TANNAKIAN TWISTS OF QUADRATIC FORMS AND ORTHOGONAL NORI MOTIVES

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ABSTRACT. We revisit classical results of Serre, Fröhlich and Saito in the theory of quadratic forms. Given a neutral Tannakian category $(\mathcal{T}, \omega)$ over a field k of characteristic $\neq 2$, another fiber functor η over a k-scheme X and an orthogonal object (M, q) in $\mathcal{T}$, we show formulas relating the torsor $\mathbf{Isom}^{\otimes}(\omega, \eta)$ to Hasse-Witt invariants of the quadratic space $\omega(M, q)$ and the symmetric bundle $\eta(M, q)$. We apply this result to various neutral Tannakian categories arising in different contexts. We first consider Nori's Tannakian category of essentially finite bundles over a reduced proper k-scheme X with a rational point, in order to formulate an analogue of the Serre-Fröhlich embedding problem for Nori's fundamental group scheme, and to decide whether or not this problem has a solution in terms of Hasse-Witt invariants. Then we consider Fontaine's Tannakian categories of B-admissible representations, in order to obtain a generalization of both the classical Serre-Fröhlich formula and Saito's analogous result for Hodge-Tate p-adic representations. Finally we consider Nori's category of mixed motives over a number field. These last two examples yield formulas relating the torsor of periods of an orthogonal motive to Hasse-Witt invariants of the associated Betti and de Rham quadratic forms and to Stiefel-Withney invariants of the associated local l-adic orthogonal representations. We give some computations for Artin motives and for the motive of a smooth hypersurface.

1. INTRODUCTION

Let k be a field of characteristic $\neq 2$, let $(\mathcal{T}, \omega)$ be a k-linear neutralized Tannakian category, and let η be another fiber functor with values in the category of vector bundles over a k-scheme X. We denote by $\mathcal{G}_\omega := \mathbf{Aut}^{\otimes}(\omega)$ the Tannaka dual and by $T_{\omega_X, \eta} := \mathbf{Isom}^{\otimes}(\omega, \eta)$ the $\mathcal{G}_\omega$-torsor over X of isomorphisms of tensor functors. An orthogonal object (M, q) in $\mathcal{T}$ is an object M endowed with a symmetric map $M \otimes M \to \mathbf{1}$ inducing an isomorphism $M \xrightarrow{\sim} M^{\vee}$, where $\mathbf{1}$ is the unit object and $M^{\vee}$ the dual of M. Such an orthogonal object yields a quadratic k-vector space $(\omega(M), q_\omega)$ and a symmetric bundle $(\eta(M), q_\eta)$ over X. An orthogonal object (M, q) may be seen as an orthogonal representation $\mathcal{G}_\omega \to \mathbf{O}(q_\omega)$. Composing with the determinant map $\mathbf{O}(q_\omega) \to \mathbb{Z}/2\mathbb{Z}$ we obtain a map $\delta_q^1 : \mathrm{Tors}(X, \mathcal{G}_\omega) \to H^1(X_{\mathrm{et}}, \mathbb{Z}/2\mathbb{Z})$. Similarly, the Pin-extension (see [13] Appendix I and [20] Appendix)

$$(1) \qquad 1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \widetilde{\mathbf{O}}(q_\omega) \longrightarrow \mathbf{O}(q_\omega) \longrightarrow 1$$

induces a map $\delta_q^2 : \mathrm{Tors}(X, \mathcal{G}_\omega) \to H^2(X_{\mathrm{et}}, \mathbb{Z}/2\mathbb{Z})$. Here $\mathrm{Tors}(X, \mathcal{G}_\omega)$ denotes the pointed set of $\mathcal{G}_\omega$-torsors over X, and δ_q^1 and δ_q^2 are cohomological invariants of degree 1 and 2 respectively. It is easily checked that the twist (in the sense of [8]) of q_ω by the $\mathcal{G}_\omega$-torsor $T_{\omega_X, \eta}$ is isometric to q_η. Applying the method of ([8] Section 6.3), we obtain the following result.

Theorem 1.1. *For any orthogonal object (M, q) in $\mathcal{T}$, we have the following identities in the étale cohomology ring $H^*(X_{\mathrm{et}}, \mathbb{Z}/2\mathbb{Z})$:*

$$\delta_q^1(T_{\omega_X, \eta}) = w_1(q_\omega) + w_1(q_\eta);$$

$$\delta_q^2(T_{\omega_X,\eta}) = w_2(q_\omega) + w_1(q_\omega) \cdot w_1(q_\omega) + w_1(q_\omega) \cdot w_1(q_\eta) + w_2(q_\eta).$$

In Section 4, we apply Theorem 1.1 to the Tannakian category giving rise to Nori's fundamental group scheme. Let X be an integral proper k-scheme with a rational point $x \in X(k)$. Then $\pi_1^N(X/k, x)$ is defined as the Tannakian dual of a certain category of essentially finite bundles. Recall that there is a 1-1 correspondence between the set of maps $\pi_1^N(X/k, x) \to G$ and the set of triples (T, G, t), where G is a finite flat k-group scheme, T a G-torsor over X and $t \in T(k)$ is a point lying over x. Let $\pi_1^N(X/k, x) \twoheadrightarrow G = \mathrm{Spec}(A)$ be a finite flat quotient. For any generator θ of the integrals of A, there is a "unit" quadratic form $\kappa_\theta : A^D \times A^D \to k$ and an orthogonal representation $G \to \mathbf{O}(\kappa_\theta)$. The map $\pi_1^N(X/k, x) \twoheadrightarrow G \to \mathbf{O}(\kappa_\theta)$ corresponds to an essentially finite symmetric bundle $(\mathcal{V}, q_\theta)$ over X. Pulling back (1) along $G \to \mathbf{O}(\kappa_\theta)$, we obtain a central extension

$$(2) \qquad 1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \widetilde{G}_\theta \overset{s_\theta}{\longrightarrow} G \longrightarrow 1.$$

Theorem 1.2. *Assume that the extension (2) is non-trivial. Then the following assertions are equivalent.*

(1) *We have*
$$w_2(\kappa_\theta) + w_1(\kappa_\theta) \cdot w_1(\kappa_\theta) + w_1(\kappa_\theta) \cdot w_1(q_\theta) + w_2(q_\theta) = 0.$$

(2) *There exist a rational point $\sigma \in G(k)$ and a commutative diagram*

where r is faithfully flat and c_σ denotes the conjugation by σ.

For G étale, there is a canonical choice θ_0 for θ, and and the form κ_{θ_0} generalize the classical unit form considered by Serre in [34]. In this case, the existence of a commutative diagram as in Theorem 1.2(2) is analogous to the existence of a solution to Serre's embedding problem [34] in the context of classical Galois theory. Theorem 1.2 expresses the obstruction to the existence of a solution in terms of Hasse-Witt invariants.

In Section 5, we apply Theorem 1.1 to Fontaine's Tannakian categories of B-admissible representations. Let k be a topological field, K/k an extension and B/k a filtered topological algebra such that $C := \mathrm{Gr}^0(B)$ is a t-Henselian field in the sense of ([14] Section 3 and [29]). We assume that B is (k, G_K)-regular and that $C^\times/C^{\times 2}$ is trivial. Let (V, q) be a quadratic k-vector space and let $\rho : G_K \to \mathbf{O}(q)(k)$ be an orthogonal continuous representation which is B-admissible. Then the Stiefel-Withney invariants $sw_1(\rho)$, $sw_2(\rho)$ and the spinor class $sp_2(\rho)$ are well defined. We consider the twisted form induced by q on $(V \otimes_k B)^{G_K}$.

Theorem 1.3. *We have the following identities in the Galois cohomology ring $H^*(G_K, \mathbb{Z}/2\mathbb{Z})$:*

(1) $w_1(q_D) = w_1(q) + sw_1(\rho)$.
(2) $w_2(q_D) = w_2(q) + w_1(q) \cdot sw_1(\rho) + sw_2(\rho) + sp_2(\rho)$.

For $k = K$ and $B = \overline{k}$ a separable closure endowed with the discrete topology and the trivial filtration, we obtain the classical Serre-Fröhlich Theorem [13]. For k a p-adic field, K/k a finite extension and $B = B_{dR}$ with its usual topology and filtration, we obtain a result of T. Saito (see [31] Theorem 1).

In Section 6, we apply Theorem 1.1 to the Tannakian category of Nori's mixed motives over a number field E with coefficients in E. Here the fiber functor ω (respectively η) is given by the Betti (respectively the de Rham) realization. For a motive M over E, we denote by $\mathcal{G}_{M/E}$ its motivic Galois group and by $\mathfrak{P}_M$ its torsor of formal periods in the sense of Kontsevich [23]. Recall that there is a canonical evaluation map $\mathfrak{P}_M \to \mathbb{C}$ whose image is the E-algebra $\mathbb{P}_M$ of classical periods of M. An orthogonal motive (M, q) yields quadratic forms q_B and q_{dR} on the Betti and the de Rham realizations, and a continuous orthogonal representation $\rho_\lambda : G_E \to \mathbf{O}(q_B)(E_\lambda)$ of the Galois group G_E on the λ-adic cohomology, for any finite place λ of E. We also consider their restrictions $\rho_{\lambda|G_{E_\lambda}}$ to the local Galois group G_{E_λ}. Similarly, for a real prime λ of E, complex conjugation on Betti cohomology gives a representation $\rho_{\lambda|G_{E_\lambda}} : G_\mathbb{R} \to \mathbf{O}(q_B)(\mathbb{R})$. We shall consider the Hasse-Witt invariants $w_i(q_B), w_i(q_{dR}) \in H^i(G_E, \mathbb{Z}/2\mathbb{Z})$ and the local Stiefel-Withney invariants $sw_i(\rho_{\lambda|G_{E_\lambda}}) \in H^i(G_{E_\lambda}, \mathbb{Z}/2\mathbb{Z})$, as well as the Global Stiefel-Withney invariant $sw_1(\rho_\lambda) := \det(\rho_\lambda) \in H^1(G_E, \mathbb{Z}/2\mathbb{Z})$. Finally, an orthogonal motive gives an orthogonal representation $\mathcal{G}_{M/E} \to \mathbf{O}(q_B)$ hence a central extension

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \widetilde{\mathcal{G}}_{M/E} \longrightarrow \mathcal{G}_{M/E} \longrightarrow 1.$$

Then $\delta_q^2(\mathfrak{P}_M)$ exactly computes the obstruction for the existence of a $\widetilde{\mathcal{G}}_{M/E}$-torsor lifting $\mathfrak{P}_M$. In particular, if $\delta_q^2(\mathfrak{P}_M) \neq 0$ then the canonical map $\mathcal{G}_E \to \mathcal{G}_{M/E}$ does not factor through $\widetilde{\mathcal{G}}_{M/E}$, where $\mathcal{G}_E$ is the full motivic Galois group. On the other hand, $\delta_q^1(\mathfrak{P}_M) \in E^*/E^{*2}$ determines the periods of the determinant motive $\det_E(M)$, in the sense that $\mathbb{P}_{\det_E(M)} = E\left(\sqrt{\delta_q^1(\mathfrak{P}_M)}\right)$.

Theorem 1.4. *Let (M, q) be an orthogonal motive over E with E-coefficients. The following identities take place in the ring $H^*(G_E, \mathbb{Z}/2\mathbb{Z})$.*

(1) *We have*
$$\delta_q^2(\mathfrak{P}_M) \;=\; w_2(q_B) + w_1(q_B) \cdot w_1(q_B) + w_1(q_B) \cdot w_1(q_{dR}) + w_2(q_{dR}).$$

(2) *Assume that either $E = \mathbb{Q}$ or that E is totally imaginary. Then one has*
$$\delta_q^2(\mathfrak{P}_M) = \sum_\lambda sw_2(\rho_{\lambda|G_{E_\lambda}}) + sp_2(\rho_{\lambda|G_{E_\lambda}})$$

in $H^2(G_E, \mathbb{Z}/2\mathbb{Z}) \subset \bigoplus_\lambda H^2(G_{E_\lambda}, \mathbb{Z}/2\mathbb{Z})$, where λ runs over the set of all places of E.

(3) *Let V be a projective smooth variety over E of even dimension n. Then there exists a unique orthogonal structure on $M = h^n(V)(n/2)$ inducing the usual quadratic form on Betti cohomology $H^n(V(\mathbb{C}), \mathbb{Q}(n/2))$.*

(4) *For $(M, q) = (h^n(V)(n/2), q)$ and any finite place λ, one has*
$$\delta_q^1(\mathfrak{P}_M) = sw_1(\rho_\lambda).$$

In order to show that Poincaré duality is motivic, so that $h^2(V)(n/2)$ inherits an orthogonal structure, we need to work out the Kunneth formula for Nori motives in Section 7. Taking this for granted, the proof of Theorem 1.4 is a simple combination of Theorem 1.1, Theorem 1.3 and p-adic Hodge theory. Note also that it is conjectured that the group of connected components of the full motivic Galois group $\mathcal{G}_E$ is the usual absolute Galois group G_E. This would force $\det_E(M)$ to be an orthogonal Artin motive, and the identity $\delta_q^1(\mathfrak{P}_M) = sw_1(\rho_\lambda)$ would immediately follow. One may avoid this conditional argument using the fact, due to T. Saito, that $sw_1(\rho_l)$ is independent on l, which in turn relies on the Weil conjectures.

Finally, we give in Section 6 examples and computations for certain orthogonal motives. We treat in details the case of certain explicit Artin motives, and we compute some these invariants for the orthogonal motives of the form $h^n(V)(n/2)$ where V is an hypersurface or a complete intersection. For example, we have the following

Corollary 1.5. *Let V be a smooth hypersurface of even dimension $n \geq 2$ in $\mathbf{P}_E^{n+1}$, defined by an homogeneous polynomial of degree d, and let $(M = h^n(V)(n/2), q)$ be the corresponding orthogonal motive. Then we have*

$$\delta_q^1(\mathfrak{P}_M) = \begin{cases} (-1)^{\frac{d-1}{2}} \cdot \mathrm{disc_d}(f) \text{ if } d \text{ is odd} \\ (-1)^{\frac{d}{2} \cdot \frac{n+2}{2}} \cdot \mathrm{disc_d}(f) \text{ if } d \text{ is even} \end{cases}$$

and

$$\delta_q^2(\mathfrak{P}_M) = w_2(q_{dR}) + \begin{cases} \frac{d-1}{2}(-1,-1) \text{ if } d \text{ is odd} \\ \frac{n}{4}(1 + \frac{d}{2})(-1,-1) \text{ if } d \text{ is even and } n \equiv 0 \bmod 4 \\ (-1, \mathrm{disc_d}(f)) + (\frac{n+2}{4})(1 + \frac{d}{2})(-1,-1) \text{ if } d \text{ is even and } n \equiv 2 \bmod 4 \end{cases}$$

where $\mathrm{disc_d}(f)$ is the divided discriminant of f in the sense of [32].

CONTENTS

2. Preliminaries

Throughout the paper k denotes a field of characteristic $\neq 2$. In this section we review basic definitions on torsors and symmetric bundles over a scheme (see [8]). One should note that a symmetric bundle over $\mathrm{Spec}(k)$ is given by a pair (V, q) where V is a k-vector space of finite dimension and q is a non degenerate quadratic form on V.

2.1. **The classifying topos of a pro-group-scheme.** Let $\mathcal{S}$ be a topos and let G be a group-object in $\mathcal{S}$. The topos B_G is the category of objects of $\mathcal{S}$ endowed with a left G-action. We denote by $\mathbf{Tors}(\mathcal{S}, G)$ the category of right G-torsors in $\mathcal{S}$. Recall that $\mathbf{Tors}(\mathcal{S}, G)$ is a groupoid, i.e. any morphism of this category is an isomorphism. There is a canonical morphism $\pi : B_G \to \mathcal{S}$ whose inverse image functor sends a a sheaf $\mathcal{F}$ on $\mathcal{S}$ to the object of B_G given by $\mathcal{F}$ with trivial G-action. In particular $\pi^* G$ is the group-object of B_G given by G with trivial left G-action. The topos B_G has a universal $\pi^* G$-torsor E_G given by G on which G acts both on the left and on the right by multiplication.

Theorem 2.1. *The functor*

$$\begin{array}{ccc} \mathbf{Homtop}_{\mathcal{S}}(\mathcal{S}, B_G)^{op} & \longrightarrow & \mathbf{Tors}(\mathcal{S}, G) \\ f & \longmapsto & f^* E_G \end{array}$$

is an equivalence of groupoids.

Here $\mathbf{Homtop}_{\mathcal{S}}(\mathcal{S}, B_G)$ denotes the category of morphisms of $\mathcal{S}$-topoi from $\mathcal{S}$ to B_G (with respect to $\mathrm{Id} : \mathcal{S} \to \mathcal{S}$ and $\pi : B_G \to \mathcal{S}$), and $(-)^{op}$ denotes the opposite category. Given a G-torsor Y, we denote the corresponding morphism of topoi by the same symbol:

$$Y : \mathcal{S} \longrightarrow B_G.$$

The morphism Y can be described as follows: For any sheaf $\mathcal{F}$ on B_G, one has

$$Y^*(\mathcal{F}) = Y \wedge^G \mathcal{F} := (Y \times \mathcal{F})/G$$

where G acts on the product $Y \times \mathcal{F}$ via the given right (respectively left) action on Y (respectively on $\mathcal{F}$). A morphism of group-objects $f : G \to G'$ in $\mathcal{S}$ (i.e. a morphism of sheaves of groups on $\mathcal{S}$) induces a morphism of topoi $B_f : B_G \to B_{G'}$.

We shall use the following variant. Let $G := \text{"}\varprojlim\text{"} G_i$ be a pro-group in $\mathcal{S}$ indexed by a filtered small category I. Recall that, for another group H, we have

$$\mathrm{Hom}_{\mathrm{Grp}(\mathcal{S})}(G, H) = \varinjlim \mathrm{Hom}_{\mathrm{Grp}(\mathcal{S})}(G_i, H)$$

where $\mathrm{Grp}(\mathcal{S})$ denotes the category of group-objects in $\mathcal{S}$. The classifying topos of the pro-group G is defined as an inverse limit in 2-category of topoi:

$$B_G := \varprojlim B_{G_i}.$$

By definition of the inverse limit, the canonical functor

$$\mathbf{Homtop}_{\mathcal{S}}(\mathcal{S}, B_G) \xrightarrow{\sim} \varprojlim \mathbf{Homtop}_{\mathcal{S}}(\mathcal{S}, B_{G_i})$$

is an equivalence. It follows that B_G classifies G-*protorsors*. A G-protorsor is given by a G_i-torsors T_i for any $i \in I$ and an isomorphism of G_j-torsors $f_\alpha : T_i \wedge^{G_i} G_j \simeq T_j$ for any $\alpha : i \to j$ in I, such that the isomorphisms f_α satisfy the obvious compatibility constraint for composite maps in I. We denote by $\mathbf{Tors}(\mathcal{S}, G)$ the groupoid of G-protorsors. Theorem 2.1 gives an equivalence

$$(3) \qquad\qquad \mathbf{Homtop}_{\mathcal{S}}(\mathcal{S}, B_G)^{op} \xrightarrow{\sim} \mathbf{Tors}(\mathcal{S}, G).$$

Let X be a scheme and let $G := "\varprojlim" G_i$ be a pro-object in the category of X-group schemes of finite type. The classifying topos B_G is defined as above by taking $\mathcal{S}$ to be the (big) fppf-topos X_{fppf}. Here G_i is of course seen as a group-object in X_{fppf} by Yoneda.

2.2. **Hasse-Witt invariants.**

2.2.1. Let k^s/k be a separable closure of k. We denote by $G_k := \text{Gal}(k^s/k)$ the absolute Galois group of k. Suppose that q is a non degenerate quadratic form of rank n over k. We choose a diagonal form $< a_1, a_2, \cdots, a_n >$ of q with $a_i \in k^*$, and consider the cohomology classes

$$(a_i) \in k^\times/(k^\times)^2 \simeq H^1(G_k, \mathbb{Z}/2\mathbb{Z}).$$

For $1 \leq m \leq n$, the mth *Hasse-Witt invariant* of q is defined to be

$$(4) \qquad\qquad w_m(q) = \sum_{1 \leq i_1 < \cdots < i_m \leq n} (a_{i_1}) \cdots (a_{i_m}) \in H^m(G_k, \mathbb{Z}/2\mathbb{Z})$$

where $(a_{i_1}) \cdots (a_{i_m})$ is the cup product. Moreover we set $w_0(q) = 1$ and $w_m(q) = 0$ for $m > n$. It can be shown that $w_m(q)$ does not depend on the choice of the diagonal form of q.

2.2.2. This definition can be generalized over more general base schemes as follows (see [8] Section 4). Let $X \to \text{Spec}(\mathbb{Z}[1/2])$ be a $\mathbb{Z}[1/2]$-scheme. A *symmetric bundle over X* is a pair (V, q) where V is a locally free $\mathcal{O}_X$-module of constant finite rank and $q : V \otimes_{\mathcal{O}_X} V \to \mathcal{O}_X$ is a symmetric map of $\mathcal{O}_X$-modules inducing an isomorphism $V \xrightarrow{\sim} V^\vee$, where $V^\vee := \underline{\text{Hom}}_{\mathcal{O}_X}(V, \mathcal{O}_X)$ is the dual of V. Given two symmetric bundles (V, q) and (W, r) of same rank over X, we consider the sheaf $\mathbf{Isom}(q, r)$ on X_{fppf} given by $\mathbf{Isom}(q, r)(Y) := \text{Isom}(q_Y, r_Y)$ for any X-scheme Y, where $\text{Isom}(q_Y, r_Y)$ is the set of isometries between the symmetric vector bundles (V_Y, q_Y) and (W_Y, r_Y) over Y, obtained from (V, q) and (W, r) by base-change along $Y \to X$. We define the orthogonal group $\mathbf{O}(q)$ as the group $\mathbf{Isom}(q, q)$. Consider the standard form $(\mathcal{O}_X^n, t_n = \sum x_i^2)$ of rank n over X. We set $\mathbf{O}(n) := \mathbf{Isom}(t_n, t_n)$.

Let (V, q) be a symmetric bundle of rank n over X. The sheaf $\mathbf{Isom}(t_n, q)$ on X_{fppf} is a $\mathbf{O}(n)$-torsor on X_{fppf}, hence Theorem 2.1 provides us with a canonical map

$$(5) \qquad\qquad \mathbf{Isom}(t_n, q) : X_{\text{fppf}} \longrightarrow B_{\mathbf{O}(n)}.$$

There is a canonical isomorphism of graded $\mathbb{Z}/2\mathbb{Z}$-algebras

$$H^*(B_{\mathbf{O}(n)}, \mathbb{Z}/2\mathbb{Z}) \simeq A[HW_1, \cdots, HW_n]$$

where HW_m has degree m and A is the étale cohomology algebra $H^*(X_{\text{et}}, \mathbb{Z}/2\mathbb{Z})$. The mth *Hasse-Witt invariant* of q is defined to be the pull-back

$$(6) \qquad\qquad w_m(q) := \mathbf{Isom}(t_n, q)^*(HW_m) \in H^m_{\text{et}}(X, \mathbb{Z}/2\mathbb{Z})$$

of HW_m along (5). When $X = \mathrm{Spec}(k)$ the definitions (4) and (6) agree.

2.3. Twists of quadratic forms. Let $X \to \mathrm{Spec}(\mathbb{Z}[1/2])$ be a $\mathbb{Z}[1/2]$-scheme, let G be a pro-group-scheme over X, let (V, q) be a symmetric bundle over X of rank n, let $\rho : G \to \mathbf{O}(q)$ be an orthogonal representation and let T be a G-protorsor over X. The equivalence (3) yields the morphism of topoi

$$X_{\mathrm{fppf}} \xrightarrow{T} B_G \xrightarrow{B_\rho} B_{\mathbf{O}(q)}$$

where B_ρ is induced by ρ. By Theorem 2.1 (again), this morphism corresponds to an $\mathbf{O}(q)$-torsor. Any such $\mathbf{O}(q)$-torsor is of the form $\mathbf{Isom}(q, r)$ for an essentially unique symmetric bundle (W, r) of rank n, which we denote by $T \wedge^{\mathcal{G}} q$. Then $T \wedge^{\mathcal{G}} q$ is called *the twist of q by the G-protorsor T*.

3. Tannakian twists of quadratic forms

For Y a scheme over the field k, we denote by $\mathrm{VB}(Y)$ the category of vector bundles over Y, i.e. the category of locally free finitely generated $\mathcal{O}_Y$-modules. We consider a k-linear neutral Tannakian category $\mathcal{T}$ endowed with a fiber functor $\omega : \mathcal{T} \to \mathrm{Vec}_k$, where Vec_k is the category of finite dimensional k-vector spaces. We denote by $\mathcal{G}_\omega := \mathbf{Aut}^{\otimes}(\omega)$ its Tannaka dual. Recall that $\mathcal{G}_\omega$ is defined via its functor of points: For any k-scheme Y, the group

$$\mathcal{G}_\omega(Y) = \mathbf{Aut}^{\otimes}(\omega)(Y) := \mathrm{Aut}^{\otimes}(\omega_Y)$$

is the group of automorphisms of the tensor functor

$$\omega_Y : \mathcal{T} \xrightarrow{\omega} \mathrm{Vec}_k \longrightarrow \mathrm{VB}(Y)$$

where the second functor is induced by pull-back along the structure map $Y \to \mathrm{Spec}(k)$. Then $\mathcal{G}_\omega$ is an affine k-group scheme ([6] Theorem 2.11). In fact $\mathcal{G}_\omega$ is most naturally defined as a pro-algebraic group. Indeed, let $\{\mathcal{T}_i \subseteq \mathcal{T}, i \in I\}$ be a partially ordered set of sub-Tannakian categories, such that $\mathcal{T} = \bigcup_{i \in I} \mathcal{T}_i$ and $\mathcal{T}_i$ is generated as a Tannakian category by a single object $X_i \in \mathcal{T}_i$. We denote by $\omega_i : \mathcal{T}_i \hookrightarrow \mathcal{T} \to \mathrm{Vec}_k$ the fiber functor induced by ω. Then $\mathcal{G}_{\omega_i} := \mathbf{Aut}^{\otimes}(\omega_i)$ is an affine algebraic group over k (i.e. an affine k-group scheme of finite type), and one has $\mathcal{G}_\omega = \varprojlim \mathcal{G}_{\omega_i}$. We consider $\mathcal{G}_\omega$ as the pro-object

$$\mathcal{G}_\omega := "\varprojlim" \mathcal{G}_{\omega_i}.$$

in the category of affine algebraic groups over k.

