A REMARK ON GAMMA FACTORS AND LOCAL WEIL-GROUPS

B. MORIN

Let $F$ be a number field and let $p$ be an archimedean prime, so that $F_p = \mathbb{R}$ or $F_p = \mathbb{C}$. We denote by $W_{F_p}$ the corresponding local Weil group, and by $B_{W_{F_p}}$ its classifying space. The aim of this short note is to show the following observation, which is based on Deninger’s description of local factors of zeta-functions in terms of zeta-regularized determinants. The cohomology with (discrete) real coefficients $H^\ast(B_{W_{F_p}}, \mathbb{R})$ of the local Weil-group $W_{F_p}$ is naturally endowed with a Hodge structure inducing an $\mathbb{R}$-action whose infinitesimal generator we denote by $\Theta$. Then the zeta-regularized determinant of $\Theta$ acting on the homology $H^\ast(B_{W_{F_p}}, \mathbb{R})$ gives the corresponding Gamma-factor $\Gamma_{F_p}(s)$. This is simply due to the fact that $B_{W_{\mathbb{C}}}$ is the infinite projective space $\mathbb{P}^\infty(\mathbb{C})$, whereas $B_{W_{\mathbb{R}}}$ can be seen as the quotient of $\mathbb{P}^\infty(\mathbb{C})$ by complex conjugation.

1. Hodge-Tate structures

We say that a semi-simple real Hodge structure $(V, V_{\mathbb{C}} = \bigoplus_{p,q} V^{p,q})$ is a Hodge-Tate structure if $V^{p,q} = 0$ whenever $p \neq q$. A semi-simple real Hodge structure on $V$ can be seen as a representation $h : S \to \text{GL}(V)$, where $S = \text{Res}_{\mathbb{C}/\mathbb{R}}$ is Deligne’s torus, such that $h(z) \cdot v = z^{-p \tau -q} \cdot v$ for $v \in V^{p,q}$ and $z \in S(\mathbb{R}) = \mathbb{C}^\times$. We consider the exact sequence

$$1 \to S^1 \to \mathbb{C}^\times \to \mathbb{R}_{>0}^\times \to 1$$

where $Z(z) = z\overline{z}$ is the norm map. If $(V, h)$ is a Hodge-Tate structure then the $\mathbb{C}^\times$-action on $V$ factors through $\mathbb{R}_{>0}^\times$, so that $r \in \mathbb{R}_{>0}^\times$ acts as $v \mapsto r^{-p} \cdot v$ for $v \in V^{p,p}$. Composing with the exponential map $\mathbb{R} \to \mathbb{R}_{>0}^\times$, we obtain an action

$$\phi : \mathbb{R} \times V \to V$$

$$(t, v) \mapsto \phi^t(v)$$

such that $\phi^t(v) = \exp(-p \cdot t) \cdot v$ if $v \in V^{p,p}$. We define an operator $\Theta$ on $V$ by the formula

$$\Theta := \lim_{t \to 0} \frac{(\phi^t)^{-1} - \text{Id}}{t}$$

Then $\Theta$ may be seen as an archimedean analogue of the geometric Frobenius (by geometric I mean $(\phi^t)^{-1}$ is involved rather than $\phi^t$). Notice that $\Theta = -\text{Id}$ on the Hodge-Tate structure $\mathbb{R}(1)$. In the next section Hodge-Tate structures are often seen as real vector spaces endowed with the operator $\Theta$ defined above. For example, $\mathbb{R}(n)$ denotes either the Hodge-Tate structure or the $\Theta$-equivariant vector-space $\mathbb{R}$ on which $\Theta$ acts as $-n \cdot \text{Id}$.

For example, let $G$ be a complex linear algebraic group, and let $B_G$ be its classifying space. Then $H^{2n+1}(B_G, \mathbb{R}) = 0$ and $H^{2n}(B_G, \mathbb{R})$ carries a Hodge-Tate structure of weight $-2n$ (see [1] Théorème 9.1.1).
2. Gamma factors and local Weil-groups

We consider the Gamma factors
\[ \Gamma_C(s) = (2\pi)^{-s} \Gamma(s) \text{ and } \Gamma_R(s) = 2^{-1/2} \pi^{-s/2} \Gamma(s/2). \]

For a prime \( p \) of a number field \( F \), we set \( \zeta_p(s) := \Gamma_{F_p}(s) \). We consider the archimedean local Weil-group \( W_{F_p} \). Recall that \( W_C := \mathbb{C}^\times = \mathbb{G}_m(\mathbb{C}) \), and that \( W_R \) is given by an extension
\[ \mathbb{C}^\times \longrightarrow W_R \longrightarrow \text{Gal}(\mathbb{C}/\mathbb{R}) \longrightarrow 1, \]
inducing the \( \text{Gal}(\mathbb{C}/\mathbb{R}) \)-action on \( \mathbb{C}^\times \) given by complex conjugation, and whose class in
\[ H^2(\text{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times) \cong \hat{H}^0(\text{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times) \cong \mathbb{R}^\times/\mathbb{R}^\times \cong \{ \pm 1 \} \]
is the non-trivial class.

