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THE WEIL-ÉTALE FUNDAMENTAL GROUP OF A NUMBER FIELD I

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Abstract. Lichtenbaum has conjectured (Ann of Math. (2) **170**(2) (2009), 657–683) the existence of a Grothendieck topology for an arithmetic scheme X such that the Euler characteristic of the cohomology groups of the constant sheaf $\mathbb{Z}$ with compact support at infinity gives, up to sign, the leading term of the zeta function $\zeta_X(s)$ at $s = 0$. In this paper we consider the category of sheaves $\bar{X}_L$ on this conjectural site for $X = Spec(\mathcal{O}_F)$ the spectrum of a number ring. We show that $\bar{X}_L$ has, under natural topological assumptions, a well-defined fundamental group whose abelianization is isomorphic, as a topological group, to the Arakelov–Picard group of F. This leads us to give a list of topological properties that should be satisfied by $\bar{X}_L$. These properties can be seen as a global version of the axioms for the Weil group. Finally, we show that any topos satisfying these properties gives rise to complexes of étale sheaves computing the expected Lichtenbaum cohomology.

1. Introduction

Lichtenbaum has conjectured in [**10**] the existence of a Grothendieck topology for an arithmetic scheme X such that the Euler characteristic of the cohomology groups of the constant sheaf $\mathbb{Z}$ with compact support at infinity gives, up to sign, the leading term of the zeta function $\zeta_X(s)$ at $s = 0$. There should exist motivic complexes of sheaves $\mathbb{Z}(n)$ giving the special value of $\zeta_X(s)$ at any non-positive integer $s = n$, and this formalism should extend to motivic L-functions. In this paper, this conjectural cohomology theory will be called the *conjectural Lichtenbaum cohomology*. This cohomology is well defined for schemes of finite type over finite fields, by the work of Lichtenbaum [**9**] and Geisser [**5**]. But the situation for flat schemes over $\mathbb{Z}$ is more difficult, and is far from being understood even in the most simple case $X = Spec(\mathbb{Z})$.

We denote by $\bar{X}$ the Arakelov compactification of a number ring $X = Spec(\mathcal{O}_F)$. Using the Weil group W_F, Lichtenbaum has defined a first candidate for his conjectural cohomology of number rings, which he calls the Weil-étale cohomology. He has shown that the resulting cohomology groups with compact support $H^i_{Wc}(X, \mathbb{Z})$, assuming that they vanish for $i \geq 3$, are indeed miraculously related to the special value of the Dedekind zeta function $\zeta_F(s)$ at $s = 0$. However, Flach has shown in [**4**] that the groups $H^i_W(\bar{X}, \mathbb{Z})$ (hence $H^i_{Wc}(X, \mathbb{Z})$) are in fact infinitely generated for any $i \geq 4$ even. This shows that Lichtenbaum's definition is

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not yet the right one, as it is mentioned in [**10**]. Giving the correct (site of) definition for the conjectural Lichtenbaum cohomology of number rings is a deep problem.

This problem cannot be attacked directly. We have first to figure out the basic properties that need to be satisfied by Lichtenbaum's conjectural site. This question only makes sense if we consider the category of sheaves of sets on Lichtenbaum's conjectural site, i.e. the associated topos, since many non-equivalent sites can produce the same topos. In this paper, this conjectural topos will be denoted by $\bar{X}_L$ and will be called the *conjectural Lichtenbaum topos*. The first goal of this paper is to figure out the basic topological properties that must be satisfied by the conjectural Lichtenbaum topos of a number ring. To this aim, we give in Section 5.2 a list of nine necessary properties for $\bar{X}_L$. Our properties globalize the usual axioms for the Weil group. From now on, we refer to them as *Properties* (1)–(9). They are all satisfied by the (naturally defined) Weil-étale topos of a smooth projective curve over a finite field, and they are in agreement with the work of Deninger (see [**3**]).

In Sections 4 and 5, we show that Properties (1)–(9) must be satisfied by $\bar{X}_L$ under natural topological assumptions. Our argument is based on the following observation. The cohomology of the topos $\bar{X}_L$ associated to the arithmetic curve $\bar{X} = \overline{Spec(\mathcal{O}_F)}$, with coefficients in $\mathbb{Z}$ and $\widetilde{\mathbb{R}}$, must be the same as the one computed in [**10**] in degrees $i \leq 3$ and must vanish in degrees $i \geq 4$. This is necessary in order to obtain the correct cohomological interpretation (due to Lichtenbaum) of the analytic class number formula. Then we show that such a topos has, under natural topological assumptions, a well-defined fundamental group whose abelianization is isomorphic, *as a topological group*, to the Arakelov–Picard group $Pic(\bar{X})$ (see Theorem 4.3). In order to prove this result, we express Pontryagin duality in terms of sheaves and we use the notion of topological fundamental groups (as a special case of the fundamental group of a connected and locally connected topos over an arbitrary base topos with a point). Moreover, this argument also applies to the case of an arbitrary connected étale $\bar{X}$-scheme $\bar{U}$. Here we find that the abelian fundamental group of the slice topos $\bar{X}_L/\bar{U}$ must be topologically isomorphic to the S-idèle class group canonically associated to $\bar{U}$.

Properties (1)–(9) give a partial description of the conjectural Lichtenbaum topos. This description, which can be seen as a global version of the axioms for the Weil group (see [**16**]), is based on an interpretation of the S-idèle class groups in terms of topological fundamental groups (see [**17**]). Section 6 is devoted to the proof of the following result.

THEOREM 1.1. (Lichtenbaum's formalism) *Assume that F is totally imaginary. Let $\gamma : \bar{X}_L \to \bar{X}_{et}$ be any topos satisfying Properties (1)–(9). We denote by $\varphi : X_L \to \bar{X}_L$ the natural open embedding, and we set $H^n_c(X_L, \widetilde{\mathbb{R}}) := H^n(\bar{X}_L, \varphi_!\widetilde{\mathbb{R}})$. The following are true.*

- $\mathbb{H}^n(\bar{X}_{et}, \tau_{\leq 2}R\gamma_*(\varphi_!\mathbb{Z}))$ *is finitely generated, zero for $n \geq 4$ and the canonical map*

$$\mathbb{H}^n(\bar{X}_{et}, \tau_{\leq 2}R\gamma_*(\varphi_!\mathbb{Z})) \otimes \mathbb{R} \longrightarrow H^n_c(X_L, \widetilde{\mathbb{R}})$$

  is an isomorphism for any $n \geq 0$.

- *There exists a fundamental class $\theta \in H^1(\bar{X}_L, \widetilde{\mathbb{R}})$. The complex of finite-dimensional vector spaces*

$$\cdots \to H^{n-1}_c(X_L, \widetilde{\mathbb{R}}) \to H^n_c(X_L, \widetilde{\mathbb{R}}) \to H^{n+1}_c(X_L, \widetilde{\mathbb{R}}) \to \cdots$$

  defined by a cup product with θ, is acyclic.

- *Let B^n be a basis of $\mathbb{H}^n(X_{et}, \tau_{\leq 2} R\gamma_*(\varphi_!\mathbb{Z}))/tors$. The leading term coefficient $\zeta_F^*(0)$ at $s = 0$ is given by the Lichtenbaum Euler characteristic*

$$\zeta_F^*(0) = \pm \prod_{n \geq 0} |\mathbb{H}^n(\bar{X}_{et}, \tau_{\leq 2} R\gamma_*(\varphi_!\mathbb{Z}))_{tors}|^{(-1)^n} / \det(H_c^n(X_L, \widetilde{\mathbb{R}}), \theta, B^*).$$

In [**14**], we construct a topos (the Weil-étale topos) which satisfies Properties (1)–(9).

2. Preliminaries

2.1. Basic properties of geometric morphisms

Let $\mathcal{S}$ and $\mathcal{S}'$ be two Grothendieck topoi. A *(geometric) morphism of topoi*

$$f := (f^*, f_*) : \mathcal{S}' \longrightarrow \mathcal{S}$$

is defined as a pair of functors (f^*, f_*), where $f^* : \mathcal{S} \to \mathcal{S}'$ is left adjoint to $f_* : \mathcal{S}' \to \mathcal{S}$ and f^* is left exact (i.e. f^* commutes with finite projective limits). We can also define such a morphism as a left exact functor $f^* : \mathcal{S} \to \mathcal{S}'$ commuting with arbitrary inductive limits. Indeed, in this case f^* has a uniquely determined right adjoint f_*.

If X is an object of $\mathcal{S}$, then the slice category $\mathcal{S}/X$ of objects of $\mathcal{S}$ over X is a topos as well. The base change functor

$$\begin{array}{ccc} \mathcal{S} & \longrightarrow & \mathcal{S}/X \\ Y & \longmapsto & Y \times X \end{array}$$

is left exact and commutes with arbitrary inductive limits, since inductive limits are universal in a topos. We obtain a morphism

$$\mathcal{S}/X \longrightarrow \mathcal{S}.$$

Such a morphism is said to be a *localization morphism* or a *local homeomorphism* (the term local homeomorphism is inspired by the case where $\mathcal{S}$ is the topos of sheaves on some topological space). For any morphism $f : \mathcal{S}' \to \mathcal{S}$ and any object X of $\mathcal{S}$, there is a natural morphism

$$f_{/X} : \mathcal{S}'/f^*X \longrightarrow \mathcal{S}/X.$$

The functor $f_{/X}^*$ is defined in the obvious way: $f_{/X}^*(Y \to X) = (f^*Y \to f^*X)$. The direct image functor $f_{/X,*}$ sends $Z \to f^*X$ to $f_*Z \times_{f_*f^*X} X \to X$, where $X \to f_*f^*X$ is the adjunction map. The morphism $f_{/X}$ is a pull-back of f, in the sense that the square

$$\begin{tikzcd} \mathcal{S}'/f^*X \arrow[r, "f_{/X}"] \arrow[d] & \mathcal{S}/X \arrow[d] \\ \mathcal{S}' \arrow[r, "f"] & \mathcal{S} \end{tikzcd}$$

is commutative and 2-cartesian. In other words, the 2-fiber product $\mathcal{S}' \times_{\mathcal{S}} \mathcal{S}/X$ can be defined as the slice topos $\mathcal{S}'/f^*X$.

A morphism $f : \mathcal{S}' \to \mathcal{S}$ is said to be *connected* if f^* is fully faithful. It is *locally connected* if f^* has an $\mathcal{S}$-indexed left adjoint $f_!$ (see [**8**, C3.3]). These definitions generalize the usual ones for topological spaces: if T is a topological space, consider the unique morphism $Sh(T) \to \underline{Sets}$, where $Sh(T)$ is the category of étale spaces over T. For example, a localization morphism $\mathcal{S}/X \to \mathcal{S}$ is always locally connected (here $f_!(Y \to X) = Y$), but is connected if and only if X is the final object of $\mathcal{S}$.

A morphism $f : \mathcal{S}' \to \mathcal{S}$ is said to be an *embedding* when f_* is fully faithful. It is an *open embedding* if f factor through $f : \mathcal{S}' \simeq \mathcal{S}/X \to \mathcal{S}$, where X is a subobject of the final object of $\mathcal{S}$. Then the essential image $\mathcal{U}$ of the functor f_* is said to be an *open subtopos* of $\mathcal{S}$. The *closed complement* $\mathcal{F}$ of $\mathcal{U}$ is the strictly full subcategory of $\mathcal{S}$ consisting of objects Y such that $Y \times X$ is the final object of $\mathcal{U}$ (i.e. f^*Y is the final object of $\mathcal{S}'$). A *closed subtopos* $\mathcal{F}$ of $\mathcal{S}$ is a strictly full subcategory which is the closed complement of an open subtopos. A morphism of topoi $i : \mathcal{E} \to \mathcal{S}$ is said to be a *closed embedding* if i factors through $i : \mathcal{E} \simeq \mathcal{F} \to \mathcal{S}$, where $\mathcal{F}$ is a closed subtopos of $\mathcal{S}$.

A *subtopos* of $\mathcal{S}$ is a strictly full subcategory $\mathcal{S}'$ of $\mathcal{S}$ such that the inclusion functor $i : \mathcal{S}' \hookrightarrow \mathcal{S}$ is the direct image of a morphism of topoi (i.e. i has a left exact left adjoint). A morphism $f : \mathcal{S}' \to \mathcal{S}$ is said to be *surjective* if f^* is faithful. Any morphism $f : \mathcal{E} \to \mathcal{S}$ can be decomposed as a surjection $\mathcal{E} \to Im(f)$ followed by a an embedding $Im(f) \to \mathcal{S}$, where $Im(f)$ is a subtopos of $\mathcal{S}$, which is called the *image of f* (see [**7**, IV. 9.1.7.2]).

2.2. *The topos $\mathcal{T}$ of locally compact topological spaces*

We denote by *Top* the category of locally compact Hausdorff topological spaces and continuous maps. This category is endowed with the open cover topology $\mathcal{J}_{op}$, which is generated by the following pretopology: a family of morphisms $(X_\alpha \to X)_{\alpha \in A}$ is in $Cov(X)$ if and only if $(X_\alpha \to X)_{\alpha \in A}$ is an open cover in the usual sense. We denote by $\mathcal{T}$ the topos of sheaves of sets on this left exact site:

$$\mathcal{T} := (\widetilde{Top, \mathcal{J}_{op}}).$$

The family of compact spaces is easily seen to be a topologically generating family for the site $(Top, \mathcal{J}_{op})$. Indeed, if X is a locally compact space, then any $x \in X$ has a compact neighborhood $K_x \subseteq X$, so $(K_x \hookrightarrow X)_{x \in X}$ is a local section cover, hence a covering family for $\mathcal{J}_{op}$. In particular, if we denote by Top^c the category of compact spaces, then $\mathcal{T} = (\widetilde{Top^c, \mathcal{J}_{op}})$. The Yoneda functor

$$\begin{array}{rccc} y: & Top & \longrightarrow & \mathcal{T} \\ & X & \longmapsto & y(X) = Hom_{Top}(-, X), \end{array}$$

which sends a space X to the sheaf represented by X, is fully faithful (since $\mathcal{J}_{op}$ is subcanonical) and commutes with arbitrary projective limits. Since the Yoneda functor is left exact, any locally compact topological group G represents a group object of $\mathcal{T}$. In what follows we consider *Top* as a (left exact) full subcategory of $\mathcal{T}$. For example, the sheaf of $\mathcal{T}$ represented by a (locally compact Hausdorff) space Z is sometimes also denoted by Z.

In this paper, we consider topoi defined over the topos of locally compact spaces since all sheaves, cohomology groups and fundamental groups that we use are defined by locally compact spaces. In order to use non-locally compact coefficients, we can consider the topos

$$\mathcal{T}' := (\widetilde{Top^h, \mathcal{J}_{op}}),$$

where Top^h is the category of Hausdorff spaces. Then for any topos $\mathcal{E}$ (connected and locally connected) over $\mathcal{T}$, we consider the base change $\mathcal{E} \times_{\mathcal{T}} \mathcal{T}'$ to obtain a (connected and locally connected) topos over $\mathcal{T}'$.

2.3. *The classifying topos of a group object*

For any topos $\mathcal{S}$ and any group object $\mathcal{G}$ in $\mathcal{S}$, we denote by $B_{\mathcal{G}}$ the category of (left) $\mathcal{G}$-objects in $\mathcal{S}$. Then $B_{\mathcal{G}}$ is a topos, as it follows from Giraud's axioms, and $B_{\mathcal{G}}$ is endowed with a canonical morphism $B_{\mathcal{G}} \to \mathcal{S}$, whose inverse image functor sends an object $\mathcal{F}$ of $\mathcal{S}$ to $\mathcal{F}$ with trivial $\mathcal{G}$-action. If there is a risk of ambiguity, we denote the topos $B_{\mathcal{G}}$ by $B_{\mathcal{S}}(\mathcal{G})$. The topos $B_{\mathcal{G}}$ is said to be the classifying topos of $\mathcal{G}$ since it classifies $\mathcal{G}$-torsors. More precisely, for any topos $f : \mathcal{E} \to \mathcal{S}$ over $\mathcal{S}$, the category $\underline{Homtop}_{\mathcal{S}}(\mathcal{E}, B_{\mathcal{G}})$ is equivalent to the category of $f^*\mathcal{G}$-torsors in $\mathcal{E}$ (see [**7**, IV. Exercise 5.9]).

2.3.1. The classifying topos of a profinite group. Let G be a discrete group, i.e. a group object of the final topos $\underline{Sets}$. We denote the category of G-sets by

$$B_G^{sm} := B_{\underline{Sets}}(G).$$

The topos B_G^{sm} is called the small classifying topos of the discrete group G.

If G is a profinite group, then the small classifying topos B_G^{sm} is defined as the category of sets on which G acts continuously.

2.3.2. The classifying topos of a topological group. Let G be a locally compact (hence Hausdorff) topological group. Then G represents a group object of $\mathcal{T}$, where $\mathcal{T} := (\widetilde{Top, \mathcal{J}_{op}})$ is defined above. Then

$$B_G := B_{\mathcal{T}}(G)$$

is the classifying topos of the locally compact topological group G. We can define the classifying topos of an arbitrary topological group by enlarging the topos $\mathcal{T}$.

2.3.3. Topological pro-groups. In this paper, a *filtered category* I is a non-empty small category such that the following holds. For any objects i and j of I, there exists a third object k and maps $i \leftarrow k \to j$. For any pair of maps $i \rightrightarrows j$ there exists a map $k \to i$ such that the diagram $k \to i \rightrightarrows j$ is commutative. Let C be any category. A *pro-object* of C is a functor $\underline{X} : I \to C$, where I is a filtered category. We can see a pro-object in C as a diagram in C. We can define the *category* $Pro(C)$ *of pro-objects in* C (see [**7**, I. 8.10]). The morphisms in this category can be made explicit as follows. Let $\underline{X} : I \to C$ and $\underline{Y} : J \to C$ be two pro-objects in C. Then we have

$$Hom_{Pro(C)}(\underline{X}, \underline{Y}) := \varprojlim_{j\in J} \varinjlim_{i\in I} Hom(X_i, Y_j).$$

A pro-object $\underline{X} : I \to C$ is *constant* if it is a constant functor, and $\underline{X} : I \to C$ is *essentially constant* if $\underline{X}$ is isomorphic (in the category $Pro(C)$) to a constant pro-object.

Definition 2.1. A *locally compact topological pro-group* $\underline{G}$ is a pro-object in the category of locally compact and Hausdorff topological groups. A locally compact topological pro-group is said to be *strict* if any transition map $G_j \to G_i$ has local sections.