3.1. The protorsor $T_{\omega_X, \eta}$. Let $(\mathcal{T}, \omega)$ be a k-linear neutral Tannakian category as above and let X be a k-scheme. We denote by

$$\omega_X : \mathcal{T} \xrightarrow{\omega} \mathrm{Vec}_k \xrightarrow{f^*} \mathrm{VB}(X)$$

the fiber functor over X induced by ω, where $f : X \to \mathrm{Spec}(k)$ is the structure map. Let $\eta : \mathcal{T} \to \mathrm{VB}(X)$ be another fiber functor over X. We also denote by $\omega_{i,X} : \mathcal{T}_i \to \mathrm{VB}(X)$ and $\eta_i : \mathcal{T}_i \to \mathrm{VB}(X)$ the fiber functors induced by ω_X and η respectively. We consider the pro-X-scheme

$$\mathbf{Isom}^{\otimes}(\omega_X, \eta) := "\varprojlim" \mathbf{Isom}^{\otimes}(\omega_{i,X}, \eta_i)$$

of isomorphisms of tensor functors from ω_X to η. Here $\mathbf{Isom}^{\otimes}(\omega_{i,X}, \eta_i)$ is defined via its functor of points. More precisely, for any X-scheme Y,

$$\mathbf{Isom}^{\otimes}(\omega_{i,X}, \eta_i)(Y) = \mathrm{Isom}^{\otimes}(\omega_{i,Y}, \eta_{i,Y})$$

is the set of morphisms of tensor functors $\omega_{i,Y} \to \eta_{i,Y}$ where $\omega_{i,Y}$ and $\eta_{i,Y}$ are the fiber functors over Y induced by ω_i and η_i respectively. Note that any morphism $\omega_{i,Y} \to \eta_{i,Y}$ necessarily is an isomorphism (see [6] Proposition 1.13 for the affine case). Then $\mathbf{Isom}^\otimes(\omega_X, \eta)$ is a projective system of faithfully flat affine X-schemes of finite type. Similarly, one has

$$\mathcal{G}_{\omega_i}(Y) = \mathrm{Aut}^\otimes(\omega_{i,Y})$$

for any k-scheme Y, hence there is a right action of $\mathcal{G}_\omega$ on $\mathbf{Isom}^\otimes(\omega_X, \eta)$ over X which is defined by composition. This action turns $\mathbf{Isom}^\otimes(\omega_X, \eta)$ into a $\mathcal{G}_\omega$-protorsor over X (see [6] Theorem 3.2). We denote by $T_{\omega_{i,X}, \eta_i} := \mathbf{Isom}^\otimes(\omega_{i,X}, \eta_i)$ the $\mathcal{G}_{\omega_i}$-torsor and by $T_{\omega_X, \eta} := \mathbf{Isom}^\otimes(\omega_X, \eta)$ the $\mathcal{G}_\omega$-protorsor just defined, and also by

$$T_{\omega_{i,X}, \eta_i} : X_{\mathrm{fppf}} \longrightarrow B_{\mathcal{G}_{\omega_i}} \text{ and } T_{\omega_X, \eta} : X_{\mathrm{fppf}} \longrightarrow B_{\mathcal{G}_\omega} := \varprojlim B_{\mathcal{G}_{\omega_i}}$$

the corresponding morphisms of topoi.

3.2. **Orthogonal objects.** We denote by $\mathbf{1}$ the unit object of $\mathcal{T}$. For an object $M \in \mathcal{T}$ we denote by $M^\vee = \underline{\mathrm{Hom}}(M, \mathbf{1})$ its dual.

Definition 3.1. *An* orthogonal object (M, q) *of* $\mathcal{T}$ *is an object* $M \in \mathcal{T}$ *endowed with a symmetric map* $q : M \otimes M \to \mathbf{1}$ *such that the induced map* $M \to M^\vee$ *is an isomorphism.*

We define the category of orthogonal objects of $\mathcal{T}$ in the obvious way: for two orthogonal objects (M, q) and (M', q'), a morphism $f : (M, q) \to (M', q')$ is a map $f : M \to M'$ in $\mathcal{T}$ such that the diagram

$$
\begin{array}{ccc}
M \otimes M & \xrightarrow{\ q\ } & \mathbf{1} \\
\downarrow{\scriptstyle f \otimes f} & & \downarrow{\scriptstyle \mathrm{Id}} \\
M' \otimes M' & \xrightarrow{\ q'\ } & \mathbf{1}
\end{array}
$$

commutes. If (M, q) is an orthogonal object of $\mathcal{T}$, then

$$q_\omega : \omega(M) \otimes \omega(M) \simeq \omega(M \otimes M) \xrightarrow{\omega(q)} \omega(\mathbf{1}) \simeq k$$

is a non-degenerate symmetric bilinear form. By Tannaka duality, an object $M \in \mathcal{T}$ can be seen as a finite dimensional representation of $\mathcal{G}_\omega$ of the form

$$\rho_M : \mathcal{G}_\omega \longrightarrow \mathbf{GL}(\omega(M))$$

where $\mathbf{GL}(\omega(M))$ is seen as a k-group scheme. If M admits an orthogonal structure (M, q), then we have a factorization

$$\mathcal{G}_\omega \longrightarrow \mathbf{O}(q_\omega) \longrightarrow \mathbf{GL}(\omega(M))$$

hence an orthogonal representation

$$\rho_q : \mathcal{G}_\omega \longrightarrow \mathbf{O}(q_\omega),$$

where $\mathbf{O}(q_\omega)$ denotes the orthogonal k-group scheme associated with q_ω.

The category of orthogonal k-linear representations of $\mathcal{G}_\omega$ is the category of orthogonal objects of the Tannakian category $\mathrm{Rep}_k(\mathcal{G}_\omega)$ of k-linear representations of $\mathcal{G}_\omega$. The fact that the functor

$$
\begin{array}{ccc}
\mathcal{T} & \longrightarrow & \mathrm{Rep}_k(\mathcal{G}_\omega) \\
M & \longmapsto & \rho_M
\end{array}
$$

is an equivalence of rigid tensor k-linear categories ([6] Theorem 2.11) immediately gives the following

Proposition 3.2. *The functor $(M, q) \longmapsto \rho_q$ is an equivalence from the category of orthogonal objects of $\mathcal{T}$ to the category orthogonal k-linear representations of $\mathcal{G}_\omega$.*

3.3. The maps δ_q^1 and δ_q^2. Let (M, q) be an orthogonal object of $\mathcal{T}$, and let X be a k-scheme. The orthogonal representation

$$\rho_q : \mathcal{G}_\omega \longrightarrow \mathbf{O}(q_\omega)$$

yields canonical maps

$$\delta_q^i : \mathrm{Tors}(X, \mathcal{G}_\omega) \longrightarrow H^i(X_{\mathrm{et}}, \mathbb{Z}/2\mathbb{Z})$$

for $i = 1, 2$. Here $\mathrm{Tors}(X, \mathcal{G}_\omega)$ is the set of isomorphism classes of $\mathcal{G}_\omega$-pro-torsors in the topos X_{fppf}. The maps δ_q^1 and δ_q^2 are defined as follows. The map

$$\delta_q^1 : \mathrm{Tors}(X, \mathcal{G}_\omega) \longrightarrow H^1(X_{\mathrm{et}}, \mathbb{Z}/2\mathbb{Z}).$$

is induced by the composite map

$$\mathcal{G}_\omega \longrightarrow \mathbf{O}(q_\omega) \overset{\det_{\mathbf{O}(q_\omega)}}{\longrightarrow} \mathbb{Z}/2\mathbb{Z}.$$

Indeed, a $\mathcal{G}_\omega$-pro-torsor T in X_{fppf} gives a map $T : X_{\mathrm{fppf}} \to B_{\mathcal{G}_\omega}$, hence a map $X_{\mathrm{fppf}} \to B_{\mathcal{G}_\omega} \to B_{\mathbb{Z}/2\mathbb{Z}}$, hence a class

$$\delta_q^1(T) \in H^1(X_{\mathrm{fppf}}, \mathbb{Z}/2\mathbb{Z}) \simeq H^1(X_{\mathrm{et}}, \mathbb{Z}/2\mathbb{Z}).$$

The map δ_q^2 is defined using the canonical central extension (see [13] Appendix I and [20] Appendix)

$$(7) \qquad\qquad 1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathbf{O}}(q_\omega) \to \mathbf{O}(q_\omega) \to 1.$$

Indeed, (7) gives a class $[C_q] \in H^2(B_{\mathbf{O}(q_\omega)}, \mathbb{Z}/2\mathbb{Z})$. For a $\mathcal{G}_\omega$-pro-torsor T, the class $\delta_q^2(T)$ is defined as the pull-back of $[C_q]$ along the morphism of topoi

$$X_{\mathrm{fppf}} \overset{T}{\to} B_{\mathcal{G}_\omega} \to B_{\mathbf{O}(q_\omega)}.$$

Another way to define $\delta_q^2(T)$ is the following. The representation ρ_q factors through $\mathcal{G}_{\omega_i} \to \mathbf{O}(q_\omega)$ for any $\mathcal{T}_i$ containing M. Pulling back (7) along $\mathcal{G}_{\omega_i} \to \mathbf{O}(q_\omega)$ gives a central extension of affine group-schemes

$$(8) \qquad\qquad 1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathcal{G}}_{\omega_i} \to \mathcal{G}_{\omega_i} \to 1.$$

and an exact sequence of pointed sets (see [16]IV.4.2.10)

$$(9) \qquad \ldots \longrightarrow H^1(X_{\mathrm{fppf}}, \widetilde{\mathcal{G}}_{\omega_i}) \longrightarrow H^1(X_{\mathrm{fppf}}, \mathcal{G}_{\omega_i}) \overset{\delta_q^2}{\longrightarrow} H^2(X_{\mathrm{fppf}}, \mathbb{Z}/2\mathbb{Z}) \longrightarrow \ldots$$

in which δ_q^2 appears as a boundary map.

3.4. Comparison formulas. Let $(\mathcal{T}, \omega)$ be a k-linear neutral Tannakian category as above, let X be a k-scheme, let $\eta : \mathcal{T} \to \mathrm{VB}(X)$ be a fiber functor over X, and let (M, q) be an orthogonal object of $\mathcal{T}$.

We consider the quadratic form q_ω over k and the following symmetric vector bundle over X:

$$q_\eta : \eta(M) \otimes_{\mathcal{O}_X} \eta(M) \simeq \eta(M \otimes M) \overset{\eta(q)}{\longrightarrow} \eta(\mathbf{1}) \simeq \mathcal{O}_X.$$

Cup-product turns $H^*(X_{\mathrm{et}}, \mathbb{Z}/2\mathbb{Z})$ into a commutative $\mathbb{Z}/2\mathbb{Z}$-algebra, and the structure map $f : X \to \mathrm{Spec}(k)$ induces a ring homomorphism

$$H^*(\mathrm{Spec}(k)_{\mathrm{et}}, \mathbb{Z}/2\mathbb{Z}) \to H^*(X_{\mathrm{et}}, \mathbb{Z}/2\mathbb{Z})$$

so that we may consider $w_i(q_\omega) \in H^i(\mathrm{Spec}(k)_{\mathrm{et}}, \mathbb{Z}/2\mathbb{Z})$ as an element of $H^i(X_{\mathrm{et}}, \mathbb{Z}/2\mathbb{Z})$.

Theorem 3.3. *For any orthogonal object (M, q) of $\mathcal{T}$, we have identities in $H^*(X_{\mathrm{et}}, \mathbb{Z}/2\mathbb{Z})$:*

$$\delta_q^1(T_{\omega_X, \eta}) = w_1(q_\omega) + w_1(q_\eta);$$

$$\delta_q^2(T_{\omega_X, \eta}) = w_2(q_\omega) + w_1(q_\omega) \cdot w_1(q_\omega) + w_1(q_\omega) \cdot w_1(q_\eta) + w_2(q_\eta).$$

Proof. We consider the twist $T_{\omega_X, \eta} \wedge^{\mathcal{G}_{\omega_X}} q_\omega$ of the form q_{ω_X} by the torsor $T_{\omega_X, \eta}$, that we simply denote by $T_{\omega_X, \eta} \wedge^{\mathcal{G}_\omega} q_\omega$ (see Section 2.3). By ([8] Corollary 6.5), Theorem 3.3 follows from the following

Lemma 3.4. *The twisted form $T_{\omega_X, \eta} \wedge^{\mathcal{G}_\omega} q_\omega$ is isometric to q_η.*

Proof. We simply denote by by $\rho : \mathcal{G}_{\omega_X} \to \mathbf{O}(q_{\omega_X})$ the morphism of X-group schemes induced by ρ_q (by base change). The form $T_{\omega_X, \eta} \wedge^{\mathcal{G}_\omega} q_\omega$ is determined by the map

$$(10) \qquad B_\rho \circ T_{\omega_X, \eta} : X_{\mathrm{fppf}} \xrightarrow{T_{\omega_X, \eta}} B_{\mathcal{G}_{\omega_X}} \xrightarrow{B_\rho} B_{\mathbf{O}(q_{\omega_X})}.$$

Let $\mathcal{T}_i \subseteq \mathcal{T}$ be a Tannakian subcategory which is generated by a single object, and such that $M \in \mathcal{T}_i$. The following diagram commutes:

In particular we have an isometry

$$T_{\omega_{i,X}, \eta_i} \wedge^{\mathcal{G}_{\omega_i}} q_\omega \simeq T_{\omega_X, \eta} \wedge^{\mathcal{G}_\omega} q_\omega$$

hence one may suppose that $\mathcal{T} = \mathcal{T}_i$ is generated by a single object. In other words, we may assume that $\mathcal{G}_\omega$ is an algebraic group and that $T_{\omega_X, \eta}$ is a $\mathcal{G}_\omega$-torsor in the usual sense.

In order to ease the notations, we set $T := T_{\omega_X, \eta}$. So the form $T \wedge^{\mathcal{G}_\omega} q_\omega$ is determined by the morphism of topoi $B_\rho \circ T$ which is in turn determined by the $\mathbf{O}(q_{\omega_X})$-torsor over X given by $(B_\rho \circ T)^*(E_{\mathbf{O}(q_{\omega_X})})$. Here $E_{\mathbf{O}(q_{\omega_X})}$ denotes the $\mathbf{O}(q_{\omega_X})$-torsor of $B_{\mathbf{O}(q_{\omega_X})}$ given by the object $\mathbf{O}(q_{\omega_X})$ on which $\mathbf{O}(q_{\omega_X})$ acts both on the left and on the right by multiplication. Similarly, the form q_η is determined by the $\mathbf{O}(q_{\omega_X})$-torsor $\mathbf{Isom}(q_{\omega_X}, q_\eta)$. Hence $T \wedge^{\mathcal{G}_\omega} q_\omega$ is isometric to q_η if and only if $(B_\rho \circ T)^*(E_{\mathbf{O}(q_{\omega_X})})$ and $\mathbf{Isom}(q_{\omega_X}, q_\eta)$ are isomorphic as $\mathbf{O}(q_{\omega_X})$-torsors. We have

$$(B_\rho \circ T)^*(E_{\mathbf{O}(q_{\omega_X})}) = T^* \circ B_\rho^*(E_{\mathbf{O}(q_{\omega_X})}) = \mathbf{Isom}^{\otimes}(\omega_X, \eta) \wedge^{\mathcal{G}_{\omega_X}} \mathbf{O}(q_{\omega_X})$$

where $\mathcal{G}_{\omega_X}$ acts on the left on $\mathbf{O}(q_{\omega_X})$ via $\rho_q : \mathcal{G}_{\omega_X} \to \mathbf{O}(q_{\omega_X})$ and on the right on $\mathbf{Isom}^{\otimes}(\omega_X, \eta)$ by composition.

Let us define a map

$$\mu : \mathbf{Isom}^{\otimes}(\omega_X, \eta) \times \mathbf{O}(q_{\omega_X}) \longrightarrow \mathbf{Isom}(q_{\omega_X}, q_\eta)$$

on points as follows. For any X-scheme Y, we define

$$\begin{array}{rcl} \mu(Y) : \ \mathrm{Isom}^{\otimes}(\omega_Y, \eta_Y) \times \mathrm{O}(q_{\omega_Y}) & \longrightarrow & \mathrm{Isom}(q_{\omega_Y}, q_{\eta_Y}) \\ (\alpha, \sigma) & \longmapsto & \alpha(M) \circ \sigma \end{array}$$

where $\mathrm{O}(q_{\omega_Y}) := \mathbf{O}(q_{\omega_X})(Y)$ and

$$\alpha(M) : \omega_Y(M) \xrightarrow{\sim} \eta_Y(M)$$

is just given by evaluating the natural transformation α at the object M. In order to show that $\mu(Y)$ is well defined, we simply need to observe that $\alpha(M)$ is an isometry, i.e. respects the quadratic forms q_{ω_Y} and q_{η_Y}. But q_{ω_Y} and q_{η_Y} are both induced by the pairing $q : M \otimes M \longrightarrow \mathbf{1}$ in $\mathcal{T}$ and α is a morphism of tensor functors, so that we have a commutative diagram

$$
\begin{array}{ccccccc}
\omega_Y(M) \otimes_{\mathcal{O}_Y} \omega_Y(M) & \xrightarrow{\simeq} & \omega_Y(M \otimes M) & \xrightarrow{\omega_Y(q)} & \omega_Y(\mathbf{1}) & \xrightarrow{\simeq} & \mathcal{O}_Y \\
\downarrow{\scriptstyle \alpha(M) \otimes \alpha(M)} & & \downarrow{\scriptstyle \alpha(M \otimes M)} & & \downarrow{\scriptstyle \alpha(\mathbf{1})} & & \downarrow{\scriptstyle \mathrm{Id}} \\
\eta_Y(M) \otimes_{\mathcal{O}_Y} \eta_Y(M) & \xrightarrow{\simeq} & \eta_Y(M \otimes M) & \xrightarrow{\eta_Y(q)} & \eta_Y(\mathbf{1}) & \xrightarrow{\simeq} & \mathcal{O}_Y
\end{array}
$$

where the left and the right hand side squares commute because α is a morphism of tensor functors (see [6] Definition 1.12). The upper row (respectively the lower row) is the form q_{ω_Y} (respectively q_{η_Y}). Hence $\alpha(M)$ is an isometry, so that $\mu(Y)$ is well defined, hence so is μ (note that $\mu(Y)$ is functorial in Y).

Moreover, $\mu(Y)$ is invariant under the action of $\mathcal{G}_{\omega_X}(Y) = \mathrm{Aut}^{\otimes}(\omega_Y)$ on the set $\mathrm{Isom}^{\otimes}(\omega_Y, \eta_Y) \times \mathrm{O}(q_{\omega_Y})$. Indeed, an element $g \in \mathcal{G}_{\omega_X}(Y)$ takes $(\alpha, \sigma) \in \mathrm{Isom}^{\otimes}(\omega_Y, \eta_Y) \times \mathrm{O}(q_{\omega_Y})$ to $(\alpha \circ g^{-1}, \rho(g) \circ \sigma)$, and one has

$$
\mu(Y)(\alpha \circ g^{-1}, \rho(g) \circ \sigma) = \alpha(M) \circ \rho(g)^{-1} \circ \rho(g) \circ \sigma = \alpha \circ \sigma = \mu(Y)(\alpha, \sigma).
$$

Hence μ is invariant under the action of $\mathcal{G}_{\omega_X}$. Equivalently, μ induces a morphism

$$
\tilde{\mu} : \mathbf{Isom}^{\otimes}(\omega_X, \eta) \wedge^{\mathcal{G}_{\omega_X}} \mathbf{O}(q_{\omega_X}) := (\mathbf{Isom}^{\otimes}(\omega_X, \eta) \times \mathbf{O}(q_{\omega_X}))/\mathcal{G}_{\omega_X} \longrightarrow \mathbf{Isom}^{\otimes}(q_\omega, q_\eta)
$$

in the topos $\mathrm{Spec}(k)_{\mathrm{fppf}}$. The group $\mathbf{O}(q_{\omega_X})$ acts on $\mathbf{Isom}^{\otimes}(\omega_X, \eta) \wedge^{\mathcal{G}_{\omega_X}} \mathbf{O}(q_{\omega_X})$ by right multiplication on the second factor and trivially on the first factor. It acts on $\mathbf{Isom}^{\otimes}(q_{\omega_X}, q_\eta)$ by composition. The map $\tilde{\mu}$ is obviously $\mathbf{O}(q_{\omega_X})$-equivariant, hence $\tilde{\mu}$ is a morphism of $\mathbf{O}(q_{\omega_X})$-torsors. The result follows, since any morphism of torsors is an isomorphism.

$\square$

$\square$

4. An analogue of the Serre-Fröhlich embedding problem for Nori's fundamental group scheme

4.1. The Serre-Fröhlich embedding problem.

Let F be a field of characteristic $\neq 2$, let G be a finite group and let $p : G_F \to G$ be a surjective map, i.e. let $\bar{F}/F$ be a separable closure and let $\bar{F}/K/F$ be a Galois sub-extension of group G. We consider the *G-unit form* $(F[G], \kappa)$ where κ is the quadratic form such that $\kappa(g, h) = \delta_{g,h}$ (Kronecker symbol) for $g, h \in G$. The action of G on itself by left multiplication extends to an orthogonal representation

$$
\rho : G_F \to G \to \mathbf{O}(\kappa). \tag{11}
$$

The Pin-extension (see [20] Appendix) gives a central extension of discrete groups

$$
1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathbf{O}}(\kappa)(\overline{F}) \to \mathbf{O}(\kappa)(\overline{F}) \to 1. \tag{12}
$$

Pulling back (12) via $G \to \mathbf{O}(\kappa)(F) \to \mathbf{O}(\kappa)(\overline{F})$ we obtain a central extension of finite discrete groups

$$
1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{G} \to G \to 1.
$$

which we suppose to be non-trivial. The embedding problem attached to $\tilde{G} \to G$ can be expressed as follows: Is there a commutative triangle

$$
\begin{array}{ccc}
 & & \widetilde{G} \\
 & \nearrow^{r} & \downarrow{s} \\
G_F & \xrightarrow{\quad p \quad} & G
\end{array}
$$

where r is surjective? In other words, is there an F-embedding $K \hookrightarrow \widetilde{K}$ where $\widetilde{K}/F$ is a Galois extension of group $\widetilde{G}$? This problem has a solution if and only if the Stiefel-Whitney class $sw_2(\rho)$ vanishes. It follows from Serre's comparison formula ([34] Théorème 1) that $sw_2(\rho) \in H^2(G_F, \mathbb{Z}/2\mathbb{Z})$ can be computed in terms of the Hasse-Witt invariant $w_2(\mathrm{Tr}_{K/F})$, where $\mathrm{Tr}_{K/F}$ is the quadratic from $x \mapsto \mathrm{Tr}_{K/F}(x^2)$. This result can be applied to decide whether or not the previous embedding problem has a solution in explicit examples.

4.2. **An analogue for Nori's fundamental group scheme.** The aim of this section is to formulate a similar embedding problem for Nori's fundamental group scheme and to deduce from the comparison formula of Theorem 3.3 a necessary and sufficient condition for the existence of a solution. More precisely, let X be a proper integral scheme over a field k of characteristic $\neq 2$, with a k-rational point $x \in X(k)$. We consider Nori's fundamental group scheme $\pi_1^N(X/k, x)$ (see [27] and [37] 6.7). Let $G = \mathrm{Spec}(A)$ be a finite flat quotient of $\pi_1^N(X/k, x)$. We denote by $p : \pi_1^N(X/k, x) \to G$ the projection.

4.2.1. *Generalization of the G-unit form.* The linear dual $A^D = \mathrm{Hom}_k(A, k)$ of A has a natural structure of Hopf k-algebra. We denote by S^D its antipode.

Definition 4.1. *For any generator θ of the k-vector space of the integrals of A we define the form*

$$
\begin{array}{cccc}
\kappa_\theta : & A^D \times A^D & \longrightarrow & k \\
 & (u, v) & \longmapsto & (S^D(u)v)(\theta)
\end{array}
$$

Then κ_θ is a non degenerate quadratic form over k. The action of A^D on itself by left multiplication corresponds to a linear representation ρ of G on A^D, dual of the regular representation of G. One can prove ([9] Proposition 5.1) that $\rho : G \to \mathbf{GL}_k(A^D)$ factors through $\mathbf{O}(\kappa_\theta)$ and thus provides an orthogonal representation of G. We refer to (A^D, κ_θ) as a *unit form*. If G is étale then there is a canonical generator θ_0 of the integrals of A, hence a canonical orthogonal representation

$$
\pi_1^N(X/k, x) \to G \to \mathbf{O}(\kappa_{\theta_0}),
$$

which generalizes the unit form (11).

4.2.2. For any generator θ of the integrals of A, we consider the representation $G \to \mathbf{O}(\kappa_\theta)$ defined above. The Pin-extension (see [20] Appendix again) induces a central extension of finite flat k-group-schemes

$$
1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{G}_\theta \xrightarrow{s_\theta} G \to 1
$$

which we suppose to be non-trivial. Notice that $\widetilde{G}_\theta := G \times_{\mathbf{O}(\kappa_\theta)} \widetilde{\mathbf{O}}(\kappa_\theta)$ depends on θ. The embedding problem we are interested is the following:

Problem 4.2. *Let θ be a generator of the integrals of A. Is there a faithfully flat morphism $r : \pi_1^N(X/k, x) \to \widetilde{G}_\theta$ which renders the following triangle*

commutative?

In the next section, we give a necessary and sufficient condition for the existence of a solution to Problem 4.2 up to conjugation by an element of $G(k)$.

4.3. **Existence of a solution in terms of Hasse-Witt invariants.** Let X be a proper integral scheme over a field k of characteristic $\neq 2$. Let $x \in X(k)$ be a rational point. In this situation, Nori defines the category $\mathrm{EF}(X)$ of essentially finite bundles. Then $\mathrm{EF}(X)$ is a full subcategory of the category $\mathrm{VB}(X)$ of vector bundles, which is closed under direct sums and tensor products. The pull-back functor along the map $x : \mathrm{Spec}(k) \to X$ gives a fiber functor $\omega : \mathrm{EF}(X) \to \mathrm{Vect}_k$, and $\mathrm{EF}(X)$ is a neutral Tannakian category. Its Tannaka dual

$$\pi_1^N(X/k, x) := \mathbf{Aut}^\otimes(\omega)$$

is by definition the fundamental group-scheme of X with respect to the rational point x. Then $\pi_1^N(X/k, x)$ is a projective limit of finite k-group schemes. It classifies pointed torsors under finite group-schemes: For any finite group-scheme G over k, one has a bijection between $\mathrm{Hom}_k(\pi_1^N(X/k, x), G)$ and the set of triples (G, T, t) where $T \to X$ is a G-torsor and $t \in T(k)$ is a rational point lying over $x \in X(k)$. The Tannakian category $\mathrm{EF}(X)$ has a distinguished fiber-functor over X: the inclusion functor

$$\eta : \mathrm{EF}(X) \longrightarrow \mathrm{VB}(X).$$

As in the previous section, we also consider the composition

$$\omega_X : \mathrm{EF}(X) \xrightarrow{\omega} \mathrm{Vect}_k \xrightarrow{f^*} \mathrm{VB}(X)$$

where $f : X \to \mathrm{Spec}(k)$ is the structure map. Now let G be a finite flat quotient of $\pi_1^N(X/k, x)$ given by the faithfully flat morphism $s : \pi_1^N(X/k, x) \twoheadrightarrow G$. The regular representation of G gives a representation $\pi_1^N(X/k, x) \to G \to \mathbf{GL}_k(V)$. By Tannaka duality, we obtain an object $\mathcal{R} \in \mathrm{EF}(X)$. Let $\mathcal{V} := \mathcal{R}^\vee$ be the dual of $\mathcal{R}$ and let $<\mathcal{V}>=<\mathcal{R}>\subseteq \mathrm{EF}(X)$ be the Tannakian category generated by $\mathcal{V}$. Let $\omega_{X|<\mathcal{V}>}$ and $\eta_{|<\mathcal{V}>}$ the restrictions of ω_X and η. Then

$$\mathbf{Aut}^\otimes(\omega_{X|<\mathcal{V}>}) \simeq G \times_k X$$

and

$$T := \mathbf{Isom}^\otimes(\omega_{X|<\mathcal{V}>}, \eta_{|<\mathcal{V}>})$$

is a G-torsor over X. Note that T has a canonical k-rational point $t : \mathrm{Spec}(k) \to T$ over x, which is given by the canonical isomorphism of tensor functors

$$x^* \circ \omega_{X|<\mathcal{V}>} \simeq x^* \circ \eta_{|<\mathcal{V}>}.$$

The triple (G, T, t) corresponds to the morphism $s : \pi_1^N(X/k, x) \twoheadrightarrow G$.

Let θ be a generator of the integrals of A, let (A^D, κ_θ) be the quadratic space of Definition 4.1 and let $\rho_\theta : G \to \mathbf{O}(\kappa_\theta)$ be the corresponding orthogonal representation. By Tannaka

duality again, ρ_θ endows the essentially finite vector bundle $\mathcal{V}$ with a canonical non-degenerate symmetric bilinear form

$$q_\theta : \mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{V} \to \mathcal{O}_X.$$

Then we consider the canonical central Pin-extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathbf{O}}(\kappa_\theta) \to \mathbf{O}(\kappa_\theta) \to 1$$

and its base change

(13) $$1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{G}_\theta \xrightarrow{s_\theta} G \to 1$$

along the map $G \to \mathbf{O}(\kappa_\theta)$. We consider the Hasse-Witt invariants $w_i(q_\theta), w_i(\kappa_\theta) \in H^i(X_{\mathrm{et}}, \mathbb{Z}/2\mathbb{Z})$ for $i = 1, 2$. Finally, if $\sigma \in G(k)$ is a rational point we denote by $c_\sigma = \sigma^{-1} \cdot (-) \cdot \sigma$ the conjugation map.