**Theorem 2.1.** For any archimedean prime \( p \mid \infty \) of the number field \( F \), we have
\[ \zeta_p(s) = \det_\infty \left( \frac{1}{2\pi} (\Theta - s \cdot \text{Id}) \mid H_*(B_{W_{F_p}}, \mathbb{R}) \right)^{-1} \]
where \( \Theta \) is induced by the natural Hodge-Tate structure on \( H^n(B_{W_{F_p}}, \mathbb{R}) \).

**Proof.** Assume first that \( p \) is a complex prime. The \( \mathbb{G}_m \)-principal homogeneous space
\[ \left( \mathbb{A}_C^{N+1} - \{ 0 \} \right) \rightarrow \mathbb{P}_C^N \]
induces a map
\[ H^n(B_{W_C}, \mathbb{R}) \xrightarrow{\sim} H^n(\mathbb{P}^N(\mathbb{C}), \mathbb{R}) \]
which is an isomorphism of Hodge structures for any \( n \leq 2N \). We obtain \( \Theta \)-equivariant isomorphisms
\[ H^*(B_{W_C}, \mathbb{R}) \cong H^*(\mathbb{P}^\infty(\mathbb{C}), \mathbb{R}) \cong \prod_{n \geq 0} \mathbb{R}(-n)[-2n] \]
and
\[ H_*(B_{W_C}, \mathbb{R}) \cong H_*(\mathbb{P}^\infty(\mathbb{C}), \mathbb{R}) \cong \bigoplus_{n \geq 0} \mathbb{R}(n)[2n] \]
where \( \mathbb{P}^\infty \) denotes the infinite projective space. The spectrum of \( \Theta \) acting on \( H_*(B_{W_C}, \mathbb{R}) \) is therefore \( \{ 0, -1, -2, \ldots \} \). Assume now that \( p \) is real. The extension (1) induces the natural action of \( \text{Gal}(\mathbb{C}/\mathbb{R}) \) on \( \mathbb{C}^\times \). It gives an (anti-holomorphic) action on the simplicial space \( B_{W_C} \), hence on its cohomology \( H^*(B_{W_C}, \mathbb{R}) \). The Hochschild-Serre spectral sequence for the extension (1) reads as follows:
\[ E_2^{ij} = H^i(\text{Gal}(\mathbb{C}/\mathbb{R}), H^j(B_{W_C}, \mathbb{R})) \Rightarrow H^{i+j}(B_{W_R}, \mathbb{R}). \]
It degenerates and gives
\[ H^n(B_{W_R}, \mathbb{R}) = H^0(\text{Gal}(\mathbb{C}/\mathbb{R}), H^n(B_{W_C}, \mathbb{R})) \]
which provides \( H^n(B_{W_R}, \mathbb{R}) \) with an Hodge-Tate structure for any \( n \in \mathbb{Z} \). The action of \( \text{Gal}(\mathbb{C}/\mathbb{R}) \) on \( H^*(B_{W_C}, \mathbb{R}) \) is the obvious one. Indeed, the action
\[ \mathbb{C}^\times \times (\mathbb{A}^{N+1}(\mathbb{C}) - \{ 0 \}) \rightarrow (\mathbb{A}^{N+1}(\mathbb{C}) - \{ 0 \}) \]
as well as the projection map \((A^{N+1}(\mathbb{C}) - \{0\}) \to \mathbb{P}^N(\mathbb{C})\) are compatible with complex conjugation. It follows that the map
\[
H^n(B_{W_C}, \mathbb{R}) \xrightarrow{\sim} H^n(\mathbb{P}^N(\mathbb{C}), \mathbb{R})
\]
is \(\text{Gal}(\mathbb{C}/\mathbb{R})\)-equivariant, where the \(\text{Gal}(\mathbb{C}/\mathbb{R})\)-action on the left hand side is induced by complex conjugation on \(W_C = \mathbb{C}^*\), and the \(\text{Gal}(\mathbb{C}/\mathbb{R})\)-action on the right hand side is induced by complex conjugation on \(\mathbb{P}^N(\mathbb{C})\). Hence the non-trivial element \(c \in \text{Gal}(\mathbb{C}/\mathbb{R})\) acts on \(H^{2n}(\mathbb{P}^N(\mathbb{C}), \mathbb{R}) = \mathbb{R}(-n)\) as \(\text{Id}\) (resp. \(-\text{Id}\)) if \(n\) is even (resp. odd). We get
\[
H^*(B_{W_\mathbb{R}}, \mathbb{R}) = \prod_{n \geq 0} \mathbb{R}(-2n)[-4n]
\]
and
\[
H^*_*(B_{W_\mathbb{R}}, \mathbb{R}) = \bigoplus_{n \geq 0} \mathbb{R}(2n)[4n]
\]
so that the spectrum of \(\Theta\) acting on \(H^*_*(B_{W_\mathbb{R}}, \mathbb{R})\) is \(\{0, -2, -4, \cdots\}\). The result then follows from (see for example [2] Proposition 3.1)
\[
\Gamma_C(s) = \det_{\infty} \left( \frac{1}{2\pi i} (\Theta - s \cdot \text{Id}) \mid \bigoplus_{n \geq 0} \mathbb{R}(n) \right)^{-1}
\]
and
\[
\Gamma_R(s) = \det_{\infty} \left( \frac{1}{2\pi i} (\Theta - s \cdot \text{Id}) \mid \bigoplus_{n \geq 0} \mathbb{R}(2n) \right)^{-1}
\]

References