If the category C is a topos, then a pro-object $\underline{X} : I \to C$ in C is said to be *strict* when the transition map $X_i \to X_j$ is an epimorphism in C for any $i \to j \in Fl(I)$. In particular, a locally compact topological pro-group $\underline{G} : I \to Gr(Top)$ pro-represents a strict pro-group object in $\mathcal{T}$:

$$y \circ \underline{G} : I \to Gr(Top) \to Gr(\mathcal{T}),$$

where $Gr(Top)$ and $Gr(\mathcal{T})$ are the categories of group objects in *Top* and $\mathcal{T}$ respectively. Indeed, a continuous map of locally compact spaces $X_i \to X_j$ has local sections if and only if it induces an epimorphism $y(X_i) \to y(X_j)$ in $\mathcal{T}$. Topos theory provides a natural way to define the limit of a strict topological pro-group without any loss of information.

Definition 2.2. The classifying topos of a locally compact strict pro-group $\underline{G} : I \to Gr(Top)$ is defined as

$$B_{\underline{G}} := \varprojlim_I B_{G_i},$$

where the projective limit is computed in the 2-category of topoi.

2.4. *The Arakelov–Picard group*

Let F be a number field and let X_∞ be the set of archimedean places of F. We denote by $\bar{X} = (Spec\, \mathcal{O}_F, X_\infty)$ the Arakelov compactification of the ring of integers in F. We consider the idèle group I_F and the idèle class group C_F of F. For any place v of F, we denote by F_v the corresponding local field and by $\mathcal{O}_{F_v}^\times$ the group of local units, i.e. the kernel of the absolute value $K_v^\times \to \mathbb{R}_{>0}$. Note that we have $\mathcal{O}_{F_v}^\times = \mathbb{S}^1$ for v complex and $\mathcal{O}_{F_v}^\times = \{\pm 1\}$ for v real. The group $\mathcal{O}_{F_v}^\times$ is always compact. The *Arakelov–Picard group* $Pic(\bar{X})$ is defined as the cokernel, endowed with the quotient topology, of the continuous map $\prod_v \mathcal{O}_{F_v}^\times \to C_F$. For any place v, the map $F_v^\times \to C_F$ induces a continuous morphism

$$W_{k(v)} := F_v^\times / \mathcal{O}_{F_v}^\times \longrightarrow Pic(\bar{X}), \tag{1}$$

where $W_{k(v)}$ is the Weil group of the 'residue field $k(v)$' at $v \in \bar{X}$. The absolute value $C_F \to \mathbb{R}_{>0}$ factors through $Pic(\bar{X})$. We obtain a canonical continuous morphism $Pic(\bar{X}) \to \mathbb{R}_{>0}$ endowed with a continuous section. The Arakelov class group $Pic^1(\bar{X})$ is the kernel of this map. In other words, we have an exact sequence of topological groups

$$0 \to Pic^1(\bar{X}) \to Pic(\bar{X}) \to \mathbb{R}_{>0} \to 0.$$

3. Pontryagin duality and topological fundamental groups

3.1. *Pontryagin duality*

Let X and Y be two objects in a topos $\mathcal{E}$. There exists an internal Hom-object $\underline{Hom}_{\mathcal{E}}(X, Y)$ in $\mathcal{E}$ such that there is a functorial isomorphism

$$Hom(Z, \underline{Hom}_{\mathcal{E}}(X, Y)) \simeq Hom(Z \times X, Y) = Hom_{\mathcal{E}/Z}(Z \times X, Z \times Y) \tag{2}$$

for any object Z of $\mathcal{E}$. Indeed, the (base change) functor $Z \to Z \times X$ commutes with (arbitrary) inductive limits since inductive limits are universal in the topos $\mathcal{E}$. Therefore, the contravariant functor

$$\begin{array}{rcl} \mathcal{E} & \longrightarrow & \underline{Set} \\ Z & \longmapsto & Hom_{\mathcal{E}}(Z \times X, Y) \end{array}$$

sends inductive limits in $\mathcal{E}$ to projective limits in $\underline{Set}$. Hence this presheaf on $\mathcal{E}$ is a sheaf for the canonical topology. Since the sheaves on a topos endowed with the canonical topology are all representable, this functor is representable by an object $\underline{Hom}_{\mathcal{E}}(X, Y)$ of $\mathcal{E}$. If G and A are both group objects in $\mathcal{E}$ such that A is abelian, then we denote by $\underline{Hom}_{\mathcal{E}}(G, A)$ the group object of $\mathcal{E}$ given by

$$\begin{array}{rcl} \mathcal{E} & \longrightarrow & \underline{Ab} \\ Z & \longmapsto & Hom_{Gr(\mathcal{E}/Z)}(Z \times G, Z \times A), \end{array}$$

where $\underline{Ab}$ and $Gr(\mathcal{E}/Z)$ denote respectively the category of (discrete) abelian groups and the category of group objects of the slice topos $\mathcal{E}/Z$.

Let $\mathcal{T}$ be the topos of sheaves on the site $(Top, \mathcal{J}_{op})$, where Top is the category *Hausdorff locally compact* topological spaces and continuous maps endowed with the open cover topology $\mathcal{J}_{op}$. Recall that the Yoneda functor

$$\begin{array}{rrcl} y: & Top & \longrightarrow & \mathcal{T} \\ & X & \longmapsto & y(X) = Hom_{Top}(-, X) \end{array}$$

sending a topological space to the sheaf represented by this space is fully faithful and commutes with arbitrary projective limits.

Let X and Y be two Hausdorff locally compact topological spaces. We denote by $\underline{Hom}_{Top}(X, Y)$ the set of continuous maps from X to Y endowed with the compact-open topology. This topological space is Hausdorff and locally compact. Then the sheaf of $\mathcal{T}$ represented by $\underline{Hom}_{Top}(X, Y)$ is precisely the internal object $\underline{Hom}_{\mathcal{T}}(y(X), y(Y))$ defined above, since $Hom_{Top}(X, Y)$ satisfies (2). Indeed, we have

$$Hom_{Top}(Z \times X, Y) = Hom_{Top}(Z, \underline{Hom}_{Top}(X, Y))$$

for any Hausdorff topological spaces Z. Hence the sheaf $\underline{Hom}_{\mathcal{T}}(y(X), y(Y))$ is represented by $\underline{Hom}_{Top}(X, Y)$, i.e. we have a canonical isomorphism in $\mathcal{T}$:

$$\underline{Hom}_{\mathcal{T}}(y(X), y(Y)) = y(\underline{Hom}_{Top}(X, Y)).$$

If G and A are two Hausdorff locally compact topological groups such that A is abelian then the abelian group of continuous morphisms $\underline{Hom}_{Top}(G, A)$ is also endowed with the compact-open topology, and we have

$$\underline{Hom}_{\mathcal{T}}(y(G), y(A)) = y(\underline{Hom}_{Top}(G, A)).$$

Note that $y(G)$ and $y(A)$ are two group objects in $\mathcal{T}$ since the Yoneda functor y commutes with finite projective limits.

Definition 3.1. Let $\mathcal{G}$ be a group object of $\mathcal{T}$. We denote by $\mathcal{G}^D$ the internal Hom-group-object of $\mathcal{T}$:

$$\mathcal{G}^D := \underline{Hom}_{\mathcal{T}}(\mathcal{G}, y(\mathbb{S}^1)),$$

where $\mathbb{S}^1$ is endowed with its standard topology. If $\mathcal{A}$ is an abelian object of $\mathcal{T}$, then the abelian object $\mathcal{A}^D$ is said to be the *dual* of $\mathcal{A}$.

For any group object $\mathcal{G}$ of $\mathcal{T}$, there is a canonical morphism

$$d_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathcal{G}^{DD}. \tag{3}$$

The discussion above shows that if $\mathcal{G} = y(G)$ is represented by a locally compact abelian topological group G, then $y(G)^D$ is represented by the usual Pontryagin dual $G^D := \underline{Hom}_{Top}(G, \mathbb{S}^1)$ of G, endowed with the compact-open topology. Therefore, the following result is given by Pontryagin duality for Hausdorff locally compact abelian groups.

THEOREM 3.2. *Let $\mathcal{A}$ be an abelian object of $\mathcal{T}$ representable by an abelian Hausdorff locally compact topological group. Then we have a canonical isomorphism*

$$d_{\mathcal{A}} : \mathcal{A} \simeq \mathcal{A}^{DD}.$$

COROLLARY 3.3. *If $y(G)$ is a group of $\mathcal{T}$ represented by a Hausdorff locally compact topological group G, then we have*

$$y(G)^{DD} \simeq y(G^{ab}),$$

where G^{ab} is the maximal Hausdorff abelian quotient of G.

Proof. Using Theorem 3.2, the result follows from

$$y(G)^D = y(G^D) = y((G^{ab})^D) = y(G^{ab})^D$$

since G^{ab} is Hausdorff and locally compact. □

3.2. Fundamental groups

Let $\mathcal{S}$ be a topos and let $t : \mathcal{E} \to \mathcal{S}$ be a connected and locally connected topos over $\mathcal{S}$ (i.e. t^* is fully faithful and has an $\mathcal{S}$-indexed left adjoint, see [**8**, C3.3]). An object L of $\mathcal{E}$ is said to be *locally constant over* $\mathcal{S}$ if there exists a covering morphism $U \to e_{\mathcal{E}}$ of the final object of $\mathcal{E}$, an object S of $\mathcal{S}$ and an isomorphism $L \times U \simeq f^*S \times U$ over U. The object U is then said to split or trivialize L. Let $LC(\mathcal{E})$ be the full subcategory of $\mathcal{E}$ consisting of locally constant objects of $\mathcal{E}$ over $\mathcal{S}$. We denote by $SLC(\mathcal{E})$ the category of (internal) sums of locally constant objects (see [**2**, Section 2] for an explicit definition). The category $SLC(\mathcal{E})$ is a topos and we have a canonical *connected* morphism

$$\mathcal{E} \to SLC(\mathcal{E}), \tag{4}$$

whose inverse image is the inclusion $SLC(\mathcal{E}) \hookrightarrow \mathcal{E}$. The fact that this morphism is connected means that its inverse image is fully faithful, which is obvious here. Note that this morphism is defined over $\mathcal{S}$.

Assume that the $\mathcal{S}$-topos $\mathcal{E}$ has a point p, i.e. a section $p : \mathcal{S} \to \mathcal{E}$ of the structure map $t : \mathcal{E} \to \mathcal{S}$. Composing p and the morphism (4), we obtain a point

$$\widetilde{p} : \mathcal{S} \to \mathcal{E} \to SLC(\mathcal{E})$$

of the topos $SLC(\mathcal{E})$ over $\mathcal{S}$. The theory of the fundamental group in the context of topos theory shows the following (see [**11**] and [**2**, Section 1]). There exists a 'pro-discrete localic group' $\mathcal{G}$ in $\mathcal{S}$ well defined up to a canonical isomorphism and an equivalence

$$B\mathcal{G} \simeq SLC(\mathcal{E}),$$

where $B\mathcal{G}$ is the classifying topos of $\mathcal{G}$, i.e. the topos of $\mathcal{G}$-objects in $\mathcal{S}$. Moreover, the equivalence above identifies the inverse image of the point $\widetilde{p} : \mathcal{S} \to SLC(\mathcal{E})$ with the forgetful functor $B\mathcal{G} \to \mathcal{S}$.

The topos $\mathcal{E}$ is said to be *locally simply connected* over $\mathcal{S}$ if there exists one covering morphism $U \to e_{\mathcal{E}}$ trivializing all locally constant objects in $\mathcal{E}$. In this case we have $SLC(\mathcal{E}) = LC(\mathcal{E})$, and the pro-discrete localic group $\mathcal{G}$ is just a group object of $\mathcal{S}$ (see [**1**] or [**2**, Section 1]). We denote this group object by $\pi_1(\mathcal{E}, p)$. We get a *connected* morphism

$$\mathcal{E} \longrightarrow LC(\mathcal{E}) \simeq B_{\pi_1(\mathcal{E},p)} \tag{5}$$

over $\mathcal{S}$, i.e. a commutative diagram.

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[dr] & B_{\pi_1(\mathcal{E},p)} \arrow[d] \\ & \mathcal{S} \end{tikzcd}$$

The morphism (5) into the classifying topos $B_{\pi_1(\mathcal{E},p)}$ corresponds to a torsor in $\mathcal{E}$ of group $\pi_1(\mathcal{E}, p)$, which is called the *universal cover* of $\mathcal{E}$ over $\mathcal{S}$.

Definition 3.4. Let $\mathcal{A}$ be an abelian object of $\mathcal{E}$. We define the *cohomology of $\mathcal{E}$ with value in $\mathcal{S}$* as

$$H^n_{\mathcal{S}}(\mathcal{E}, \mathcal{A}) = R^n(t_*)\mathcal{A}.$$

The fundamental group represents the first cohomology group over an arbitrary base topos. More precisely, we have the following result.

PROPOSITION 3.5. *Let $\mathcal{E}$ be a connected, locally connected and locally simply connected topos over $\mathcal{S}$ endowed with a point p. For any abelian object $\mathcal{A}$ of $\mathcal{S}$, $t^*\mathcal{A}$ is a constant abelian object of $\mathcal{E}$ over $\mathcal{S}$ and we have*

$$H^1_{\mathcal{S}}(\mathcal{E}, t^*\mathcal{A}) \simeq \underline{Hom}_{\mathcal{S}}(\pi_1(\mathcal{E}, p), \mathcal{A}),$$

where the right-hand side is the internal Hom-group-object in $\mathcal{S}$ defined as above.

3.2.1. Examples. Let X be a Hausdorff topological space. We denote by $Sh(X)$ the topos of sheaves of sets on X. There exists a unique map

$$t : Sh(X) \to \underline{Set}.$$

The topological space X is connected if and only if t is connected (i.e. if t^* is fully faithful). Let F be a sheaf on X (i.e. an étale space $\widetilde{F} \to X$). If X is locally connected then $\widetilde{F}$ is locally connected and $\widetilde{F}$ is the coproduct in $Sh(X)$ of its connected components. The functor $F \to \pi_0(\widetilde{F})$ is left adjoint to t^* hence t is a locally connected map of topoi. Conversely, if t is a locally connected map then X is locally connected as a topological space. A sheaf F on X is locally constant if and only if $\widetilde{F} \to X$ is an étale cover. Assume that X is locally simply connected and let $\{U_i \subseteq X,\ i \in I\}$ be an open covering such that U_i is simply connected. Then any locally constant sheaf on X is trivialized by $U := \coprod U_i \to X$. A point $x \in X$ yields a morphism $p_x : \underline{Set} \to Sh(X)$ (and conversely). The inverse image of this morphism is the stalk functor $F \to F_x$. The category $LC(Sh(X))$ is precisely the category of étale covers of X and the group $\pi_1(Sh(X),\ p_x)$ is the usual fundamental group $\pi_1(X, x)$. In this special case, the equivalence of categories

$$\begin{array}{ccc} LC(Sh(X)) & \longrightarrow & B\pi_1(X, x) \\ F & \longmapsto & F_x \end{array}$$

is the usual Galois theory for topological spaces. Here $B\pi_1(X, x)$ is the classifying topos of the discrete group $\pi_1(X, x)$, i.e. the category of $\pi_1(X, x)$-sets.

Let $\mathcal{S}$ be a topos and let G be a group of $\mathcal{S}$. We denote by B_G the topos of G-objects in $\mathcal{S}$. The canonical morphism

$$t : B_G \longrightarrow \mathcal{S}$$

is connected, locally connected and locally simply connected. Indeed, t is connected since t^* is obviously fully faithful. Moreover, t is locally connected since t^* has a $\mathcal{S}$-indexed left adjoint given by the quotient functor

$$t_!\mathcal{F} = \mathcal{F}/G := \varinjlim(G \times \mathcal{F} \rightrightarrows \mathcal{F}).$$

Note that the inductive limit or coequalizer $\varinjlim(G \times \mathcal{F} \rightrightarrows \mathcal{F})$, where the maps are given by multiplication and projection, always exists in the topos $\mathcal{S}$. Finally, $E_G \to \{*\}$ trivializes any object, hence t is locally simply connected. There is a canonical point $p : \mathcal{S} \to B_G$, whose inverse image is the forgetful functor. In this case, the inclusion $LC(B_G) \hookrightarrow B_G$ is an equivalence (in fact an isomorphism) and the fundamental group $\pi_1(B_G,\ p)$ is G.

3.2.2. Topological fundamental groups. Let $t : \mathcal{E} \to \mathcal{T}$ be a connected and locally connected topos over $\mathcal{T}$ endowed with a $\mathcal{T}$-valued point p. The fundamental group $\pi_1(\mathcal{E},\ p)$ will be called the *topological fundamental group* of $\mathcal{E}$.

COROLLARY 3.6. *Let $t : \mathcal{E} \to \mathcal{T}$ be a connected, locally connected and locally simply connected topos over $\mathcal{T}$ endowed with a $\mathcal{T}$-valued point p. Let $y\mathbb{S}^1$ be the sheaf of $\mathcal{T}$ represented by the standard topological group $\mathbb{S}^1$, and define $\widetilde{\mathbb{S}}^1 := t^*y\mathbb{S}^1$. We have*

$$H^1_{\mathcal{T}}(\mathcal{E}, \widetilde{\mathbb{S}}^1) \simeq \pi_1(\mathcal{E},\ p)^D.$$

If $\pi_1(\mathcal{E},\ p)$ is represented by a locally compact group, then $H^1_{\mathcal{T}}(\mathcal{E}, \widetilde{\mathbb{S}^1})$ is represented by the usual Pontryagin dual $\pi_1(\mathcal{E},\ p)^D$ and we have

$$H^1_{\mathcal{T}}(\mathcal{E}, \widetilde{\mathbb{S}}^1)^D \simeq \pi_1(\mathcal{E},\ p)^{DD} = \pi_1(\mathcal{E},\ p)^{ab},$$

where $\pi_1(\mathcal{E},\ p)^{ab}$ is the maximal abelian Hausdorff quotient of $\pi_1(\mathcal{E},\ p)$.