Theorem 4.3. *Assume that the extension (13) is non-trivial. Then the following assertions are equivalent.*

(1) *We have*

$$w_2(\kappa_\theta) + w_1(\kappa_\theta) \cdot w_1(\kappa_\theta) + w_1(\kappa_\theta) \cdot w_1(q_\theta) + w_2(q_\theta) = 0.$$

(2) *There exist a rational point $\sigma \in G(k)$ and a commutative diagram*

where r is faithfully flat.

Proof. In order to ease the notations, we fix θ and we set $\widetilde{G} := \widetilde{G}_\theta$, $s := s_\theta$, $\kappa := \kappa_\theta$ and $q = q_\theta$. We have an exact sequence of pointed sets (see [16] IV.4.2.10)

$$H^1(X, \widetilde{G}) \longrightarrow H^1(X, \widetilde{G}) \xrightarrow{\delta^2} H^2(X_{\mathrm{et}}, \mathbb{Z}/2\mathbb{Z}).$$

The quadratic space $\omega(\mathcal{V}, q)$ is the unit form (A^D, κ), whereas $\eta(\mathcal{V}, q) = (\mathcal{V}, q)$. By Theorem 3.3, we have

$$\delta^2(T) = w_2(\kappa) + w_1(\kappa) \cdot w_1(\kappa) + w_1(\kappa) \cdot w_1(q) + w_2(q).$$

By the exact sequence above, $\delta^2(T) = 0$ iff there exists a $\widetilde{G}$-torsor $\widetilde{T}$ over X and a $\widetilde{G}$-equivariant map $\widetilde{T} \to T$.

We show (2) $\Longrightarrow$ (1). Suppose that there exists a commutative diagram as in (2). The map

$$\pi_1^N(X/k, x) \to G \xrightarrow{c_\sigma} G$$

corresponds to the triple $(G, T, t \cdot \sigma)$, where $(t \cdot \sigma) \in T(k)$ is the point

$$\mathrm{Spec}(k) \xrightarrow{(t,\sigma)} T \times_k G = T \times_X G_X \xrightarrow{\mu} T.$$

The commutative diagram (2) therefore gives a triple $(\widetilde{G}, \widetilde{T}, \widetilde{t})$ mapping to $(G, T, t \cdot \sigma)$ hence a $\widetilde{G}$-torsor $\widetilde{T}$ over X and a $\widetilde{G}$-equivariant map $\widetilde{T} \to T$. We obtain $\delta^2(T) = 0$, hence (1).

We show (1) $\Longrightarrow$ (2). We have $\delta^2(T) = 0$ hence there exists a $\widetilde{G}$-torsor $\widetilde{T}$ over X together with a $\widetilde{G}$-equivariant map $\widetilde{T} \to T$. Taking the base change along $x \in X(k)$, we obtain a

$\widetilde{G}$-torsor $\widetilde{T}_x := \widetilde{T} \times_X x$ over k lifting T_x. Note that T_x is the trivial G-torsor. The exact sequence of pointed sets

$$H^1(k, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^1(k, \widetilde{G}) \longrightarrow H^1(k, G)$$

shows that there exists $\alpha \in H^1(k, \mathbb{Z}/2\mathbb{Z})$ mapping to $\widetilde{T}_x$. Recall from ([16] III.3.4.5) the natural action of $H^1(k, \mathbb{Z}/2\mathbb{Z})$ on $H^1(k, \widetilde{G})$. We have

$$[\widetilde{T}_x] = \alpha \cdot [\widetilde{T}_0]$$

where $\widetilde{T}_0$ is the trivial $\widetilde{G}$-torsor over k.

Let A be the image of α in $H^1(X, \mathbb{Z}/2\mathbb{Z})$. The action (see [16] III.3.4.5) of $H^1(X, \mathbb{Z}/2\mathbb{Z})$ on the set $H^1(X, \widetilde{G})$ is compatible with base-change. Let $\widetilde{T}'$ be a $\widetilde{G}$-torsor over X such that we have $[\widetilde{T}'] = A \cdot [\widetilde{T}]$ in $H^1(X, \widetilde{G})$. Taking the pull-back $\widetilde{T}'_x$ of $\widetilde{T}'$ along $x \in X(k)$, we have

$$[\widetilde{T}'_x] = x^*[\widetilde{T}'] = x^*(A \cdot [\widetilde{T}]) = (x^*A) \cdot (x^*[\widetilde{T}]) = \alpha \cdot [\widetilde{T}_x] = \alpha \cdot \alpha \cdot [\widetilde{T}_0] = [\widetilde{T}_0].$$

Therefore, replacing $\widetilde{T}$ with $\widetilde{T}'$, we may suppose that $\widetilde{T}_x$ is a trivial $\widetilde{G}$-torsor over k. Choose a rational point $\widetilde{t} \in \widetilde{T}_x(k)$, and let t_0 be its image in $T_x(k)$. Since T_x is a G-torsor, there exists a unique $\sigma \in G(k)$ such that $t_0 = t \cdot \sigma$. Hence the triple $(\widetilde{G}, \widetilde{T}, \widetilde{t})$ maps to (G, T, t_0), which corresponds to the map $\pi_1^N(X/k, x) \to G \overset{c_\sigma}{\to} G$. We obtain the commutative diagram in (2).

It remains to show that the map $r : \pi_1^N(X/k, x) \to \widetilde{G}$ in the diagram is faithfully flat. The map r factors as follows $\pi_1^N(X/k, x) \twoheadrightarrow \pi \to \widetilde{G}$, where π is a finite flat k-group scheme and $\pi_1^N(X/k, x) \twoheadrightarrow \pi$ is faithfully flat. It is therefore enough to show that the induced map $r' : \pi \to \widetilde{G}$ is faithfully flat. The map r' gives a morphism of exact sequences

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & N & \longrightarrow & \pi & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle r'} & & \downarrow{\scriptstyle c_\sigma} & & \\
1 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \widetilde{G} & \overset{s}{\longrightarrow} & G & \longrightarrow & 1
\end{array}
$$

We have an exact sequence (see [8] Proposition 3.8)

$$H^1(B_G, \mathbb{Z}/2\mathbb{Z}) \to H^1(\mathrm{Spec}(k)_{\mathrm{fppf}}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Ext}_k(G, \mathbb{Z}/2\mathbb{Z}) \to H^2(B_G, \mathbb{Z}/2\mathbb{Z}).$$

But the map $H^1(B_G, \mathbb{Z}/2\mathbb{Z}) \to H^1(\mathrm{Spec}(k)_{\mathrm{fppf}}, \mathbb{Z}/2\mathbb{Z})$ is surjective, since it is induced by the canonical section $\mathrm{Spec}(k)_{\mathrm{fppf}} \to B_G$ of the structure map $B_G \to \mathrm{Spec}(k)_{\mathrm{fppf}}$. Hence the canonical map $\mathrm{Ext}_k(G, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H^2(B_G, \mathbb{Z}/2\mathbb{Z})$ is injective. Similarly, we have an injection $\mathrm{Ext}_k(\pi, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H^2(B_\pi, \mathbb{Z}/2\mathbb{Z})$. The class $[E] \in \mathrm{Ext}_k(G, \mathbb{Z}/2\mathbb{Z})$ is non-trivial by assumption, but dies in $\mathrm{Ext}_k(\widetilde{G}, \mathbb{Z}/2\mathbb{Z})$, and a fortiori in $\mathrm{Ext}_k(\pi, \mathbb{Z}/2\mathbb{Z})$ (here we denote by $[E]$ the class of the extension denoted by (13) in the statement of the theorem). We consider the exact sequence (see [8] Corollary 3.13)

$$\to H^0(B_G, \underline{\mathrm{Hom}}(N, \mathbb{Z}/2\mathbb{Z})) \to H^2(B_G, \mathbb{Z}/2\mathbb{Z}) \to H^2(B_\pi, \mathbb{Z}/2\mathbb{Z}).$$

Then $[E]$ lies in $\mathrm{Ker}\left(H^2(B_G, \mathbb{Z}/2\mathbb{Z}) \to H^2(B_\pi, \mathbb{Z}/2\mathbb{Z})\right)$, and

$$\beta \in \mathrm{Hom}(N, \mathbb{Z}/2\mathbb{Z})^{G(k)} = H^0(B_G, \underline{\mathrm{Hom}}(N, \mathbb{Z}/2\mathbb{Z}))$$

maps to $[E] \in H^2(B_G, \mathbb{Z}/2\mathbb{Z})$, which is non-trivial since $\mathrm{Ext}_k(G, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H^2(B_G, \mathbb{Z}/2\mathbb{Z})$ is injective. Hence β is non-trivial too, hence surjective. It follows that r' is an epimorphism in $\mathrm{Spec}(k)_{\mathrm{fppf}}$, i.e. r' is faithfully flat. $\qquad\square$

Remark 4.4. *We may replace $\rho : G \to \mathbf{O}(\kappa)$ with an arbitrary orthogonal representation $G \to \mathbf{O}(u)$, and consider Problem 4.2 in this more general situation. This problem is the analogue of Fröhlich's embedding problem ([13], Section 7) for Nori's fundamental group scheme. We can prove mutatis mutandis Theorem 4.3 in this case.*

5. Orthogonal B-admissible representations

Various comparison formulas have been obtained, in different set-ups, between the Hasse-Witt invariants of a form and the Hasse-Witt invariants of its twists (see [34], [13], [31]). In this section we show that they all can be seen as consequences of Theorem 3.3 when applied to a well choosen tannakian category. Fontaine's theory of B-admissible representations and regular G-rings provides us with a large range of such categories.

5.1. **B-twist of a form.** Let k be a topological field of characteristic $\neq 2$ and let G be a topological group. We denote by $\mathrm{Rep}_k(G)$ the category of continuous finite dimensional k-linear representations of G, and by $\omega : \mathrm{Rep}_k(G) \to \mathrm{Vec}_k$ be the forgetful functor. Then $(\mathrm{Rep}_k(G), \omega)$ a neutralized k-linear tannakian category. We let B be a topological commutative k-algebra (recall that the homomorphism $k \to B$ is continuous), equipped with a continuous action of G by k-automorphisms. We set $K = B^G$. For any k-representation V of G we set

$$D_B(V) = (B \otimes_k V)^G$$

where G acts diagonally on $B \otimes_k V$. By scalar extension we obtain a map

$$(14) \qquad\qquad \alpha_V : B \otimes_K D_B(V) \to B \otimes_k V.$$

The map α_V is B-linear and G-equivariant. Here G acts trivially on $D_B(V)$ and diagonally on $B \otimes_K D_B(V)$. We now recall the definition of a (k, G)-regular algebra ([12] Definition 2.8).

Definition 5.1. *We say that B is (k, G)-regular if the following conditions hold:*
  (1) *B is a domain.*
  (2) *$B^G = Fr(B)^G$.*
  (3) *If $b \in B$, $b \neq 0$, such that kb is stable by G, then b is invertible in B.*

We assume from now on that B is (k, G)-regular. This implies that K is a field and that α_V is injective for any k-representation of G (see [12] Theorem 2.13).

Definition 5.2. *A k-representation V of G is said to be B-admissible if α_V is an isomorphism.*

We denote by $\mathrm{Rep}_k^B(G)$ the full subcategory of $\mathrm{Rep}_k(G)$ consisting of representations of G which are B-admissible. This category is a tannakian sub-category of $\mathrm{Rep}_k(G)$ with fiber functor given by the restriction of ω. Moreover the restriction of D_B to $\mathrm{Rep}_k^B(G)$ defines a fiber functor $\eta : \mathrm{Rep}_k^B(G) \to \mathrm{Vec}_K$ ([12] Theorem 2.13). We let $\mathcal{G}_B$ be the Tannaka dual $\mathbf{Aut}^\otimes(\omega)$ and T_B be the $\mathcal{G}_{B,K}$-torsor $\mathbf{Isom}^\otimes(\omega_K, \eta)$. There is a continuous morphism $G \to \mathcal{G}_B(k)$ and the functor sending $\rho : \mathcal{G}_B \to \mathbf{GL}(V)$ to $\bar\rho : G \to \mathcal{G}_B(k) \to \mathbf{GL}_V(k)$ induces an equivalence of tensor categories ([12] Proposition 1.2.3)

$$\mathrm{Rep}_k(\mathcal{G}_B) \overset{\sim}{\longrightarrow} \mathrm{Rep}_k^B(G).$$

We identify these categories.

An orthogonal object of $\mathrm{Rep}_k^B(G)$ is given by a continuous orthogonal representation $\rho : G \to \mathbf{O}(q)(k)$ where (V, q) is a quadratic form over k and $\mathbf{O}(q)(k)$ is endowed with topology induced by that of k. Following the method of Section 3, we associate with (V, q, ρ), via the

fiber functors ω and η, non degenerate quadratic forms on k and K, on the one hand the form (V, q) obtained through the forgetful functor ω and the form $(D_B(V), q_D)$ obtained through η, where q_D is the restriction to $D_B(V)$ of the form $B \otimes q$. The quadratic space $(D_B(V), q_D)$ will be called the *B-twist of* (V, q).

Since q_D is the twist of q by the $\mathcal{G}_{B,K}$ torsor T_B (see [11], definition 2.4), then Theorem 3.4 provides us with a comparison formula for the Hasse-Witt invariants of these forms. More precisely, let (V, q, ρ) be an orthogonal B-admissible k-representation of G and let $< V >$ be the subtannakian category generated by V. Then ω and η induce fiber functors ω' and η' on $< V >$. We consider the subalgebra B_V of B consisting of all the periods of all the objects of $< V >$ ([11], Section 1.7)). The algebra B_V/k is a (k, G)-regular algebra of finite type. We set $T_{B,V} = \mathrm{Spec}(B_V)$. By [11], Theorem 1.7.3 we know that $T_{B,V}$ represents the functor $\mathbf{Isom}^{\otimes}(\omega'_K, \eta')$. Therefore Theorem 3.4 expresses the invariants $\delta_q^1(T_B) = \delta_q^1(T_{B,V})$ and $\delta_q^2(T_B) = \delta_q^2(T_{B,V})$ of this algebra in terms of the Hasse-Witt invariants of the forms q and q_D.

Our aim is now to express $\delta_q^1(T_B)$ and $\delta_q^2(T_B)$ in terms of invariants attached to ρ. We start by introducing some notations. As before B/k is a topological algebra and K/k is a field extension. We fix a separable closure $\bar{K}$ of K and we take for group G the profinite group $G_K = \mathrm{Gal}(\bar{K}/K)$. We suppose that B is a (k, G_K) regular algebra, such that $B^{G_K} = K$. Moreover we assume that B/K is a filtered equivariant algebra. In other words there exists a decreasing filtration by K-subvector spaces $\{B^i, i \in \mathbb{Z}\}$, stable under the action of G_K, such that $K \subset B^0$ and $B^i B^j \subset B^{i+j}$ for $i, j \in \mathbb{Z}$. If V is a k-representation of G_K, the filtration on B induces natural filtrations on $B \otimes_k V$, $D_B(V)$ and $B \otimes_K D_B(V)$. More precisely for any integer i, we set

$$(B \otimes_k V)^i = B^i \otimes_k V, \quad D_B(V)^i = (B^i \otimes_k V)^{G_K},$$

and we endow $B \otimes_K D_B(V)$ with the tensor product filtration given by $(B \otimes_K D_B(V))^i = \sum_{k+l=i} B^k \otimes_K D_B(V)^l$.

We recall that a Hausdorff topological field F is called t-Henselian if for every etale morphism $X \to Y$ of F-varieties the induced map $X(F) \to Y(F)$ is a local homeomorphism, where $X(F)$ and $Y(F)$ are endowed with the topology of F ([14] Section 3 and [29]).

Definition 5.3. *We say that the algebra B/K is well filtered if the following conditions hold:*

(1) *There exists a G_K-equivariant $\bar{K}$-structure on B^0 extending the K-algebra structure.*

(2) *$C := Gr^0(B)$ is a t-Henselian field when endowed with the quotient topology such that $C^* = C^{*2}$.*

If B/K is filtered a B-admissible representation V of G_K is said to be *filtered* if α_V and α_V^{-1} both respect the filtrations.

Let $\rho : G_K \to \mathbf{O}(q)(k)$ be a continuous, orthogonal and B-admissible representation. We define *the first Stiefel-Whitney class* $sw_1(\rho)$ as $\det(\rho) \in H^1(G_K, \mathbb{Z}/2\mathbb{Z})$. We now consider the central extension of group schemes

$$(15) \qquad\qquad 1 \to \mathbb{Z}/2\mathbb{Z} \to \tilde{\mathbf{O}}(q_C) \to \mathbf{O}(q_C) \to 1,$$

where $q_C = q \otimes_k C$. We associate to (15) an exact sequence of groups

$$(16) \qquad 1 \to \mathbb{Z}/2\mathbb{Z} \to \tilde{\mathbf{O}}(q)(C) \to \mathbf{O}(q)(C) \to H^1(\mathrm{Spec}(C)_{\mathrm{et}}, \mathbb{Z}/2\mathbb{Z}).$$

Since B/K is well filtered and since the morphism $\tilde{\mathbf{O}}(q_C) \to \mathbf{O}(q_C)$ is etale we deduce from (16) an exact sequence of topological groups

$$(17) \qquad\qquad 1 \to \mathbb{Z}/2\mathbb{Z} \to \tilde{\mathbf{O}}(q)(C) \to \mathbf{O}(q)(C) \to 1,$$

with the trivial G_K-action. Since the map $k \to C$ is a continuous and G_K-equivariant ring homomorphism, the map ρ defines a 1-continuous cocycle of G_K with values in C and thus a class $[\rho] \in H^1(G_K, \mathbf{O}(q)(C))$. We define *the second Stiefel-Whitney class* $sw_2(\rho)$ *of* ρ as the image of $[\rho]$ by the boundary map of continuous cohomology

$$\delta^2 : H^1(G_K, \mathbf{O}(q)(C)) \to H^2(G_K, \mathbb{Z}/2\mathbb{Z}).$$

Composing ρ with the spinor norm homomorphism

$$sp : \mathbf{O}(q)(k) \to k^*/k^{*2} \to K^*/K^{*2} = H^1(G_K, \mathbb{Z}/2\mathbb{Z}),$$

we obtain a homomorphism $sp \circ \rho \in \mathrm{Hom}(G_K, H^1(G_K, \mathbb{Z}/2\mathbb{Z}))$. *The spinor class* $sp_2(\rho)$ is defined as the image of $sp \circ \rho$ by the map

$$\mathrm{Hom}(G_K, H^1(G_K, \mathbb{Z}/2\mathbb{Z})) \simeq H^1(G_K, \mathbb{Z}/2\mathbb{Z}) \otimes H^1(G_K, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^2(G_K, \mathbb{Z}/2\mathbb{Z})$$

defined by cup-product.

Theorem 5.4. *Let B be a (k, G_K)-regular algebra and let $\rho : G_K \to \mathbf{O}(q)(k)$ be a continuous, orthogonal and B-admissible representation. Then we have:*

(1) $w_1(q_D) = w_1(q) + sw_1(\rho)$.

(2) *Moreover, if B/K is well filtered and the representation is filtered then*

$$w_2(q_D) = w_2(q) + w_1(q)sw_1(\rho) + sw_2(\rho) + sp_2(\rho).$$

Proof. It follows from Theorem 3.3 that in order to prove the first equality it suffices to prove the equality

$$\delta^1_q(T_B) = sw_1(\rho)$$

in $H^1(G_K, \mathbb{Z}/2\mathbb{Z})$. The morphism of group schemes $\rho : \mathcal{G}_B \to \mathbf{O}(q)$ induces a map $\rho_* : H^1(G_K, \mathcal{G}_B) \to H^1(G_K, \mathbf{O}(q))$. The image of T_B by ρ_* is the $\mathbf{O}(q)$-torsor $\mathbf{Isom}(q_K, q_D)$ over K ([CCMT], Section 4). It defines a class in the continuous cohomology set $H^1(G_K, \mathbf{O}(q)(\bar{K}))$, when $\mathbf{O}(q)(\bar{K})$ is endowed with the discrete topology. One defines a 1-cocycle representative of $\rho_*(T_B)$ by choosing a $\bar{K}$-point of $\mathbf{Isom}(q_K, q_D)$, given by an isometry

$$f : \bar{K} \otimes_k V \to \bar{K} \otimes_K D_B(V),$$

and considering the map $c : G_K \to \mathbf{O}(q)(\bar{K})$, $g \to f^{-1} \circ (g \otimes 1) \circ f \circ (g^{-1} \otimes 1)$. We have the equality

$$(18) \qquad\qquad \delta^1_q(T_B) = \det([c]),$$

where $[c] \in H^1(G_K, \mathbf{O}(q)(\bar{K}))$ is the class of c.

Scalar extension by B induces a G_K-equivariant group homomorphism $\mathbf{O}(q)(\bar{K}) \to \mathbf{O}(q)(B)$ which is continuous, with respect to the discrete topology on $\mathbf{O}(q)(\bar{K})$ and the topology induced by B on $\mathbf{O}(q)(B)$. Therefore this group homomorphism induces a map of pointed sets

$$i_B : H^1(G_K, \mathbf{O}(q)(\bar{K})) \to H^1(G_K, \mathbf{O}(q)(B)),$$

and the class $i_B([c])$ is represented by the cocycle c_B associated with the B-point f_B of $\mathbf{Isom}(q_K, q_D)$ obtained by scalar extension from f

$$f_B : B \otimes_k V \to B \otimes_K D_B(V).$$

Since ρ is B-admissible we obtained an other B-point of $\mathbf{Isom}(q, q_D)$ by considering

$$\alpha_V^{-1} : B \otimes_k V \to B \otimes_K D_B(V).$$

Let $c'_B : G_K \to \mathbf{O}(q)(B)$ be the map $(g \to \alpha_V \circ (g \otimes 1) \circ \alpha_V^{-1} \circ (g^{-1} \otimes 1)$. We observe that α_V is G_K-equivariant when G_K acts diagonally on $B \otimes V$ and via B on $B \otimes D_B(V)$. Therefore we have

$$(19) \quad \alpha_V \circ (g \otimes 1) \circ \alpha_V^{-1} \circ (g^{-1} \otimes 1)(b \otimes v) = \alpha_V \circ (g \otimes 1) \circ \alpha_V^{-1}(g^{-1}(b \otimes \rho(g)(v)) = b \otimes \rho(g)(v).$$

Since $k \to B$ is continuous and preserves G_K-action, the group homomorphism $\rho : G_K \to \mathbf{O}(q)(k)$ defines a 1-cocycle of the continuous cohomology set $H^1(G_K, \mathbf{O}(q)(B))$ and (19) shows the equality $c'_B = \rho$. Moreover since the 1-cocycles c_B and c'_B are both defined through B- points they are cohomologous. Thus we have proved that

$$(20) \qquad\qquad\qquad i_B([c]) = [\rho]$$

in $H^1(G_K, \mathbf{O}(q)(B))$. Since the group homomorphism $\det : \mathbf{O}(q)(k) \to \mathbb{Z}/2\mathbb{Z}$ factors through $\mathbf{O}(q)(B)$ we conclude that

$$(21) \qquad\qquad \delta_q^1(T_B) = \det([c]) = \det([i_B([c]) = \det(\rho) = sw_1(\rho)$$

We now assume that B is well filtered and the representation ρ is filtered. It follows from Theorem 3.3 and the previous description of $\rho_*(T_B)$ that it is sufficient to show the equality

$$\delta^2([c]) = sw_2(\rho) + sp_2(\rho),$$

where δ^2 is the boundary map associated with the exact sequence of groups

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \tilde{\mathbf{O}}(q)(\bar{K}) \to \mathbf{O}(q)(\bar{K}) \to 1$$

with the natural G_K-action. Moreover it follows from a general result of [31], Lemma 3, that $sw_2(\rho) + sp_2(\rho) = \delta_C^2(\rho)$, where δ_C^2 is the boundary map associated with the exact sequence of topological groups (17)

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \tilde{\mathbf{O}}(q)(C) \to \mathbf{O}(q)(C) \to 1.$$

Therefore, in order to prove (ii) we have to show, in $H^2(G_K, \mathbb{Z}/2\mathbb{Z})$, that

$$\delta^2([c]) = \delta_C^2(\rho).$$

We denote by $\mathbf{O}^\bullet(q)(B)$ the group of isometries $B \otimes_k V \to B \otimes_k V$ which respect the filtration of $B \otimes_k V$. Since B is well filtered we note that G_K acts on $\mathbf{O}^\bullet(q)(B)$. It is now easy to check that since f_B and α_V both preserve the filtration thus the 1-cocycles c_B and c'_B have values in $\mathbf{O}^\bullet(q)(B)$ and define the same class in $H^1(G_K, \mathbf{O}^\bullet(q)(B))$. We conclude to the equality

$$(22) \qquad\qquad\qquad i_B^\bullet([c]) = [\rho]$$

in $H^1(G_K, \mathbf{O}^\bullet(q)(B)))$. Any isometry $B \otimes_k V \to B \otimes_k V$ which respects the filtration induces an isometry $Gr^0(B) \otimes V \to Gr^0(B) \otimes V$. This yields to a group homomorphism $\mathbf{O}^\bullet(q)(B) \to \mathbf{O}(q)(C)$ and thus a map $H^1(G_K, \mathbf{O}^\bullet(q)(B)) \to H^1(G_K, \mathbf{O}(q)(C)$. Using this map we deduce from (22) that

$$(23) \qquad\qquad\qquad i_C([c]) = [\rho]$$

in $H^1(G_K, \mathbf{O}(q)(C)$, where i_C is induced on the cohomology sets by the group homomorphism $\mathbf{O}(q)(\bar{K}) \to \mathbf{O}(q)(C)$. From (23) and from the commutativity of the diagram

$$\begin{array}{ccccccccc}
\{1\} & \longrightarrow & (\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & \widetilde{\mathbf{O}}(q)(\bar{K}) & \longrightarrow & \mathbf{O}(q)(\bar{K}) & \longrightarrow & \{1\} \\
& & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \widetilde{i}_C} & & \downarrow{\scriptstyle i_C} & & \\
\{1\} & \longrightarrow & (\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & \widetilde{\mathbf{O}}(q)(C) & \longrightarrow & \mathbf{O}(q)(C) & \longrightarrow & \{1\}
\end{array}$$

we deduce that $\delta^2([c]) = \delta_C^2(\rho)$ as required

$\square$

5.2. Examples.
We consider several examples of (k, G_K)-algebras where we can apply the formulas of Theorem 5.4.

We introduce a small amount of notations. We set

$$H^*(K, \mathbb{Z}/2\mathbb{Z}) = \bigoplus_{0 \leq i \leq 2} H^i(G_K, \mathbb{Z}/2\mathbb{Z})$$

endowed by cup product with a structure of abelian group. For a quadratic form q over K we define its Hasse-Witt invariant by $w(q) = 1 + w_1(q) + w_2(q) \in H^*(K, \mathbb{Z}/2\mathbb{Z})$. For a virtual quadratic form $q_1 - q_2$ over K we write $w(q_1 - q_2) = w(q_1) w(q_2)^{-1}$. For a B-admissible orthogonal representation ρ of G_K we define respectively the Stiefel-Whitney and the spinor norm classes of ρ by $sw(\rho) = 1 + sw_1(\rho) + sw_2(\rho)$ and $sp(\rho) = 1 + sp_2(\rho)$

5.2.1. *Fröhlich-Serre formulas.* We let k be a field of characteristic $\neq 2$, $K = k$ and $B = \bar{k}$ the separable closure of k endowed with the discrete topology. Then $\bar{k}/k$ is a (k, G_k)-algebra, which satisfies the hypotheses of Section 4.1. The $\bar{k}$-admissible orthogonal representations of G_k are the group homomorphisms $\rho : G_k \to \mathbf{O}(q)(k)$, with open kernel, where q is a non degenerate quadratic form on the k-vector space V of finite dimension. It is clear that $\bar{k}$ is a well filtered when endowed with the filtration given by $\bar{k}^0 = \bar{k}$ and $\bar{k}^1 = \{0\}$ and that any orthogonal representation is filtered. The $\bar{k}$-twist of (V, q) is the k-vector space $(\bar{k} \otimes_k V)^{G_k}$, with the form obtained by restriction from $\bar{k} \otimes_k q$. Theorems 2 and 3 of Fröhlich [13] are the equalities of Theorem 5.4 in this special case. More precisely:

Corollary 5.5. *Let $\rho : G_k \to \mathbf{O}(q)(k)$ be an orthogonal representation with open kernel and let $\underline{q}$ be the form $(\bar{k} \otimes q)^{G_k}$. Then*

$$sw(\rho) + sp(\rho) = w(\underline{q} - q).$$

Let H be an open subgroup of G_k. We let V be the k-vector space with basis the left-cosets gH of H in G_k and q be the quadratic form on V which has $\{gH\}$ as an orthonormal basis. The action of G_k by permuting the cosets gH extends to an orthogonal representation ρ of G_k. The Hasse-Witt invariants of q are trivial since q is the form $x_1^2 + ... x_n^2$. The $\bar{k}$-twist of q is the trace form $\mathrm{Tr}_{L/k}$ where L is the fixed field of H. Moreover one can compute $sp_2(\rho)$ in this case and prove that it is equal to the cup product $(2)(d_{L/k})$ where $d_{L/k}$ is the relative discriminant of L/k. Therefore Corollary 5.5 provides us with the equality

$$sw_2(\rho) + (2)(d_{L/k}) = w_2(\mathrm{Tr}_{L/k}),$$

which is Serre's formula [34].

