representation $h : \mathbb{S} \to \mathbf{GL}(V)$, where $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}$ is Deligne's torus, such that $h(z)\cdot v = z^{-p}\overline{z}^{-q}\cdot v$ for $v \in V^{p,q}$ and $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^*$. We consider the exact sequence

$$1 \to \mathbb{S}^1 \to \mathbb{C}^{\times} \xrightarrow{N} \mathbb{R}^{\times}_{>0} \to 1$$

where $N(z) = z\overline{z}$ is the norm map. If (V, h) is a Hodge-Tate structure then the $\mathbb{C}^{\times}$-action on V factors through $\mathbb{R}^{\times}_{+}$, so that $r \in \mathbb{R}^{\times}_{+}$ acts as $v \mapsto r^{-p}\cdot v$ for $v \in V^{pp} := V \cap V^{p,p}$. Composing with the exponential map $\mathbb{R} \to \mathbb{R}^{\times}_{+}$, we obtain an action

$$\begin{array}{cccc} \phi : & \mathbb{R} \times V & \longrightarrow & V \\ & (t, v) & \longmapsto & \phi^t(v) \end{array}$$

such that $\phi^t(v) = \exp(-p\cdot t)\cdot v$ if $v \in V^{pp}$. We define an operator Θ on V by the formula

$$\Theta := \lim_{t\to 0}\frac{(\phi^t)^{-1} - \mathrm{Id}}{t}$$

Then Θ may be seen as an archimedean analogue of the *geometric* Frobenius (by geometric I mean $(\phi^t)^{-1}$ is involved rather than ϕ^t). Notice that $\Theta = -\mathrm{Id}$ on the Hodge-Tate structure $\mathbb{R}(1)$. In the next section Hodge-Tate structures are often seen as real vector spaces endowed with the operator Θ defined above. For example, $\mathbb{R}(n)$ denotes either the Hodge-Tate structure or the Θ-equivariant vector-space $\mathbb{R}$ on which Θ acts as $-n\cdot\mathrm{Id}$.

For example, let G be a complex linear algebraic group, and let B_G be its classifying space. Then $H^{2n+1}(B_G, \mathbb{R}) = 0$ and $H^{2n}(B_G, \mathbb{R})$ carries a Hodge-Tate structure of weight $-2n$ (see [1] Théorème 9.1.1).

2. Gamma factors and local Weil-groups

We consider the Gamma factors

$$\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s) \text{ and } \Gamma_{\mathbb{R}}(s) = 2^{-1/2}\pi^{-s/2}\Gamma(s/2).$$

For $\mathfrak{p}$ an archimdean prime of a number field F, we set $\zeta_{\mathfrak{p}}(s) := \Gamma_{F_{\mathfrak{p}}}(s)$. We consider the archimedean local Weil-group $W_{F_{\mathfrak{p}}}$. Recall that $W_{\mathbb{C}} := \mathbb{C}^{\times} = \mathbb{G}_m(\mathbb{C})$, and that $W_{\mathbb{R}}$ is given by an extension

$$1 \longrightarrow \mathbb{C}^{\times} \longrightarrow W_{\mathbb{R}} \longrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \longrightarrow 1, \tag{1}$$

inducing the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action on $\mathbb{C}^{\times}$ given by complex conjugation, and whose class in

$$H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^{\times}) \simeq \widehat{H}^0(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^{\times}) \simeq \mathbb{R}^{\times}/\mathbb{R}^{\times 2} \simeq \{\pm 1\}$$

is the non-trivial class.

Theorem 2.1. *For any archimedean prime $\mathfrak{p} \mid \infty$ of the number field F, we have*

$$\zeta_{\mathfrak{p}}(s) = \det\nolimits_{\infty}\left(\frac{1}{2\pi}(\Theta - s\cdot \mathrm{Id}) \mid H_*(B_{W_{F_{\mathfrak{p}}}}, \mathbb{R})\right)^{-1}$$

where Θ is induced by the natural Hodge-Tate structure on $H^n(B_{W_{F_{\mathfrak{p}}}}, \mathbb{R})$.