Proof. This follows from Proposition 3.5, Definition 3.1 and Corollary 3.3. □

4. Application to the arithmetic fundamental group

Let $\bar{X} = (Spec\ \mathcal{O}_F, X_\infty)$ be the Arakelov compactification of the ring of integers in a number field F. Following the computations of Lichtenbaum (see [**10**]), we are looking for a topos $\bar{X}_L$ defined over $\mathcal{T}$ whose cohomology is related the Dedekind zeta function $\zeta_F(s)$. This conjectural topos $\bar{X}_L$ will be called the *conjectural Lichtenbaum topos*. The topos $\bar{X}_L$ should be defined over $\mathcal{T}$, since the coefficients for this conjectural cohomology theory should contain the category of locally compact abelian topological groups. If we denote by

$$t : \bar{X}_L \longrightarrow \mathcal{T}$$

the structure map, then we define the sheaf of continuous real valued functions $\widetilde{\mathbb{R}}$ to be $t^*(y\mathbb{R})$, where $y\mathbb{R}$ is the abelian object of $\mathcal{T}$ represented by the standard topological group $\mathbb{R}$.

Following the computations of Lichtenbaum, the cohomology of $\bar{X}_L$ must satisfy

$$H^i(\bar{X}_L, \mathbb{Z}) = \mathbb{Z}, 0, Pic^1(\bar{X})^D \quad \text{for } i = 0, 1, 2 \text{ respectively,} \tag{6}$$

and

$$H^i(\bar{X}_L, \widetilde{\mathbb{R}}) = \mathbb{R}, \mathbb{R}, 0 \quad \text{for } i = 0, 1, 2 \text{ respectively.} \tag{7}$$

Recall that for any (Grothendieck) topos $\mathcal{E}$, there is a unique morphism $e : \mathcal{E} \to \underline{Set}$. The cohomology of the topos $\mathcal{E}$ with coefficients in $\mathcal{A}$ is defined by

$$H^n(\mathcal{E}, \mathcal{A}) := R^n(e_*)\mathcal{A}.$$

Since the base topos of the topos $\bar{X}_L$ is $\mathcal{T}$ instead of $\underline{Set}$, it is natural to consider the cohomology of $\bar{X}_W$ with value in $\mathcal{T}$. More precisely, the category $\mathcal{T}$ is thought of as a universe of sets, and we define

$$H^n_{\mathcal{T}}(\bar{X}_L, \mathcal{A}) := R^n(t_*)\mathcal{A}$$

for any abelian object $\mathcal{A}$ of $\bar{X}_L$. The unique morphism $\mathcal{T} \to \underline{Sets}$ is strongly acyclic (i.e. its direct image is exact) and this point of view is inoffensive. We should have

$$H^i_{\mathcal{T}}(\bar{X}_L, \mathbb{Z}) = \mathbb{Z}, 0, Pic^1(\bar{X})^D \quad \text{for } i = 0, 1, 2 \text{ respectively,} \tag{8}$$

where $\mathbb{Z}$ and $Pic^1(\bar{X})^D$ are the sheaves of $\mathcal{T}$ represented by the discrete abelian groups $\mathbb{Z}$ and $Pic^1(\bar{X})^D$. Respectively, the $\mathcal{T}$-cohomology of $\bar{X}_L$ with value in $\widetilde{\mathbb{R}}$ should be given by

$$H^i_{\mathcal{T}}(\bar{X}_L, \widetilde{\mathbb{R}}) = y(\mathbb{R}), y(\mathbb{R}), 0 \quad \text{for } i = 0, 1, 2 \text{ respectively,} \tag{9}$$

where $y(\mathbb{R})$ is the abelian object of $\mathcal{T}$ represented by the standard topological group $\mathbb{R}$.

Hypothesis 4.1. The topos $\bar{X}_L$ is connected, locally connected, locally simply connected over $\mathcal{T}$, and endowed with a point $p : \mathcal{T} \to \bar{X}_L$.

It is natural to expect that $\bar{X}_L$ is connected and locally connected over $\mathcal{T}$. A point $p : \mathcal{T} \to \bar{X}_L$ should be given by any valuation of the number field F. However, it is not clear that $\bar{X}_L$ should be locally simply connected over $\mathcal{T}$ (for example, $\bar{X}_{et}$ is not locally simply connected over $\underline{Sets}$ in general). But this assumption can be avoided using the more advanced notion of localic groups (or pro-groups). We make this assumption to simplify the following computations.

Hypothesis 4.2. The cohomology of $\bar{X}_L$ with value in $\mathcal{T}$ satisfies (8) and (9).

4.1. The abelian arithmetic fundamental group

THEOREM 4.3. *Let $\bar{X}_L$ be a topos over $\mathcal{T}$ satisfying Hypotheses 4.1 and 4.2. Then we have an isomorphism of topological groups*

$$\pi_1(\bar{X}_L, p)^{DD} \simeq Pic(\bar{X}),$$

where $Pic(\bar{X})$ denotes the Arakelov–Picard group of the number field F. In particular, if $\pi_1(\bar{X}_L, p)$ is represented by a locally compact topological group, then we have an isomorphism of topological groups

$$\pi_1(\bar{X}_L, p)^{ab} \simeq Pic(\bar{X}).$$

Proof. By Hypothesis 4.1 and Section 3.2, the fundamental group $\pi_1(\bar{X}_L, p)$ is well defined as a group object of $\mathcal{T}$. The basic idea is to use Corollary 3.6 to recover the abelian fundamental group. We have

$$H^1_{\mathcal{T}}(\bar{X}_L, \mathcal{A}) = \underline{Hom}_{\mathcal{T}}(\pi_1(\bar{X}_L, p), \mathcal{A})$$

for any abelian object $\mathcal{A}$ of $\mathcal{T}$. The exact sequence of topological groups

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{S}^1 \to 0$$

induces an exact sequence

$$0 \to \mathbb{Z} \to \widetilde{\mathbb{R}} \to \widetilde{\mathbb{S}}^1 \to 0$$

of abelian sheaves in $\bar{X}_L$, where $\widetilde{\mathbb{S}}^1$ denotes $t^*(y(\mathbb{S}^1))$. Consider the induced long exact sequence of $\mathcal{T}$-cohomology

$$0 = H^1_{\mathcal{T}}(\bar{X}_L, \mathbb{Z}) \to H^1_{\mathcal{T}}(\bar{X}_L, \widetilde{\mathbb{R}}) \to H^1_{\mathcal{T}}(\bar{X}_L, \widetilde{\mathbb{S}}^1) \to H^2_{\mathcal{T}}(\bar{X}_L, \mathbb{Z}) \to H^2_{\mathcal{T}}(\bar{X}_L, \widetilde{\mathbb{R}}) = 0.$$

We obtain an exact sequence in $\mathcal{T}$:

$$0 \to \mathbb{R} \to H^1_{\mathcal{T}}(\bar{X}_L, \widetilde{\mathbb{S}}^1) \to Pic^1(\bar{X})^D \to 0.$$

It follows that

$$H^1_{\mathcal{T}}(\bar{X}_L, \widetilde{\mathbb{S}}^1) = \underline{Hom}_{\mathcal{T}}(\pi_1(\bar{X}_L, p), y(\mathbb{S}^1)) = \pi_1(\bar{X}_L, p)^D$$

is representable by an abelian Hausdorff locally compact topological group. Indeed, $H^1_{\mathcal{T}}(\bar{X}_L, \widetilde{\mathbb{S}}^1)$ is representable locally on $Pic^1(\bar{X})^D$. But $Pic^1(\bar{X})^D$ is discrete (recall that $Pic^1(\bar{X})$ is compact) and the Yoneda embedding $y : Top \to \mathcal{T}$ commutes with coproducts (see [**4**, Corollary 1]), hence the sheaf $H^1_{\mathcal{T}}(\bar{X}_L, \widetilde{\mathbb{S}}^1)$ is representable by a topological space T. The functor $y : Top \to \mathcal{T}$ is fully faithful and commutes with finite projective limits. Hence the space T is endowed with a structure of an abelian topological group since $y(T) = H^1_{\mathcal{T}}(\bar{X}_L, \widetilde{\mathbb{S}}^1)$ is an abelian object of $\mathcal{T}$. The connected component of the identity in T is isomorphic to $\mathbb{R}$, since $Pic^1(\bar{X})^D$ is discrete. Hence T is Hausdorff and locally compact. Therefore $\pi_1(\bar{X}_L, p)^{DD} = y(T^D)$ is representable by an abelian Hausdorff locally compact topological group as well.

By Pontryagin duality, we obtain the exact sequence in $\mathcal{T}$,

$$0 \to Pic^1(\bar{X}) \to \pi_1(\bar{X}_L, p)^{DD} \to \mathbb{R} \to 0. \tag{10}$$

LEMMA 4.4. *We have $H^n(B_{\mathbb{R}}, Pic^1(\bar{X})) = 0$ for any $n \geq 2$.*

Proof. Let r_1 and r_2 be the sets of real and complex places of the number field F, respectively. We have the exact sequence of topological groups (with trivial $\mathbb{R}$-action)

$$0 \to \mathbb{R}^{r_1+r_2-1}/\log(\mathcal{O}_F^\times/\mu_F) \to Pic^1(\bar{X}) \to Cl(F) \to 0,$$

where $\log(\mathcal{O}_F^\times/\mu_F)$ denotes the image of the logarithmic embedding of the units modulo torsion $\mathcal{O}_F^\times/\mu_F$ in the kernel $\mathbb{R}^{r_1+r_2-1}$ of the sum map $\Sigma : \mathbb{R}^{r_1+r_2} \to \mathbb{R}$. The class group $Cl(F)$ is finite hence we have $H^n(B_{\mathbb{R}}, Cl(F)) = 0$ for any $n \geq 1$ (see [**4**, Proposition 9.6]). Hence we have

$$H^n(B_{\mathbb{R}}, \mathbb{R}^{r_1+r_2-1}/\log(\mathcal{O}_F^\times/\mu_F)) \simeq H^n(B_{\mathbb{R}}, Pic^1(\bar{X}))$$

for any $n \geq 1$. Now consider the exact sequence

$$0 \to \mathcal{O}_F^\times/\mu_F \to \mathbb{R}^{r_1+r_2-1} \to \mathbb{R}^{r_1+r_2-1}/\log(\mathcal{O}_F^\times/\mu_F) \to 0.$$

We have $H^n(B_{\mathbb{R}}, \mathcal{O}_F^\times/\mu_F) = 0$ for any $n \geq 1$, since $\mathcal{O}_F^\times/\mu_F$ is discrete (see [**4**, Proposition 9.6]). We obtain

$$H^n(B_{\mathbb{R}}, Pic^1(\bar{X})) = H^n(B_{\mathbb{R}}, \mathbb{R}^{r_1+r_2-1}/\log(\mathcal{O}_F^\times/\mu_F)) = H^n(B_{\mathbb{R}}, \mathbb{R}^{r_1+r_2-1}) = 0$$

for any $n \geq 2$ (again, see [**4**, Proposition 9.6]). □

In particular $H^2(B_{\mathbb{R}}, Pic^1(\bar{X})) = 0$, hence (see [**6**, VIII Proposition 8.2]) the extension (10) of abelian groups in $\mathcal{T}$ is isomorphic to the exact sequence

$$0 \to Pic^1(\bar{X}) \to Pic(\bar{X}) \to \mathbb{R} \to 0,$$

where $Pic(\bar{X}) \to \mathbb{R}$ is the canonical continuous morphism. In particular there is an isomorphism $\pi_1(\bar{X}_L, p)^{DD} \simeq Pic(\bar{X})$ in $\mathcal{T}$. This shows that $\pi_1(\bar{X}_L, p)^{DD}$ and $Pic(\bar{X})$ are isomorphic as topological groups, since $y : Top \to \mathcal{T}$ is fully faithful. The last claim of the theorem then follows from Corollary 3.3. □

4.2. *The morphism flow and the fundamental class*

COROLLARY 4.5. *Let $\bar{X}_L$ be a topos over $\mathcal{T}$ satisfying Hypotheses 4.1 and 4.2. Then there is a canonical morphism over $\mathcal{T}$,*

$$\pi : \bar{X}_L \longrightarrow B_{Pic(\bar{X})}. \tag{11}$$

In particular, there is a canonical morphism

$$\mathfrak{f} : \bar{X}_L \longrightarrow B_{\mathbb{R}}. \tag{12}$$

Proof. There is a morphism $\bar{X}_L \to B_{\pi_1(\bar{X}_L, p)}$ over $\mathcal{T}$ (the universal cover defined by the point p). Composing with the morphism of the classifying topos induced by the map (see (3))

$$\pi_1(\bar{X}_L, p) \longrightarrow \pi_1(\bar{X}_L, p)^{DD} \simeq Pic(\bar{X}),$$

we get the morphism π. Note that (11) does not depend on p since $\pi_1(\bar{X}_L, p)^{DD}$ is abelian. The morphism $\mathfrak{f}$ is then given by the canonical morphism of topological groups

$$Pic(\bar{X}) \longrightarrow \mathbb{R},$$

or (more directly) by the Pontryagin dual of the map

$$\mathbb{R} = H^1_{\mathcal{T}}(\bar{X}_L, \widetilde{\mathbb{R}}) \longrightarrow H^1_{\mathcal{T}}(\bar{X}_L, \widetilde{\mathbb{S}}^1) = \pi_1(\bar{X}_L, p)^D. \qquad \square$$

COROLLARY 4.6. *Let $\bar{X}_L$ be a topos over $\mathcal{T}$ satisfying Hypotheses 4.1 and 4.2. Then there is a* fundamental class $\theta \in H^1(X_L, \widetilde{\mathbb{R}})$. *If the fundamental group $\pi_1(\bar{X}_L, p)$ is representable by a locally compact group, then*

$$\theta \in H^1(X_L, \widetilde{\mathbb{R}}) = Hom_{cont}(Pic(\bar{X}), \mathbb{R})$$

is the canonical continuous morphism $\theta : Pic(\bar{X}) \to \mathbb{R}$.

Proof. The canonical map $\pi : \bar{X}_L \to B_{Pic(\bar{X})}$ induces a map

$$\pi^* : H^1(B_{Pic(\bar{X})}, \widetilde{\mathbb{R}}) \longrightarrow H^1(\bar{X}_L, \widetilde{\mathbb{R}}).$$

The direct image of the unique map $\mathcal{T} \to \underline{Sets}$ is exact, hence we have

$$H^1(B_{Pic(\bar{X})}, \widetilde{\mathbb{R}}) = H^0(\mathcal{T}, H^1_{\mathcal{T}}(B_{Pic(\bar{X})}, \widetilde{\mathbb{R}})) = Hom_{Top}(Pic(\bar{X}), \mathbb{R}).$$

Therefore, the usual continuous morphism $\alpha : Pic(\bar{X}) \to \mathbb{R}$ is a distinguished element $\alpha \in H^1(B_{Pic(\bar{X})}, \widetilde{\mathbb{R}})$. We define the fundamental class as

$$\theta := \pi^*(\alpha) \in H^1(\bar{X}_L, \widetilde{\mathbb{R}}).$$

Note that the fundamental class φ can also be defined by

$$\theta := \mathfrak{f}^*(Id_{\mathbb{R}}) \in H^1(\bar{X}_L, \widetilde{\mathbb{R}}),$$

where $Id_{\mathbb{R}}$ is the distinguished non-zero element of $H^1(B_{\mathbb{R}}, \widetilde{\mathbb{R}}) = Hom_{Top}(\mathbb{R}, \mathbb{R})$.

Finally, if the fundamental group $\pi_1(\bar{X}_L, p)$ is representable by a locally compact group, then the map

$$\pi^* : H^1(B_{Pic(\bar{X})}, \widetilde{\mathbb{R}}) \longrightarrow H^1(\bar{X}_L, \widetilde{\mathbb{R}})$$

is an isomorphism, and θ can be identified with α. Indeed, Theorem 4.3 yields in this case that

$$\begin{aligned} H^1(\bar{X}_L, \widetilde{\mathbb{R}}) &= Hom_{cont}(\pi_1(\bar{X}_L, p), \mathbb{R}) \\ &= Hom_{cont}(\pi_1(\bar{X}_L, p)^{ab}, \mathbb{R}) \\ &= Hom_{cont}(Pic(\bar{X}), \mathbb{R}). \qquad \square \end{aligned}$$

4.3. *The fundamental group and unramified class field theory*

There exist complexes $R_W(\varphi_!\mathbb{Z})$ and $R_W(\mathbb{Z})$ of sheaves on the Artin–Verdier étale topos whose hypercohomology is the conjectural Lichtenbaum cohomology with and without compact support respectively (see [**13**]). This suggests the existence of a canonical morphism of topoi

$$\gamma : \bar{X}_L \longrightarrow \bar{X}_{et}$$

such that $R\gamma_*\mathbb{Z} = R_W(\mathbb{Z})$, where $\bar{X}_{et}$ denotes the Artin–Verdier étale topos of X. On the one hand, the complex $R_W(\mathbb{Z})$ yields a canonical map

$$H^n(\bar{X}_{et}, \mathbb{Z}) \longrightarrow H^n_L(\bar{X}, \mathbb{Z})$$

for any $n \geq 0$. In degree $n = 2$, this map

$$Pic(X)^D = (\pi_1(\bar{X}_{et})^{ab})^D = H^2(\bar{X}_{et}, \mathbb{Z}) \longrightarrow H^2_L(\bar{X}, \mathbb{Z}) := Pic^1(\bar{X})^D \tag{13}$$

is the dual map of the canonical morphism $Pic^1(\bar{X}) \to Pic(X) = Cl(F)$. On the other hand, the morphism γ would induce a morphism of abelian fundamental groups

$$\pi_1(\bar{X}_L, p)^{DD} \longrightarrow \pi_1(\bar{X}_{et}, q)^{DD} \simeq \pi_1(\bar{X}_{et})^{ab}, \tag{14}$$

where q is a geometric point of $\bar{X}$ such that the following diagram commutes.

$$\begin{tikzcd} \bar{X}_L \arrow[r, "\gamma"] & \bar{X}_{et} \\ \mathcal{T} \arrow[u, "p"] \arrow[r, "e_{\mathcal{T}}"] & \underline{Sets} \arrow[u, "q"] \end{tikzcd}$$

Note that q is uniquely determined by p since the unique map $e_{\mathcal{T}} : \mathcal{T} \to \underline{Sets}$ has a canonical section s (see [**7**, IV. 4.10]). Indeed, we have $e_{\mathcal{T}} \circ s = Id$ hence

$$q \simeq q \circ e_{\mathcal{T}} \circ s \simeq \gamma \circ p \circ s. \tag{15}$$

The map (14) needs to be compatible with the canonical map (13). In other words, the following morphism should be the *reciprocity map of class field theory*:

$$Pic(\bar{X}) \simeq \pi_1(\bar{X}_L, p)^{DD} \longrightarrow \pi_1(\bar{X}_{et})^{ab}. \tag{16}$$

More precisely, the diagram

$$\begin{tikzcd} Pic(\bar{X}) \arrow[r] \arrow[d] & Pic(X) = Cl(F) \arrow[d] \\ \pi_1(\bar{X}_L, p)^{DD} \arrow[r, "(14)"] & \pi_1(\bar{X}_{et})^{ab} \end{tikzcd}$$

should be commutative, where $Pic(\bar{X}) \to Pic(X) = Cl(F)$ is the canonical map, $Cl(F) \to \pi_1(\bar{X}_{et})^{ab}$ is the isomorphism of unramified class field theory and $Pic(\bar{X}) \to \pi_1(\bar{X}_L, p)^{DD}$ is the isomorphism defined in Theorem 4.3.