The references for the next two sections are [11] and [12]. Throughout these sections k and K denote complete discrete valuation fields of characteristic 0, whose residue fields are perfect of characteristic p. We assume that $k \subset K$. We let C_p be the completion of an algebraic closure $\bar{K}$ of K. We note that since C_p is Henselian it is t-Henselian ([4] Lemma 5-3, or [14]

Proposition 3.1.4). There exist many examples of (k, G_K) regular rings in this set-up. For instance C_p itself is (k, G_K) regular. The C_p-admissible representations are those for which the inertia acts through a finite quotient. We can filter C_p by setting $C_p^0 = C_p$ and $C_p^1 = \{0\}$. Therefore for any C_p-admissible orthogonal representation $\rho : G_K \to \mathbf{O}(q)(k)$ we will deduce from Theorem 5.4 that

$$sw(\rho) + sp(\rho) = w(q' - q),$$

where q' is the form $(C_p \otimes_k q)_K^G$. We treat in more details the case of Hodge-Tate and de Rham representations.

5.2.2. *Orthogonal Hodge-Tate representations.* We fix a basis t of $\mathbb{Z}_p(1)$, so that $g(t) = \chi(g)t$ for any $g \in G_K$, where χ is the cyclotomic character. We consider the ring of Hodge-Tate periods

$$B_{HT} = C_p[t, t^{-1}] = \bigoplus_{k \in \mathbb{Z}} C_p t^k.$$

This is a graded C_p-algebra endowed with a semilinear action of G_K. Moreover B_{HT} is a (k, G_K) regular ring such that $B_{HT}^{G_K} = K$. We endow B_{HT} with a filtration by setting

$$B_{HT}^i = \bigoplus_{k \geq i} C_p t^k,$$

and thus $Gr^0(B) \simeq C_p$. One checks that B_{HT} is a well filtered algebra. We denote by $(D_{HT}(V), q_{HT})$ the B_{HT}-twist of an orthogonal Hodge-Tate representation (V, q, ρ) of G_K.

Corollary 5.6. *Let $\rho : G_K \to \mathbf{O}(q)(k)$ be a filtered Hodge-Tate k-representation of G_K. Let q_K be the form $K \otimes q$ and q_{HT} be the B_{HT}-twist of q. Then*

$$sw(\rho) + sp(\rho) = w(q_{HT} - q_K).$$

Let MF_K be the category of filtered K-vector spaces. This is a K-linear tensor, addive category, with an identity object and an internal homomorphism. Therefore, following definition 3.1, we may define an orthogonal object of this category. One easily checks that a quadratic object of MF_K is a filtered quadratic K-vector space as defined in [31], Section 1. The fiber functor $D_{HT} : \mathrm{Rep}_k^{dR}(G_K) \to \mathrm{Vec}_k$ factorises through $\mathrm{Rep}_k^{dR}(G_K) \to MF_K$. Hence if (V, q, ρ) is an orthogonal Hodge-Tate representation of G_K, then $(D_{dR}(V), q_{dR})$ is a filtered quadratic K-vector space. Moreover, if the representation is filtered, the comparison isomorphism α_V respects the filtration and so induces an isometry

$$\alpha_V^0 : Gr^0(B_{HT} \otimes_K D_{HT}(V)) \simeq Gr^0(B_{HT} \otimes_k V)$$

which can be written

$$\alpha_V^0 : \bigoplus_i Gr^i(D_{HT}(V)) \otimes_K C_p(-i) \simeq C_p \otimes_k V.$$

Therefore the orthogonal representations considered by Saito in [31] are precisely the filtered orthogonal Hodge-Tate k-representations of G_K and Corollary 5.6 provides a new proof of [31], Theorem 1.

5.2.3. *Orthogonal de Rham representations.* We consider the ring of periods $B = B_{dR}$ endowed with "its natural topology" (see [12], 5.2.2). Then B_{dR}/k is a (k, G_K)-regular ring such that $B_{dR}^{G_K} = K$. The field B_{dR} is the fraction field of the discrete valuation ring B_{dR}^+. One can check that B_{dR}, filtered by the i-th powers of the maximal ideal of B_{dR}^+, is well filtered and that any de Rham representation of G_K is filtered [3] Proposition 6.3.7. The ring $G^0(B_{dR})$ is by definition the residue field of B_{dR}^+. It identifies with C_p and the topology induced on C_p coincides with the usual topology. For any continuous, orthogonal, de Rham representation $\rho : G_K \to \mathbf{O}(q)(k)$, we can define the B_{dR}-twist of q, that we denote by q_{dR}.

Corollary 5.7. *Let $\rho : G_K \to \mathbf{O}(q)(k)$ be an orthogonal de Rham k-representation of G_K. Let $q_K = K \otimes_k q$ and let q_{dR} be the B_{dR}- twist of q. Then*

$$sw(\rho) + sp(\rho) = w(q_{dR} - q_K).$$

Let Grad_K be the category of graded finite dimensional K-vector spaces. The fiber functor D_{HT} factorizes through a fiber functor $D_{HT} : Rep_k^{HT}(G_K) \to \mathrm{Grad}_K$. Let (V, q, ρ) be an orthogonal de Rham representation. Since $D_{dR}(V)$ is filtered we can consider $gr(D_{dR}(V))$ as an object of Grad_K et prove that $gr(D_{dR}(V))$ and $D_{HT}(V)$ are isomorphic in this category. By applying the forgetful functor we obtain an isomorphism of quadratic space

$$gr(D_{dR}(V)) \simeq D_{HT}(V).$$

Let X/K be a proper and smooth scheme of even dimension n. The cup product defines a non degenerate quadratic form q on the $\mathbb{Q}_p$-vector space $V = H^n(X_{\bar{K}}, \mathbb{Q}_p)(n/2)$. The action of G_K on V provides us with a continuous orthogonal de Rham representation $\rho : G_K \to \mathbf{O}(q)(\mathbb{Q}_p)$ to which the equality of Corollary 5.7 applies. It follows from a theorem of Faltings that the B_{dR}-twist $(D_{dR}(V), q_{dR})$ of (V, q) coincides in this case, up to isometry, with the K-vector space $H_{dR}^n(X/K)$, endowed with the quadratic form defined by the cup product.

5.2.4. *Orthogonal de Rham representation in the case of non perfect residue field.* We let K be a complete discrete valuation fields of characteristic 0, with residue field k_K of characteristic p, such that $[k_K : k_K^p] = p^d$ with $d \geq 0$. We fix $\bar{K}/K$ an algebraic closure of K and we set $G_K = \mathrm{Gal}(\bar{K}/K)$. Let C be the completion of $\bar{K}$ for the p-adic topology. In [2] O. Brinon has constructed a ring of periods B_{dR} which is a generalization to the case of imperfect residue fields of the algebra considered in Section 4.2.3. One should note that B_{dR} is not necessarily a field when $d \geq 1$. This is a toplogical $(\mathbb{Q}_p, G_K)$-regular algebra such that $B_{dR}^{G_K} = K$. Therefore for any B_{dR}-admissible othogonal representation $\rho : G_K \to \mathbf{O}(q)(\mathbb{Q}_p)$ we have

$$w_1(q_{dR}) = w_1(q) + sw_1(\rho).$$

B_{dR} has a structure of $\bar{K}$-algebra and is endowed with a decreasing, exhaustive and separated filtration, stable under the action of G_K. It follows from [2] Proposition 2.19 that $Gr^0(B_{dR})$ is a ring of polynomials over C in d variables and so that $H^1(\mathrm{Spec}(C)_{\mathrm{et}}, \mathbb{Z}/2\mathbb{Z}) = \{0\}$. Hence we obtain from (16) an exact sequence of groups

$$(24) \qquad\qquad 1 \to \mathbb{Z}/2\mathbb{Z} \to \tilde{\mathbf{O}}(q)(C) \to \mathbf{O}(q)(C) \to 1.$$

However for such a C we don't know if the map $\tilde{\mathbf{O}}(q)(C) \to \mathbf{O}(q)(C)$ is a local homeomorphism and thus if B_{dR} is well filtered.

6. Orthogonal E-motives

In this section we apply Theorem 3.3 to the tannakian category of Nori's mixed motives over a number field (see [28] and [18]). Here the fiber functors ω and η are the Betti and the de Rham realization respectively, and the torsor $T_{\omega,\eta}$ is given by the spectrum of the algebra of Kontsevich's formal periods together with its natural action of the motivic Galois group.

6.1. Nori motives.

6.1.1. Let $E/\mathbb{Q}$ be a number field with a given complex embedding $\sigma : E \to \mathbb{C}$. We denote by NMM_E the category of cohomological Nori mixed motives over E with $\mathbb{Q}$-coefficients (see [28], [24], [18]). It is a neutral Tannakian category with Tannaka dual $\mathcal{G}_{\mathrm{mot}}(E)$. More precisely, Betti cohomology induces a fiber functor

$$H_B^* : \mathrm{NMM}_E \longrightarrow \mathrm{Vec}_{\mathbb{Q}}$$

and the motivic Galois group $\mathcal{G}_{\mathrm{mot}}(E)$ is defined as the automorphism group-scheme $\mathbf{Aut}^{\otimes}(H_B^*)$ of the tensor functor H_B^*. By Tannaka duality, an object M is determined by the $\mathbb{Q}$-linear representation

$$(25) \qquad \mathcal{G}_{\mathrm{mot}}(E) \longrightarrow \mathbf{GL}(H_B^*(M))$$

where $\mathbf{GL}(H_B^*(M))$ is seen as group scheme over $\mathbb{Q}$. De Rham cohomology induces another fiber functor

$$H_{dR}^* : \mathrm{NMM}_E \longrightarrow \mathrm{Vec}_E.$$

If X is a variety over E, $Y \subset X$ a subvariety and i an integer, then there is an object $h^i(X,Y) \in \mathrm{NMM}_E$ such that $H_B^*(h^i(X,Y)) = H^i(X_\sigma(\mathbb{C}), Y_\sigma(\mathbb{C}), \mathbb{Q})$ and $H_{dR}^*(h^i(X,Y)) = H_{dR}^i(X,Y)$. Here $X_\sigma(\mathbb{C})$ denotes the set of complex points of X over E with respect to the complex embedding σ. We write $h^i(X) := h^i(X, \emptyset)$ and $h^i(X,Y)(j) = h^i(X,Y) \otimes \mathbf{1}(j)$. The category of Artin motives NMM_E^0 is the Tannakian sub-category of NMM_E generated by objects of the form $h^0(\mathrm{Spec}(F))$ where F/E is a finite extension. Then NMM_E^0 is equivalent to the category of $\mathbb{Q}$-linear representations of the absolute Galois group G_E. For any prime number p, étale p-adic cohomology yields a tensor functor

$$H_p^* : \mathrm{NMM}_E \longrightarrow \mathrm{Rep}_{\mathbb{Q}_p}(G_E)$$

where $\mathrm{Rep}_{\mathbb{Q}_p}(G_E)$ denotes the Tannakian category of continuous $\mathbb{Q}_p$-linear representations of G_E. Artin's comparison isomorphism yields an isomorphism of tensor functors $\mathrm{Ou} \circ H_p^* \simeq H_B^* \otimes \mathbb{Q}_p$, where Ou is the forgetful functor $\mathrm{Rep}_{\mathbb{Q}_p}(G_E) \to \mathrm{Vec}_{\mathbb{Q}_p}$. Note also that p-adic Hodge theory gives an isomorphism of tensor functor

$$(26) \qquad \left(\mathrm{Ou} \circ H_p^*\right) \otimes_{\mathbb{Q}_p} B_{dR,\lambda} \simeq (H_{dR}^* \otimes_E E_\lambda) \otimes_{E_\lambda} B_{dR,\lambda}$$

for any prime number p and any finite place λ of E lying over p, where $B_{dR,\lambda}$ is the field of p-adic periods (see Section 5.2.3). Moreover, (26) is compatible with the natural Galois actions and filtrations. For an archimedean place λ of E we have a tensor functor

$$\begin{array}{rccc} H_\lambda^* : & \mathrm{NMM}_E & \longrightarrow & \mathrm{Rep}_{E_\lambda}(G_{E_\lambda}) \\ & h^i(X,Y)(j) & \longmapsto & H^i\left(X_\lambda(\mathbb{C}), Y_\lambda(\mathbb{C}), (2i\pi)^j \mathbb{Q}\right) \otimes_{\mathbb{Q}} E_\lambda \end{array}$$

where $X_\lambda(\mathbb{C})$ is the set of $\mathbb{C}$-points of X over E with respect to the complex embedding $\tilde{\lambda} : E \to \mathbb{C}$, for any choice of a representative $\tilde{\lambda}$ of λ. Finally, the tensor functor H_p^* induces a continuous morphism

$$G_E \longrightarrow \mathcal{G}_{\mathrm{mot}}(E)(\mathbb{Q}_p).$$

Here $\mathcal{G}_E(\mathbb{Q})(\mathbb{Q}_p)$ is endowed with its natural topology (i.e. the projective limit of the p-adic topology on the $\mathbb{Q}_p$-points of the algebraic quotients of $\mathcal{G}_E(\mathbb{Q})$).

6.1.2. We refer to [36] for generalities about scalar extensions of Tannakian categories, and we define the category $\mathrm{NMM}_E(E)$ of Nori mixed motives with E-coefficients as the scalar extension $\mathrm{NMM}_E(E) := \mathrm{NMM}_E \otimes_{\mathbb{Q}} E$. Then $\mathrm{NMM}_E(E)$ is an E-linear Tannakian category endowed with a universal $\mathbb{Q}$-linear tensor functor

$$t : \mathrm{NMM}_E \longrightarrow \mathrm{NMM}_E(E).$$

In particular, a $\mathbb{Q}$-linear tensor functor $\mathrm{NMM}_E \to \mathcal{C}$ into some E-linear tensor category $\mathcal{C}$ induces an essentially unique E-linear tensor functor $\mathrm{NMM}_E(E) \to \mathcal{C}$. We still denote by $h^i(X, Y)(j)$ its image in $\mathrm{NMM}_E(E)$.

The Betti realization functor above induces a fiber functor

$$H^*_{B,E} : \mathrm{NMM}_E(E) \longrightarrow \mathrm{Vec}_E$$

and we have

$$\mathcal{G}_E := \mathbf{Aut}^{\otimes}(H^*_{B,E}) \simeq \mathbf{Aut}^{\otimes}(H^*_B) \otimes_{\mathbb{Q}} E =: \mathcal{G}_{\mathrm{mot}}(E) \otimes_{\mathbb{Q}} E.$$

The realization functor H^*_{dR} above induces a fiber functor

$$H^*_{dR,E} : \mathrm{NMM}_E(E) \longrightarrow \mathrm{Vec}_E.$$

The p-adic realization functor induces

$$H^*_\lambda : \mathrm{NMM}_E(E) \longrightarrow \mathrm{Rep}_{E_\lambda}(G_E)$$

for any place λ of E lying over p. Artin's comparison theorem gives an isomorphism of tensor functors

$$(27) \qquad\qquad \mathrm{Ou} \circ H^*_\lambda \simeq H^*_{B,E} \otimes_E E_\lambda,$$

where Ou is the forgetful functor $\mathrm{Rep}_{E_\lambda}(G_E) \to \mathrm{Vec}_{E_\lambda}$. For any finite place λ of E, (26) induces by extension of scalars an isomorphism of tensor functors

$$(28) \qquad\qquad (\mathrm{Ou} \circ H^*_\lambda) \otimes_{E_\lambda} B_{dR,\lambda} \simeq H^*_{dR,E_\lambda} \otimes_{E_\lambda} B_{dR,\lambda}$$

which is moreover compatible with the natural G_{E_λ}-actions and filtrations, where $H^*_{dR,E_\lambda} := H^*_{dR,E} \otimes_E E_\lambda$. Finally, for any archimedean place $\lambda \mid \infty$ the tensor functor H^*_λ above induces

$$H^*_\lambda : \mathrm{NMM}_E(E) \longrightarrow \mathrm{Rep}_{E_\lambda}(G_{E_\lambda}).$$

6.2. The torsor of formal periods.

6.2.1. The space of effective formal periods $\mathcal{P}_+$ is the $\mathbb{Q}$-vector space generated by the symbols (X, D, ω, γ), where X is a variety over $\mathbb{Q}$, $D \subset X$ a subvariety, $\omega \in H^d_{dR}(X, D)$ a de Rham cohomology class and $\gamma \in H_d(X(\mathbb{C}), D(\mathbb{C}), \mathbb{Q})$ a singular homology class modulo the following relations:

(1) (linearity) (X, D, ω, γ) is linear in both ω and γ.
(2) (change of variables) If $f : (X, D) \to (X', D')$ is a morphism of pairs over $\mathbb{Q}$, $\gamma \in H_d(X(\mathbb{C}), D(\mathbb{C}), \mathbb{Q})$ and $\omega' \in H^d_{dR}(X', D')$ then

$$(X, D, f^*\omega', \gamma) = (X', D', \omega', f_*\gamma)$$

.

(3) (Stokes formula) For every triple $Z \subset Y \subset X$ we have
$$(Y, Z, \omega, \delta\gamma) = (X, Y, d\omega, \gamma)$$
where $\delta : H_d(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q}) \to H_{d-1}(Y(\mathbb{C}), Z(\mathbb{C}), \mathbb{Q})$ is the boundary operator and $d : H_{dR}^{d-1}(Y, Z) \to H_{dR}^d(X, Y)$ is the differential.

Then $\mathcal{P}_+$ is turned into a $\mathbb{Q}$-algebra via
$$(X, D, \omega, \gamma) \cdot (X', D', \omega', \gamma') := (X \times X', X \times D' \cup D \times X', \omega \wedge \omega', \gamma \times \gamma').$$

The space of formal periods $\mathcal{P} := \mathcal{P}_+[(\mathbb{G}_m, \{1\}, dT/T, \mathbf{S}^1)^{-1}]$ is defined as a localization of $\mathcal{P}_+$, where $\mathbf{S}^1 \subset \mathbb{G}_m(\mathbb{C})$ is the unit circle. Then
$$
\begin{array}{ccc}
H_{dR}^d(X, D) \times H_d(X(\mathbb{C}), D(\mathbb{C}), \mathbb{Q}) & \longrightarrow & \mathbb{C} \\
(\omega, \gamma) & \longmapsto & \int_\gamma \omega
\end{array}
$$
induces a map of $\mathbb{Q}$-algebras
$$\mathrm{ev} : \mathcal{P} \to \mathbb{C}.$$

Definition 6.1. *The image of* ev *is the space* $\mathbb{P} \subset \mathbb{C}$ *of Kontsevich-Zagier periods.*

Below is Grothendieck's period conjecture in the formulation of Kontsevich-Zagier, see ([23] Conjecture 1) and ([23] Conjecture 4.1).

Conjecture 6.2. *(Kontsevich-Zagier) The map* $\mathrm{ev} : \mathcal{P} \to \mathbb{C}$ *is injective.*

We denote by $\mathcal{P}_{\mathrm{alg}}$ the sub-algebra of $\mathcal{P}$ generated by symbols (X, D, ω, γ) where $\dim(X) = 0$. Then the map ev induces an isomorphism $\mathcal{P}_{\mathrm{alg}} \xrightarrow{\sim} \overline{\mathbb{Q}} \subset \mathbb{C}$, where $\overline{\mathbb{Q}} \subset \mathbb{C}$ denotes the set of complex numbers which are algebraic over $\mathbb{Q}$. Let z be such a complex number. Let $m_z(T)$ be its minimal polynomial over $\mathbb{Q}$ and let $F = \mathbb{Q}[T]/(m_z(T))$ be the corresponding number field. We set $X = \mathrm{Spec}(F)$. Then $\bar{T} \in F$ is a class in $H_{dR}^0(X) = F$. The complex number $z \in \mathbb{C}$ gives a point in $X(\mathbb{C}) = \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{C})$, i.e. a 0-cycle γ_z in $X(\mathbb{C})$, such that $\mathrm{ev}(\bar{T}, \gamma_z) = z$.

6.2.2. We set $\mathfrak{P} := \mathrm{Spec}(\mathcal{P})$. Then $\mathfrak{P}$ carries a natural action of $\mathcal{G}_{\mathrm{mot}}(\mathbb{Q})$. By ([18] Theorem 2.10), there is an isomorphism of $\mathcal{G}_{\mathrm{mot}}(\mathbb{Q})$-torsors
$$\tag{29} \mathfrak{P} := \mathrm{Spec}(\mathcal{P}) \simeq \mathbf{Isom}^\otimes(H_B^*, H_{dR}^*).$$
Here $\mathbf{Isom}^\otimes(H_B^*, H_{dR}^*)$ is defined via its functor of points: For any commutative $\mathbb{Q}$-algebra R we have
$$\mathbf{Isom}^\otimes(H_B^*, H_{dR}^*)(R) = \mathrm{Isom}^\otimes(H_B^* \otimes_{\mathbb{Q}} R, H_{dR}^* \otimes_{\mathbb{Q}} R)$$
where the tensor functor $H_B^* \otimes_{\mathbb{Q}} R$ (resp. $H_{dR}^* \otimes_{\mathbb{Q}} R$) is the composition of H_B^* (resp. H_{dR}^*) with $(-) \otimes_{\mathbb{Q}} R$, and $\mathrm{Isom}^\otimes(-, -)$ denotes the set of isomorphisms of tensor functors. Let $E/\mathbb{Q}$ be a number field with a complex embedding σ. There is a unique map $\mathfrak{P} \to \mathrm{Spec}(E)$ such that $E \to \mathcal{P} \xrightarrow{\mathrm{ev}} \mathbb{C}$ coincides with σ. Considering $\mathcal{G}_{\mathrm{mot}}(E)$ as a subgroup of $\mathcal{G}_{\mathrm{mot}}(\mathbb{Q})$ leads to an exact sequence of pointed sets
$$1 \to \mathcal{G}_{\mathrm{mot}}(E) \to \mathcal{G}_{\mathrm{mot}}(\mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{C}) \to *$$
where $\mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{C})$ is pointed by σ. Restricting the group of operators, $\mathfrak{P}$ becomes a $(\mathcal{G}_{\mathrm{mot}}(E) \otimes_{\mathbb{Q}} E)$-torsor over $\mathrm{Spec}(E)$. Moreover, composition with the functor $t : \mathrm{NMM}_E \to \mathrm{NMM}_E(E)$ yields an equivalence from the category of R-valued fiber functors on $\mathrm{NMM}_E(E)$ to the category of R-valued fiber functors on NMM_E. In particular, we have $\mathcal{G}_E \simeq \mathcal{G}_{\mathrm{mot}}(E) \otimes_{\mathbb{Q}} E$ and the identification (29) induces an isomorphism of $\mathcal{G}_E$-torsors
$$\mathfrak{P} \simeq \mathbf{Isom}^\otimes(H_{B,E}^*, H_{dR,E}^*)$$

over $\mathrm{Spec}(E)$.

If M is an object of $\mathrm{NMM}_E(E)$, we denote by $\langle M \rangle$ the Tannakian subcategory of $\mathrm{NMM}_E(E)$ generated by M, i.e. the smallest strictly full E-linear abelian subcategory of $\mathrm{NMM}_E(E)$ containing M, which is stable by tensor products, sums, duals and sub-quotients. We denote by $H^*_{B,E|<M>}$ and $H^*_{dR,E|<M>}$ the restriction of the fiber functors $H^*_{B,E}$ and $H^*_{dR,E}$ to $\langle M \rangle$. We set $\mathcal{G}_{M/E} := \mathbf{Aut}^{\otimes}(H^*_{B,E|<M>})$ and we consider the $\mathcal{G}_{M/E}$-torsor $\mathbf{Isom}^{\otimes}(H^*_{B,E|<M>}, H^*_{dR,E|<M>})$. Note that the sheaf $\mathbf{Isom}^{\otimes}(H^*_{B,E|<M>}, H^*_{dR,E|<M>})$ on $\mathrm{Spec}(E)_{\mathrm{fppf}}$ is represented by an affine scheme. The inclusion $\langle M \rangle \subset \mathrm{NMM}_E(E)$ induces a faithfully flat map $\mathcal{G}_E \to \mathcal{G}_{M/E}$ and there is a canonical isomorphism of $\mathcal{G}_{M/E}$-torsors

$$\mathbf{Isom}^{\otimes}(H^*_{B,E}, H^*_{dR,E}) \wedge^{\mathcal{G}_E} \mathcal{G}_{M/E} \simeq \mathbf{Isom}^{\otimes}(H^*_{B,E|<M>}, H^*_{dR,E|<M>}).$$

We consider the E-sub-algebra $\mathcal{P}_M \subset \mathcal{P}$ such that (29) induces an isomorphism of $\mathcal{G}_{M/E}$-torsors

$$(30) \qquad \mathfrak{P}_M := \mathrm{Spec}(\mathcal{P}_M) \simeq \mathbf{Isom}^{\otimes}(H^*_{B,E|<M>}, H^*_{dR,E|<M>}).$$

Finally, we consider the restriction of the evaluation map

$$\mathrm{ev}_{|\mathcal{P}_M} : \mathcal{P}_M \hookrightarrow \mathcal{P} \to \mathbb{C},$$

and we the denote by

$$\mathbb{P}_M := \mathrm{Im}(\mathrm{ev}_{|\mathcal{P}_M})$$

the E-algebra of periods of M.

6.3. Invariants of orthogonal E-motives.

An *orthogonal Nori motive (M, q) over E with E-coefficients*, or simply an orthogonal E-motive, is an orthogonal object of $\mathrm{NMM}_E(E)$, i.e. an object M endowed with a symmetric map $q : M \otimes M \to \mathbf{1}$ inducing an isomorphism $M \overset{\sim}{\to} M^\vee$. Let (M, q) be such a motive.

6.3.1. *The Hasse-Witt invariants $w_i(q_B)$ and $w_i(q_{dR})$.* Recall that $M_B := H^*_{B,E}(M)$ and $M_{dR} := H^*_{dR,E}(M)$ denote the Betti and de Rham realizations of M. We denote by

$$q_B : M_B \otimes M_B \to E \quad \text{and} \quad q_{dR} : M_{dR} \otimes M_{dR} \to E$$

the non-degenerate symmetric E-bilinear forms obtained by applying the tensor functors $H^*_{B,E}$ and $H^*_{dR,E}$ to the map $q : M \otimes M \to \mathbf{1}$. We shall consider the Hasse-Witt invariants $w_i(q_B) \in H^i(G_E, \mathbb{Z}/2\mathbb{Z})$ and $w_i(q_{dR}) \in H^i(G_E, \mathbb{Z}/2\mathbb{Z})$ of these quadratic forms.