Proof. Assume first that $\mathfrak{p}$ is a complex prime. The $\mathbb{G}_m$-principal homogeneous space

$$\left(\mathbf{A}^{N+1}_{\mathbb{C}} - \{0\}\right) \to \mathbf{P}^N_{\mathbb{C}}$$

induces a map

$$H^n(B_{W_{\mathbb{C}}}, \mathbb{R}) \xrightarrow{\sim} H^n(\mathbf{P}^N(\mathbb{C}), \mathbb{R})$$

which is an isomorphism of Hodge structures for any $n \leq 2N$. We obtain Θ-equivariant isomorphisms

$$H^*(B_{W_{\mathbb{C}}}, \mathbb{R}) \simeq H^*(\mathbf{P}^{\infty}(\mathbb{C}), \mathbb{R}) \simeq \prod_{n\geq 0} \mathbb{R}(-n)[-2n]$$

and

$$H_*(B_{W^1_{\mathbb{C}}}, \mathbb{R}) \simeq H_*(\mathbf{P}^{\infty}(\mathbb{C}), \mathbb{R}) \simeq \bigoplus_{n\geq 0} \mathbb{R}(n)[2n]$$

where $\mathbf{P}^{\infty}$ denotes the infinite projective space. The spectrum of Θ acting on $H_*(B_{W_{\mathbb{C}}}, \mathbb{R})$ is therefore $\{0, -1, -2, \cdots\}$. Assume now that $\mathfrak{p}$ is real. The extension (1) induces the natural action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $\mathbb{C}^{\times}$. It gives an (anti-holomorphic) action on the simplicial space $B_{W_{\mathbb{C}}}$, hence on its cohomology $H^*(B_{W_{\mathbb{C}}}, \mathbb{R})$. The Hoschild-Serre spectral sequence for the extension (1) reads as follows:

$$E_2^{i,j} = H^i(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), H^j(B_{W_{\mathbb{C}}}, \mathbb{R})) \Longrightarrow H^{i+j}(B_{W_{\mathbb{R}}}, \mathbb{R}).$$

It degenerates and gives

$$H^n(B_{W_{\mathbb{R}}}, \mathbb{R}) = H^0(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), H^n(B_{W_{\mathbb{C}}}, \mathbb{R})))$$

which provides $H^n(B_{W_{\mathbb{R}}}, \mathbb{R})$ with an Hodge-Tate structure for any $n \in \mathbb{Z}$. The action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $H^*(B_{W_{\mathbb{C}}}, \mathbb{R})$ is the obvious one. Indeed, the action

$$\mathbb{C}^{\times} \times \left(\mathbf{A}^{N+1}(\mathbb{C}) - \{0\}\right) \longrightarrow \left(\mathbf{A}^{N+1}(\mathbb{C}) - \{0\}\right)$$

as well as the projection map $\left(\mathbf{A}^{N+1}(\mathbb{C}) - \{0\}\right) \to \mathbf{P}^N(\mathbb{C})$ are compatible with complex conjugation. It follows that the map

$$H^n(B_{W_\mathbb{C}}, \mathbb{R}) \xrightarrow{\sim} H^n(\mathbf{P}^N(\mathbb{C}), \mathbb{R})$$

is $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant, where the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action on the left hand side is induced by complex conjugation on $W_\mathbb{C} = \mathbb{C}^*$, and the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action on the right hand side is induced by complex conjugation on $\mathbf{P}^N(\mathbb{C})$. Hence the non-trivial element $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $H^{2n}(\mathbf{P}^N(\mathbb{C}), \mathbb{R}) = \mathbb{R}(-n)$ as Id (resp. $-$Id) if n is even (resp. odd). We get

$$H^*(B_{W_\mathbb{R}}, \mathbb{R}) = \prod_{n \geq 0} \mathbb{R}(-2n)[-4n]$$

and

$$H_*(B_{W_\mathbb{R}}, \mathbb{R}) = \bigoplus_{n \geq 0} \mathbb{R}(2n)[4n]$$

so that the spectrum of Θ acting on $H_*(B_{W_\mathbb{R}}, \mathbb{R})$ is $\{0, -2, -4, \cdots\}$. The result then follows from (see for example [2] Proposition 3.1)

$$\Gamma_\mathbb{C}(s) = \det{}_\infty \left(\frac{1}{2\pi}(\Theta - s \cdot \mathrm{Id}) \mid \bigoplus_{n \geq 0} \mathbb{R}(n) \right)^{-1}$$

and

$$\Gamma_\mathbb{R}(s) = \det{}_\infty \left(\frac{1}{2\pi}(\Theta - s \cdot \mathrm{Id}) \mid \bigoplus_{n \geq 0} \mathbb{R}(2n) \right)^{-1}.$$

□

References

[1] Deligne, P. : *Théorie de Hodge. III.* Inst. Hautes Études Sci. Publ. Math. No. 44 (1974), 5–77.
[2] Deninger, C. : *Arithmetic Geometry and Analysis on Foliated Spaces.* arXiv:math/0505354