4.4. *The fundamental group and the closed embedding i_v*

For any closed point v of $\bar{X}$, i.e. any non-trivial valuation of the number field F, we denote by $W_{k(v)} := F_v^{\times}/\mathcal{O}_{F_v}^{\times}$ the Weil group of the residue field $k(v)$ at v, where $\mathcal{O}_{F_v}^{\times}$ is the kernel of the valuation $F_v^{\times} \to \mathbb{R}^{\times}$.

Let $U \subseteq X$ be an open sub-scheme. The conjectural Lichtenbaum cohomology with compact support is defined as (see the Introduction of [**10**]):

$$H_c^*(U, \mathcal{A}) := H^*(\bar{X}_L, \varphi_! \mathcal{A}),$$

where

$$\varphi : U_L := \bar{X}_L/\gamma^* U \longrightarrow \bar{X}_L$$

is the canonical open embedding. Consider the exact sequence

$$0 \to \varphi_! \varphi^* \mathcal{A} \to \mathcal{A} \to i_* i^* \mathcal{A} \to 0, \tag{17}$$

where $i : \mathrm{F} \to \bar{X}_L$ is the embedding of the closed complement of the open subtopos $\varphi : U_L \to \bar{X}_L$. The morphism i is a closed embedding so that i_* is exact. We obtain

$$H^n(\mathrm{F}, i^* \mathcal{A}) = H^n(\bar{X}_L, i_* i^* \mathcal{A}). \tag{18}$$

Using (17) and (18), we see that the conjectural Lichtenbaum cohomology with and without compact support determines the cohomology of the closed sub-topos F (with coefficients in $\mathbb{Z}$ and $\widetilde{\mathbb{R}}$), and we find

$$H^*(\mathrm{F}, i^* \mathcal{A}) = H^*(\mathrm{F}, \mathcal{A}) = H^*\Big(\coprod_{v \in \bar{X} - U} B_{W_{k(v)}}, \mathcal{A} \Big)$$

for $\mathcal{A} = \mathbb{Z}$ and $\mathcal{A} = \widetilde{\mathbb{R}}$. This suggests the existence of an equivalence

$$\mathrm{F} \simeq \coprod_{v \in \bar{X} - U} B_{W_{k(v)}}. \tag{19}$$

The equivalence (19) is indeed satisfied (see [**12**, Chapter 7]) by the Weil-étale topos in characteristic p (which is the correct Lichtenbaum topos in this case). Moreover, (19) is also predicted by Deninger's program (see [**12**, Chapter 9]). Hence the equivalence (19) should hold. Using [**7**, IV. Corollary 9.4.3], [**13**, Proposition 6.2] and the universal property of sums of topoi, we can prove that (19) is equivalent to the existence of a pull-back diagram of topoi:

$$\begin{tikzcd} B_{W_{k(v)}} \arrow[r] \arrow[d, "i_v"'] & B^{sm}_{G_{k(v)}} \arrow[d, "u_v"] \\ \bar{X}_L \arrow[r, "\gamma"] & \bar{X}_{et} \end{tikzcd}$$

for any v not in U. For an ultrametric place v, the morphism

$$u_v : B^{sm}_{G_{k(v)}} \simeq Spec(k(v))_{et} \longrightarrow \bar{X}_{et}$$

is defined by the scheme map $v \to \bar{X}$ (see [**13**, Proposition 6.2]) and by a geometric point of $\bar{X}$ over v. If v is archimedean, $G_{k(v)} = \{1\}$ and $u_v : \underline{Sets} \to \bar{X}_{et}$ is the point of the étale topos corresponding to $v \in \bar{X}$. In particular, for any closed point v of $\bar{X}$, we have a *closed embedding* of topoi

$$i_v : B_{W_{k(v)}} \longrightarrow \bar{X}_L, \tag{20}$$

where $B_{W_{k(v)}}$ is the classifying topos of $W_{k(v)}$. For any closed point v of $\bar{X}$, the composition

$$B_{W_{k(v)}} \longrightarrow \bar{X}_L \longrightarrow B_{Pic(\bar{X})}$$

should be the morphism of classifying topoi $B_{W_{k(v)}} \to B_{Pic(\bar{X})}$ induced by the canonical morphism of topological groups (see (1))

$$W_{k(v)} \longrightarrow Pic(\bar{X}).$$

Finally, the existence of the morphism (20) is also suggested by the following argument. For an ultrametric place v, $B_{W_{k(v)}}$ is the Lichtenbaum topos of $Spec(k(v))$. Hence the existence of the morphism (20) follows from the fact that the map

$$\bar{X} \rightsquigarrow \bar{X}_L,$$

sending a (regular) arithmetic scheme to the topos of sheaves on the Grothendieck site conjectured in [**10**], should be a pseudo-functor from the category of (regular) arithmetic schemes to the 2-category of topoi.

5. Expected properties of the conjectural Lichtenbaum topos

The conjectural Lichtenbaum cohomology is in fact known for any étale $\bar{X}$-scheme $\bar{U}$, and the arguments of the previous section give the value of the abelian arithmetic fundamental group of $\bar{U}$. More precisely, we should have

$$\pi_1(\bar{U}_L,\, p_{\bar{U}})^{DD} \simeq C_{\bar{U}},$$

where $C_{\bar{U}}$ is the S-idèle class group naturally associated to $\bar{U}$ (see (21) below). Moreover, the study of the complexes $R_W(\mathbb{Z})$ and $R_W(\widetilde{\mathbb{R}})$ defined in [**13**] yields the functorial behavior of these isomorphisms. The relation between the arithmetic fundamental group and the étale fundamental group is given by the natural maps between étale cohomology groups and conjectural Lichtenbaum cohomology groups (see Section 4.3). Finally, the structure of the conjectural Lichtenbaum topos at the closed points is dictated by the conjectural Lichtenbaum cohomology with compact support (see Section 4.4). Putting those facts together, we obtain a (partial) description of the conjectural Lichtenbaum topos. This description is also suggested by our previous study of the Weil-étale topos in characteristic p (see [**12**, Chapter 8]) and by the work of Deninger (see [**12**, Chapter 9]).

5.1. Notation

We refer to [**13**] for the definition of the Artin–Verdier étale site of $\bar{X} = \overline{Spec(\mathcal{O}_F)}$. The Artin–Verdier étale topos $\bar{X}_{et}$ is the category of sheaves of sets on the Artin–Verdier étale site. Let $\bar{U} = (Spec\ \mathcal{O}_{K,S_0},\, U_\infty)$ be a connected étale $\bar{X}$-scheme then we consider the S-idèle class group of K endowed with the quotient topology:

$$C_{\bar{U}} := C_{K,S} = coker\left(\prod_{w\in\bar{U}} \mathcal{O}_{K_w}^{\times} \to C_K\right). \tag{21}$$

Here S is the set of places of K not corresponding to a point of $\bar{U}$, K_w is the completion of K at the place w and $\mathcal{O}_{K_w}^{\times}$ is the kernel of the valuation $K_w^{\times} \to \mathbb{R}^{\times}$. Note that $C_{\bar{U}}$ is a Hausdorff locally compact group canonically associated to $\bar{U}$.

We define the *Weil group* $W_{k(w)}$ of the 'residue field $k(w)$' at any closed point w of $\bar{U}$ as

$$W_{k(w)} := K_w^{\times}/\mathcal{O}_{K_w}^{\times}.$$

For any closed point $w \in \bar{U}$, the map $K_w^{\times} \to C_K$ induces a continuous morphism

$$W_{k(w)} := K_w^{\times}/\mathcal{O}_{K_w}^{\times} \longrightarrow C_{\bar{U}}. \tag{22}$$

Note that we have $W_{k(w)} \simeq \mathbb{Z}$ for w ultrametric and $W_{k(w)} \simeq \mathbb{R}_+^\times$ for w archimedean. We denote by $G_{k(w)} := D_w/I_w$ the Galois group of the residue field $k(w)$, where D_w and I_w are, respectively, the decomposition and the inertia subgroups of G_K at w. Hence $G_{k(w)}$ is the trivial group for w archimedean. There is a canonical morphism

$$W_{k(w)} \longrightarrow G_{k(w)} \tag{23}$$

for any closed point $w \in \bar{U}$. We consider the big classifying topos $B_{W_{k(w)}}$ and the small classifying topos $B^{sm}_{G_{k(w)}}$, i.e. the category of continuous $G_{k(w)}$-sets. In particular, $B^{sm}_{G_{k(w)}}$ is just the final topos $\underline{Sets}$ for w archimedean. The map (23) induces a morphism of toposes:

$$\alpha_v : B_{W_{k(w)}} \longrightarrow B^{sm}_{G_{k(w)}}.$$

We denote by $\mathcal{T}$ the topos of sheaves on the site $(Top, \mathcal{J}_{op})$, where Top is the category of Hausdorff locally compact spaces endowed with the open cover topology. If we need to use constant sheaves represented by non-locally compact spaces, then we can define $\mathcal{T}' := (Top^h, \mathcal{J}_{op})$, where Top^h is the category of Hausdorff spaces, and consider the base change

$$\bar{X}_L \times_{\mathcal{T}} \mathcal{T}'$$

to obtain a connected and locally connected topos over $\mathcal{T}'$.

Finally, if $\underline{\mathcal{G}}$ is a strict pro-group object of $\mathcal{T}$ given by a covariant functor $\underline{\mathcal{G}} : I \to Gr(\mathcal{T})$ where $Gr(\mathcal{T})$ denotes the category of groups in $\mathcal{T}$ and I is a small filtered category. We consider the pro-abelian group object $\underline{\mathcal{G}}^{DD}$ of $\mathcal{T}$ defined as the composite functor

$$(-)^{DD} \circ \underline{\mathcal{G}} : I \longrightarrow Gr(\mathcal{T}) \longrightarrow Ab(\mathcal{T}).$$

Let $t : \mathcal{E} \to \mathcal{T}$ be a connected and locally connected topos over $\mathcal{T}$, i.e. t is a connected and locally connected morphism. In particular, t^* has a left adjoint $t_!$. An object X of $\mathcal{E}$ is said to be *connected over* $\mathcal{T}$ if $t_!X$ is the final object of $\mathcal{T}$. A *$\mathcal{T}$-point of $\mathcal{E}$* is a section $s : \mathcal{T} \to \mathcal{E}$ of the structure map t, i.e. $t \circ s$ is isomorphic to $Id_{\mathcal{T}}$.

5.2. *Expected properties*

(1) *The conjectural Lichtenbaum topos $\bar{X}_L$ should be naturally associated to $\bar{X}$. There should be a canonical connected morphism from $\bar{X}_L$ to the Artin–Verdier étale topos:*

$$\gamma : \bar{X}_L \longrightarrow \bar{X}_{et}.$$

(2) *The conjectural Lichtenbaum topos $\bar{X}_L$ should be defined over $\mathcal{T}$. The structure map*

$$t : \bar{X}_L \longrightarrow \mathcal{T}$$

should be connected and locally connected, and $\bar{X}_L$ should have a $\mathcal{T}$-point p. For any connected étale $\bar{X}$-scheme $\bar{U}$, the object $\gamma^*\bar{U}$ of $\bar{X}_L$ should be connected over $\mathcal{T}$.

It follows that the slice topos

$$\bar{U}_L := \bar{X}_L/\gamma^*\bar{U} \longrightarrow \bar{X}_L \longrightarrow \mathcal{T}$$

is connected and locally connected over $\mathcal{T}$, for any connected étale $\bar{X}$-scheme $\bar{U}$, and has a $\mathcal{T}$-point

$$p_{\bar{U}} : \mathcal{T} \longrightarrow \bar{U}_L.$$

Then the fundamental group $\pi_1(\bar{U}_L, p_{\bar{U}})$ is well defined as a prodiscrete localic group in $\mathcal{T}$. Moreover, $\pi_1(\bar{U}_L, p_{\bar{U}})$ *should be pro-representable by a locally compact strict pro-group*, and we consider this fundamental group as a locally compact pro-group. By Corollary 3.3, we have

$$\pi_1(\bar{U}_L, p_{\bar{U}})^{DD} = \pi_1(\bar{U}_L, p_{\bar{U}})^{ab} = \pi_1(\bar{U}_L)^{ab}.$$

We have a canonical connected morphism

$$\bar{U}_L := \bar{X}_L/\gamma^*\bar{U} \longrightarrow \bar{X}_{et}/\bar{U} = \bar{U}_{et}$$

inducing a morphism

$$\varphi_{\bar{U}} : \pi_1(\bar{U}_L, p_{\bar{U}}) \longrightarrow \pi_1(\bar{U}_{et}, q_{\bar{U}}),$$

where $q_{\bar{U}}$ is defined by $p_{\bar{U}}$ as in (15). We obtain a morphism

$$\varphi_{\bar{U}}^{DD} : \pi_1(\bar{U}_L)^{ab} = \pi_1(\bar{U}_L, p_{\bar{U}})^{DD} \longrightarrow \pi_1(\bar{U}_{et}, p_{\bar{U}})^{DD} = \pi_1(\bar{U}_{et})^{ab}.$$

(3) *We should have a canonical isomorphism*

$$r_{\bar{U}} : C_{\bar{U}} \simeq \pi_1(\bar{U}_L)^{ab}$$

such that the composition

$$\varphi_{\bar{U}}^{DD} \circ r_{\bar{U}} : \ C_{\bar{U}} \simeq \pi_1(\bar{U}_L)^{ab} \longrightarrow \pi_1(\bar{U}_{et})^{ab}$$

is the reciprocity law of class field theory. This reciprocity morphism is defined by the topological class formation

$$(\pi_1(\bar{U}_{et}, q_{\bar{U}}), \varinjlim C_{\bar{V}}),$$

where $\bar{V}$ runs over the filtered system of pointed étale cover of $(\bar{U}, q_{\bar{U}})$ (see [**15**, Proposition 8.3.8] and [**15**, Theorem 8.3.12]).

(4) *The isomorphism $r_{\bar{U}}$ should be covariantly functorial for any map $f : \bar{V} \to \bar{U}$ of connected étale $\bar{X}$-schemes.* More precisely, such a map induces a morphism of toposes:

$$f_L : \bar{V}_L := \bar{X}_L/\bar{V} \longrightarrow \bar{U}_L := \bar{X}_L/\bar{U}$$

hence a morphism of abelian pro-groups in $\mathcal{T}$,

$$\widetilde{f}_L : \pi_1(\bar{V}_L)^{ab} \longrightarrow \pi_1(\bar{U}_L)^{ab}.$$

Then the following diagram should be commutative:

$$\begin{array}{ccc} \pi_1(\bar{V}_L)^{ab} & \xrightarrow{r_{\bar{V}}} & C_{\bar{V}} \\ {\scriptstyle \widetilde{f}_L}\downarrow & & \downarrow{\scriptstyle N} \\ \pi_1(\bar{U}_L)^{ab} & \xrightarrow{r_{\bar{U}}} & C_{\bar{U}} \end{array}$$

where N is induced by the norm map.

(5) *For any Galois étale cover $\bar{V} \to \bar{U}$ (of étale $\bar{X}$-schemes), the conjugation action on $\pi_1(\bar{V}_L)^{ab}$ should correspond to the Galois action on $C_{\bar{V}}$.* In other words, the following diagram should be commutative:

$$\begin{array}{ccc} \pi_1(\bar{U}_L, p_{\bar{U}}) \times \pi_1(\bar{V}_L)^{ab} & \xrightarrow{(\varphi_{\bar{U}}, r_{\bar{V}})} & \pi_1(\bar{U}_{et}, q_{\bar{U}}) \times C_{\bar{V}} \\ \downarrow & & \downarrow \\ \pi_1(\bar{V}_L)^{ab} & \xrightarrow{r_{\bar{V}}} & C_{\bar{V}} \end{array}$$

where the vertical arrows are the conjugation action of $\pi_1(\bar{U}_L, p_{\bar{U}})$ on $\pi_1(\bar{V}_L)^{ab}$ and the natural action of $\pi_1(\bar{U}_{et}, p_{\bar{U}})$ on $C_{\bar{V}}$.

(6) *The isomorphism $r_{\bar{U}}$ should be contravariantly functorial for an étale cover.* More precisely, let $\bar{V} \to \bar{U}$ be a *finite* étale map. Then the following diagram should be commutative:

$$\begin{array}{ccc} \pi_1(\bar{V}_L)^{ab} & \xrightarrow{r_{\bar{V}}} & C_{\bar{V}} \\ \uparrow \mathrm{tr} & & \uparrow \\ \pi_1(\bar{U}_L)^{ab} & \xrightarrow{r_{\bar{U}}} & C_{\bar{U}} \end{array}$$

where the map $C_{\bar{U}} \to C_{\bar{V}}$ is the inclusion, and tr is the transfer map defined in Proposition 5.6 below.

(7) *For any closed point v of $\bar{X}$, we should have pull-back of topoi.*

$$\begin{array}{ccc} B_{W_{k(v)}} & \xrightarrow{\alpha_v} & B^{sm}_{G_{k(v)}} \\ i_v \downarrow & & \downarrow u_v \\ \bar{X}_L & \xrightarrow{\gamma} & \bar{X}_{et} \end{array}$$

Here the morphism

$$u_v : B^{sm}_{G_{k(v)}} \simeq Spec(k(v))_{et} \longrightarrow \bar{X}_{et}$$

is defined by a geometric point of $\bar{X}$ over v and by the scheme map $v \to \bar{X}$. The map α_v is induced by the canonical morphism $W_{k(v)} \to G_{k(v)}$. It follows that the morphism i_v is a *closed embedding*.