6.3.2. *The global Stiefel-Whitney invariants $sw_i(\rho_{\lambda|G_{E_\lambda}})$.* Let $\lambda \nmid \infty$ be a finite prime of E. Recall that we denote $M_\lambda := H^*_\lambda(M)$. Applying the tensor functor H^*_λ to the orthogonal motive (M, q) we obtain a quadratic form

$$q_\lambda : M_\lambda \otimes M_\lambda \longrightarrow E_\lambda$$

and a continuous E_λ-linear orthogonal representation of the Galois group G_E on (M_λ, q_λ), which we denote by

$$\rho_\lambda : G_E \longrightarrow \mathbf{O}(q_\lambda)(E_\lambda)$$

where $\mathbf{O}(q_\lambda)(E_\lambda) \subset GL_{E_\lambda}(H^*_\lambda(M))$ is endowed with the λ-adic topology. If one forgets the Galois action, we have an isometry $(M_\lambda, q_\lambda) \simeq (M_B, q_B) \otimes_E E_\lambda$. Composing ρ_λ with the determinant map, we obtain a continuous morphism

$$\det(\rho_\lambda) : G_E \longrightarrow \mathbf{O}(q_\lambda)(E_\lambda) \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

The first Stiefel-Whitney invariant $sw_1(\rho_\lambda) \in H^1(G_E, \mathbb{Z}/2\mathbb{Z})$ is the cohomology class of $\det(\rho_\lambda)$.

Let $\mathbb{C}_\lambda$ be the completion of $\overline{E}_\lambda$, given with its natural topology. Consider the continuous representation

$$\overline{\rho}_\lambda : G_E \xrightarrow{\rho_\lambda} \mathbf{O}(q_\lambda)(E_\lambda) \longrightarrow \mathbf{O}(q_\lambda)(\mathbb{C}_\lambda).$$

We have a central extension of topological groups

$$(31) \qquad\qquad 1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \widetilde{\mathbf{O}}(q_\lambda)(\mathbb{C}_\lambda) \longrightarrow \mathbf{O}(q_\lambda)(\mathbb{C}_\lambda) \longrightarrow 1$$

inducing a central extension in the topos $\widetilde{\mathrm{Top}}$ of sheaves on the site of all topological spaces endowed with the open cover topology (see [16]VIII.8.1). Pulling-back (31) via $\overline{\rho}_\lambda$, we obtain central extension of topological groups

$$(32) \qquad\qquad 1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \widetilde{G_E} \longrightarrow G_E \longrightarrow 1$$

inducing a central extension in $\widetilde{\mathrm{Top}}$. The second Stiefel-Whitney invariant $sw_2(\rho_\lambda) \in H^2(G_E, \mathbb{Z}/2\mathbb{Z})$ is the cohomology class (see [16]VIII.8.2) of the central extension (32).

6.3.3. *The local Stiefel-Whitney invariants $sw_i(\rho_{\lambda|G_{E_\lambda}})$.* For a finite prime $\lambda \nmid \infty$, we choose an E-embedding $\overline{E} \to \overline{E_\lambda}$ and we consider the restriction of ρ_λ to G_{E_λ}, i.e. the representation

$$\rho_{\lambda|G_{E_\lambda}} : G_{E_\lambda} \longrightarrow G_E \longrightarrow \mathbf{O}(q_\lambda)(E_\lambda)$$

and we define $sw_i(\rho_{\lambda|G_{E_\lambda}}) \in H^i(G_{E_\lambda}, \mathbb{Z}/2\mathbb{Z})$ as above. Then $sw_i(\rho_{\lambda|G_{E_\lambda}})$ is the image of $sw_i(\rho_\lambda)$ under the restriction map $H^i(G_E, \mathbb{Z}/2\mathbb{Z}) \to H^i(G_{E_\lambda}, \mathbb{Z}/2\mathbb{Z})$. For an archimedean prime $\lambda \mid \infty$, we denote by $M_\lambda := H^*_\lambda(M)$ and by

$$q_\lambda : M_\lambda \otimes M_\lambda \longrightarrow E_\lambda$$

the quadratic form over E_λ obtained by applying the tensor functors H^*_λ to the map q. If λ is a real place, then we obtain an orthogonal representation

$$\rho_{\lambda|G_{E_\lambda}} : G_{E_\lambda} \longrightarrow \mathbf{O}(q_\lambda)(E_\lambda)$$

and $sw_i(\rho_{\lambda|G_{E_\lambda}}) \in H^i(G_{E_\lambda}, \mathbb{Z}/2\mathbb{Z})$ is its classical Stiefel-Whitney invariant.

6.3.4. *The local spinor class $sp_2(\rho_{\lambda|G_{E_\lambda}})$.* Let λ be a place of E. The spinor norm is the map

$$sp : \mathbf{O}(q_\lambda)(E_\lambda) = H^0(G_{E_\lambda}, \mathbf{O}(q_\lambda)(\overline{E_\lambda})) \longrightarrow H^1(G_{E_\lambda}, \mathbb{Z}/2\mathbb{Z}) = E_\lambda^\times / E_\lambda^{\times 2}$$

induced by the exact sequence of group-schemes

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathbf{O}}(q_\lambda) \to \mathbf{O}(q_\lambda) \to 1.$$

Composing $\rho_{\lambda|G_{E_\lambda}}$ with the spinor norm we obtain an element $sp \circ \rho_{\lambda|G_{E_\lambda}}$ in the group

$$\mathrm{Hom}(G_{E_\lambda}, E_\lambda^\times / E_\lambda^{\times 2}) = \mathrm{Hom}(G_{E_\lambda}, H^1(G_{E_\lambda}, \mathbb{Z}/2\mathbb{Z})) \simeq H^1(G_{E_\lambda}, \mathbb{Z}/2\mathbb{Z}) \otimes H^1(G_{E_\lambda}, \mathbb{Z}/2\mathbb{Z}).$$

The spinor class $sp_2(\rho_{\lambda|G_{E_\lambda}})$ is defined as the image of $sp \circ \rho_{\lambda|G_{E_\lambda}}$ by the cup product

$$H^1(G_{E_\lambda}, \mathbb{Z}/2\mathbb{Z}) \otimes H^1(G_{E_\lambda}, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cup} H^2(G_{E_\lambda}, \mathbb{Z}/2\mathbb{Z}).$$

6.3.5. *The classes $\delta_q^1(\mathfrak{P}_M)$ and $\delta_q^2(\mathfrak{P}_M)$.* As in section 3.3, an orthogonal Nori motive (M, q) over E with E-coefficients yields an orthogonal representation

$$\rho_q : \mathcal{G}_E \longrightarrow \mathbf{O}(q_B)$$

inducing maps

$$\delta_q^i : H^1(\mathrm{Spec}(E)_{\mathrm{fppf}}, \mathcal{G}_E) \longrightarrow H^i(G_E, \mathbb{Z}/2\mathbb{Z})$$

for $i = 1, 2$. The protorsor $\mathfrak{P}$ gives a canonical class in $H^1(\mathrm{Spec}(E)_{\mathrm{fppf}}, \mathcal{G}_E)$ and we shall consider its image $\delta_q^i(\mathfrak{P})$. The orthogonal representation ρ_q factors through

$$\mathcal{G}_{M/E} \longrightarrow \mathbf{O}(q_B)$$

hence gives

$$\delta_q^i : H^1(\mathrm{Spec}(E)_{\mathrm{fppf}}, \mathcal{G}_{M/E}) \longrightarrow H^i(G_E, \mathbb{Z}/2\mathbb{Z}).$$

By definition, we have

$$\delta_q^i(\mathfrak{P}) = \delta_q^i(\mathfrak{P}_M) \in H^i(G_E, \mathbb{Z}/2\mathbb{Z}).$$

6.4. Interpretation of $\delta_q^1(\mathfrak{P}_M)$ and $\delta_q^2(\mathfrak{P}_M)$.

6.4.1. *The class $\delta_q^1(\mathfrak{P}_M)$ and the periods of $\det_E(M)$.* Let $M \in \mathrm{NMM}_E(E)$ be a motive given by the representation $\mathcal{G}_E \to \mathbf{GL}(M_B)$. The motive $\det_E(M)$ is the motive associated with the determinant representation

$$\mathcal{G}_E \longrightarrow \mathbf{GL}(\Lambda_E^{\dim_E(M_B)} M_B)$$

Using the canonical isomorphism $\mathbf{GL}(\Lambda_E^{\dim_E(M_B)} M_B) \simeq \mathbf{G}_m$, where $\mathbf{G}_m$ is the multiplicative group over E, we see that $\det_E(M)$ is the motive corresponding to

$$(33) \qquad\qquad \mathcal{G}_E \longrightarrow \mathbf{GL}(M_B) \xrightarrow{\det} \mathbf{G}_m.$$

An orthogonal structure q on M induces an orthogonal structure $\det(q)$ on $\det_E(M)$. Recall that the class $\delta_q^1(\mathfrak{P}_M)$ belongs to the group $H^1(\mathrm{Spec}(E), \mathbb{Z}/2\mathbb{Z}) \simeq E^\times/E^{\times 2}$.

Proposition 6.3. *Let (M, q) be an orthogonal motive.*

(1) *We have*

$$\mathbb{P}_{\det_E(M)} = E\left(\sqrt{\delta_q^1(\mathfrak{P}_M)}\right).$$

(2) *If $\delta_q^1(\mathfrak{P}_M)$ is non-trivial, then we have $\mathcal{G}_{\det_E(M)/E} \simeq \mathbb{Z}/2\mathbb{Z}$ and an isomorphism*

$$\mathfrak{P}_{\det_E(M)} \simeq \mathrm{Spec}\left(E\left(\sqrt{\delta_q^1(\mathfrak{P}_M)}\right)\right)$$

of $\mathbb{Z}/2\mathbb{Z}$-torsor over $\mathrm{Spec}(E)$.

Proof. Consider the composite map

$$(34) \qquad\qquad \mathcal{G}_E \longrightarrow \mathbf{O}(q_B) \xrightarrow{\det} \mathbb{Z}/2\mathbb{Z} \simeq \mathbf{O}(\det(q)_B).$$

Here $\mathbb{Z}/2\mathbb{Z} \simeq \mathbf{O}(\det(q)_B)$ is the orthogonal group of the Betti realization of $(\det_E(M), \det(q))$. The motivic Galois group $\mathcal{G}_{\det_E(M)/E}$ comes with a monomorphism

$$\mathcal{G}_{\det_E(M)/E} \hookrightarrow \mathbb{Z}/2\mathbb{Z}$$

so that $\mathcal{G}_{\det_E(M)/E}$ is either trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The canonical morphism $\mathcal{G}_{M/E} \to \mathcal{G}_{\det_E(M)/E}$ induces $H^1(E, \mathcal{G}_{M/E}) \to H^1(E, \mathcal{G}_{\det_E(M)/E})$, which maps $\mathfrak{P}_M$ to $\mathfrak{P}_{\det_E(M)}$. Moreover, δ_q^1 is the map $H^1(E, \mathcal{G}_{M/E}) \to H^1(E, \mathbb{Z}/2\mathbb{Z})$ induced by

$$\mathcal{G}_{M/E} \to \mathcal{G}_{\det_E(M)/E} \hookrightarrow \mathbb{Z}/2\mathbb{Z}.$$

In particular, if $\mathcal{G}_{\det_E(M)/E}$ is trivial then $\delta_q^1(\mathfrak{P}_M)$ is trivial, $\mathfrak{P}_{\det_E(M)}$ is trivial and $\mathbb{P}_{\det_E(M)} = E$. So one may assume that $\mathcal{G}_{\det_E(M)/E} \simeq \mathbb{Z}/2\mathbb{Z}$. In this case we have $\delta_q^1(\mathfrak{P}_M) = \mathfrak{P}_{\det_E(M)}$. If $\mathfrak{P}_{\det_E(M)}$ is the trivial $\mathbb{Z}/2\mathbb{Z}$-torsor, then $\mathbb{P}_{\det_E(M)} = E$ and $\delta_q^1(\mathfrak{P}_M)$ is trivial. So one may assume that the $\mathbb{Z}/2\mathbb{Z}$-torsor $\mathfrak{P}_{\det_E(M)}$ is non-trivial. In this case we have $\mathfrak{P}_{\det_E(M)} = \mathrm{Spec}\left(E\left(\sqrt{\delta_q^1(\mathfrak{P}_M)}\right)\right)$ and the evaluation map $\mathcal{P}_{\det_E(M)} \to \mathbb{P}_{\det_E(M)}$ is an isomorphism. $\qquad\square$

6.4.2. *The class $\delta_q^2(\mathfrak{P}_M)$ and the lifting problem* Lift(M,q).

Definition 6.4. *Let $\widetilde{\mathcal{G}} \to \mathcal{G}$ be a map of group schemes and let T be a $\mathcal{G}$-torsor. We say that T admits a $\widetilde{\mathcal{G}}$-lifting if there exist a $\widetilde{\mathcal{G}}$-torsor $\widetilde{T}$ and an isomorphism of $\mathcal{G}$-torsors $\widetilde{T} \wedge^{\widetilde{\mathcal{G}}} \mathcal{G} \simeq T$.*

Let (M,q) be an orthogonal motive given by the representation

$$\mathcal{G}_E \to \mathcal{G}_{M/E} \to \mathbf{O}(q_B).$$

Pulling back the canonical central extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathbf{O}}(q_B) \to \mathbf{O}(q_B) \to 1$$

we obtain a central extension

$$(35) \qquad\qquad 1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathcal{G}}_{M/E} \to \mathcal{G}_{M/E} \to 1.$$

We shall also consider its base change

$$(36) \qquad 1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathcal{G}}_{M/E} \otimes_E E_\lambda \to \mathcal{G}_{M/E} \otimes_E E_\lambda \to 1.$$

We would like to address the following lifting problem for torsors.

Notation 6.5. *We say that* Lift(M,q) *has a solution if the $\mathcal{G}_{M/E}$-torsor $\mathfrak{P}_M$ admits a $\widetilde{\mathcal{G}}_{M/E}$-lifting. We say that* Lift(M,q) *has a solution over E_λ if the $\left(\mathcal{G}_{M/E} \otimes_E E_\lambda\right)$-torsor $\mathfrak{P}_M \otimes_E E_\lambda$ admits a $\left(\widetilde{\mathcal{G}}_{M/E} \otimes_E E_\lambda\right)$-lifting.*

We identify $H^*(\mathrm{Spec}(E)_{\mathrm{fppf}}, \mathbb{Z}/2\mathbb{Z})$ with Galois cohomology $H^*(G_E, \mathbb{Z}/2\mathbb{Z})$. Recall that the class $\delta_q^2(\mathfrak{P}_M)$ belongs to the groups $H^2(G_E, \mathbb{Z}/2\mathbb{Z})$ which is described by class field theory: One has the exact sequence

$$(37) \qquad 0 \longrightarrow H^2(G_E, \mathbb{Z}/2\mathbb{Z}) \longrightarrow \bigoplus_\lambda H^2(G_{E_\lambda}, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\Sigma} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

and $H^2(G_{E_\lambda}, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ for λ finite and for λ real. Finally, we denote by

$$\begin{array}{ccc} H^*(G_E, \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & H^*(G_{E_\lambda}, \mathbb{Z}/2\mathbb{Z}) \\ x & \longmapsto & x_\lambda \end{array}$$

the canonical restriction map, for any place λ of E.

Proposition 6.6. *Let (M,q) be an orthogonal motive.*

   (1) *The following assertions are equivalent.*

(a) $\delta_q^2(\mathfrak{P}_M) = 0$.
(b) *The problem* $\mathrm{Lift}(M,q)$ *has a solution.*
(c) *The problem* $\mathrm{Lift}(M,q)$ *has a solution over* E_λ *for any place* λ *of* E.

(2) *Let* λ *be a finite place. The problem* $\mathrm{Lift}(M,q)$ *has a solution over* E_λ *if and only if* $\delta_q^2(\mathfrak{P}_M)_\lambda = 0$.

(3) *If there exists a finite prime* $\lambda \nmid \infty$ *such that* $sw_2(\rho_\lambda) \neq 0$ *then the central extension* *(36) is non-trivial (and a fortiori (35) is non-trivial).*

Proof. The exact sequence (35) induces an exact sequence of pointed sets

$$H^1(\mathrm{Spec}(E)_{\mathrm{fppf}}, \widetilde{\mathcal{G}}_{M/E}) \longrightarrow H^1(\mathrm{Spec}(E)_{\mathrm{fppf}}, \mathcal{G}_{M/E}) \xrightarrow{\delta_q^2} H^2(\mathrm{Spec}(E)_{\mathrm{fppf}}, \mathbb{Z}/2\mathbb{Z})$$

and similarly for (36). Assertions (1) and (2) follow from these exact sequences of pointed sets, their compatibility with base change and from the exact sequence (37). It remains to prove assertion (3). Assume that (36) is a trivial extension. Then the extension of topological groups

$$(38) \qquad\qquad 1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathcal{G}}_{M/E}(\mathbb{C}_\lambda) \to \mathcal{G}_{M/E}(\mathbb{C}_\lambda) \to 1$$

is also trivial. But (32) is the pull-back of (38) via the continuous morphism $G_E \to \mathcal{G}_{M/E}(\mathbb{C}_\lambda)$, so that (32) is trivial as well.

$\square$

6.4.3. *The class* $\delta_q(\mathfrak{P}_M)$ *as an obstruction to the existence of isometry* $q_B \simeq q_{dR}$. Let E be a totally imaginary number field and let (M,q) be an arbitrary orthogonal E-motive. Recall that E has strict cohomological dimension 2. We consider the multiplicative group

$$H^*(G_E, \mathbb{Z}/2\mathbb{Z})^\times := \{1 + a_1 + a_2 \in \sum_{0 \leq i \leq 2} H^i(G_E, \mathbb{Z}/2\mathbb{Z}),\ a_i \in H^i(G_E, \mathbb{Z}/2\mathbb{Z})\}$$

of the cohomology ring $H^*(G_E, \mathbb{Z}/2\mathbb{Z})$. We consider the elements

$$\begin{aligned}
\delta_q(\mathfrak{P}_M) &= 1 + \delta_q(\mathfrak{P}_M) + \delta_q^2(\mathfrak{P}_M) \\
w(q_B) &= 1 + w_1(q_B) + w_2(q_B) \\
w(q_{dR}) &= 1 + w_1(q_{dR}) + w_2(q_{dR})
\end{aligned}$$

of $H^*(G_E, \mathbb{Z}/2\mathbb{Z})^\times$. We already record the following corollary of Theorem 6.10.

Corollary 6.7. *Assume that* E *is a totally imaginary. The forms* q_B *and* q_{dR} *are isometric if and only if* $\delta_q(\mathfrak{P}_M) = 1$.

Proof. Using the notation introduced above, Theorem 6.10(1) reads as follows:

$$(39) \qquad\qquad \delta_q(\mathfrak{P}_M) = w(q_B)^{-1} \cdot w(q_{dR}).$$

By (39) we have $\delta_q(\mathfrak{P}_M) = 1$ if and only if $w(q_B) = w(q_{dR})$. This is equivalent to $w(q_B)_\lambda = w(q_{dR})_\lambda$ for any place λ, hence equivalent to $q_B \otimes_E E_\lambda \simeq q_{dR} \otimes_E E_\lambda$ for any λ, which is in turn equivalent to $q_B \simeq q_{dR}$.

$\square$

6.4.4. *The motivic lifting problem* $\mathrm{MotLift}(M, q)$.

Notation 6.8. *Let (M, q) be an orthgonal E-motive given by the representation $\rho_q : \mathcal{G}_E \to \mathbf{O}(q_B)$. We say that $\mathrm{MotLift}(M, q)$ has a solution if there exists a representation $\tilde{\rho}$ such that the following diagram commutes:*

$$
\begin{array}{ccc}
 & & \widetilde{\mathbf{O}}(q_B) \\
 & \overset{\tilde{\rho}_q}{\nearrow} & \downarrow \\
\mathcal{G}_E & \xrightarrow{\ \rho_q\ } & \mathbf{O}(q_B)
\end{array}
$$

Proposition 6.9. *If the $\mathrm{MotLift}(M, q)$ has a solution, then $\delta_q^2(\mathfrak{P}_M) = 0$, $sw_2(\rho_\lambda) = 0$ for any finite prime λ, and $sw_2(\rho_{\lambda|G_{E_\lambda}}) = sp_2(\rho_{\lambda|G_{E_\lambda}}) = 0$ for any prime λ of E.*

6.5. **Examples of orthogonal Nori motives.** Let V/E be a smooth projective algebraic variety of even dimension n. By Corollary 8.11, the motive $h^n(V)(n/2) \in \mathrm{NMM}_E(E)$ has a canonical orthogonal structure q such that the forms q_B and q_{dR} are the perfect pairings

$$
q_B : H_B^n(V(\mathbb{C}), (2i\pi)^{n/2}\mathbb{Q})_E \otimes H_B^n(V(\mathbb{C}), (2i\pi)^{n/2}\mathbb{Q})_E \xrightarrow{\cup} H_B^{2n}(V(\mathbb{C}), (2i\pi)^n\mathbb{Q})_E \xrightarrow{tr} E
$$

and

$$
q_{dR} : H_{dR}^n(V/E) \otimes H_{dR}^n(V/E) \xrightarrow{\wedge} H_{dR}^{2n}(V/E) \xrightarrow{tr} E
$$

given by cup-product and trace maps. Similarly, for $\lambda \mid p$ a finite prime of E, the λ-adic realization yields the orthogonal Galois representation ρ_λ of G_E given by p-adic étale cohomology

$$
q_\lambda : H^n(V_{\overline{E}}, \mathbb{Q}_p(n/2))_{E_\lambda} \otimes H^n(V_{\overline{E}}, \mathbb{Q}_p(n/2))_{E_\lambda} \xrightarrow{\cup} H^{2n}(V_{\overline{E}}, \mathbb{Q}_p(n/2))_{E_\lambda} \xrightarrow{tr} E_\lambda
$$

with its natural Galois action. If λ is a real prime of E, then $\rho_{\lambda|G_{E_\lambda}}$ is the G_{E_λ}-equivariant quadratic form

$$
H^n(V_\lambda(\mathbb{C}), (2i\pi)^{n/2}\mathbb{Q})_{E_\lambda} \otimes H^n(V_\lambda(\mathbb{C}), (2i\pi)^{n/2}\mathbb{Q})_{E_\lambda} \xrightarrow{\cup} H_B^{2n}(V_\lambda(\mathbb{C}), (2i\pi)^n\mathbb{Q})_{E_\lambda} \xrightarrow{tr} E_\lambda
$$

where $V_\lambda(\mathbb{C})$ is the set of complex points of V over E with respect with the complex embedding $E \xrightarrow{\lambda} \mathbb{R} \to \mathbb{C}$.

6.6. **The main result.** Recall that we denote by

$$
\begin{array}{ccc}
H^*(G_E, \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & H^*(G_{E_\lambda}, \mathbb{Z}/2\mathbb{Z}) \\
x & \longmapsto & x_\lambda
\end{array}
$$

the canonical restriction map, for any prime λ. In particular, if $x \in H^2(G_E, \mathbb{Z}/2\mathbb{Z})$ then we have $x = \sum_\lambda x_\lambda$ in $H^2(G_E, \mathbb{Z}/2\mathbb{Z})$.

Theorem 6.10. *Let (M, q) be an orthogonal Nori E-motive.*

(1) *We have*

$$
\begin{aligned}
\delta_q^1(\mathfrak{P}_M) &= w_1(q_B) + w_1(q_{dR}) \\
\delta_q^2(\mathfrak{P}_M) &= w_2(q_B) + w_1(q_B) \cdot w_1(q_B) + w_1(q_B) \cdot w_1(q_{dR}) + w_2(q_{dR}).
\end{aligned}
$$

(2) *Assume that (M, q) is of the form $(h^n(V)(n/2), q)$ as in Section 6.5. Then we have*

$$
\delta_q^1(\mathfrak{P}_M) = sw_1(\rho_\lambda).
$$

(3) *Let (M, q) be an arbitrary orthogonal motive. Assume that either $E = \mathbb{Q}$ or E is totally imaginary. Then we have*

$$\delta_q^2(\mathfrak{P}_M) = \sum_\lambda sw_2(\rho_{\lambda|G_{E_\lambda}}) + sp_2(\rho_{\lambda|G_{E_\lambda}}).$$

Proof. In view of the isomorphism of $\mathcal{G}_{M/E}$-torsors $\mathfrak{P}_M \simeq \mathbf{Isom}^\otimes(H^*_{B,E<M>}, H^*_{dR,E<M>})$, Assertion (1) follows from Theorem 3.3.

Let λ be a finite place of E. By Artin's comparison theorem (27), we have an isometry $(M_B, q_B) \otimes_E E_\lambda \simeq (M_\lambda, q_\lambda)$. We denote by $(M_{dR,E_\lambda}, q_{dR,E_\lambda})$ the E_λ-quadratic space $(M_{dR}, q_{dR}) \otimes_E E_\lambda$. It follows from the first assertion that we have identities

$$\delta_q^1(\mathfrak{P})_\lambda = w_1(q_\lambda) + w_1(q_{dR,E_\lambda}) \tag{40}$$

$$\delta_q^2(\mathfrak{P})_\lambda = w_2(q_\lambda) + w_1(q_\lambda) \cdot w_1(q_\lambda) + w_1(q_\lambda) \cdot w_1(q_{dR,E_\lambda}) + w_2(q_{dR,E_\lambda}) \tag{41}$$

since the Hasse-Witt invariants are compatible with base-change.

By (28) the representation $\rho_{\lambda|G_{E_\lambda}}$ is de Rham and $(M_{dR,E_\lambda}, q_{dR,E_\lambda})$ is a λ-adic twist of (M_λ, q_λ) by $\rho_{\lambda|G_{E_\lambda}}$ in the sense of Section 5.1 (see for example [3], Theorem 2.2.3). By (40), (41) and Corollary 5.7, we obtain

$$\delta_q^1(\mathfrak{P})_\lambda = sw_1(\rho_{\lambda|G_{E_\lambda}}) \tag{42}$$

$$\delta_q^2(\mathfrak{P})_\lambda = sw_2(\rho_{\lambda|G_{E_\lambda}}) + sp_2(\rho_{\lambda|G_{E_\lambda}}) \tag{43}$$

for any $\lambda \nmid \infty$. In order to prove the assertion (3), it remains to consider the case $E = \mathbb{Q}$ and $\lambda = \infty$, and to prove that we have

$$w_2(q_\infty) + w_1(q_\infty) \cdot w_1(q_\infty) + w_1(q_\infty) \cdot w_1(q_{dR,\mathbb{R}}) + w_2(q_{dR,\mathbb{R}}) = sw_2(\rho_{\infty|G_\mathbb{R}}) + sp_2(\rho_{\infty|G_\mathbb{R}}) \tag{44}$$

where $(M_\infty, q_\infty) = (M_B, q_B) \otimes_\mathbb{Q} \mathbb{R}$ and $\rho_{\infty|G_\mathbb{R}} : G_\mathbb{R} \to \mathbf{O}(q_\infty)(\mathbb{R})$ is induced by the action of the complex conjugation on M_B. But in this case we have an isometry

$$(M_\infty \otimes_\mathbb{R} \mathbb{C})^{G_\mathbb{R}} \simeq M_{dR,\mathbb{R}}$$

and thus $q_{dR,\mathbb{R}}$ is the Fröhlich twist of q_∞ by $\rho_{\infty|G_\mathbb{R}}$, so that (44) is given by Corollary 5.5.

We show Assertion (2). By ([32] Corollary 3.3), the global Stiefel-Whitney invariant $sw_1(\rho_\lambda)$ does not dependent on the finite place λ. Let μ be a fixed finite place of E. For any λ finite, we have

$$sw_1(\rho_\mu)_\lambda = sw_1(\rho_\lambda)_\lambda = sw_1(\rho_{\lambda|G_{E_\lambda}}) = \delta_q^1(\mathfrak{P})_\lambda.$$

But the map

$$H^1(G_E, \mathbb{Z}/2\mathbb{Z}) \longrightarrow \prod_{\lambda \nmid \infty} H^1(G_{E_\lambda}, \mathbb{Z}/2\mathbb{Z})$$

is injective, by the Chebotarev density theorem. In view of (42), we obtain $\delta_q^1(\mathfrak{P}_M) = sw_1(\rho_\mu)$, as required. $\qquad \square$

We observe the following corollary, where we suppose to simplify that $E = \mathbb{Q}$ or E totally imaginary.

Corollary 6.11. *If the map $\mathcal{G}_E \to \mathbf{O}(q_B)$ given by the orthogonal motive (M, q) admits a lifting $\mathcal{G}_E \to \widetilde{\mathbf{O}}(q_B)$, then $\delta_q^2(\mathfrak{P}_M) = 0$ and $sw_2(\rho_{\lambda|G_{E_\lambda}}) = sp_2(\rho_{\lambda|G_{E_\lambda}}) = 0$ for any prime λ of E.*

Proof. Assume that $\mathcal{G}_E \to \mathbf{O}(q_B)$ admits a lifting $\mathcal{G}_E \to \widetilde{\mathbf{O}}(q_B)$. Then $\mathcal{G}_E \to \mathcal{G}_{M/E}$ factors through $\widetilde{\mathcal{G}}_{M/E}$, and $\mathfrak{P} \wedge^{\mathcal{G}_E} \widetilde{\mathcal{G}}_{M/E}$ lifts $\mathfrak{P}_M$ so that $\delta_q^2(\mathfrak{P}_M) = 0$. Moreover, the extension of topological group

$$(45) \qquad\qquad 1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathcal{G}}_E(\mathbb{C}_\lambda) \to \mathcal{G}_E(\mathbb{C}_\lambda) \to 1.$$

is trivial (it has a global section), hence so is the extension (32). It follows that $sw_2(\rho_\lambda) = 0$ for any prime λ. In view of Theorem 6.10(3), we get $sw_2(\rho_{\lambda|G_{E_\lambda}}) = sp_2(\rho_{\lambda|G_{E_\lambda}}) = 0$ for any λ. $\qquad\square$

6.7. **The case $M = h^n(V)(n/2)$.** Let V be a smooth projective variety over E of even dimension n. We endow $M = h^n(V)(n/2)$ with its canonical orthogonal structure q. For any prime p, we consider $sw_1(\rho_p) \in H^1(G_E, \mathbb{Z}/2\mathbb{Z}) \simeq E^\times/E^{\times 2}$. We denote by $b_i = \dim_{\mathbb{Q}} H_B^i(V_\sigma(\mathbb{C}), \mathbb{Q})$ the ith Betti number and, following [30], we set

$$r(V) = \sum_{i<n}(-1)^i \cdot b_i \qquad \left(\text{respectively } r(V) = \sum_{i\le n}(-1)^i \cdot b_i\right)$$

if $n \equiv 0 \bmod 4$ (respectively if $n \equiv 2 \bmod 4$).

Corollary 6.12. *If V is a smooth projective variety over E of even dimension n, then we have*

$$\mathbb{P}_{\det_E(h^n(V)(n/2))} = E\left(\sqrt{sw_1(\rho_p)}\right)$$

for any prime $p < \infty$, and

$$\delta_q^2(\mathfrak{P}_M) = w_2(q_B) + (-1)^{r(V)} \cdot (-1)^{r(V)} + (-1)^{r(V)} \cdot w_1(q_{dR}) + w_2(q_{dR}).$$

Proof. The first assertion immediately follows from Theorem 6.10 and Proposition 6.3. As for the second assertion, it is enough to showing that

$$w_1(q_B) = \mathrm{disc}(q_B) = (-1)^{r(V)}$$

in $E^\times/E^{\times 2}$ by Theorem 6.10(1). This is equivalent to the equality (written additively)

$$w_1(q_{dR}) = sw_1(\rho_p) + r(V) \cdot \{-1\}$$

by Theorem 6.10(2), which is ([30] Theorem 2) (see also [33], Remark 2.4). $\qquad\square$

Remark 6.13. *It is conjectured that there is a short exact sequence*

$$1 \to \mathcal{G}_E^0 \to \mathcal{G}_E \to G_E \to 1$$

where $\mathcal{G}_E^0$ is the connected component of the identity in $\mathcal{G}_E$, G_E is the absolute Galois group and the map $\mathcal{G}_E \to G_E$ is induced by the inclusion of Artin motives inside the category of all motives. Assume that there is such an exact sequence and let (M, q) be an orthogonal motive, given by the orthogonal representation

$$\rho_M : \mathcal{G}_E \to \mathbf{O}(q_B).$$

Then the determinant motive $\det_E(M)$ is given by the representation

$$\rho_{\det_E(M)} : \mathcal{G}_E \to \mathbf{O}(\det(q)_B) = \mathbb{Z}/2\mathbb{Z}$$

which must factor as follows:

$$\rho_{\det_E(M)} : \mathcal{G}_E \to G_E \xrightarrow{\alpha} \mathbb{Z}/2\mathbb{Z}.$$

Hence $\det_E(M)$ *would be an orthogonal Artin motive, and we would obtain* $\det(\rho_\lambda) = \alpha$ *and* $\mathbb{P}_{\det_E(M)} = E\left(\sqrt{sw_1(\rho_\lambda)}\right)$ *for any finite prime* λ *of* E.

7. EXAMPLES

7.1. **Artin motives.** In the case of orthogonal Artin motives, Theorem 6.10 reduces to classical results of Serre and Fröhlich, see ([34] Theorem 1) and ([13] Theorems 2 and 3) over number fields. The aim of this section is to compute these invariants for some explicit Artin motives.

Let (M, q) be an orthogonal Artin motive given by a representation of the form

$$\rho_q : \mathcal{G}_E \longrightarrow G_E \xrightarrow{\rho_0} \mathbf{O}(q_B)$$

We have $\mathfrak{P}_M = \mathrm{Spec}(K)$ for K/E a finite Galois extension. In this situation Fröhlich defines [13] the twisted form $(q_B \otimes_E K)^{G_{K/E}}$.

Corollary 7.1. *We have an isometry* $q_{dR} \simeq (q_B \otimes_E K)^{G_{K/E}}$ *and the following identities in* $H^*(G_E, \mathbb{Z}/2\mathbb{Z})$:

$$\begin{aligned}
\delta_q^1(\mathrm{Spec}(K)) &= w_1(q_B) + w_1(q_{dR}) = sw_1(\rho_0); \\
\delta_q^2(\mathrm{Spec}(K)) &= w_2(q_B) + w_1(q_B) \cdot w_1(q_B) + w_1(q_B) \cdot w_1(q_{dR}) + w_2(q_{dR}) \\
&= sw_2(\rho_0) + sp_2(\rho_0).
\end{aligned}$$

Proof. The local representation $\rho_{\lambda|G_{E_\lambda}}$ identifies with the composition

$$G_{E_\lambda} \longrightarrow G_E \xrightarrow{\rho_0} \mathbf{O}(q_B)(E) \longrightarrow \mathbf{O}(q_B)(E_\lambda)$$

for any place λ of E. We obtain

$$sw_2(\rho_0) = \sum_\lambda sw_2(\rho_{\lambda|G_{E_\lambda}}) \text{ and } sp_2(\rho_0) = \sum_\lambda sp_2(\rho_{\lambda|G_{E_\lambda}})$$

and $sw_1(\rho_0) = sw_1(\rho_\lambda)$ for any λ finite. The result then follows immediately from Theorem 6.10 and its proof. $\qquad\square$

Let F/E be finite extension let $M = h^0(\mathrm{Spec}(F))(0) \in \mathrm{NMM}_E(E)$. Consider the canonical orthogonal structure q on M. In this case one has $M_B = E^{\mathrm{Hom}_E(F, \mathbb{C})}$ (w.r.t. $\sigma : E \to \mathbb{C}$) and

$$q_B : \begin{array}{ccc} E^{\mathrm{Hom}_E(F, \mathbb{C})} \otimes E^{\mathrm{Hom}_E(F, \mathbb{C})} & \longrightarrow & E \\ ((v_\tau)_\tau, (w_\tau)_\tau) & \longmapsto & \sum_\tau v_\tau \cdot w_\tau \end{array}$$

is the standard form. Moreover, one has $M_{dR} = F$ and

$$q_{dR} : \begin{array}{ccc} F \otimes F & \longrightarrow & E \\ (x, y) & \longmapsto & \mathrm{Trace}_{F/E}(x \cdot y) \end{array}$$

is the usual trace form. The representation $\rho_q : \mathcal{G}_E \to \mathbf{O}(q_B)$ factors as follows

$$\rho_q : \mathcal{G}_E \longrightarrow G_E \xrightarrow{\rho_0} \mathbf{O}(q_B)$$

where G_E acts on $M_B = E^{\mathrm{Hom}_E(F, \mathbb{C})} = E^{\mathrm{Hom}_E(F, \overline{E})}$ by permuting the factors. Let K be the Galois closure of (some embedding of) F/E in $\overline{E} \subset \mathcal{P}$. We have $\mathfrak{P}_M = \mathrm{Spec}(K)$. We denote by $d_{F/E}$ the discriminant.

Corollary 7.2. *For $M = h^0(\text{Spec}(F))$, one has*

$$\delta_q^1(\text{Spec}(K)) = w_1(q_{dR}) = d_{F/E} = sw_1(\rho_0);$$

$$\delta_q^2(\text{Spec}(K)) = w_2(q_{dR}) = sw_2(\rho_0) + sp_2(\rho_0).$$

Proof. The from q_B is the standard form, hence $w_1(q_B) = w_2(q_B) = 0$. □

7.1.1. *The lifting problem* $\text{Lift}(h^0(\text{Spec}(F)), q)$. As above, we consider an extension of number fields F/E with discriminant $d_{F/E}$ and let $M = h^0(\text{Spec}(F))$. Let K/E be the Galois closure of F in $\overline{E}$. Recall that $\mathcal{G}_{M/E} = G_{K/E}$. The terms $\delta_q^2(\text{Spec}(K))$ and $sw_2(\rho_0)$ have a Galois theoretic interpretation. We consider the exact sequence of groups

$$(46) \qquad\qquad 1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \widetilde{\mathbf{O}}(q_B)(\overline{E}) \longrightarrow \mathbf{O}(q_B)(\overline{E}) \longrightarrow 1.$$

Then the group-scheme $\widetilde{\mathcal{G}}_{M/E}$ can be seen as the group of $\overline{E}$-points

$$\widetilde{\mathcal{G}}_{M/E}(\overline{E}) = G_{K/E} \times_{\mathbf{O}(q_B)(\overline{E})} \widetilde{\mathbf{O}}(q_B)(\overline{E})$$

endowed with its natural G_E-action. We also consider

$$\widetilde{G}_{K/E} := G_{K/E} \times_{\mathbf{O}(q_B)(\overline{E})} \widetilde{\mathbf{O}}(q_B)(\overline{E})$$

endowed with the trivial action of G_E, as a constant group scheme over E. Note that the group-schemes $\widetilde{\mathcal{G}}_{M/E}$ and $\widetilde{G}_{K/E}$ are isomorphic if and only if $\widetilde{\mathcal{G}}_{M/E}$ is constant, i.e. iff the natural G_E-action is trivial. We say that the lifting problem $\text{Lift}^\delta(h^0(\text{Spec}(F)), q)$ has a solution if there exists a $\widetilde{G}_{K/E}$-torsor lifting $\mathfrak{P}_M = \text{Spec}(K)$, i.e. if it is possible to embed the Galois extension K/E into a Galois extension $\widetilde{K}/E$ with Galois group $\widetilde{G}_{K/E}$. Then $\text{Lift}^\delta(h^0(\text{Spec}(F)), q)$ has a solution if and only if $sw_2(\rho_0) = 0$ while $\text{Lift}(h^0(\text{Spec}(F)), q)$ has a solution if and only if $\delta_q^2(\text{Spec}(K)) = 0$. It follows that if $sp_2(\rho_0) \neq 0$ then $\widetilde{\mathcal{G}}_{M/E}$ is not constant. Moreover, it follows from [13] and from [10] in a more general set up, that $sp_2(\rho) = (2)(d_{F/E})$.

7.1.2. *Explicit examples.* We assume in this section that $F/\mathbb{Q}$ is of degree 4 and that $G_{K/\mathbb{Q}} = \mathfrak{S}_4$. The group $\widetilde{\mathfrak{S}}_4$ is characterized by the fact that a transposition in $\mathfrak{S}_4$ lifts to an element of order 2 of $\widetilde{\mathfrak{S}}_4$ whereas a product of 2 disjoint transpositions lifts to an element of order 4. Since 2 is a square of $\mathbb{R} = \mathbb{Q}_\infty$ we note that $(2, d_F)_\infty = 1$. We write w_2 for the Hasse Witt invariant $w_2(\text{Tr}_{F/\mathbb{Q}})$ and we denote by (r, s) the signature of the field F. Proposition 6.5 gives $w_{2,\infty} = -1$ if $(r, s) = (0, 2)$ and $w_{2,\infty} = 1$ otherwise. The following result is taken from [21].

Proposition 7.3. *Let $p \geq 3$ be a prime number. Then $\omega_{2,p} = (2, d_F)_p = 1$ if p is non ramified in F. Moreover we have the following equalities:*

(1) $w_{2,p} = (2, d_F)_p = (-1)^{\frac{p^2-1}{8}}$ *if* $p\mathcal{O}_F = \mathfrak{p}_1^2\mathfrak{p}_1'\mathfrak{p}_2''$,

(2) $w_{2,p} = (2, d_F)_p = 1$ *if* $p\mathcal{O}_F = \mathfrak{p}_1^3\mathfrak{p}_1'$,

(3) $w_{2,p} = -(2, d_F)_p = -(-1)^{\frac{p^2-1}{8}}$ *if* $p\mathcal{O}_F = \mathfrak{p}_1^2\mathfrak{p}_2'$,

(4) $w_{2,p} = (-1)^{\frac{p-1}{2}}$ *and* $(2, d_F)_p = (-1)^{\frac{p^2-1}{8}}$ *if* $p\mathcal{O}_F = \mathfrak{p}_1^4$,

(5) $w_{2,p} = (-1)^{\frac{p+1}{2}}$ *and* $(2, d_F)_p = 1$ *if* $p\mathcal{O}_F = \mathfrak{p}_2^2$,

(6) $w_{2,p} = (-1)^{\frac{p-1}{2}}(d_F, p)_p$ *and* $(2, d_F)_p = 1$ *if* $p\mathcal{O}_F = \mathfrak{p}_1^2\mathfrak{p}_1'^2$.

In the following examples P is an irreducible polynomial of $\mathbb{Q}[X]$, α is a root of P in $\mathbb{C}$, $F = \mathbb{Q}(\alpha)$ and $G_{K/\mathbb{Q}} = \mathfrak{S}_4$.

- $P = X^4 + X - 1$, $d_F = -283$ and the signature of F is $(2,1)$. Since the ideal $283\mathcal{O}_F$ decomposes into a product $\mathfrak{p}_1^2\mathfrak{p}_1'\mathfrak{p}_2''$ it follows from Proposition 7.3(1) that $w_{283,p} = (2, d_F)_{283} = -1$. Since 283 is the unique finite prime ramified in F, we have $w_{2,p} = (2, d_F)_p$ for every $p \neq 2$. We deduce from the product formula that w_2 and $(2, d_F)$ coincide at every prime p of $\mathbb{Q}$ and so are equal. We conclude that

$$w_2(\mathrm{Tr}_{F/\mathbb{Q}}) = sp_2(\rho) \neq 0, \ sw_2(\rho) = 0$$

  hence $\mathrm{Lift}(h^0(\mathrm{Spec}(F)), q)$ has no solution and $\mathrm{Lift}^\delta(h^0(\mathrm{Spec}(F)), q)$ has a solution.

- $P = X^4 + X^3 - 2X - 1$ and $d_F = -5^2 11$. One checks that $11\mathcal{O}_K = \mathfrak{p}_2\mathfrak{p}_1'^2$ and $5\mathcal{O}_K = \mathfrak{p}_2^2$. By elementary computations on Hilbert symbols, it follows from Proposition 7.3(3) and 7.3(5) that

$$(2, d_F)_{11} = -w_{2,11} = -1, \ (2, d_F)_5 = -w_{2,5} = 1.$$

  We conclude that

$$w_2(\mathrm{Tr}_{F/\mathbb{Q}}) \neq 0, \ sp_2(\rho) \neq 0, \ sw_2(\rho) \neq 0$$

  hence $\mathrm{Lift}(h^0(\mathrm{Spec}(F)), q)$ has no solution and $\mathrm{Lift}^\delta(h^0(\mathrm{Spec}(F)), q)$ has no solution.

- $P = X^4 - 2X^2 - 4X - 1$, $d_F = -2^8 11$ and the signature of F is $(2,1)$. One checks that $11\mathcal{O}_K = \mathfrak{p}_1^2\mathfrak{p}_2'$. Hence, by Proposition 7.3(3), we know that $w_{2,11} = -(2, d_F)_{11} = 1$. We conclude that

$$w_2(\mathrm{Tr}_{F/\mathbb{Q}}) = 0, \ sp_2(\rho) \neq 0, \ sw_2(\rho) \neq 0$$

  hence $\mathrm{Lift}(h^0(\mathrm{Spec}(F)), q)$ has a solution and $\mathrm{Lift}^\delta(h^0(\mathrm{Spec}(F)), q)$ has no solution.

- $P = X^4 - X^3 - X + 1$, $d_F = 2777$ and the signature of F is $(4,0)$. Since $2777\mathcal{O}_K = \mathfrak{p}_1^2\mathfrak{p}_2'\mathfrak{p}_2''$, by Proposition 7.3 we obtain that $(2, 2777)_{2777} = \omega_{2,2777} = 1$. We conclude that

$$w_2(\mathrm{Tr}_{F/\mathbb{Q}}) = 0, \ sp_2(\rho) = 0, \ sw_2(\rho) = 0$$

  hence both lifting problems $\mathrm{Lift}(h^0(\mathrm{Spec}(F)), q)$ and $\mathrm{Lift}^\delta(h^0(\mathrm{Spec}(F)), q)$ have a solution.

7.2. Complete intersections.
We consider a smooth and proper variety V over the number field E of even dimension n.

7.2.1. *The Betti form q_B.* For any integer $k \geq 0$ and for $R = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$, we let $H^k(V; R)$ be the singular cohomology group $H^k(V_\sigma(\mathbb{C}); R)$ with coefficients in R. We set $H^k(V) = H^k(V; \mathbb{Z})$. We consider the quadratic form on $\mathbb{Q}$

$$q_B : H^n(V; \mathbb{Q}) \times H^n(V; \mathbb{Q}) \to \mathbb{Q}$$

obtained by composing the cup product with evaluation on the fundamental class.

Proposition 7.4. *Let (r, s) be the signature of the extended form $\mathbb{R} \otimes_{\mathbb{Q}} q_B$. Then q_B is isometric to the quadratic form*

$$\sum_{1 \leq i \leq r} x_i^2 - \sum_{1 \leq j \leq s} x_{r+j}^2.$$

In particular, we have $w_1(q_B) = s(-1)$ and $w_2(q_B) = \binom{s}{2}(-1, -1)$.

Proof. We set $q = q_B$ and $q' = \sum_{1 \leq i \leq r} x_i^2 - \sum_{1 \leq j \leq s} x_{r+j}^2$. For a quadratic form t on $\mathbb{Q}$ and any place v of $\mathbb{Q}$ we denote by t_v the extended form $\mathbb{Q}_v \otimes_{\mathbb{Q}} t$ on $\mathbb{Q}_v$. In order to prove the proposition it suffices to show that q and q' are locally isometric for every place of $\mathbb{Q}$. Since q and q' have the same signature then q_∞ and q'_∞ are isometric over $\mathbb{R}$. The form q is induced by extension from the integral form

$$q_0 : H^n(V)_{\mathrm{cotor}} \times H^n(V)_{\mathrm{cotor}} \to \mathbb{Z}$$

where $H^n(V)_{\mathrm{cotor}}$ is the maximal quotient of $H^n(V)$ which is free as a $\mathbb{Z}$-module.

It follows from Poincaré duality that q_0 is unimodular. Therefore the determinant of q is congruent to ± 1 modulo $\mathbb{Q}^{*2}$. The group homomorphism $\mathbb{Q}^* \to \mathbb{R}^*$ induces an injective map $\{\pm 1\}\mathbb{Q}^{*2}/\mathbb{Q}^{*2} \to \mathbb{R}^*/\mathbb{R}^{*2}$ such that $w_1(q) \to w_1(q_{\mathbb{R}})$. This implies that

$$w_1(q) = s(-1)$$

and so $w_1(q) = w_1(q')$. We now examine $w_2(q) \in H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$. For any prime $p \leq \infty$ we let $w_2(q)_p$ be the p-component of $w_2(q)$ in $H^2(\mathbb{Q}_p, \mathbb{Z}/2\mathbb{Z})$. Indeed $w_2(q)_p = w_2(q_p)$. For $p = \infty$ we note that $w_2(q)_\infty = w_2(q')_\infty = \binom{s}{2}(-1, -1)$. Let $p \neq 2$ be a prime number. The form q_p is induced from the $\mathbb{Z}_p$-unimodular form $q_{0,p}$. Since $\mathbb{Z}_p$ is a local ring, this form has an orthogonal basis and we may write $q_{0,p} = <a_1, a_2, ..., a_n>$ for $a_i \in \mathbb{Z}_p^*$. Therefore

$$w_2(q_p) = \sum_{i<j}(a_i, a_j).$$

Since the a_i's are units we conclude that $w_2(q_p) = 0$. It now follows from the definition of q' that $w_2(q'_p) = 0$. Hence we have proved that $w_2(q)_v = w_2(q')_v$ for any plave $v \neq 2$. By the product formula we deduce that $w_2(q)_2 = w_2(q')_2$. Therefore the forms q and q' have the same rank, the same signature and for any place v of $\mathbb{Q}$ then $w_i(q_v) = w_i(q'_v)$, for $i \in \{1, 2\}$. It follows that q and q' are isometric. In particular they have the same Hasse-Witt invariants. $\square$

As before we denote by q_{dR} the quadratic form on $H_{dR}^n(V/E)$ induced by the cup product and we set

$$d_V = \mathrm{disc}(q_{dR}) = w_1(q_{dR}).$$

As a consequence of Proposition 7.4 we observe that the comparison formula of Theorem 6.10, applied to the quadratic motive $(M = h^n(V)(n/2), q)$, should be written

$$\tag{47} \delta_q^1(\mathfrak{P}_M) = sw_1(\rho_p) = d_V + s(-1)$$

and

$$\tag{48} \delta_q^2(\mathfrak{P}_M) = w_2(q_{dR}) + s(d_V, -1) + \binom{s+1}{2}(-1, -1),$$

where in these formulas (-1) (resp. $(-1, -1)$) has to be understood as the image of -1 (resp. $(-1, -1)$) in E^*/E^{*2} (resp. by the restriction map $H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}) \to H^2(E, \mathbb{Z}/2\mathbb{Z})$).

7.2.2. *The Hasse-Witt invariants of q_B.* We assume that V is a smooth complete intersection in the projective space $\mathbf{P}_E^{n+c}$. More precisely V is a variety of even dimension n defined by c homogeneous polynomials $f_1, ..., f_c$ with coefficients in E. For $1 \leq i \leq c$ we let d_i be the degree of f_i. We note that V_σ is a complete intersection in $\mathbf{P}_\mathbb{C}^{n+c}$ defined by the polynomials f_i seen via σ as polynomials with coefficients in $\mathbb{C}$. In this situation the rank of q_B and the Hasse Witt invariants $w_1(q_B)$ and $w_2(q_B)$ can be explicitly expressed in terms of the integers $n, d_1, ..., d_c$. We will treat in some details the case where V is an hypersurface where the formulas are very simple. However for any given complete intersection explicit formulas can be obtained as well but they may be rather complicated. As an example we will consider a case where $c = 2$. When V is an hypersurface ($c = 1$) we write f for f_1 and d for d_1. For any integer k we denote by $b_k(V)$ the dimension of $\mathbb{Q}$ vector space $H^k(V, \mathbb{Q})$. For reason of simplicity we will often denoted by V the complex variety V_σ and by $\mathbf{P}^n$ the projective space $\mathbf{P}_\mathbb{C}^n$.

We need some preliminary results.

Lemma 7.5. *For any integer k , $0 \leq k \leq 2n$ then $H^k(V)$ is torsion free. Moreover one has:*
  (1) *$b_k(V) = 0$ if k is odd and $b_k(V) = 1$ if k is even and $k \neq n$.*
  (2) *$b_n(V) = \chi(V) - n$ where $\chi(V) = \sum_{0 \leq k \leq 2n}(-1)^k b_k(V)$.*
  (3) *$b_n(V) = 2 + \frac{1}{d}[(1-d)^{n+2} - 1]$ when V is an hypersurface.*

Proof. It follows from Lefschetz Theorem (see [7], Chapter 5, (2.6) and (2.11)) that

$$H^k(V) = H^k(\mathbf{P}^n) \ \text{ for } k \neq n.$$

This implies that $H^k(V)$ is torsion free for any integer k and the equalities of (1). In order to show (2) it suffices to observe that

$$\chi(V) = \chi(\mathbf{P}^n) + b_n(V) - b_n(\mathbf{P}^n).$$

Since the Euler characteristic of an hypersurface is given by $\chi(V) = n + 2 + \frac{1}{d}[(1-d)^{n+2} - 1]$, then (3) follows from (2). $\qquad\square$

There exist several effective procedures to compute $\chi(V)$ when V is a complete intersection. For instance this can be done by evaluating the n-th Chern class of V on the fundamental class. This leads to prove that $\chi(V)$ is the coefficient of h^{n+c} in

$$(49) \qquad \frac{(1+h)^{n+c+1}d_1...d_c h^c}{(1+d_1 h)(1+d_2 h)...(1+d_c h)}.$$

Our aim is now to compute the signature of $q_{B,\mathbb{R}}$. We consider the index $\tau_n(V) = r - s$ of the quadratic lattice $H^n(V)$. This index has been extensively studied (see [25] and [26] for instance). If we put $d = d_1 d_2 ... d_c$, we have

$$(50) \qquad \tau_n(V) \equiv \begin{cases} 0 \bmod 8 \text{ if } \binom{\frac{n}{2}+t}{t} \text{ is even,} \\ d \bmod 8 \text{ if } \binom{\frac{n}{2}+t}{t} \text{ is odd.} \end{cases}$$

where $d_1, ...d_t$ are even and $d_{t+1}, ...d_c$ are odd integers. We define the integer m by

$$(51) \qquad m = \begin{cases} \chi(V) - n \text{ if } \binom{\frac{n}{2}+t}{t} \text{ is even,} \\ \chi(V) - n - d \ \text{ if } \binom{\frac{n}{2}+t}{t} \text{ is odd.} \end{cases}$$

The following proposition is an immediate consequence of (51), Proposition 7.4 and Lemma 7.5.

Proposition 7.6. *Let V be a smooth complete intersection in $\mathbf{P}_E^{n+c}$ of even dimension n. Then m is even and we have*

$$w_1(q_B) = m'(-1) \ \text{ and } \ w_2(q_B) = \binom{m'}{2}(-1,-1), \ \text{ with } m = 2m'.$$

If V is an hypersurface defined by a polynomial of degree d we deduce from (51) that

$$\text{(52)} \qquad \tau_n(V) \equiv \begin{cases} 0 \bmod 8 \text{ if } d \text{ is even and } n \equiv 2 \bmod 4, \\ d \bmod 8 \text{ otherwise.} \end{cases}$$

We denote by $\{u\}$ the integral part of $u \in \mathbb{Q}$.