On the one hand, the pull-back above induces a closed embedding

$$i_w : B_{W_{k(w)}} \longrightarrow \bar{U}_L$$

for any $\bar{U}$ étale over $\bar{X}$ and any closed point w of $\bar{U}$. On the other hand we have a canonical morphism

$$\bar{U}_L \to B_{\pi_1(\bar{U}_L, p_{\bar{U}})} \to B_{\pi_1(\bar{U}_L)^{ab}} \simeq B_{C_{\bar{U}}}.$$

(8) *For any closed point w of a connected étale $\bar{X}$-scheme $\bar{U}$, the composition*

$$B_{W_{k(w)}} \longrightarrow \bar{U}_L \longrightarrow B_{C_{\bar{U}}}$$

should be the morphism of classifying topoi induced by the canonical morphism of topological groups $W_{k(w)} \to C_{\bar{U}}$.

Define the sheaf of continuous real valued functions on $\bar{X}_L$ as $\widetilde{\mathbb{R}} := t^* y\mathbb{R}$, where $y\mathbb{R}$ is the sheaf of $\mathcal{T}$ represented by the standard topological group $\mathbb{R}$.

(9) *For any étale $\bar{X}$-scheme $\bar{U}$, we should have $H^n(\bar{U}_L, \widetilde{\mathbb{R}}) = 0$ for any $n \geq 2$.*

The following result shows that the properties listed above are consistent. A proof is given in [**14**].

THEOREM 5.1. *There exists a topos satisfying Properties (1)–(9) listed above.*

Note that the isomorphism $r_{\bar{U}} : C_{\bar{U}} \simeq \pi_1(\bar{U}_L)^{ab}$ can be understood in two different ways. On the one hand, we can consider $\pi_1(\bar{U}_L)^{ab}$ as a usual topological group defined as the projective limit of the topological pro-group $\pi_1(\bar{U}_L)^{DD}$. Then $r_{\bar{U}}$ is just an isomorphism of topological groups. On the other hand, we can consider $\pi_1(\bar{U}_L)^{ab}$ and $C_{\bar{U}}$ as topological pro-groups (see Section 5.3.1 below) and assume that $r_{\bar{U}}$ is an isomorphism of topological pro-groups. The second point of view is stronger than the first.

5.3. *Explanations*

In this section, we define the morphisms used in the previous description. First of all, the fundamental group $\pi_1(\bar{U}_L, p_{\bar{U}})$ is assumed to be pro-representable by a locally compact strict pro-group. In other words, we assume that there exist a locally compact strict pro-group $\underline{G}$ indexed over a small filtered category (in the usual sense, see Definition 2.1) and an equivalence $SLC_{\mathcal{T}}(\bar{U}_L) \simeq B_{\underline{G}}$ compatible with the point $p_{\bar{U}}$, where $SLC_{\mathcal{T}}(\bar{U}_L)$ and $B_{\underline{G}}$ are defined as in [**2**, Section 2] and as in Definition 2.2, respectively.

The fact that the fundamental groups $\pi_1(\bar{U}_L, p_{\bar{U}})$ and $\pi_1(\bar{U}_{et}, q_{\bar{U}})$ should be defined as topological pro-groups and the previous description of the Lichtenbaum topos suggests that the groups $C_{\bar{U}}$ are in fact topological pro-groups and that all the maps between these topological pro-groups are compatible with this additional structure. We show below that it is indeed the case. This detail can be ignored if we consider the limit of those topological pro-groups computed in the category of topological groups, and the morphisms between these pro-groups as usual continuous morphisms.

5.3.1. *The S-idèle class group as a pro-group.*

Let $\bar{U} = (Spec\, \mathcal{O}_{K,S_0}, U_\infty)$ be a connected étale $\bar{X}$-scheme. We denote by S_∞ the set of archimedean places of K not corresponding to a point of U_∞, i.e. $U_\infty \coprod S_\infty$ is the set of archimedean places of K. If we set $S = (S_0 \cup S_\infty)$ then we have

$$\bar{U} = \overline{Spec\, \mathcal{O}_K} - S$$

and $C_{\bar{U}} = C_{K,S}$ is the S-idèle class group of K. Assume for simplicity that $S_\infty \neq \emptyset$. Then there is an exact sequence of topological groups

$$0 \to \prod_{w \in S_0} \mathcal{O}_{K_w}^\times \to C_{K,S} \to C_{K,S_\infty} \to 0, \tag{24}$$

where C_{K,S_∞} is the following extension of the finite group $Cl(K)$:

$$0 \to \left(\prod_{w \in S_\infty} K_w^\times \prod_{w \in U_\infty} \mathbb{R}_+^\times \right) \Big/ \mathcal{O}_K^\times \to C_{K,S_\infty} \to Cl(K) \to 0.$$

Note that C_{K,S_∞} has a finite filtration such that the quotients of the form Fil^n/Fil^{n+1} are either finite or connected. Recall that, for w ultrametric, $\mathcal{O}_{K_w}^\times$ is given with the filtration

$$\mathcal{O}_{K_w}^\times = U_w^0 \supseteq U_w^1 \supseteq U_w^2 \supseteq \cdots$$

so that $\mathcal{O}_{K_w}^\times$ is the profinite group

$$\mathcal{O}_{K_w}^\times = \varprojlim U_w^0/U_w^n.$$

Hence the exact sequence (24) provides $C_{\bar{U}}$ with a structure of a topological pro-group. More precisely, we have

$$C_{K,S} = \varprojlim C_{K,S}/\Omega,$$

where Ω runs over the system of open subgroups of $\prod_{w\in S_0} \mathcal{O}_{K_w}^\times$.

Definition 5.2. We define $C_{\bar{U}}$ as the topological pro-group

$$C_{\bar{U}} := \{C_{K,S}/\Omega,\ \text{ for } \Omega \text{ open in } \textstyle\prod_{w\in S_0} \mathcal{O}_{K_w}^\times\}.$$

The pro-group $C_{\bar{U}}$ can also be seen as the locally compact group $C_{K,S}$ endowed with the filtration

$$C_{K,S} \supseteq \prod_{w\in S_0} \mathcal{O}_{K_w}^\times \supseteq \prod_{w\in S_0} U_w^1 \supseteq \prod_{w\in S_0} U_w^2 \supseteq \cdots . \tag{25}$$

Indeed the sequence $\{\Omega^n := \prod_{w\in S_0} U_w^n$, for $n \geq 0\}$ is cofinal in the system of open $\Omega \subseteq \prod_{w\in S_0} \mathcal{O}_{K_w}^\times$. Hence the pro-group $C_{\bar{U}}$ can be defined as follows:

$$C_{\bar{U}} := \{C_{K,S}/\Omega^n \text{ for } n \geq 0\}.$$

PROPOSITION 5.3. *For any map $\bar{V} \to \bar{U}$ of connected étale $\bar{X}$-schemes, the map $N : C_{\bar{V}} \to C_{\bar{U}}$, induced by the usual norm map, is compatible with the pro-group structures of $C_{\bar{V}}$ and $C_{\bar{U}}$.*

For any Galois étale cover $\bar{V} \to \bar{U}$, the usual Galois action of $Gal(\bar{V}/\bar{U})$ on $C_{\bar{V}}$ is compatible with the pro-group structure of $C_{\bar{V}}$.

For any finite étale map $\bar{V} \to \bar{U}$, the natural morphism $C_{\bar{U}} \to C_{\bar{V}}$ is compatible with the pro-group structures of $C_{\bar{V}}$ and $C_{\bar{U}}$.

For any connected étale $\bar{X}$-schemes $\bar{U}$, the reciprocity morphism

$$r_{\bar{U}} : C_{\bar{U}} \longrightarrow \pi_1(\bar{U}_{et})^{ab}$$

is a morphism of topological pro-groups.

Proof. Concerning the first three statements, we just have to remark that those morphisms are all compatible with the filtration (25), which is clear. The reciprocity morphism $r_{\bar{U}}$ is defined by the topological class formation $(\pi_1(\bar{U}_{et}, q_{\bar{U}}), \varinjlim C_{\bar{V}})$, where $\bar{V}$ runs over the filtered system of pointed étale covers of $(\bar{U}, q_{\bar{U}})$ (see [**15**, Proposition 8.3.8 and Theorem 8.3.12]). Recall that the group U_v^n is mapped, by class field theory, onto the nth-ramification subgroup

$$(G_v^n)^{ab} \subset G_{K_v}^{ab} \subset G_{K,S}^{ab} = \pi_1(\bar{U}_{et}, q_{\bar{U}})^{ab}.$$

Hence $r_{\bar{U}}$ is a morphism of topological pro-groups. $\square$

5.3.2. The morphism $\varphi_{\bar{U}}$ has dense image. By Property (1), the map $\gamma : \bar{X}_L \to \bar{X}_{et}$ is connected, i.e. γ^* is fully faithful. It follows immediately that the morphism

$$\gamma_{\bar{U}} : \bar{U}_L := \bar{X}_L/\gamma^*\bar{U} \longrightarrow \bar{X}_{et}/U = \bar{U}_{et}$$

is connected as well. Chose a $\mathcal{T}$-point $p_{\bar{U}}$ of $\bar{U}_L$ and let $q_{\bar{U}}$ be the geometric point of $\bar{U}$ defined by $p_{\bar{U}}$ as in Section 4.3. We have a commutative square

$$\begin{tikzcd}
\bar{U}_L \arrow[r, "\gamma_{\bar{U}}"] \arrow[d] & \bar{U}_{et} \arrow[d] \\
B_{\pi_1(\bar{U}_L, p_{\bar{U}})} \arrow[r, "B_{\varphi_{\bar{U}}}"] & B^{sm}_{\pi_1(\bar{U}_{et}, q_{\bar{U}})}
\end{tikzcd}$$

where the vertical maps are both connected. Indeed, the inverse image of the morphism $\bar{U}_L \to B_{\pi_1(\bar{U}_L, p_{\bar{U}})}$ (respectively of the morphism $\bar{U}_{et} \to B^{sm}_{\pi_1(\bar{U}_{et}, q_{\bar{U}})}$) is the inclusion of the full subcategory of sums of locally constant objects $SLC_{\mathcal{T}}(\bar{U}_L) \hookrightarrow \bar{U}_L$ (respectively $SLC(\bar{U}_{et}) \hookrightarrow \bar{U}_{et}$). Hence the previous diagram shows that

$$B_{\varphi_{\bar{U}}} : B_{\pi_1(\bar{U}_L, p_{\bar{U}})} \longrightarrow B^{sm}_{\pi_1(\bar{U}_{et}, q_{\bar{U}})}$$

is connected as well. This morphism is induced by the morphism of strict topological pro-groups:

$$\varphi_{\bar{U}} : \pi_1(\bar{U}_L, p_{\bar{U}}) \longrightarrow \pi_1(\bar{U}_{et}, q_{\bar{U}}).$$

Consider $\pi_1(\bar{U}_L, p_{\bar{U}})$ as a projective system of locally compact groups $(W_\alpha)_{\alpha\in A}$ and $\pi_1(\bar{U}_{et}, q_{\bar{U}})$ as a projective system of finite groups $(G_\beta)_{\beta\in B}$. Then $\varphi_{\bar{U}}$ is given by a family, indexed over B, of compatible morphisms $W_\alpha \to G_\beta$. More precisely, we have

$$\varphi_{\bar{U}} \in Hom((W_\alpha)_{\alpha\in A}, (G_\beta)_{\beta\in B}) := \varprojlim_{\beta\in B} \varinjlim_{\alpha\in A} Hom_c(W_\alpha, G_\beta).$$

Definition 5.4. We say that $\varphi_{\bar{U}}$ *has dense image* if all those maps $W_\alpha \to G_\beta$ are surjective.

The fact that the morphism $B_{\varphi_{\bar{U}}}$ is connected implies that $\varphi_{\bar{U}}$ has dense image in that sense. Indeed, assume that one of the maps $W_\alpha \to G_\beta$ is not surjective. Then the functor $\varphi^* : B^{sm}_{G_\beta} \to B_{W_\alpha}$, sending a G_β-set E to the (sheaf represented by the) discrete W_α-space E on which W_α acts via $W_\alpha \to G_\beta$, is not fully faithful. But we have the commutative diagram of categories

$$\begin{tikzcd}
B_{\pi_1(\bar{U}_L, p_{\bar{U}})} & B^{sm}_{\pi_1(\bar{U}_{et}, q_{\bar{U}})} \arrow[l, "B^*_{\varphi_{\bar{U}}}"'] \\
B_{W_\alpha} \arrow[u] & B^{sm}_{G_\beta} \arrow[l, "\varphi^*"'] \arrow[u]
\end{tikzcd}$$

where the vertical arrows are fully faithful functors. Hence the fact that φ^* is not fully faithful implies that $B^*_{\varphi_{\bar{U}}}$ is not fully faithful. We have obtained the following result.

PROPOSITION 5.5. *Let $\bar{X}_L$ be a topos satisfying Properties (1)–(9). Then for any connected étale $\bar{X}$-scheme $\bar{U}$ the morphism of topological pro-groups $\varphi_{\bar{U}}$ has dense image.*

Let $\bar{V} \to \bar{U}$ be a finite Galois étale cover of étale $\bar{X}$-schemes with $Gal(\bar{V}/\bar{U}) = G$, and consider the injective morphism of the topological pro-group

$$\pi_1(\bar{V}_L, p_{\bar{V}}) \hookrightarrow \pi_1(\bar{U}_L, p_{\bar{U}}).$$

In other words, if we see the fundamental groups of $\bar{V}_L$ and of $\bar{U}_L$ as projective systems of topological groups $(W'_\alpha)_{\alpha\in A}$ and $(W_\alpha)_{\alpha\in A}$ (indexed over the same filtered category A), the previous map is given by a family of compatible injective morphisms of topological groups $W'_\alpha \to W_\alpha$. We can consider the quotient pro-object of $\mathcal{T}$:

$$\pi_1(\bar{U}_L, p_{\bar{U}})/\pi_1(\bar{V}_L, p_{\bar{V}}) := (yW_\alpha/yW'_\alpha)_{\alpha\in A}.$$

Then this projective system is in fact an essentially constant pro-group and we have an isomorphism in $\mathcal{T}$:

$$\pi_1(\bar{U}_L, p_{\bar{U}})/\pi_1(\bar{V}_L, p_{\bar{V}}) \simeq y(G).$$

More generally, for any finite étale map $\bar{V} \to \bar{U}$ of étale $\bar{X}$-schemes the pro-object of $\mathcal{T}$,

$$\pi_1(\bar{U}_L, p_{\bar{U}})/\pi_1(\bar{V}_L, p_{\bar{V}}),$$

is essentially constant, endowed with an action of the pro-group object $\pi_1(\bar{U}_L, p_{\bar{U}})$, and we have an isomorphism of finite $\pi_1(\bar{U}_L, p_{\bar{U}})$-sets:

$$\pi_1(\bar{U}_L, p_{\bar{U}})/\pi_1(\bar{V}_L, p_{\bar{V}}) \simeq \pi_1(\bar{U}_{et}, q_{\bar{U}})/\pi_1(\bar{V}_{et}, q_{\bar{V}}).$$

Therefore, for any finite étale map $\bar{V} \to \bar{U}$, the induced morphism

$$\pi_1(\bar{V}_L, p_{\bar{V}}) \longrightarrow \pi_1(\bar{U}_L, p_{\bar{U}})$$

is given by a *compatible family of closed topological subgroups of finite index* $W'_\alpha \hookrightarrow W_\alpha$. Moreover, we can choose an index $\alpha_0 \in A$ such that for any map $\alpha \to \alpha_0$ in A, the map $W_\alpha / W'_\alpha \to W_{\alpha_0}/W'_{\alpha_0}$ is a bijective map of finite sets. It follows that the usual transfer maps

$$\mathrm{tr}_\alpha : W_\alpha^{ab} \longrightarrow W_\alpha^{\prime ab}$$

are well defined and that they make the following square commutative.

$$\begin{array}{ccc} W_\alpha^{\prime ab} & \xleftarrow{\mathrm{tr}_\alpha} & W_\alpha^{ab} \\ \downarrow & & \downarrow \\ W_{\alpha_0}^{\prime ab} & \xleftarrow{\mathrm{tr}_{\alpha_0}} & W_{\alpha_0}^{ab} \end{array}$$

We obtain a morphism of locally compact topological pro-groups

$$\mathrm{tr} : \pi_1(\bar{U}_L, p_{\bar{U}})^{ab} \longrightarrow \pi_1(\bar{V}_L, p_{\bar{V}})^{ab}.$$

If $\bar{V} \to \bar{U}$ is a Galois étale cover, then W'_α is normal in W_α for any $\alpha \in A$, hence W_α acts on $W_\alpha^{\prime ab}$ by conjugation. This action is certainly functorial in α hence $\pi_1(\bar{U}_L, p_{\bar{U}})$ acts on $\pi_1(\bar{V}_L, p_{\bar{V}})^{ab}$ by conjugation. More precisely, we consider the topological pro-group

$$\begin{array}{rccc} \pi_1(\bar{U}_L, p_{\bar{U}}) \times \pi_1(\bar{V}_L, p_{\bar{V}})^{ab} : & A & \longrightarrow & Gr(\mathcal{T}) \\ & \alpha & \longmapsto & W_\alpha \times W_\alpha^{\prime ab}. \end{array}$$

Then we have a morphism of topological pro-groups:

$$\pi_1(\bar{U}_L, p_{\bar{U}}) \times \pi_1(\bar{V}_L, p_{\bar{V}})^{ab} \longrightarrow \pi_1(\bar{V}_L, p_{\bar{V}})^{ab}.$$

We have shown the following result.

PROPOSITION 5.6. *Let $\bar{V} \to \bar{U}$ be a finite étale map of étale $\bar{X}$-schemes. We have a morphism of abelian topological pro-groups*

$$\mathrm{tr} : \pi_1(\bar{U}_L, p_{\bar{U}})^{ab} \longrightarrow \pi_1(\bar{V}_L, p_{\bar{V}})^{ab}.$$

If $\bar{V} \to \bar{U}$ is Galois, then $\pi_1(\bar{U}_L, p_{\bar{U}})$ acts on $\pi_1(\bar{V}_L, p_{\bar{V}})^{ab}$ by conjugation:

$$\pi_1(\bar{U}_L, p_{\bar{U}}) \times \pi_1(\bar{V}_L, p_{\bar{V}})^{ab} \longrightarrow \pi_1(\bar{V}_L, p_{\bar{V}})^{ab}.$$

5.4. *The Weil-étale topos in characteristic p*

Let Y be a smooth projective curve over a finite field k. Assume that Y is geometrically connected. The Weil-étale topos Y_W is defined as the category of W_k-equivariant étale sheaves on the geometric curve $Y \times_k \bar{k}$. Then we can prove that Y_W satisfies all the properties (1)–(9) above (replacing $\mathcal{T}$ with <u>Sets</u>). Moreover, Y_W is universal for these properties, i.e. it is the smallest topos satisfying those properties. In other words, if $\mathcal{S}$ is a topos satisfying properties (1)–(9), then there exists an essentially unique morphism $\mathcal{S} \to Y_W$ compatible with this structure (i.e. making all the diagrams commutative).