Corollary 7.7. *Let V be a smooth hypersurface of degree d and even dimension $n \geq 2$ in the projective space $\mathbf{P}_E^{n+1}$. Then*

(1) $w_1(q_B) = \frac{n}{2}(d-1)(-1)$

(2) $w_2(q_B) = \begin{cases} \frac{d-1}{2}(-1,-1) \text{ if } d \text{ is odd,} \\ \{\frac{n+2}{4}\}(1+\frac{d}{2})(-1,-1) \text{ if } d \text{ is even.} \end{cases}$

Proof. We let r_n be the integer defined by the equality $[(1-d)^{n+2} - 1] = dr_n$. If d is odd, it follows from Lemma 7.5 and (52) that

$$\text{(53)} \qquad s \equiv 1 + \frac{r_n}{2} - \frac{d}{2} \bmod 4.$$

From the equality $(d-1)^{n+2} = 1 + dr_n$ we deduce that d and r_n are both odd and that one and only one of them is congruent to 1 mod 4. Suppose that $d = 1 + 4v$. Since $1 + dr_n \equiv 0 \bmod 8$, then $r_n \equiv -1 + 4v \bmod 8$ and therefore $s \equiv 0 \bmod 4$. If now $d \equiv 3 \bmod 4$ and so $r_n = 1 + 4t$, we obtain that $d \equiv -1 + 4t \bmod 8$ and so that $s \equiv 2 \bmod 4$.

We assume now that d is even. First we check that

$$\text{(54)} \qquad 1 + \frac{r_n}{2} \equiv -\frac{n}{2} + (n+1)\frac{(n+2)}{2}\frac{d}{2} \bmod 4.$$

Suppose that $n \equiv 2 \bmod 4$ and write $n = 2 + 4u$. It follows once again from Lemma 7.5 and (52) that $s \equiv 1 + 2(1+u)(1+\frac{d}{2}) \bmod 4$. Therefore we conclude that $s \equiv 3 \bmod 4$ if $n \equiv 2 \bmod 8$ and $d \equiv 0 \bmod 4$ and $s \equiv 1 \bmod 4$ otherwise. Finally we suppose $n \equiv 0 \bmod 4$ and we put $n = 4u$. In this case we know that $s \equiv 1 + \frac{r_n}{2} - \frac{d}{2} \bmod 4$ which implies that $s \equiv 2u(1 + \frac{d}{2}) \bmod 4$. Hence $s \equiv 2 \bmod 4$ if $d \equiv 0 \bmod 4$ and n not divisible by 8 and $s \equiv 0 \bmod 4$ otherwise. The proposition is proved by a case by case checking. $\qquad \square$

In various situations more precise computations of the index leads to the determination of the form q_B itself. This is the case when V be quadric of $\mathbf{P}^{n+1}$. By Lemma 7.5 the unimodular lattice $H^n(V)$ is of rank 2. By Proposition 7.7 we have $w_1(q_B) = \frac{n}{2}(-1)$ and $w_2(q_B) = 0$. If $n \equiv 2 \bmod 4$, then $\tau_n(V) = 0$ and q_B is isometric to the form $x^2 - y^2$. If $n \equiv 0 \bmod 4$, $\tau_n(V) = 2$ and q_B is isometric to $x^2 + y^2$.

We now consider the case of a cubic surface V ($d = 3$ and $n = 2$). The rank of the lattice $H^2(V)$ given by Lemma 7.5 is equal to 7. Moreover by (52) we know that $\tau_n(V) \equiv 3 \bmod 8$ and so $\tau_n(V)$ is either equal to 3 or -5. We now observe that, under our hypotheses, (52) can

be improved. More precisely it follows from (2) in [25] that $\tau_n(V) \equiv d + 8$ mod 16. Hence $\tau_n(V) = -5$. We conclude that q_B is isometric to

$$x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2 - x_6^2.$$

We treat now as an example the case of a smooth surface V defined in $\mathbf{P}^4$ by the polynomials f_1 and f_2 of degree d_1 and d_2 ($n = 2$ and $c = 2$). It follows from (49) that:

$$(55) \qquad \chi(V) = d_1 d_2 [d_1^2 + d_2^2 + d_1 d_2 - 5(d_1 + d_2) + 10].$$

Corollary 7.8. *Let V be a smooth surface of $\mathbf{P}_E^4$ defined by the homogeneous polynomials f_1 and f_2 of degree d_1 and d_2. Then*

(1) $w_1(q_B) = (d_1 d_2 - 1)(-1)$

(2) $w_2(q_B) = \begin{cases} (\{\frac{d_1}{2}\} + \{\frac{d_2}{2}\} + \{\frac{d_1}{2}\}\{\frac{d_2}{2}\})(-1, -1) \text{ if } d_1 \text{ and } d_2 \text{ are both odd,} \\ (1 + \{\frac{d_1}{2}\}\{\frac{d_2}{2}\})(-1, -1) \text{ if } d_1 \text{ or } d_2 \text{ otherwise.} \end{cases}$

Proof. The equalities are obtained from (55) and Lemma 7.6 by an easy computation. $\qquad \square$

We may use the results of Corollary 7.7 to improve the comparison formulas of Theorem 6.10 when (M, q) is the quadratic motive attached to an hypersurface.

Corollary 7.9. *Let V be a smooth hypersurface of even dimension $n \geq 2$ in $\mathbf{P}_E^{n+1}$, defined by an homogeneous polynomial f of degree d and let $(M = h^n(V)(n/2), q)$ be the associated motive. Then*

$$d_V = w_1(q_{dR}) = \varepsilon'(n, d)\mathrm{disc}_\mathrm{d}(f) \text{ in } E^*/E^{*2},$$

where $\mathrm{disc}_\mathrm{d}(f)$ is the divided discriminant of f and where $\varepsilon'(n, d) = (-1)^{\frac{(d-1)}{2}}$ if d is odd and is $(-1)^{(1+\frac{n}{2})(1+\frac{d}{2})+1}$ if d is even.

Proof. This is an immediate consequence of Corollary 7.7 and [32]. $\qquad \square$

Remark 7.10. *Suppose for simplicity that $E = \mathbb{Q}$. When $n = 0$, we deduce from (53) that $s \equiv 0$ mod 4 and so that $w_1(q_B) = w_2(q_B) = 0$. This result and Corollary 7.9 can be easily understood in this case. Suppose that V is defined by the polynomial $f = X_0^d + a_{d-1}X_0^{d-1}X_1 + ... + a_0 X_1^d$. We set $g(X) = f(X = \frac{X_0}{X_1}, 1)$. The motive $h^0(V)$ is isomorphic to the Artin motive $\mathrm{Spec}(F)$ where $F \simeq \mathbb{Q}[X]/(g(X))$. As we have seen in Section 6.1, the form q_B is the form $x_1^2 + ... + x_d^2$ and thus its Hasse-Witt invariants are trivial. Moreover the form q_{dR} identifies with the trace form $\mathrm{Tr}_{F/\mathbb{Q}}$. Therefore*

$$(56) \qquad d_V = \mathrm{disc}(F/\mathbb{Q}) = \mathrm{disc}(\mathrm{Tr}_{F/\mathbb{Q}}) = \prod_{i<j}(x_i - x_j)^2,$$

where $x_1, ..., x_d$ are the roots of g in $\mathbb{C}$. This equality can be expressed in terms of resultants. More precisely, by [15], Chapter 12, (1.29) and (1.31),

$$\prod_{i<j}(x_i - x_j)^2 = (-1)^{\frac{d(d-1)}{2}} R_{d,d-1}(g, g') = \frac{\varepsilon'(0, d)}{d^{d-2}} R_{d-1,d-1}(D_0(f), D_1(f)) = \varepsilon'(0, d)\mathrm{disc}_\mathrm{d}(f).$$

We conclude that Corollary 7.9 generalizes in higher dimension the classical equality (56).

Corollary 7.11. *Let V be a smooth hypersurface of even dimension $n \geq 2$ in $\mathbf{P}_E^{n+1}$, defined by an homogeneous polynomial of degree d and let $(M = h^n(V)(n/2), q)$ be the associated motive. Then we have*

$$\delta_q^1(\mathfrak{P}_M) = \begin{cases} (-1)^{\frac{d-1}{2}} \cdot \mathrm{disc_d}(f) \ \textit{if } d \textit{ is odd} \\ (-1)^{\frac{d}{2} \cdot \frac{n+2}{2}} \cdot \mathrm{disc_d}(f) \ \textit{if } d \textit{ is even} \end{cases}$$

and

$$\delta_q^2(\mathfrak{P}_M) = w_2(q_{dR}) + \begin{cases} \frac{d-1}{2}(-1,-1) \ \textit{if } d \textit{ is odd} \\ \frac{n}{4}(1+\frac{d}{2})(-1,-1) \ \textit{if } d \textit{ is even and } n \equiv 0 \bmod 4 \\ (-1, \mathrm{disc_d}(f)) + (\frac{n+2}{4})(1+\frac{d}{2})(-1,-1) \ \ \textit{if } d \textit{ is even and } n \equiv 2 \bmod 4. \end{cases}$$

Proof. The first assertion follows from [32] and Theorem 6.10(2). The second assertion follows from Theorem 6.10(1) and Lemma 7.7. $\qquad\square$

Remark 7.12. *Let us assume for simplicity that $n \equiv 0 \bmod 4$. Following a conjecture of Saito [33], proved for $l > n+1$, there should exist integers $\alpha(n,d)$ and $\beta(n,d)$, which can be explicitly expressed in terms of n and d, such that for any prime number l, one has*

$$\delta_q^2(\mathfrak{P}_M) = sw_2(\rho_l) + \alpha(n,d)c_l + \beta(n,d)(-1,-1) + (2, d_V),$$

where c_l is the unique element of $H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$ which ramifies exactly in l and ∞.

7.3. The lifting problem $\mathrm{Lift}(M, q)$ over $\mathbb{R}$.

Let (M, q) be an orthogonal $\mathbb{Q}$-motive, let (M_B, q_B) be its Betti realization and let ∞ be the real place of $\mathbb{Q}$. Recall that $(M_\infty, q_\infty) := (M_B, q_B) \otimes_{\mathbb{Q}} \mathbb{R}$ and that we have an orthogonal representation $\rho_{\infty|G_{\mathbb{R}}} : G_{\mathbb{R}} \to \mathbf{O}(M_\infty)$. Under this action M_∞ decomposes into a direct sum $M_\infty = M_\infty^+ \oplus M_\infty^-$ where the complex conjugation (i.e. the non-trivial element in $G_{\mathbb{R}}$) acts as the identity Id (resp. as $-\mathrm{Id}$) on M_∞^+ (resp. on M_∞^-). Our aim is to compute the local invariants $sw_2(\rho_{\infty|G_{\mathbb{R}}})$ and $sp_2(\rho_{\infty|G_{\mathbb{R}}})$, which belong to the group $H^2(G_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$.

Proposition 7.13. *Let b^- be the dimension of M_∞^-. Then we have*

$$sw_2(\rho_{\infty|G_{\mathbb{R}}}) = \frac{b^-(b^- - 1)}{2} \ \ \textit{and} \ \ sp_2(\rho_{\infty|G_{\mathbb{R}}}) = 0.$$

Proof. In order to ease the notation, we write ρ for $\rho_{\infty|G_{\mathbb{R}}}$. By choosing orthogonal bases of M_∞^+ and M_∞^-, we decompose (M_∞, q_∞) into an orthogonal sum of dimension 1 quadratic spaces, stable under the action of $G_{\mathbb{R}}$. Therefore the orthogonal character χ_ρ of ρ can be written as a sum of degree one orthogonal characters,

$$(57) \qquad\qquad\qquad\qquad \chi_\rho = b^+ \chi_0 + b^- \chi_1,$$

where χ_0 (resp. χ_1) is the trivial (resp. non trivial) character of $G_{\mathbb{R}}$ and b^+ is the dimension of M_∞^+. The Stiefel-Whitney class of an orthogonal representation only depends on the character of the representation. Moreover we obtain from [13] the following addition formula.

Lemma 7.14. *Let ρ and ρ' be orthogonal representations of a finite group Γ, of characters χ_ρ and $\chi_{\rho'}$. Then we have*

$$sw_2(\chi_\rho + \chi_{\rho'}) = sw_2(\chi_\rho) + sw_2(\chi_{\rho'}) + \det(\chi_\rho) \cdot \det(\chi_{\rho'}).$$

Moreover $sw_2(\chi) = 0$ for $\chi \in \mathrm{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$.

It follows from the lemma that $sw_2(\rho) = sw_2(\chi_\rho) = sw_2(b^-\chi_1)$ and that

$$sw_2(a\chi_1) = sw_2((a-1)\chi_1) + (\chi_1)(\chi_1^{a-1}) = sw_2((a-1)\chi_1) + (a-1)(\chi_1)(\chi_1), \ \forall a.$$

Therefore $sw_2(\rho) = \frac{b^-(b^- - 1)}{2}(\chi_1)(\chi_1)$. Since $(\chi_1)(\chi_1)$ identifies with the class of the Hamilton quaternion algebra over $\mathbb{R}$, under the group monomorphism $H^2(G_\mathbb{R}, \mathbb{Z}/2\mathbb{Z}) \to Br(\mathbb{R})$, we conclude that $(\chi_1)(\chi_1)$ is the non trivial element of $H^2(G_\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$.

We now recall the definition of $sp_2(t)$ for an orthogonal representation of $t : G_\mathbb{R} \to \mathbf{O}(q_t)$ as given in [13]. We let $sp : \mathbf{O}(q_t) \to \mathrm{Hom}(G_\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$ be the group homomorphism obtained by composing the spinor norm $\mathbf{O}(q_t) \to \mathbb{R}^\times/\mathbb{R}^{\times 2}$ with the canonical isomorphim $\mathbb{R}^\times/\mathbb{R}^{\times 2} \simeq \mathrm{Hom}(G_\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$. Let $sp[t] : G_\mathbb{R} \to \mathrm{Hom}(G_\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$ be the group homomorphism $g \to sp(t(g))$. This map satisfies the relations

$$(58) \qquad\qquad sp[t + t'] = sp[t] + sp[t'] \ \ (\text{addition in } \ \mathrm{Hom}(G_\mathbb{R}, \mathrm{Hom}(G_\mathbb{R}, \mathbb{Z}/2\mathbb{Z})))$$

and

$$(59) \qquad\qquad\qquad sp[\phi] = 0 \ \ \forall \ \phi \ \in \mathrm{Hom}(G_\mathbb{R}, \pm 1)$$

The map $sp[t]$ gives rise to a bilinear form

$$c_t : G_\mathbb{R} \times G_\mathbb{R} \to \mathbb{Z}/2\mathbb{Z}.$$

It is a 2-cocycle which represents $sp_2(t)$ in $H^2(G_\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$. The triviality of $sp_2(\rho)$ now follows from (57), (58) and (59). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 7.15. *The lifting problem* $\mathrm{Lift}(M, q)$ *has a solution over* $\mathbb{R}$ *if and only if* $\frac{b^-(b^- - 1)}{2}$ *is even.*

8. The Künneth formula for Nori motives

A Künneth decomposition for motives appears in [1], Corollary 4.4.5. For the seek of completeness we prove such a formula in this section. For generalities on Nori motives, we refer to [28], [24] and [18].

Following Nori's construction we attach to any diagram D and any representation $T : D \to$ $\mathrm{Vec}_\mathbb{Q}$ a $\mathbb{Q}$-linear abelian category $C(D, T)$, a representation $\tilde{T} : D \to C(D, T)$ and a forgetful functor f_T such that $f_T \circ \tilde{T} = T$ and $\tilde{T}$ is universal for this property. For any such pair (D, T), we may consider the pair $(D^{op}, T^\vee)$ where D^{op} is the opposite diagram and $T^\vee : D^{op} \to \mathrm{Vec}_\mathbb{Q}$ assigns to any vertex p of D the dual $\mathbb{Q}$-vector space of $T(p)$ and to any edge m from p to q the transpose of $T(m)$. Let k be a subfield of the field of complex numbers $\mathbb{C}$. The category of effective homological Nori motives (respectively of effective cohomological Nori motives) is defined as $EHM := C(D, H_*)$ (resp. $ECM := C(D^{op}, H_*^\vee)$), where D is the diagram of effective good pairs consisting of triples (X, Y, i) where X is a k-variety, $Y \hookrightarrow X$ a closed subvariety and i an integer such that the singular homology $H_j(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q}) = 0$ if $j \neq i$, and H_* is the representation which maps (X, Y, i) to the relative singular homology $H_i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Q})$. For a good pair (X, Y, i) we write $h_i(X, Y)$ (resp. $h^i(X, Y)$) for the corresponding object in EHM (resp. ECM).

We denote by $\mathrm{Ind}(EHM)$ the category of ind-objects of EHM, i.e. the strictly full subcategory of the category of presheaves of sets on EHM which are isomorphic to filtered colimits of representables. Then $\mathrm{Ind}(EHM)$ is an abelian category with enough injectives, and the Yoneda functor $y : EHM \to \mathrm{Ind}(EHM)$ is an exact fully faithful functor. An object of $\mathrm{Ind}(EHM)$ lying in the essential image of y is called essentially constant. Then we consider

the abelian category $\mathbf{Ch}(\mathrm{Ind}(EHM))$ (resp. $\mathbf{Ch}_+(\mathrm{Ind}(EHM))$) of (resp. bounded below) homological chain complexes in $\mathrm{Ind}(EHM)$.

Let X be an affine variety over k (i.e. an affine scheme of finite type over k). A good filtration of X is a sequence of closed subvarieties

$$\emptyset = X_{-1} \subset X_0 \subset ... \subset X_{n-1} \subset X_n = X$$

such that (X_j, X_{j-1}, j) is a good pair for $0 \leq j \leq n$. Nori's basic Lemma ensures that the set of good filtrations is non-empty and filtered. It is moreover functorial in the sense that given a morphism $f : X \to Y$ of affine varieties, there exist good filtrations X_* on X and Y_* on Y such that $f(X_i) \subset Y_i$ for any i. We set

$$C_*(X_*) := [h_n(X_n, X_{n-1}) \to h_{n-1}(X_{n-1}, X_{n-2}) \to h_0(X_0, \emptyset)] \in \mathbf{Ch}_+(EHM)$$

where $h_0(X_0, \emptyset)$ sits in degree 0. We obtain a functor

$$C_* : \quad \mathbf{Aff}_k \quad \longrightarrow \quad \mathbf{Ch}_+(\mathrm{Ind}(EHM))$$
$$X \quad \longmapsto \quad \varinjlim C_*(X_*)$$

where the colimit is taken over the system of good filtrations X_* of X, and $\mathbf{Aff}_k$ denotes the category of affine k-schemes of finite type.

Let X be any variety (i.e. a separated scheme of finite type over k). A *rigidified affine cover* $(\mathcal{U}, r)$ is given by a finite open affine cover $\mathcal{U} = \{U_i \subset X, i \in I\}$ of X, and a *surjective* map of sets $r : X(\mathbb{C}) \twoheadrightarrow I$ such that $x \in U_{r(x)}(\mathbb{C})$ for any $x \in X(\mathbb{C})$, where $X(\mathbb{C})$ denotes the (discrete) set of complex points of X. A morphism $(X, \mathcal{U}, r) \to (X', \mathcal{U}', r')$ of varieties with rigidified affine covers is a pair (f, ϕ) given by a morphism of schemes $f : X \to X'$ inducing a map of sets $\phi : I \to I'$ (i.e. such that the obvious square commutes) such that $f(U_i) \subseteq U'_{\phi(i)}$ for every $i \in I$. Note that, if $(\mathcal{U}, r)$ and $(\mathcal{U}', r')$ are two rigidified affine covers of X, there exist at most one morphism $(X, \mathcal{U}', r') \to (X, \mathcal{U}, r)$ whose underlying map $X \to X$ is the identity. If such a morphism exists, we say that $(\mathcal{U}', r')$ refines $(\mathcal{U}, r)$.

Given a variety X endowed with a rigidified affine cover $(\mathcal{U}, r)$, we consider the simplicial Cech complex $\check{C}_\bullet(\mathcal{U})$, a simplicial affine scheme. Applying the functor C_* above, we get a simplicial object $C_*\check{C}_\bullet(\mathcal{U})$ of $\mathbf{Ch}_+(\mathrm{Ind}(EHM))$, and taking the associated complex we get a double complex in $\mathrm{Ind}(EHM)$. We denote by $\mathrm{Toc}(C_*\check{C}_\bullet(\mathcal{U})) \in \mathbf{Ch}_+(\mathrm{Ind}(EHM))$ the total complex. In other words, we consider the following functor
(60)
$$\mathrm{Toc} : \mathrm{Fun}(\Delta^{op}, \mathbf{Ch}_+(\mathrm{Ind}(EHM))) \longrightarrow \mathbf{Ch}_+(\mathbf{Ch}_+(\mathrm{Ind}(EHM))) \xrightarrow{\mathrm{Tot}} \mathbf{Ch}_+(\mathrm{Ind}(EHM))$$

where the first functor sends a simplicial object to the corresponding unnormalized homological complex and the second functor sends a double complex to its total complex. Then $\mathrm{Toc}(C_*\check{C}_\bullet(\mathcal{U}))$ is functorial in $(\mathcal{U}, r)$. Moreover, if $(\mathcal{V}, s)$ refines $(\mathcal{U}, r)$ then

$$\mathrm{Toc}(C_*\check{C}_\bullet(\mathcal{V})) \to \mathrm{Toc}(C_*\check{C}_\bullet(\mathcal{U}))$$

is a quasi-isomorphism. Hence $\mathrm{Toc}(C_*\check{C}_\bullet(\mathcal{U})) \in \mathbf{D}_b(\mathrm{Ind}(EHM))$ does not depend on $(\mathcal{U}, r)$ up to a canonical isomorphism. Here $\mathbf{D}_b(\mathrm{Ind}(EHM))$ of course denotes the derived category of bounded homological complexes in $\mathrm{Ind}(EHM)$.

This defines a functor

$$\tilde{M} : \quad \mathbf{Var}_k \quad \longrightarrow \quad \mathbf{D}_b(\mathrm{Ind}(EHM))$$
$$X \quad \longmapsto \quad \mathrm{Tot}(C_*\check{C}_\bullet(\mathcal{U}))$$

The functor
$$\mathbf{D}_b(EHM) \to \mathbf{D}_b(\mathrm{Ind}(EHM))$$
is fully faithful and its essential image $\tilde{\mathbf{D}}_b(EHM)$ is precisely the full subcategory consisting in chain complexes with essentially constant homology. Note that $\tilde{\mathbf{D}}_b(EHM)$ is strictly larger than $\mathbf{D}_b(EHM)$ but these two categories are canonically equivalent; we make no distinction between them. Let us denote by $\mathbf{Var}_k$ the category of varieties over E. We obtain a functor
$$M : \mathbf{Var}_k \to \mathbf{D}_b(EHM)$$
such that the image of $M(X)$ in $\mathbf{D}_b(\mathrm{Vec}_\mathbb{Q})$ computes the singular homology of X.

Notation 8.1. *For any variety X/k we define the object $h_i(X)$ of EHM by*
$$h_i(X) = H_i(M(X)).$$

The tensor product on EHM extends to a tensor product on $\mathrm{Ind}(EHM)$. Any object of EHM is flat (since EHM has a fiber functor to the category of finite dimensional $\mathbb{Q}$-vector spaces) hence so is any object of $\mathrm{Ind}(EHM)$. Therefore, derived tensor products on $\mathbf{D}_b(EHM)$ and $\mathbf{D}_b(\mathrm{Ind}(EHM))$ are well defined and $\mathbf{D}_b(EHM) \to \mathbf{D}_b(\mathrm{Ind}(EHM))$ is monoidal.

Theorem 8.2. *We have a functorial isomorphism in $\mathbf{D}_b(EHM)$*
$$M(X) \otimes^L M(Y) \xrightarrow{\sim} M(X \times_k Y)$$
inducing the usual Kunneth isomorphism on singular homology.

Proof. Since the fully faithful functor $\mathbf{D}_b(EHM) \to \mathbf{D}_b(\mathrm{Ind}(EHM))$ commutes with $\otimes^L$, it is enough to prove the result in $\mathbf{D}_b(\mathrm{Ind}(EHM))$. Given arbitrary varieties X and Y endowed with rigidified open affine covers $(\mathcal{U}, r)$ and $(\mathcal{V}, s)$, the complexes $\mathrm{Toc}(C_* \check{C}_\bullet(\mathcal{U}))$ and $\mathrm{Toc}(C_* \check{C}_\bullet(\mathcal{V}))$ in $\mathbf{Ch}_+(\mathrm{Ind}(EHM))$ represent $M(X)$ and $M(Y)$. Hence $M(X) \otimes^L M(Y)$ is represented by the total complex
$$\mathrm{Tot}(\mathrm{Toc}(C_* \check{C}_\bullet(\mathcal{U})) \otimes \mathrm{Toc}(C_* \check{C}_\bullet(\mathcal{V}))) \in \mathbf{Ch}_+(\mathrm{Ind}(EHM))$$
associated with the double complex $\mathrm{Toc}(C_* \check{C}_\bullet(\mathcal{U})) \otimes \mathrm{Toc}(C_* \check{C}_\bullet(\mathcal{V}))$. It is therefore enough to exhibit a quasi-isomorphism
$$(61) \qquad \mathrm{Tot}(\mathrm{Toc}(C_* \check{C}_\bullet(\mathcal{U})) \otimes \mathrm{Toc}(C_* \check{C}_\bullet(\mathcal{V}))) \xrightarrow{\sim} \mathrm{Toc}(C_* \check{C}_\bullet(\mathcal{U} \times \mathcal{V}))$$
which is functorial in $(\mathcal{U}, r)$, $(\mathcal{V}, s)$, X and Y. Here $(\mathcal{U} \times \mathcal{V}, r \times s)$ is the obvious rigidified open affine cover of $X \times Y$.

We first consider the affine case. Consider the following diagram of functors:

$$
\begin{array}{ccc}
\mathbf{Aff}_k \times \mathbf{Aff}_k & \xrightarrow{C_* \times C_*} & \mathbf{Ch}_+(\mathrm{Ind}(EHM)) \times \mathbf{Ch}_+(\mathrm{Ind}(EHM)) \\
\downarrow{\scriptstyle - \times -} & & \downarrow{\scriptstyle \mathrm{Tot}(- \otimes -)} \\
\mathbf{Aff}_k & \xrightarrow{\quad C_* \quad} & \mathbf{Ch}_+(\mathrm{Ind}(EHM))
\end{array}
$$

Lemma 8.3. *(Nori, [28] Section 4.5) The square above is quasi-commutative in the sense that there is a natural transformation*
$$\mathrm{Tot}(C_*(-) \otimes C_*(-)) \longrightarrow C_*(- \times -)$$
inducing a quasi-isomorphism
$$\mathrm{Tot}(C_*(X) \otimes C_*(Y)) \to C_*(X \times Y)$$

in $\mathbf{Ch}(\mathrm{Ind}(EHM))$ *for every* $X, Y \in \mathbf{Aff}_k$.