We give below a sketch of the proof of these facts. By [**12**, Theorem 8.5] we have a canonical equivalence

$$Y_W \simeq Y_{et} \times_{B_{G_k}} B_{W_k}, \tag{26}$$

where Y_{et} denotes the étale topos of Y, i.e. the category of sheaves of sets on the étale site of Y. Consider the first projection

$$\gamma : Y_W \simeq Y_{et} \times_{B_{G_k}} B_{W_k} \longrightarrow Y_{et}.$$

For any étale Y-scheme U, we thus have

$$Y_W/\gamma^* yU \simeq (Y_{et}/yU) \times_{B_{G_k}} B_{W_k} \simeq U_{et} \times_{B_{G_k}} B_{W_k} \simeq U_W. \tag{27}$$

If U is connected étale over Y, then U_{et} and U_W are both connected and locally connected over <u>Sets</u>. Any geometric point p_U of U yields a <u>Sets</u>-valued point of U_{et} and of U_W, and we have an isomorphism of pro-discrete groups

$$\pi_1(U_W, p) \simeq \pi_1(U_{et}, p) \times_{G_k} W_k. \tag{28}$$

The group $\pi_1(U_{et}, p) \times_{G_k} W_k$ is often called the Weil group of U. For any closed point y of Y we have a closed embedding

$$B_{G_{k(y)}} \simeq Spec(k(y))_{et} \longrightarrow Y_{et}.$$

The inverse image of this closed subtopos under γ is given by the fiber product

$$Y_W \times_{Y_{et}} B_{G_{k(y)}} \simeq B_{W_k} \times_{B_{G_k}} Y_{et} \times_{Y_{et}} B_{G_{k(y)}} \simeq B_{W_k} \times_{B_{G_k}} B_{G_{k(y)}} \simeq B_{W_{k(y)}}. \tag{29}$$

Then properties (1)–(9) and the fact that Y_W is universal for those properties follow from (26)–(29) and class field theory for function fields. Note that Weil's interpretation of class field theory for function fields can be restated as follows: the reciprocity morphism gives an isomorphism between the S-idèle class group and the abelian Weil-étale fundamental group. Finally, note that the canonical morphism

$$Y_W \longrightarrow B_{W_k}$$

gives rise to the Frobenius-equivariant l-adic cohomology (see [**12**, Chapter 8, Section 4.3]).

5.5. *The Lichtenbaum topos and Deninger's dynamical system*

Property (3) of section 5.2 yields a canonical morphism flow

$$\mathfrak{f} : \bar{X}_L \longrightarrow B_{Pic(\bar{X})} \longrightarrow B_{\mathbb{R}}.$$

A topos is Grothendieck's generalization of a space, hence $\bar{X}_L$ can be seen as a generalized space. Then the morphism of topoi $\mathfrak{f}$ can be interpreted as follows. The topos $\bar{X}_L$ is a generalized space endowed with an action of the topological group $\mathbb{R}$.

Properties (7) and (8) above give a closed embedding $i_v : B_{W_{k(v)}} \to \bar{X}_L$ such that the composition

$$i_v : B_{W_{k(v)}} \longrightarrow \bar{X}_L \longrightarrow B_{\mathbb{R}}$$

is the morphism induced by the canonical morphism $l_v : W_{k(v)} \to \mathbb{R}$. For an ultrametric place v of F, the morphism l_v sends the canonical generator of $W_{k(v)}$ to $-\log N(v)$. The closed embedding i_v should be thought of as a *closed orbit of the flow of length* $\log N(v)$. For an archimedean place v, the composite morphism $\mathfrak{f} \circ i_v : B_{W_{k(v)}} \to B_{\mathbb{R}}$ is an isomorphism of topoi, and i_v should be thought of as a *(closed) inclusion of a fixed point of the flow*. Thus the morphism $\mathfrak{f}$ encodes all the numbers $\log N(v)$. This suggests that the morphism $\mathfrak{f}$, or more precisely the derived functor $R\mathfrak{f}_*$, could yield a geometric cohomology theory (i.e. a cohomology allowing a cohomological interpretation of the zeta function itself). In other words, we can dream that the conjectural Lichtenbaum topos of $\bar{X}$ (if it exists) will play the role of Deninger's dynamical system (see [**3**], for example). In any case, the correct conjectural Lichtenbaum topos is far from being constructed. We refer to [**12**, Chapter 9] for some details concerning the analogy between the conjectural Lichtenbaum topos and Deninger's dynamical system.

5.6. *The base topos $B_{\mathbb{R}}$ and the field with one element $\mathbb{F}_1$*

Let Y be a smooth scheme of finite type over $\mathbb{F}_q$. Assume for simplicity that Y is geometrically connected. The Weil-étale topos Y_W is given with a canonical morphism

$$f_Y : Y_W \longrightarrow B^{sm}_{W_{\mathbb{F}_q}}$$

over the small classifying topos $B^{sm}_{W_{\mathbb{F}_q}}$. The Weil-étale topos of Y is thought of as a space endowed with an action of the group $W_{\mathbb{F}_q}$. Indeed, Y_W is the étale topos associated to the Frobenius-equivariant geometric scheme $\overline{Y} := Y \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$. The étale topos $\overline{Y}_{et}$ of the geometric scheme $\overline{Y}$ is obtained as the localization

$$\overline{Y}_{et} = Y_W \times_{B^{sm}_{W_{\mathbb{F}_q}}} \underline{Set} = Y_W/f_Y^* EW_{\mathbb{F}_q},$$

where $\underline{Set} \to B^{sm}_{W_{\mathbb{F}_q}}$ is the canonical point of $B^{sm}_{W_{\mathbb{F}_q}}$. We denote the *geometric topos* by

$$Y_{geo} := Y_W/f_Y^* EW_{\mathbb{F}_q}.$$

We have an exact sequence of fundamental groups

$$1 \to \pi_1(Y_{geo}, p) \to \pi_1(Y_W, p) \to W_{\mathbb{F}_q} \to 1. \tag{30}$$

Over $Spec(\mathbb{Z})$, the role of $B_{W_{\mathbb{F}_q}}$ is played by $B_{\mathbb{R}^\times_+} = B_{\mathbb{R}}$. Let $\bar{X}$ be the compactification of $Spec(\mathcal{O}_F)$. We have a canonical morphism

$$\mathfrak{f} : \bar{X}_L \longrightarrow B_{Pic(\bar{X})} \longrightarrow B_{\mathbb{R}^\times_+}.$$

We can imagine that the base topos $B_{\mathbb{R}^\times_+}$ is the classifying topos of the Weil group $W_{\mathbb{F}_1} = \mathbb{R}^\times_+$ of some arithmetic object $\mathbb{F}_1$. Then the localized topos

$$\bar{X}_{\mathrm{geo}} := \bar{X}_L \times_{B_{W_{\mathbb{F}_1}}} \mathcal{T} = \bar{X}_L/\mathfrak{f}^* EW_{\mathbb{F}_1},$$

where $\mathcal{T} \to B_{\mathbb{R}}$ is the canonical point of $B_{\mathbb{R}}$, would play the role of the geometric étale topos $Y_{geo} := \overline{Y}_{et}$. Intuitively, $\bar{X}_{geo}$ corresponds to Deninger's space without the $\mathbb{R}$-action. We have an exact sequence of fundamental groups

$$1 \to \pi_1(\bar{X}_{geo}, p) \to \pi_1(\bar{X}_L, p) \to W_{\mathbb{F}_1} \to 1.$$

This exact sequence is analogous to (30).

6. Cohomology

In this section we consider the curve $\bar{X} = \overline{Spec(\mathcal{O}_F)}$, where the number field F is totally imaginary. Let $\gamma : \bar{X}_L \to \bar{X}_{et}$ be any topos satisfying Properties (1)–(9) given in Section 5.2. We show that these properties yield a natural proof of the fact that the complex of étale sheaves $\tau_{\leq 2} R\gamma_*(\varphi_! \mathbb{Z})$ produces the special value of $\zeta_F(s)$ at $s = 0$ up to sign.

6.1. The base change from the Weil-étale cohomology to the étale cohomology

Recall that we denote by $C_{\bar{U}} = C_{K,S}$ the S-idèle class group canonically associated to $\bar{U}$. We consider the sheaves on $\bar{U}_L$ defined by $\widetilde{\mathbb{R}} := t^*_{\bar{U}}(y\mathbb{R})$ and $\widetilde{\mathbb{S}}^1 := t^*_{\bar{U}}(y\mathbb{S}^1)$, where $y\mathbb{S}^1$ and $y\mathbb{R}$ are the sheaves on $\mathcal{T}$ represented by the topological groups $\mathbb{S}^1$ and $\mathbb{R}$, and $t_{\bar{U}} : \bar{U}_L \to \mathcal{T}$ is the canonical map (defined for Property (2)).

PROPOSITION 6.1. *For any connected étale $\bar{X}$-scheme $\bar{U}$, we have*

$$H^n(\bar{U}_L, \widetilde{\mathbb{R}}) = \begin{cases} \mathbb{R} & \text{for } n = 0 \\ Hom_c(C_{\bar{U}}, \mathbb{R}) & \text{for } n = 1 \\ 0 & \text{for } n \geq 2. \end{cases}$$

Proof. The result for $n=0$ follows from the connectedness of $\bar{U}_L \to \mathcal{T}$ given by Property (2). Indeed, we have

$$H^0(\bar{U}_W, t^*\widetilde{\mathbb{R}}) := (e_{\mathcal{T}*} \circ t_*)\, t^*\widetilde{\mathbb{R}} = e_{\mathcal{T}*}\widetilde{\mathbb{R}} = \mathbb{R},$$

where $e_{\mathcal{T}}$ denotes the unique map $e_{\mathcal{T}} : \mathcal{T} \to \underline{Sets}$. By Property (3), the result for $n=1$ follows from

$$\begin{aligned} H^1(\bar{U}_L, \widetilde{\mathbb{R}}) &= Hom_c(\pi_1(\bar{U}_L), \mathbb{R}) \\ &:= \varinjlim Hom_c(\pi_1(\bar{U}_L), \mathbb{R}) \\ &= \varinjlim Hom_c(\pi_1(\bar{U}_L)^{ab}, \mathbb{R}) \\ &= Hom_c(C_{\bar{U}}, \mathbb{R}). \end{aligned}$$

The result for $n \geq 2$ is given by Property (9). □

The maximal compact subgroup of $C_{\bar{U}}$, i.e. the kernel of the absolute value map $C_{\bar{U}} \to \mathbb{R}_{>0}$, is denoted by $C^1_{\bar{U}}$. Thus we have an exact sequence of topological groups

$$1 \to C^1_{\bar{U}} \to C_{\bar{U}} \to \mathbb{R}_{>0} \to 1.$$

The Pontraygin dual $(C^1_{\bar{U}})^D$ is discrete since $C^1_{\bar{U}}$ is compact.

PROPOSITION 6.2. *For any connected étale $\bar{X}$-scheme $\bar{U}$, we have canonically*

$$H^n(\bar{U}_L, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \textit{for } n=0 \\ 0 & \textit{for } n=1 \\ (C^1_{\bar{U}})^D & \textit{for } n=2. \end{cases}$$

Proof. As above, the result for $n=0$ follows from the connectedness of $\bar{U}_L \to \mathcal{T}$ given by Property (2). By Property (3), the result for $n=1$ follows from

$$H^1(\bar{U}_L, \mathbb{Z}) = Hom_c(\pi_1(\bar{U}_L)^{ab}, \mathbb{Z}) = 0.$$

By Property (3) we have canonical isomorphisms

$$\begin{aligned} H^1(\bar{U}_L, \widetilde{\mathbb{S}}^1) &= Hom_c(\pi_1(\bar{U}_L)^{ab}, \mathbb{S}^1) \\ &:= \varinjlim Hom_c(\pi_1(\bar{U}_L)^{ab}, \mathbb{S}^1) \\ &= Hom_c(\varprojlim \pi_1(\bar{U}_L)^{ab}, \mathbb{S}^1) \\ &= Hom_c(C_{\bar{U}}, \mathbb{S}^1) = C^D_{\bar{U}}. \end{aligned}$$

The exact sequence of topological groups

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{S}^1 \to 0$$

induces an exact sequence

$$0 \to \mathbb{Z} \to \widetilde{\mathbb{R}} \to \widetilde{\mathbb{S}}^1 \to 0$$

of abelian sheaves on $\bar{U}_L$. The induced long exact sequence

$$0 = H^1(\bar{U}_L, \mathbb{Z}) \to H^1(\bar{U}_L, \widetilde{\mathbb{R}}) \to H^1(\bar{U}_L, \widetilde{\mathbb{S}}^1) \to H^2(\bar{U}_L, \mathbb{Z}) \to H^2(\bar{U}_L, \widetilde{\mathbb{R}}) = 0$$

is canonically identified with

$$0 \to Hom_c(C_{\bar{U}}, \mathbb{R}) \to Hom_c(C_{\bar{U}}, \mathbb{S}^1) \to H^2(\bar{U}_L, \mathbb{Z}) \to 0$$

and we obtain $H^2(\bar{U}_L, \mathbb{Z}) = (C^1_{\bar{U}})^D$. □

Recall that we have a canonical morphism $\gamma : \bar{X}_L \to \bar{X}_{et}$. We consider the truncated functor $\tau_{\leq 2} R\gamma_*$ of the total derived functor $R\gamma_*$.

COROLLARY 6.3. *We have* $\gamma_*\mathbb{Z} = \mathbb{Z}$, $R^1(\gamma_*)\mathbb{Z} = 0$ *and* $R^2(\gamma_*)\mathbb{Z}$ *is the étale sheaf associated to the abelian presheaf*

$$\begin{array}{rccc} \mathcal{P}^2\gamma_*\mathbb{Z}: & Et_{\bar{X}} & \longrightarrow & \underline{Ab} \\ & \bar{U} & \longmapsto & (C^1_{\bar{U}})^D. \end{array}$$

Proof. The sheaf $R^n(\gamma_*)\mathbb{Z}$ is the sheaf associated to the presheaf $\bar{U} \mapsto H^n(\bar{X}_L/\gamma^*\bar{U}, \mathbb{Z})$. Hence the corollary follows immediately from Proposition 6.2. Note that it follows from Property (4) that the restriction map

$$\mathcal{P}^2\gamma_*\mathbb{Z}(\bar{U}) = (C^1_{\bar{U}})^D \to \mathcal{P}^2\gamma_*\mathbb{Z}(\bar{V}) = (C^1_{\bar{V}})^D$$

is the Pontryagin dual of the canonical morphism $C^1_{\bar{V}} \to C^1_{\bar{U}}$ (induced by the norm map), for any $\bar{V} \to \bar{U}$ in $Et_{\bar{X}}$. □

The cohomology of sheaf $R^2\gamma_*\mathbb{Z}$ associated to $\mathcal{P}^2\gamma_*\mathbb{Z}$ is computed in Section 6.3. The étale sheaf $R^2\gamma_*\mathbb{Z}$ is acyclic for the global sections functor on $\bar{X}_{et}$. More precisely, we have

$$H^n(\bar{X}_{et};\, R^2\gamma_*\mathbb{Z}) = \begin{cases} Hom(\mathcal{O}_F^\times, \mathbb{Q}) & \text{for } n = 0, \\ 0 & \text{for } n \geq 1. \end{cases}$$

Recall that $Pic(\bar{X}) = C_{\bar{X}}$ is the Arakelov–Picard group of F, and that μ_F is the group of roots of unity in F. We compute below the hypercohomology of the complex of abelian étale sheaves $\tau_{\leq 2}R\gamma_*\mathbb{Z}$.

THEOREM 6.4. *We have*

$$\mathbb{H}^n(\bar{X}_{et}, \tau_{\leq 2}R\gamma_*\mathbb{Z}) = \begin{cases} \mathbb{Z} & \textit{for } n = 0 \\ 0 & \textit{for } n = 1 \\ Pic^1(\bar{X})^D & \textit{for } n = 2 \\ \mu_F^D & \textit{for } n = 3 \\ 0 & \textit{for } n \geq 4. \end{cases}$$

Recall that the Artin–Verdier étale cohomology of $\mathbb{Z}$ is given by

$$H^i(\bar{X}_{et}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0 \\ 0 & \text{for } i = 1 \\ Cl(F)^D & \text{for } i = 2 \\ Hom(\mathcal{O}_F^*, \mathbb{Q}/\mathbb{Z}) & \text{for } i = 3 \\ 0 & \text{for } i \geq 4. \end{cases}$$

Proof. The hypercohomology spectral sequence

$$H^i(\bar{X}_{et}, H^j(\tau_{\leq 2} R\gamma_* \mathbb{Z})) \Rightarrow \mathbb{H}^{i+j}(\bar{X}_{et}, \tau_{\leq 2} R\gamma_* \mathbb{Z})$$

first gives $\mathbb{H}^0(\bar{X}_0, \tau_{\leq 2} R\gamma_* \mathbb{Z}) = \mathbb{Z}$ and $\mathbb{H}^1(\bar{X}_{et}, \tau_{\leq 2} R\gamma_* \mathbb{Z}) = 0$. On the other hand we have

$$\mathbb{H}^n(\bar{X}_{et}, \tau_{\leq 2} R\gamma_* \mathbb{Z}) = \mathbb{H}^n(\bar{X}_{et}, R\gamma_* \mathbb{Z}) = H^n(\bar{X}_L, \mathbb{Z})$$

for any $n \leq 2$. In particular, we have

$$\mathbb{H}^2(\bar{X}_{et}, \tau_{\leq 2} R\gamma_* \mathbb{Z}) = H^2(\bar{X}_L, \mathbb{Z}) = Pic^1(\bar{X})^D.$$

Therefore, the spectral sequence above yields an exact sequence

$$\begin{aligned} &0 \to Cl(F)^D \\ &\quad \to Pic^1(\bar{X})^D \to Hom(\mathcal{O}_F^*, \mathbb{Q}) \to Hom(\mathcal{O}_F^*, \mathbb{Q}/\mathbb{Z}) \to \mathbb{H}^3(\bar{X}_L, \tau_{\leq 2} R\gamma_* \mathbb{Z}) \to 0. \end{aligned}$$

Here the maps $Cl(F)^D \to Pic^1(\bar{X})^D \to Hom(\mathcal{O}_F^*, \mathbb{Q})$ are explicitly given. The first is induced by the morphism from the Weil-étale fundamental group to the étale fundamental group

$$\pi_1(\bar{X}_L)^{ab} \longrightarrow \pi_1(\bar{X}_{et})^{ab}$$

given by Property (3), and the second is induced by the canonical morphism $\mathcal{P}^2\gamma_*\mathbb{Z} \to R^2(\gamma_*)\mathbb{Z}$ (the map from a presheaf to its associated sheaf). It follows that the cokernel of the map

$$Pic^1(\bar{X})^D \to Hom(\mathcal{O}_F^*, \mathbb{Q})$$

is

$$Hom(\mathcal{O}_F^*, \mathbb{Q})/Hom(\mathcal{O}_F^*, \mathbb{Z}) \simeq Hom(\mathcal{O}_F^*/\mu_F, \mathbb{Q}/\mathbb{Z}).$$

We obtain an exact sequence

$$0 \to Hom(\mathcal{O}_F^*/\mu_F, \mathbb{Q}/\mathbb{Z}) \to Hom(\mathcal{O}_F^*, \mathbb{Q}/\mathbb{Z}) \to \mathbb{H}^3(\bar{X}_{et}, \tau_{\leq 2} R\gamma_* \mathbb{Z}) \to 0. \tag{31}$$

Let us denote by $\alpha : Hom(\mathcal{O}_F^*/\mu_F, \mathbb{Q}/\mathbb{Z}) \to Hom(\mathcal{O}_F^*, \mathbb{Q}/\mathbb{Z})$ the first map of the exact sequence (31). We need to show that α is the canonical map. There is a decomposition

$$Hom(\mathcal{O}_F^*, \mathbb{Q}/\mathbb{Z}) = Hom(\mathcal{O}_F^*/\mu_F, \mathbb{Q}/\mathbb{Z}) \times \mu_F^D$$

and the composition (where p is the projection)

$$p \circ \alpha : Hom(\mathcal{O}_F^*/\mu_F, \mathbb{Q}/\mathbb{Z}) \to Hom(\mathcal{O}_F^*, \mathbb{Q}/\mathbb{Z}) \to \mu_F^D$$

must be 0 since $Hom(\mathcal{O}_F^*/\mu_F, \mathbb{Q}/\mathbb{Z})$ is divisible and μ_F^D finite. It follows that the image of α is contained in the subgroup

$$Hom(\mathcal{O}_F^*/\mu_F, \mathbb{Q}/\mathbb{Z}) \subset Hom(\mathcal{O}_F^*, \mathbb{Q}/\mathbb{Z})$$

hence α induces an injective morphism

$$\widetilde{\alpha} : Hom(\mathcal{O}_F^*/\mu_F, \mathbb{Q}/\mathbb{Z}) \hookrightarrow Hom(\mathcal{O}_F^*/\mu_F, \mathbb{Q}/\mathbb{Z}).$$

Since those two groups are both finite sums of $\mathbb{Q}/\mathbb{Z}$, this map $\widetilde{\alpha}$ needs to be an isomorphism. Indeed, the n-torsion subgroups are both finite of the same cardinality for any n, hence an injective map must be bijective. Hence α has the same image as the canonical map, i.e. the image of the map induced by the quotient map $\mathcal{O}_F^* \to \mathcal{O}_F^*/\mu_F$. Hence the exact sequence (31) yields a canonical identification

$$\mathbb{H}^3(\bar{X}_{et}, \tau_{\leq 2}R\gamma_*\mathbb{Z}) = \mu_F^D.$$

Finally, we have

$$\mathbb{H}^n(\bar{X}_{et}, \tau_{\leq 2}R\gamma_*\mathbb{Z}) = 0 \quad \text{for } n \geq 4$$

since the diagonals of the hypercohomology spectral sequence are all trivial for $n \geq 4$. □

Let $\varphi : X_L \to \bar{X}_L$ be the open embedding. By Property (7), we have an open/closed decomposition

$$\varphi : X_L \to \bar{X}_L \leftarrow \coprod_{v \in X_\infty} B_{W_{k(v)}} : i_\infty.$$

In particular, we have adjoint functors $\varphi_!$, φ^*, φ_*. If we consider the induced functors on abelian sheaves, we have adjoint functors i_∞^*, $i_{\infty *}$, $i_\infty^!$, showing that $i_{\infty *}$ is exact (on abelian objects). We obtain an exact sequence of sheaves

$$0 \to \varphi_!\varphi^*\mathcal{A} \to \mathcal{A} \to \prod_{v|\infty} i_{v*}i_v^*\mathcal{A} \to 0 \tag{32}$$

for any abelian object $\mathcal{A}$ of $\bar{X}_L$.

THEOREM 6.5. *We have canonically*

$$\mathbb{H}^n(\bar{X}_{et}, \tau_{\leq 2}R\gamma_*(\varphi_!\mathbb{Z})) = \begin{cases} 0 & \textit{for } n = 0 \\ \left(\prod_{X_\infty} \mathbb{Z}\right)\Big/\mathbb{Z} & \textit{for } n = 1 \\ Pic^1(\bar{X})^D & \textit{for } n = 2 \\ \mu_F^D & \textit{for } n = 3 \\ 0 & \textit{for } n \geq 4. \end{cases}$$

Proof. Let $v \in X_\infty$. By Property (7), we have $\gamma \circ i_v = u_v \circ \alpha_v$, where $\alpha_v : B_{W_{k(v)}} \to \underline{Sets}$ is the unique map. The morphisms i_v and u_v are both closed embeddings so that i_{v*} and u_{v*} are both exact, hence we have

$$R(\gamma_*)i_{v*}\mathbb{Z} = u_{v*}R(\alpha_{v*})\mathbb{Z}.$$

This complex is concentrated in degree 0 since $R^n(\alpha_{v*})\mathbb{Z} = H^n(B_{W_{k(v)}}, \mathbb{Z}) = 0$ for any $n \geq 1$, and we have $R^0(\gamma_*)i_{v*}\mathbb{Z} = u_{v*}\mathbb{Z}$. Hence the theorem follows from Theorem 6.4, exact sequence (32), and $H^*(\bar{X}_{et}, u_{v*}\mathbb{Z}) = H^*(\underline{Sets}, \mathbb{Z})$. □

Remark 6.6. A complex quasi-isomorphic to $\tau_{\leq 2}R\gamma_*(\varphi_!\mathbb{Z})$ was constructed in [**13**] using a more complicated method.

6.2. *Dedekind zeta functions at $s = 0$*

We denote by $(\bigoplus_{v|\infty} W_{k(v)})^1$ the kernel of the canonical morphism $\bigoplus_{v|\infty} W_{k(v)} \to \mathbb{R}_{>0}$.

THEOREM 6.7. *We have canonical isomorphisms*

$$H^n(\bar{X}_L, \varphi_!\widetilde{\mathbb{R}}) = \begin{cases} \left(\prod_{v|\infty} \mathbb{R}\right) \Big/ \mathbb{R} & \text{for } n = 1 \\ Hom_c((\bigoplus_{v|\infty} W_{k(v)})^1, \mathbb{R}) & \text{for } n = 2 \\ 0 & \text{for } n \neq 1, 2. \end{cases}$$

Proof. The direct image i_{v*} is exact hence the group $H^n(\bar{X}_L, \prod_{v|\infty} i_{v*}\widetilde{\mathbb{R}})$ is canonically isomorphic to

$$\prod_{v|\infty} H^n(B_{W_{k(v)}}\widetilde{\mathbb{R}}) = \begin{cases} \prod_{v|\infty} \mathbb{R} & n = 0 \\ Hom_c\left(\sum_{v|\infty} W_{k(v)}, \mathbb{R}\right) & n = 1 \\ 0 & n \geq 2. \end{cases}$$

Using the exact sequence (32), the result for $n \geq 3$ follows from Property (9). By (32) we have the exact sequence

$$0 \to H^0(\bar{X}_L, \varphi_!\widetilde{\mathbb{R}}) \to H^0(\bar{X}_W, \widetilde{\mathbb{R}}) = \mathbb{R} \to \prod_{v|\infty} H^n(B_{W_{k(v)}}\widetilde{\mathbb{R}})$$
$$= \prod_{v|\infty} \mathbb{R} \to H^1(\bar{X}_L, \varphi_!\widetilde{\mathbb{R}}) \to 0,$$

where the central map is the diagonal embedding. The result follows for $n = 0, 1$. For $n = 2$, we have the exact sequence

$$Hom_c(Pic(\bar{X}), \mathbb{R}) \to Hom_c\left(\sum_{v|\infty} W_{k(v)}, \mathbb{R}\right) \to H^2(\bar{X}_L, \varphi_!\widetilde{\mathbb{R}}) \to 1,$$

where, by Property (8), the first map is induced by the canonical morphism $\sum_{v|\infty} W_{k(v)} \to Pic(\bar{X})$. We obtain a canonical isomorphism

$$H^1(\bar{X}_L, \varphi_!\widetilde{\mathbb{R}}) = Hom_c\left(\left(\sum_{v|\infty} W_{k(v)}\right)^1, \mathbb{R}\right). \qquad \square$$

Definition 6.8. We define the *fundamental class* $\theta \in H^1(\bar{X}_L, \widetilde{\mathbb{R}})$ as the canonical morphism

$$\theta \in H^1(\bar{X}_L, \widetilde{\mathbb{R}}) = Hom_c(Pic(\bar{X}), \mathbb{R}).$$

Recall that, for any closed point $v \in \bar{X}$, we have $H^n(B_{W_{k(v)}}, \widetilde{\mathbb{R}}) = \mathbb{R},\ Hom_c(W_{k(v)}, \mathbb{R})$ and 0 for $n = 0, 1$ and $n \geq 2$ respectively.

Definition 6.9. For any closed point $v \in \bar{X}$, the *v-fundamental class* is the canonical morphism $\theta_v : W_{k(v)} \to \mathbb{R}$:

$$\theta_v \in H^1(B_{W_{k(v)}}, \widetilde{\mathbb{R}}) = Hom_c(W_{k(v)}, \mathbb{R}).$$

The morphism obtained by cup product with θ_v is the canonical isomorphism

$$\begin{array}{llcl} \cup\theta_v : & H^0(B_{W_{k(v)}}, \widetilde{\mathbb{R}}) = \mathbb{R} & \longrightarrow & H^1(B_{W_{k(v)}}, \widetilde{\mathbb{R}}) = Hom_c(W_{k(v)}, \mathbb{R}) \\ & 1 & \longmapsto & \theta_v. \end{array}$$

THEOREM 6.10. *The morphism $\cup\theta$ obtained by cup product with the fundamental class θ is the canonical isomorphism*

$$\begin{array}{cccc} \cup\theta: & H^1(\bar{X}_L,\varphi_!\widetilde{\mathbb{R}})=(\prod_{v|\infty}\mathbb{R})/\mathbb{R} & \longrightarrow & H^2(\bar{X}_L,\varphi_!\widetilde{\mathbb{R}})=Hom_c((\sum_{v|\infty}W_{k(v)})^1,\mathbb{R}) \\ & v & \longmapsto & \theta_v\circ p_v, \end{array}$$

where $p_v:(\sum_{w|\infty}W_{k(w)})^1\to W_{k(v)}$ is given by the projection.

Proof. By Property (8), the morphism

$$\begin{array}{ccc} H^1(B_{Pic(\bar{X})},\widetilde{\mathbb{R}})=H^1(\bar{X}_L,\widetilde{\mathbb{R}}) & \longrightarrow & H^1(\coprod_{v\in X_\infty}B_{W_{k(v)}},\widetilde{\mathbb{R}})=\prod_{v|\infty}H^1(B_{W_{k(v)}},\widetilde{\mathbb{R}}) \\ \theta & \longmapsto & (\theta_v)_{v|\infty}, \end{array}$$

which is induced by $\coprod_{v\in X_\infty}B_{W_{k(v)}}\to\bar{X}_L\to B_{Pic(\bar{X})}$, sends fundamental class to fundamental class. Hence the cup-product morphism $\cup\theta$ is induced by

$$(\cup\theta_v)_{v|\infty}:H^0(\bar{X}_L,i_{\infty*}\widetilde{\mathbb{R}})=\prod_{v|\infty}\mathbb{R}\longrightarrow H^1(\bar{X}_L,i_{\infty*}\widetilde{\mathbb{R}})=\prod_{v|\infty}Hom_c(B_{W_{k(v)}},\widetilde{\mathbb{R}})$$

and the result follows. Note that $\cup\theta$ is well defined, since $\cup\theta(\sum_{v|\infty}v)=\sum_{v|\infty}\theta_v$ is the canonical map $\sum_{v|\infty}W_{k(v)}\to\mathbb{R}$, which vanishes on $(\sum_{v|\infty}W_{k(v)})^1$. □

THEOREM 6.11. *For any $n\geq 1$, the morphism*

$$R_n:\mathbb{H}^n(\bar{X}_{et},\tau_{\leq 2}R\gamma_*(\varphi_!\mathbb{Z}))\otimes\mathbb{R}\longrightarrow H^n(\bar{X}_L,\varphi_!\widetilde{\mathbb{R}}),$$

induced by the morphism of sheaves $\varphi_!\mathbb{Z}\to\varphi_!\widetilde{\mathbb{R}}$, is an isomorphism.

Proof. We denote by

$$\kappa_n:\mathbb{H}^n(\bar{X}_{et},\tau_{\leq 2}R\gamma_*(\varphi_!\mathbb{Z}))\longrightarrow H^1(\bar{X}_L,\varphi_!\widetilde{\mathbb{R}})$$

the morphism induced by $\varphi_!\mathbb{Z}\to\varphi_!\widetilde{\mathbb{R}}$. The result is obvious for $n\neq 1,2$. The result is also clear for $n=1$, since κ_1 is canonically identified with

$$H^0\Big(\coprod_{v\in X_\infty}B_{W_{k(v)}},\mathbb{Z}\Big)\Big/H^0(\bar{X}_L,\mathbb{Z})\to H^0\Big(\coprod_{v\in X_\infty}B_{W_{k(v)}},\widetilde{\mathbb{R}}\Big)\Big/H^0(\bar{X}_L,\widetilde{\mathbb{R}}),$$

hence R_1 is the identity. Assume that $n=2$. On the one hand, we have canonically

$$\mathbb{H}^2(\bar{X}_{et},\tau_{\leq 2}R\gamma_*(\varphi_!\mathbb{Z}))=H^2(\bar{X}_L,\mathbb{Z})=H^1(\bar{X}_L,\widetilde{\mathbb{S}}^1)/H^1(\bar{X}_L,\widetilde{\mathbb{R}}).$$

On the other hand, we have

$$H^2(\bar{X}_L,\varphi_!\widetilde{\mathbb{R}})=H^1\Big(\coprod_{v|\infty}B_{W_{k(v)}},\widetilde{\mathbb{R}}\Big)\Big/H^1(\bar{X}_L,\widetilde{\mathbb{R}})=H^1\Big(\coprod_{v|\infty}B_{W_{k(v)}},\widetilde{\mathbb{S}}^1\Big)\Big/H^1(\bar{X}_L,\widetilde{\mathbb{R}})$$

and the map R_2 is induced by

$$H^1(\bar{X}_L,\widetilde{\mathbb{S}}^1)=Hom_c(Pic(\bar{X}),\mathbb{S}^1)\longrightarrow H^1\Big(\coprod_{v|\infty}B_{W_{k(v)}},\widetilde{\mathbb{S}}^1\Big)=Hom_c\Big(\sum_{v|\infty}W_{k(v)},\mathbb{S}^1\Big)$$

which is in turn induced by the canonical morphism $\sum_{v|\infty}W_{k(v)}\to Pic(\bar{X})$, as it follows from Property (8). We have the exact sequence

$$Hom_c(Pic(\bar{X}),\mathbb{R})\to Hom_c(Pic(\bar{X}),\mathbb{S}^1)\to Hom_c(Pic^1(\bar{X}),\mathbb{S}^1)\to 0,$$

hence κ_2 is the morphism

$$\kappa_2 : Hom_c(Pic^1(\bar{X}), \mathbb{S}^1) \longrightarrow Hom((\oplus_{v|\infty} W_{k(v)})^1, \mathbb{S}^1) = Hom((\oplus_{v|\infty} W_{k(v)})^1, \mathbb{R}),$$

where the first map is the Pontryagin dual of $(\oplus_{v|\infty} W_{k(v)})^1 \to Pic^1(\bar{X})$. Recall that we have the exact sequence of topological groups

$$0 \to (\oplus_{v|\infty} W_{k(v)})^1/(\mathcal{O}_F^\times/\mu_F) \to Pic^1(\bar{X}) \to Cl(F) \to 0.$$

Then it is straightforward to check that there is a canonical identification

$$\mathbb{H}^2(\bar{X}_{et}, \tau_{\leq 2} R\gamma_*(\varphi_!\mathbb{Z})) \otimes \mathbb{R} = Hom(\mathcal{O}_F^\times, \mathbb{R})$$

and that the map R_2 is the morphism

$$R_2 : Hom(\mathcal{O}_F^\times, \mathbb{R}) \longrightarrow Hom_c((\oplus_{v|\infty} W_{k(v)})^1, \mathbb{R})$$

which is the inverse of the isomorphism induced by the natural map $\mathcal{O}_F^\times \to (\oplus_{v|\infty} W_{k(v)})^1$. In other words

$$R_2^{-1} : Hom_c((\oplus_{v|\infty} W_{k(v)})^1, \mathbb{R}) \longrightarrow Hom(\mathcal{O}_F^\times, \mathbb{R})$$

is induced by the natural map $\mathcal{O}_F^\times \to (\oplus_{v|\infty} W_{k(v)})^1$. □

The morphisms $\cup\theta$, R_1 and R_2 have been made explicit during the proof of Theorem 6.10 and in Theorem 6.11. The result of Corollary 6.12 follows.

COROLLARY 6.12. *We have a commutative diagram*

$$\begin{tikzcd}
(\sum_{v|\infty} \mathbb{R})/\mathbb{R} \arrow[r, "D"] \arrow[d, "R_1"] & Hom(\mathcal{O}_F^\times, \mathbb{R}) \arrow[d, "R_2"] \\
(\sum_{v|\infty} \mathbb{R})/\mathbb{R} \arrow[r, "\cup\theta"] & Hom_c((\oplus_{v|\infty} W_{k(v)})^1, \mathbb{R})
\end{tikzcd}$$

where D is the transpose of the usual regulator map

$$\mathcal{O}_F^\times \otimes \mathbb{R} \longrightarrow \left(\sum_{v|\infty} \mathbb{R}\right)^+.$$

We denote by $\varphi : X_L \to \bar{X}_L$ the natural open embedding, and by $H_c^n(X_L, \mathcal{A}) := H^n(\bar{X}_L, \varphi_!\mathcal{A})$ the cohomology with compact support with coefficients in the abelian sheaf $\mathcal{A}$.

THEOREM 6.13. (Lichtenbaum's formalism) *Assume that F is totally imaginary. Let $\bar{X}_L$ be any topos satisfying Properties (1)–(9) above. We denote by $\tau_{\leq 2} R\gamma_*$ the truncated functor of the total derived functor $R\gamma_*$, where γ is the morphism given by Property (1). Then the following are true.*

- $\mathbb{H}^n(\bar{X}_{et}, \tau_{\leq 2} R\gamma_*(\varphi_!\mathbb{Z}))$ *is finitely generated and zero for $n \geq 4$.*
- *The canonical map*

$$\mathbb{H}^n(\bar{X}_{et}, \tau_{\leq 2} R\gamma_*(\varphi_!\mathbb{Z})) \otimes \mathbb{R} \longrightarrow H_c^n(X_L, \widetilde{\mathbb{R}})$$

is an isomorphism for any $n \geq 0$.

- *There exists a fundamental class* $\theta \in H^1(\bar{X}_L, \widetilde{\mathbb{R}})$. *The complex of finite-dimensional vector spaces*

$$\cdots \to H_c^{n-1}(X_L, \widetilde{\mathbb{R}}) \to H_c^n(X_L, \widetilde{\mathbb{R}}) \to H_c^{n+1}(X_L, \widetilde{\mathbb{R}}) \to \cdots$$

  defined by cup product with θ, *is acyclic.*

- *The vanishing order of the Dedekind zeta function* $\zeta_F(s)$ *at* $s = 0$ *is given by*

$$ord_{s=0}\zeta_F(s) = \sum_{n \geq 0} (-1)^n \, n \dim_{\mathbb{R}} H_c^n(X_L, \widetilde{\mathbb{R}}).$$

- *The leading term coefficient* $\zeta_F^*(s)$ *at* $s = 0$ *is given by the Lichtenbaum Euler characteristic*

$$\zeta_F^*(s) = \pm \prod_{n \geq 0} |\mathbb{H}^n(\bar{X}_{et}, \tau_{\leq 2} R\gamma_*(\varphi_! \mathbb{Z}))_{tors}|^{(-1)^n} / \det(H_c^n(X_L, \widetilde{\mathbb{R}}), \theta, B^*),$$

  where B^n *is a basis of* $\mathbb{H}^n(\bar{X}_{et}, \tau_{\leq 2} R\gamma_*(\varphi_! \mathbb{Z}))/tors$.

In particular, those results hold for the Weil-étale topos $\bar{X}_W$ *defined in Section 6.*

Proof. This follows from Theorems 6.5, 6.10, 6.11, Corollary 6.12 and from the analytic class number formula. □

6.3. *The sheaf* $R^2\gamma_*\mathbb{Z}$

The étale sheaf $R^2\gamma_*\mathbb{Z}$ is the sheaf associated to the presheaf

$$\begin{array}{rccc} \mathcal{P}^2\mathbb{Z}: & Et_{\bar{X}} & \longrightarrow & \underline{Ab} \\ & \bar{U} & \longmapsto & (C^1_{\bar{U}})^D. \end{array}$$

Recall that if $\bar{U}$ is connected of function field $K(\bar{U})$, then $C_{\bar{U}}$ is the S-idèle class group of $K(\bar{U})$, where S is the set of places of $K(\bar{U})$ not corresponding to a point of $\bar{U}$. In other words, if we set $K = K(U)$ then $C_{\bar{U}} = C_{K,S}$ is the S-idèle class group of K defined by the exact sequence

$$\prod_{v \in U} \mathcal{O}_{K_v}^\times \to C_K \to C_{K,S} \to 0.$$

The compact group $C^1_{\bar{U}}$ is then defined as the kernel of the canonical map $C_{\bar{U}} \to \mathbb{R}^\times$. Note that such a finite set S does not necessarily contain all the archimedean places. The restriction maps of the presheaf $\mathcal{P}^2\mathbb{Z}$ are induced by the canonical maps $C_{\bar{V}} \to C_{\bar{U}}$ (well defined for any étale map $\bar{V} \to \bar{U}$ of connected étale $\bar{X}$-schemes). By class field theory, we have a covariantly functorial exact sequence of compact topological groups

$$0 \to D^1_{\bar{U}} \to C^1_{\bar{U}} \to \pi_1(\bar{U}_{et})^{ab} \to 0,$$

where $\pi_1(\bar{U}_{et})^{ab}$ is the abelian étale fundamental group of $\bar{U}$ and $D^1_{\bar{U}}$ is the connected component of 1 in $C^1_{\bar{U}}$. Here $\pi_1(\bar{U})^{ab}$ is defined as the abelianization of the profinite fundamental group of the Artin–Verdier étale topos $\bar{X}_{et}/y\bar{U} \simeq \bar{U}_{et}$. If we denote the function field of $\bar{U}$ by $K(\bar{U})$ then this group is just the Galois group of the maximal abelian extension of $K(\bar{U})$ unramified at every place of $K(\bar{U})$ corresponding to a point of $\bar{U}$.

By Pontryagin duality, we obtain a contravariantly functorial exact sequence of discrete abelian groups

$$0 \to \pi_1^{ab}(\bar{U}_{et})^D \to (C^1_{\bar{U}})^D \to (D^1_{\bar{U}})^D \to 0, \tag{33}$$

i.e. an exact sequence of abelian étale presheaves on $\bar{X}$. On the one hand, the sheaf associated to the presheaf

$$\begin{array}{ccc} Et_{\bar{X}} & \longrightarrow & \underline{Ab} \\ \bar{U} & \longmapsto & \pi_1^{ab}(\bar{U}_{et})^D = H^2(\bar{U}_{et}, \mathbb{Z}) \end{array}$$

vanishes and on the other hand the associated sheaf functor is exact. Therefore, the exact sequence (33) shows that $R^2\mathbb{Z}$ is the sheaf associated to the presheaf

$$\begin{array}{cccc} P^2\mathbb{Z}: & Et_{\bar{X}} & \longrightarrow & \underline{Ab} \\ & \bar{U} & \longmapsto & (D^1_{\bar{U}})^D. \end{array}$$

The structure of the connected component $D^1_{\bar{U}}$ of the S-idèle class group $C^1_{\bar{U}}$ is not known in general.

We consider the following open subscheme of $\bar{X}$:

$$Y := (X, X(\mathbb{R})).$$

Let $\bar{U} \to Y$ be a connected étale Y-scheme with function field $K := K(\bar{U})$. Note that U_∞ contains only real places. If v is a real place of K, then we denote by $\mathcal{O}^\times_{K_v} = \pm 1$ the kernel of the valuation $K_v{}^\times \to \mathbb{R}_{>0}$. The Leray spectral sequence associated to the morphism $Spec(K)_{et} \to \bar{U}_{et}$ gives an exact sequence

$$0 \to H^2(\bar{U}_{et}, \mathbb{Z}) \to H^2(G_K, \mathbb{Z}) \to \sum_{v \in \bar{U}^0} H^2(I_v, \mathbb{Z}) \to H^3(\bar{U}_{et}, \mathbb{Z}) \to H^3(G_K, \mathbb{Z}) = 0, \tag{34}$$

which is functorial in $\bar{U}$ with respect to the natural morphisms between Galois groups. We have the following canonical identifications:

$$H^2(\bar{U}_{et}, \mathbb{Z}) = \pi_1(\bar{U}_{et})^D, \quad H^2(G_K, \mathbb{Z}) = G_K^D, \quad H^2(I_v, \mathbb{Z}) = I_v^D.$$

Then the central map of the exact sequence (34) is induced by the natural maps $I_v^{ab} \to G_K^{ab}$, and the first map is given by the natural surjection $G_K^{ab} \to \pi_1(\bar{U}_{et})^{ab}$. Then global and local class field theory give the exact sequence

$$0 \to \pi_1(\bar{U}_{et})^D \to (C_K/D_K)^D \to \sum_{v \in \bar{U}^0} (\mathcal{O}^\times_{K_v})^D \to H^3(\bar{U}_{et}, \mathbb{Z}) \to 0.$$

Here the functoriality is given by the norm maps, and $(C_K/D_K)^D \to \sum_{v \in \bar{U}^0} (\mathcal{O}^\times_{K_v})^D$ is the dual of the canonical map

$$\rho_{\bar{U}} : \prod_{v \in \bar{U}^0} \mathcal{O}^\times_{K_v} \longrightarrow C_K/D_K.$$

We obtain an isomorphism

$$H^3(\bar{U}_{et}, \mathbb{Z}) \simeq Ker(\rho_{\bar{U}})^D,$$

which is functorial with respect to the presheaf structure on $H^3(-,\mathbb{Z})$ and with the norm maps on the right-hand side. Note that

$$Ker(\rho_{\bar{U}}) = \prod_{v\in\bar{U}^0} \mathcal{O}_{K_v}^{\times} \cap D_K = \prod_{v\in\bar{U}^0} \mathcal{O}_{K_v}^{\times} \cap D_K^1 \subset C_K.$$

For any connected étale scheme $\bar{U}$ over $Y=(X, X(\mathbb{R}))$, we have a functorial exact sequence of compact abelian topological groups

$$0 \to \prod_{v\in\bar{U}^0} \mathcal{O}_{K(\bar{U})_v}^{\times} \cap D_{K(\bar{U})}^1 \to D_{K(\bar{U})}^1 \to D_{\bar{U}}^1 \to 0,$$

hence a contravariantly functorial exact sequence of discrete abelian groups

$$0 \to (D_{\bar{U}}^1)^D \to (D_{K(\bar{U})}^1)^D \to \left(\prod_{v\in\bar{U}^0} \mathcal{O}_{K(\bar{U})_v}^{\times} \cap D_{K(\bar{U})}^1\right)^D = H^3(\bar{U}_{et},\mathbb{Z}) \to 0.$$

But the sheaf associated to the presheaf

$$\begin{array}{ccc} Et_Y & \longrightarrow & \underline{Ab} \\ \bar{U} & \longmapsto & H^3(\bar{U}_{et},\mathbb{Z}) \end{array}$$

vanishes. It follows that the sheaf $R^2\gamma_*\mathbb{Z}$ restricted to Y_{et} is the sheaf associated to the presheaf

$$\begin{array}{cccc} P: & Et_Y & \longrightarrow & \underline{Ab} \\ & \bar{U} & \longmapsto & (D_{K(\bar{U})}^1)^D. \end{array}$$

Let $\xi : Spec(F)_{et} \to Y_{et}$ be the morphism induced by the inclusion of the generic point. Let $\mathcal{F}$ be the presheaf on $Et_{Spec(F)}$ sending a finite extension K/F to $(D_K^1)^D$. By Tate's theorem (see [**15**, Theorem 8.2.5]), we have a functorial isomorphism of compact groups

$$D_K^1 = (\mathbb{V} \otimes_{\mathbb{Z}} \mathcal{O}_K^{\times}) \times \left(\prod_{r_2(K)} \mathbb{S}^1\right),$$

where $\mathbb{V}=\mathbb{Q}^D$ is the solenoid and $r_2(K)$ is the set of complex places of K. We obtain a functorial isomorphism of abelian groups

$$(D_K^1)^D \simeq Hom(\mathcal{O}_K^{\times},\mathbb{Q}) \oplus \left(\bigoplus_{r_2(K)} \mathbb{Z}\right).$$

It follows that $(D_K^1)^D$ satisfies Galois descent, hence $\mathcal{F}$ is a sheaf on the étale site of $Spec(F)$ and $P=\xi_*\mathcal{F}$ is a sheaf on Et_Y. We obtain the following description of $R^2\gamma_*\mathbb{Z}|Y$.

PROPOSITION 6.14. *For any $\bar{U}$ connected étale over $Y=(X, X(\mathbb{R}))$, we have*

$$R^2\gamma_*\mathbb{Z}(\bar{U}) = (D_{K(\bar{U})}^1)^D \simeq Hom(\mathcal{O}_{K(\bar{U})}^{\times},\mathbb{Q}) \oplus \left(\sum_{r_2(K(\bar{U}))} \mathbb{Z}\right).$$

Let $\phi : Y_{et} \to \bar{X}_{et}$ be the open immersion. Consider the adjunction map

$$R^2\gamma_*\mathbb{Z} \longrightarrow \phi_*\phi^* R^2\gamma_*\mathbb{Z}. \tag{35}$$

For any $\bar{U}$ connected and étale over $\bar{X}$, the map $D^1_{K(\bar{U})} \to D^1_{\bar{U}}$ is surjective hence the induced map

$$P^2\mathbb{Z}(\bar{U}) = (D^1_{\bar{U}})^D \longrightarrow \phi_*\phi^* R^2\gamma_*\mathbb{Z}(\bar{U}) = R^2\gamma_*\mathbb{Z}(\bar{U} \times_{\bar{X}} Y) = (D^1_{K(\bar{U})})^D$$

is injective. Applying the (exact) associated functor, we see that the adjunction map (35) is also injective.

Recall that $R^2\gamma_*\mathbb{Z}$ is the sheaf associated to the presheaf on $Et_{\bar{X}}$ defined by $\mathcal{P}^2\mathbb{Z}(\bar{U}) = (C^1_{\bar{U}})^D$, and consider the presheaf

$$\begin{array}{rccc} \phi_p\phi^p\mathcal{P}^2\mathbb{Z}: & Et_{\bar{X}} & \longrightarrow & \underline{Ab} \\ & \bar{U} & \longmapsto & (C^1_{\bar{U}\times_{\bar{X}} Y})^D. \end{array}$$

For any $\bar{U}$ connected and étale over $\bar{X}$ and such that $\bar{U}$ does not contain all the places of $K(\bar{U})$, we have an exact sequence

$$0 \to \prod_{U_\infty - U(\mathbb{R})} \mathbb{S}^1 \to C^1_{\bar{U}\times_{\bar{X}} Y} \to C^1_{\bar{U}} \to 0$$

inducing an exact sequence of discrete abelian groups

$$0 \to (C^1_{\bar{U}})^D \to (C^1_{\bar{U}\times_{\bar{X}} Y})^D \to \prod_{U_\infty - U(\mathbb{R})} \mathbb{Z} \to 0.$$

In other words, we have an exact sequence of presheaves on $Et'_{\bar{X}}$:

$$0 \to \mathcal{P}^2\mathbb{Z} \to \phi_p\phi^p\mathcal{P}^2\mathbb{Z} \to \prod_{v\in\bar{X}-Y} u_{v*}\mathbb{Z} \to 0,$$

where $Et'_{\bar{X}}$ is the full subcategory of $Et_{\bar{X}}$ consisting of connected objects $\bar{U}$ such that $\bar{U}$ does not contain all the places of $K(\bar{U})$, and the adjoint functors ϕ^p and ϕ_p are functors between categories of presheaves. But $Et'_{\bar{X}}$ is a topologically generating full subcategory of the étale site $Et_{\bar{X}}$. Applying the associated sheaf functor, we get an exact sequence of sheaves

$$0 \to R^2\gamma_*\mathbb{Z} \to \phi_*\phi^* R^2\gamma_*\mathbb{Z} \to \prod_{v\in\bar{X}-Y} u_{v*}\mathbb{Z} \to 0 \tag{36}$$

since the sheaf associated to $\phi_p\phi^p\mathcal{P}^2\mathbb{Z}$ is just $\phi_*\phi^* R^2\gamma_*\mathbb{Z}$. In order to check this last claim, we consider the open–closed decomposition

$$\phi : Y_{et} \longrightarrow \bar{X}_{et} \longleftarrow \coprod_{v\in\bar{X}-Y} \underline{Sets} : u,$$

where the gluing functor $u^*\phi_*$ sends a sheaf $\mathcal{F}$ on Y_{et} to the collection of the stalks $(\mathcal{F}_v)_{v\in\bar{X}-Y}$ (here, $\mathcal{F}_v$ is the stalk of $\mathcal{F}$ at the geometric point $v : Spec(\mathbb{C}) \to Y$). It follows easily that $a(\phi_p P) = \phi_* a(P)$ for any presheaf P on Y, where a denotes the associated sheaf functor. Hence we have

$$a(\phi_p\phi^p\mathcal{P}^2\mathbb{Z}) \simeq \phi_* a(\phi^p\mathcal{P}^2\mathbb{Z}) \simeq \phi_*\phi^* a(\mathcal{P}^2\mathbb{Z}) \simeq \phi_*\phi^* R^2\gamma_*\mathbb{Z}.$$

In view of Proposition 6.14, we obtain the following result, where $r_2(K(\bar{U})) - U_\infty$ denotes the set of complex places of $K(\bar{U})$ which do not correspond to a point of $\bar{U}$.

THEOREM 6.15. *For any connected étale $\bar{X}$-scheme $\bar{U}$, we have*

$$R^2\gamma_*\mathbb{Z}(\bar{U}) = \Big(D^1_{K(\bar{U})}\Big/ \prod_{U_\infty - U(\mathbb{R})} \mathbb{S}^1\Big)^D$$
$$\simeq Hom(\mathcal{O}^\times_{K(\bar{U})}, \mathbb{Q}) \oplus \sum_{r_2(K(\bar{U}))-U_\infty} \mathbb{Z}.$$

COROLLARY 6.16. *Assume that F is totally imaginary. Then the étale sheaf $R^2\gamma_*\mathbb{Z}$ is acyclic for the global sections functor.*

Proof. In view of the exact sequence (36), it is enough to show that $\phi_*\phi^*R^2\gamma_*\mathbb{Z}$ is acyclic for the global sections functor. Here we denote by $\xi : Spec(F)_{et} \to \bar{X}_{et}$ the morphism induced by the inclusion of the generic point, and by $\mathcal{F}$ the presheaf on $Et_{Spec(F)}$ sending a finite extension K/F to $(D^1_K)^D$. Then we have $\phi_*\phi^*R^2\gamma_*\mathbb{Z} = \xi_*\mathcal{F}$. But for any finite Galois extension K/F of group G, the G-module $(D^1_K)^D$ is the product of a $\mathbb{Q}$-vector space by an induced G-module. It follows that the sheaf $\xi_*\mathcal{F}$ is acyclic for the global sections functor. □

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