Proof. We have $\mathrm{Tot}(C_*(X) \otimes C_*(Y)) \simeq \varinjlim \mathrm{Tot}(C_*(X_*) \otimes C_*(Y_*))$ where the colimit is taken over the system of good filtrations X_* and Y_* on X and Y respectively. Given such good filtrations X_* and Y_*, there exists a good filtration Z_* on $X \times Y$ such that $\cup_{i+j=k} X_i \times Y_j \subseteq Z_k$. The obvious map from

$$\mathrm{Tot}_k(C_*(X_*) \otimes C_*(Y_*)) = \bigoplus_{i+j=k} h_i(X_i, X_{i-1}) \otimes h_j(Y_j, Y_{j-1}) \simeq \bigoplus_{i+j=k} h_i(X_i \times Y_j, X_i \times Y_{j-1} \cup X_{i-1} \times Y_j).$$

to $C_k(Z_*) = H_k(Z_k, Z_{k-1})$ gives morphisms of complexes

$$\mathrm{Tot}(C_*(X_*) \otimes C_*(Y_*)) \to C_*(Z_*) \to C_*(X \times Y)$$

which does not depend on the choice of the filtration Z_*. This induces a quasi-isomorphism

$$\mathrm{Tot}(C_*(X) \otimes C_*(Y)) \to C_*(X \times Y)$$

which is moreover functorial in X and Y. This proves the lemma and thus the theorem when X and Y are both affine. $\qquad\square$

Take now arbitrary varieties X and Y, and choose rigidified open affine covers $(\mathcal{U}, r)$ and $(\mathcal{V}, s)$, with $\mathcal{U} = \{U_i \subset X, \ i \in I\}$ and $\mathcal{V} = \{V_j \subset Y, \ j \in J\}$. We denote by $\check{C}_\bullet(\mathcal{U}) : \Delta^{op} \to \mathbf{Aff}_k$ and $\check{C}_\bullet(\mathcal{V}) : \Delta^{op} \to \mathbf{Aff}_k$ the corresponding simplicial affine schemes. Consider the diagram

$$
\begin{array}{ccccc}
\Delta^{op} \times \Delta^{op} & \xrightarrow{\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V})} & \mathbf{Aff}_k \times \mathbf{Aff}_k & \xrightarrow{C_* \times C_*} & \mathbf{Ch}_+(\mathrm{Ind}(EHM)) \times \mathbf{Ch}_+(\mathrm{Ind}(EHM)) \\
\downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle - \times -} & & \downarrow{\scriptstyle \mathrm{Tot}(-\otimes-)} \\
\Delta^{op} \times \Delta^{op} & \longrightarrow & \mathbf{Aff}_k & \xrightarrow{\quad C_* \quad} & \mathbf{Ch}_+(\mathrm{Ind}(EHM))
\end{array}
$$

where the left square commutes. The bottom line gives a bisimplicial object of $\mathbf{Ch}_+(\mathrm{Ind}(EHM))$, which we denote by

$$C_*(\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V})) : \Delta^{op} \times \Delta^{op} \longrightarrow \mathbf{Ch}_+(\mathrm{Ind}(EHM)).$$

Composing the top line with $\mathrm{Tot}(-\otimes-)$ gives another bisimplicial object of $\mathbf{Ch}_+(\mathrm{Ind}(EHM))$, which we denote by

$$\mathrm{Tot}(C_* \check{C}_\bullet(\mathcal{U}) \otimes C_* \check{C}_\bullet(\mathcal{V}))) : \Delta^{op} \times \Delta^{op} \longrightarrow \mathbf{Ch}_+(\mathrm{Ind}(EHM)).$$

We denote by

(62) $\mathrm{T}(C_*(\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V})))$ and $\mathrm{T}(\mathrm{Tot}(C_* \check{C}_\bullet(\mathcal{U}) \otimes C_* \check{C}_\bullet(\mathcal{V}))) \in \mathbf{Ch}_+(\mathbf{Ch}_+(\mathrm{Ind}(EHM)))$

the complexes of objects of $\mathbf{Ch}_+(\mathrm{Ind}(EHM))$ associated with these bisimplicial objects of $\mathbf{Ch}_+(\mathrm{Ind}(EHM))$. More precisely, if $A_{\bullet\bullet}$ is a bisimplicial object in an abelian category $\mathcal{A}$, we denote by $\mathrm{C}(A_{\bullet\bullet})$ the double complex associated with $A_{\bullet\bullet}$ and by $\mathrm{T}(A_{\bullet\bullet})$ its total complex, so that $\mathrm{T}(A_{\bullet\bullet})_n = \bigoplus_{p+q=n} A_{p,q}$. In our situation $\mathcal{A} = \mathbf{Ch}_+(\mathrm{Ind}(EHM))$.

We can see (62) as double complexes and consider their total complexes

$$\mathrm{Tot}(\mathrm{T}(C_*(\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V})))) \text{ and } \mathrm{Tot}(\mathrm{T}(\mathrm{Tot}(C_* \check{C}_\bullet(\mathcal{U}) \otimes C_* \check{C}_\bullet(\mathcal{V})))) \in \mathbf{Ch}_+(\mathrm{Ind}(EHM)).$$

Lemma 8.4. *There is a canonical quasi-isomorphism*

$$\mathrm{Tot}(\mathrm{T}(\mathrm{Tot}(C_* \check{C}_\bullet(\mathcal{U}) \otimes C_* \check{C}_\bullet(\mathcal{V})))) \xrightarrow{\mathcal{N}} \mathrm{Tot}(\mathrm{T}(C_*(\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V})))).$$

Proof. Consider

$$\mathrm{T} : \mathrm{Fun}(\Delta^{op} \times \Delta^{op}, \mathbf{Ch}_+(\mathrm{Ind}(EHM))) \longrightarrow \mathbf{Ch}_+(\mathbf{Ch}_+(\mathrm{Ind}(EHM)))$$

as a functor from bisimplicial objects of $\mathbf{Ch}_+(\mathrm{Ind}(EHM))$ to $\mathbf{Ch}_+(\mathbf{Ch}_+(\mathrm{Ind}(EHM)))$. The natural transformation of Lemma 8.3 gives a morphism of bisimplicial objects:

$$\mathrm{Tot}(C_* \check{C}_\bullet(\mathcal{U}) \otimes C_* \check{C}_\bullet(\mathcal{V})) \longrightarrow C_*(\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V})).$$

Applying the functor T, we get a morphism

$$D_* = \mathrm{T}(\mathrm{Tot}(C_* \check{C}_\bullet(\mathcal{U}) \otimes C_* \check{C}_\bullet(\mathcal{V}))) \longrightarrow \mathrm{T}(C_*(\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V}))) = D'_*$$

in $\mathbf{Ch}_+(\mathbf{Ch}_+(\mathrm{Ind}(EHM)))$. By Lemma 8.3, the morphism in $\mathbf{Ch}_+(\mathrm{Ind}(EHM))$

$$D_n = \bigoplus_{p+q=n} \mathrm{Tot}(C_* \check{C}_p(\mathcal{U}) \otimes C_* \check{C}_q(\mathcal{V})) \longrightarrow \bigoplus_{p+q=n} C_*(\check{C}_p(\mathcal{U}) \times \check{C}_q(\mathcal{V})) = D'_n$$

is a quasi-isomorphism for any $n \geq 0$. The result then follows from a standard spectral sequence argument. $\square$

Lemma 8.5. *There is a canonical isomorphism*

$$\mathrm{Tot}(\mathrm{Toc}(C_* \check{C}_\bullet(\mathcal{U})) \otimes \mathrm{Toc}(C_* \check{C}_\bullet(\mathcal{V}))) \simeq \mathrm{Tot}(\mathrm{T}(\mathrm{Tot}(C_* \check{C}_\bullet(\mathcal{U}) \otimes C_* \check{C}_\bullet(\mathcal{V})))).$$

Proof. These two complexes can be both identified with the total complex associated with the quadruple complex whose generic term is

$$(C_p \check{C}_q(\mathcal{U}) \otimes C_u \check{C}_v(\mathcal{V}))_{p,q,u,v \in \mathbb{N}^4}.$$

Indeed, we have

$$
\begin{aligned}
\mathrm{Tot}(\mathrm{T}(\mathrm{Tot}(C_* \check{C}_\bullet(\mathcal{U}) \otimes C_* \check{C}_\bullet(\mathcal{V}))))_t &= \bigoplus_{n+m=t} (\mathrm{T}(\mathrm{Tot}(C_* \check{C}_\bullet(\mathcal{U}) \otimes C_* \check{C}_\bullet(\mathcal{V})))_n)_m \\
&= \bigoplus_{n+m=t} \bigoplus_{p+q=n} \mathrm{Tot}(C_* \check{C}_p(\mathcal{U}) \otimes C_* \check{C}_q(\mathcal{V}))_m \\
&= \bigoplus_{n+m=t} \bigoplus_{p+q=n} \bigoplus_{u+v=m} C_u \check{C}_p(\mathcal{U}) \otimes C_v \check{C}_q(\mathcal{V}) \\
&= \bigoplus_{u+v+p+q=t} C_u \check{C}_p(\mathcal{U}) \otimes C_v \check{C}_q(\mathcal{V}),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Tot}(\mathrm{Toc}(C_* \check{C}_\bullet(\mathcal{U})) \otimes \mathrm{Toc}(C_* \check{C}_\bullet(\mathcal{V})))_t &= \bigoplus_{n+m=t} \mathrm{Toc}(C_* \check{C}_\bullet(\mathcal{U})) \otimes \mathrm{Toc}(C_* \check{C}_\bullet(\mathcal{V}))_{n,m} \\
&= \bigoplus_{n+m=t} \mathrm{Toc}(C_* \check{C}_\bullet(\mathcal{U}))_n \otimes \mathrm{Toc}(C_* \check{C}_\bullet(\mathcal{V}))_m \\
&= \bigoplus_{n+m=t} (\bigoplus_{p+q=n} C_p \check{C}_q(\mathcal{U})) \otimes (\bigoplus_{u+v=m} C_u \check{C}_v(\mathcal{V})) \\
&= \bigoplus_{n+m=t} \bigoplus_{p+q=n} \bigoplus_{u+v=m} C_p \check{C}_q(\mathcal{U}) \otimes C_u \check{C}_v(\mathcal{V}) \\
&= \bigoplus_{u+v+p+q=t} C_p \check{C}_q(\mathcal{U}) \otimes C_u \check{C}_v(\mathcal{V}).
\end{aligned}
$$

$\square$

Consider now the diagonal simplicial object associated with the bisimplicial object $C_*(\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V}))$, which we denote by

$$\mathrm{diag}(C_*(\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V}))) : \Delta^{op} \longrightarrow \Delta^{op} \times \Delta^{op} \longrightarrow \mathbf{Ch}_+(\mathrm{Ind}(EHM)).$$

We denote by $\mathrm{D}(C_*(\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V}))) \in \mathbf{Ch}_+(\mathbf{Ch}_+(\mathrm{Ind}(EHM)))$ the associated (unnormalized) homological complex, so that

$$\mathrm{D}(C_*(\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V})))_n = C_*(\check{C}_n(\mathcal{U}) \times \check{C}_n(\mathcal{V})).$$

Lemma 8.6. *(Eilenberg-Zilber Theorem) There is a canonical quasi-isomorphism*

$$\mathrm{Tot}(\mathrm{T}(C_*(\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V})))) \xrightarrow{\nabla} \mathrm{Tot}(\mathrm{D}(C_*(\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V})))).$$

Proof. We consider the bisimplicial object $C_*(\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V}))$ in the abelian category $\mathcal{A} :=$ $\mathbf{Ch}_+(\mathrm{Ind}(EHM))$. By the Eilenberg-Zilber Theorem ([W] 8.5.1), the shuffle map (see [W] 8.5.4)

$$\mathrm{T}(C_*(\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V}))) \longrightarrow \mathrm{D}(C_*(\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V}))),$$

a map in $\mathbf{Ch}_+(\mathcal{A}) = \mathbf{Ch}_+(\mathbf{Ch}_+(\mathrm{Ind}(EHM)))$, is a quasi-isomorphism. We conclude with a spectral sequence argument. $\qquad\square$

We consider the rigidified open affine cover $(\mathcal{U} \times \mathcal{V}, r \times s)$ of $X \times Y$, where $\mathcal{U} \times \mathcal{V} := \{U_i \times V_j \subset X \times Y, (i, j) \in I \times J\}$.

Lemma 8.7. *There is a canonical isomorphism*

$$\mathrm{Tot}(\mathrm{D}(C_*(\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V})))) \simeq \mathrm{Toc}(C_*(\check{C}_\bullet(\mathcal{U} \times \mathcal{V})))$$

Proof. The result follows from the fact that there is a canonical isomorphism of simplicial object of $\mathbf{Ch}_+(\mathrm{Ind}(EHM))$:

$$(63) \qquad \mathrm{diag}(C_*(\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V}))) = C_*(\check{C}_\bullet(\mathcal{U} \times \mathcal{V})).$$

Indeed, we have

$$
\begin{aligned}
\mathrm{diag}(C_*(\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V})))_n &= C_*(\check{C}_n(\mathcal{U}) \times \check{C}_n(\mathcal{V})) \\
&= C_*\Big(\bigsqcup_{(i_0,\ldots,i_n)} U_{i_0} \cap \ldots \cap U_{i_n} \times \bigsqcup_{(j_0,\ldots,j_n)} V_{j_0} \cap \ldots \cap V_{j_n} \Big) \\
&\simeq C_*\Big(\bigsqcup_{((i_0,j_0),\ldots,(i_n,j_n))} (U_{i_0} \times V_{j_0}) \cap \ldots \cap (U_{i_n} \times V_{j_n}) \Big) \\
&= C_*(\check{C}_n(\mathcal{U} \times \mathcal{V})).
\end{aligned}
$$

The desired isomorphism is obtained by applying the functor (60) to the isomorphism of simplicial objects (63). $\qquad\square$

We obtain (61) by composing the quasi-isomorphisms of the previous lemmas. More precisely, we have the quasi-isomorphism

$$
\begin{aligned}
\mathrm{Tot}(\mathrm{Toc}(C_*\check{C}_\bullet(\mathcal{U})) \otimes \mathrm{Toc}(C_*\check{C}_\bullet(\mathcal{V}))) &\simeq \mathrm{Tot}(\mathrm{T}(\mathrm{Tot}(C_*\check{C}_\bullet(\mathcal{U}) \otimes C_*\check{C}_\bullet(\mathcal{V})))) \\
&\xrightarrow{\mathcal{N}} \mathrm{Tot}(\mathrm{T}(C_*(\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V})))) \\
&\xrightarrow{\nabla} \mathrm{Tot}(\mathrm{D}(C_*(\check{C}_\bullet(\mathcal{U}) \times \check{C}_\bullet(\mathcal{V})))) \\
&\simeq \mathrm{Toc}(C_*(\check{C}_\bullet(\mathcal{U} \times \mathcal{V})))
\end{aligned}
$$

Note that $\mathcal{N}$ is induced by Nori's Lemma 8.3 (i.e. the Künneth formula for affine varieties) and that ∇ is induced by the Eilenberg-Zilber shuffle map. This quasi-isomorphism is functorial in $(X, \mathcal{U}, r)$ and $(Y, \mathcal{V}, s)$ by construction. It induces a canonical isomorphism

$$M(X) \otimes^L M(Y) \xrightarrow{\sim} M(X \times_k Y)$$

in $\mathbf{D}_b(EHM)$, inducing the usual Künneth formula on singular homology.   □

Corollary 8.8. *For any varieties X and Y, there is a canonical isomorphism in EHM*

$$\bigoplus_{p+q=n} h_p(X) \otimes h_q(Y) \xrightarrow{\sim} h_n(X \times Y).$$

Proof. This follows immediately from the previous theorem since any object of EHM is flat.
                    □

Applying Nori's universal construction to the diagram of arbitrary pairs of varieties, or proceeding as we did for homological motives (see Notation 8.1), we may define $h^i(X) \in ECM$ for any variety X.

Corollary 8.9. *For any varieties X and Y, there is a canonical isomorphism in ECM.*

$$\bigoplus_{p+q=n} h^p(X) \otimes h^q(Y) \xleftarrow{\sim} h^n(X \times Y).$$

Proof. Let $C(D, T)$ be the diagram category associated to a diagram D and a representation $T : D \to \text{Vec}_{\mathbb{Q}}$. First we exhibit a canonical equivalence of categories

(64) $$C(D, T) \simeq C(D^{op}, T^{\vee})^{op}.$$

To this aim, we consider the following square

$$
\begin{array}{ccc}
D & \xrightarrow{\tilde{T}} & C(D, T) \\
\downarrow{\scriptstyle G} & & \downarrow{\scriptstyle f_T} \\
C(D^{op}, T^{\vee})^{op} & \xrightarrow{f_{T^{\vee}}^{\vee}} & \text{Vec}_{\mathbb{Q}}
\end{array}
$$

where G is the representation of D obtained by composing the functors

$$D \to D^{op} \to C(D^{op}, T^{\vee})^{op}.$$

Since the square is commutative we know from ([28], Theorem 1.6), that there exists a unique $\mathbb{Q}$-linear functor $L(G) : C(D, T) \to C(D^{op}, T^{\vee})^{op}$ such that

$$G = L(G) \circ \tilde{T} \text{ and } f_T = f_{T^{\vee}}^{\vee} \circ L(G).$$

The functor $L(G)$ is an equivalence of categories. A quasi inverse for $L(G)$ is constructed by considering the square attached to the pair $(D^{op}, T^{\vee})$.

We need to give a module-theoretic interpretation of the functor $L(G)$. Recall that the category $C(D, T)$ is defined in [28] by

$$C(D, T) = \text{colim}_F C(F, T|F)$$

where F runs through all the finite subdiagrams F of D and the 2-colimit is taken in the 2-category of small categories. If F is finite, then $C(F, T|F)$ is the category of finitely generated left $\text{End}(T|F)$-modules where

$$\text{End}(T|F) = \{u = (u_p)_{p \in F} \in \prod_{p \in F} \text{End}(T(p)) | u_q \circ T(m) = T(m) \circ u_p, \forall m : p \to q, p, q \in F\}.$$

For M an $\mathrm{End}(T|F)$-module, we have ring homomorphisms

$$\mathrm{End}(T|F) \xrightarrow{\ f\ } \mathrm{End}_{\mathbb{Q}}(M) \xrightarrow{\ \vee\ } \mathrm{End}_{\mathbb{Q}}(M^{\vee})^{op}$$

where f is given by the action of $\mathrm{End}(T|F)$ on M and where $\vee$ associates to any $u \in \mathrm{End}(M)$ its transposed $u^{\vee}$. Moreover the map $u = (u_p)_{p \in F} \to u^{\vee} = (u_p^{\vee})_{p \in F}$ induces a ring isomorphism $\mathrm{End}(T|F) \simeq \mathrm{End}(T^{\vee}|F)^{op}$. Therefore we obtain by composition a ring homomorphism

$$\mathrm{End}(T^{\vee}|F) \to \mathrm{End}(T|F)^{op} \to \mathrm{End}_{\mathbb{Q}}(M^{\vee})$$

and thus a structure of $\mathrm{End}(T^{\vee}|F)$-module on $M^{\vee}$. We let α_F be the functor

$$\begin{array}{ccc} C(F,T|F) & \longrightarrow & C(F^{op},T^{\vee}|F)^{op} \\ M & \longmapsto & M^{\vee} \end{array}$$

and we denote by $\alpha : C(D,T) \to C(D^{op},T^{\vee})^{op}$ the functor defined by the $(\alpha_F)_{F \subset D}$. It is easy to check that $G = \alpha \circ \tilde{T}$ and $f_T = f_{T^{\vee}}^{\vee} \circ \alpha$. It follows from the universal property of $C(D,T)$ that $L(G)$ and α are isomorphic.

Applying these observations to the diagram of good pairs, we obtain an equivalence

$$\alpha : EHM \xrightarrow{\ \sim\ } ECM^{op}.$$

We have $\alpha(h_i(X)) = h^i(X)$ for any integer i and any variety X/k. Moreover the functor α sends direct sums to direct sums. Thus in order to deduce Corollary 8.9 from Corollary 8.8 it suffices to prove that α sends tensor products to tensor products. The categories EHM and ECM are both diagram categories attached to a graded diagram endowed with a commutative product structure with unit and a graded unital multiplicative representation. Using the definition of the tensor product on such categories as defined in ([18], Proposition B.16) and in ([19], Chapter VII, Section 7.1) and using the description of α in terms of modules given above, we check that α respects tensor products as required. $\qquad\square$

We denote by NMM_k the category of cohomological Nori motives.

Corollary 8.10. *Let V/k be a smooth projective variety of dimension n. There is a map in* NMM_k

$$(65) \qquad h^i(V)(j) \otimes h^{2n-i}(V)(n-j) \longrightarrow h^{2n}(V)(n) \xrightarrow{\ \mathrm{Tr}\ } \mathbf{1}$$

inducing the usual pairings in Betti and de Rham cohomology for any $i,j \in \mathbb{Z}$. In particular, (65) yields an canonical isomorphism

$$h^i(V)(j) \xrightarrow{\ \sim\ } h^{2n-i}(V)(n-j)^{\vee}.$$

Proof. Corollary 8.9 gives the canonical map in NMM_k

$$h^i(V)(j) \otimes h^{2n-i}(V)(n-j) \longrightarrow h^{2n}(V \times V)(n) \longrightarrow h^{2n}(V)(n)$$

inducing the same map on Betti cohomology. Here $h^{2n}(V \times V) \to h^{2n}(V)$ is induced by the diagonal map.

Let us show the existence of the trace map

$$h^{2n}(V)(n) \xrightarrow{\ \mathrm{Tr}\ } \mathbf{1}.$$

We follow [18] with a slight modification. Since $h^i(V_1 \bigsqcup V_2) = h^i(V_1) \oplus h^i(V_2)$, we may assume that V is connected. Let $V \to \mathbf{P}_k^N$ be a closed embedding. We consider the finite extension $K := \mathcal{O}_V(V)$ of k, so that V/K is geometrically connected. We obtain a closed embedding

$V \to \mathbf{P}_k^N \times_k K = \mathbf{P}_K^N$, inducing a map $\varphi : h^{2n}(\mathbf{P}_K^N) \to h^{2n}(V)$. But $H_B^*(\varphi)$ is an isomorphism, hence so is φ. By ([18] Lemma 1.12(1)) we have a canonical isomorphism $h^{2n}(\mathbf{P}_k^N) \simeq \mathbf{1}(-n)$. We obtain

$$h^{2n}(\mathbf{P}_K^N) = h^{2n}(\mathbf{P}_k^N \times \mathrm{Spec}(K)) \simeq h^{2n}(\mathbf{P}_k^N) \otimes h^0(\mathrm{Spec}(K)) \simeq h^0(\mathrm{Spec}(K))(-n).$$

One is therefore reduced to define the trace map $h^0(\mathrm{Spec}(K)) \to \mathbf{1}$ in the category of Artin motives, i.e. in the category of $\mathbb{Q}$-linear representations of G_k. But $h^0(\mathrm{Spec}(K))$ corresponds to the linear G_k-action on $\mathbb{Q}^{\mathrm{Hom}_k(K,\overline{k})}$ by permutation of the factors, hence the sum map

$$\begin{array}{ccc} \mathbb{Q}^{\mathrm{Hom}_k(K,\overline{k})} & \longrightarrow & \mathbb{Q} \\ (v_\sigma)_\sigma & \longmapsto & \sum_\sigma v_\sigma \end{array}$$

is G_k-equivariant, where G_k acts trivially on k. This defines a map $h^0(\mathrm{Spec}(K)) \overset{\Sigma}{\to} \mathbf{1}$. The trace map of the lemma is then given by composition

$$h^{2n}(V)(n) \overset{\sim}{\longleftarrow} h^{2n}(\mathbf{P}_K^N)(n) \simeq h^0(\mathrm{Spec}(K)) \overset{\Sigma}{\longrightarrow} \mathbf{1}.$$

This map induces the usual trace map in Betti cohomology.

So we have the map

$$m : h^i(V)(j) \otimes h^{2n-i}(V)(n-j) \longrightarrow h^{2n}(V)(n) \overset{\mathrm{Tr}}{\longrightarrow} \mathbf{1}$$

and this map induces the usual pairing on Betti cohomology. Applying the tensor functors H_B^* and H_{dR}^*, we obtain product maps

$$H_B^*(m) : H^i(V(\mathbb{C}), (2i\pi)^j \mathbb{Q}) \otimes H^{2n-i}(V(\mathbb{C}), (2i\pi)^{n-j}\mathbb{Q}) \longrightarrow H^{2n}(V(\mathbb{C}), (2i\pi)^n \mathbb{Q}) \overset{\mathrm{Tr}}{\longrightarrow} \mathbb{Q}$$

and

$$H_{dR}^*(m) : H^i(V/k) \otimes H^{2n-i}(V/k) \longrightarrow H^{2n}(V/k) \overset{\mathrm{Tr}}{\longrightarrow} k.$$

On the other hand, we denote by

$$m_B : H^i(V(\mathbb{C}), (2i\pi)^j \mathbb{Q}) \otimes H^{2n-i}(V(\mathbb{C}), (2i\pi)^{n-j}\mathbb{Q}) \longrightarrow H^{2n}(V(\mathbb{C}), (2i\pi)^n \mathbb{Q}) \overset{\mathrm{Tr}}{\longrightarrow} \mathbb{Q}$$

and

$$m_{dR} : H_{dR}^i(V/k) \otimes H_{dR}^{2n-i}(V/k) \longrightarrow H^{2n}(V/k) \overset{\mathrm{Tr}}{\longrightarrow} k$$

the usual product map in Betti cohomology (given by cup-product) and in de Rham cohomology (given by wedge product) respectively. The usual period isomorphism yields an isomorphism of tensor functors $H_B^* \otimes_{\mathbb{Q}} \mathbb{C} \simeq H_{dR}^* \otimes_k \mathbb{C}$, hence the square

$$\begin{array}{ccc} H^i(V(\mathbb{C}), (2i\pi)^j \mathbb{Q})_{\mathbb{C}} \otimes H^{2n-i}(V(\mathbb{C}), (2i\pi)^{n-j}\mathbb{Q})_{\mathbb{C}} & \overset{H_B^*(m)\otimes\mathbb{C}}{\longrightarrow} & \mathbb{C} \\ \downarrow{\simeq} & & \downarrow{\mathrm{Id}} \\ H_{dR}^i(V/k)_{\mathbb{C}} \otimes H_{dR}^{2n-i}(V/k)_{\mathbb{C}} & \overset{H_{dR}^*(m)\otimes\mathbb{C}}{\longrightarrow} & \mathbb{C} \end{array}$$

commutes, where the vertical isomorphisms are given by the period isomorphism. But the square

$$\begin{array}{ccc} H^i(V(\mathbb{C}), (2i\pi)^j \mathbb{Q})_{\mathbb{C}} \otimes H^{2n-i}(V(\mathbb{C}), (2i\pi)^{n-j}\mathbb{Q})_{\mathbb{C}} & \overset{m_B\otimes\mathbb{C}}{\longrightarrow} & \mathbb{C} \\ \downarrow{\simeq} & & \downarrow{\mathrm{Id}} \\ H_{dR}^i(V/k)_{\mathbb{C}} \otimes H_{dR}^{2n-i}(V/k)_{\mathbb{C}} & \overset{m_{dR}\otimes\mathbb{C}}{\longrightarrow} & \mathbb{C} \end{array}$$

commutes as well. By construction of m, we have $H_B^*(m) = m_B$. It follows that $H_{dR}^*(m) \otimes_k \mathbb{C} = m_{dR} \otimes_k \mathbb{C}$, hence that $H_{dR}^*(m) = m_{dR}$. The first claim of the corollary is proved.

By adjunction, the map m gives a map $d : h^i(V)(j) \to h^{2n-i}(V)(n-j)^\vee$ inducing the usual map $H_B^*(d) : H^i(V(\mathbb{C}), (2i\pi)^j\mathbb{Q}) \to H^{2n-i}(V(\mathbb{C}), (2i\pi)^{n-j}\mathbb{Q})^\vee$ in Betti cohomology. But $H_B^*(d)$ is an isomorphism, hence so is d, since H_B^* is conservative. $\qquad\square$

We record the following special case of Corollary 8.10.

Corollary 8.11. *Let V/k be a smooth projective algebraic variety of even dimension n over the number field k. Then $h^n(V)(n/2) \in \mathrm{NMM}_k$ has a canonical orthogonal structure q such that the forms q_B and q_{dR} are the perfect pairings*

$$(66) \quad q_B : H_B^n(V(\mathbb{C}), (2i\pi)^{n/2}\mathbb{Q}) \otimes H_B^n(V(\mathbb{C}), (2i\pi)^{n/2}\mathbb{Q}) \overset{\cup}{\longrightarrow} H_B^{2n}(V(\mathbb{C}), (2i\pi)^n\mathbb{Q}) \overset{\mathrm{Tr}}{\longrightarrow} E$$

and

$$(67) \qquad\qquad q_{dR} : H_{dR}^n(V/k) \otimes H_{dR}^n(V/k) \overset{\wedge}{\longrightarrow} H_{dR}^{2n}(V/k) \overset{\mathrm{Tr}}{\longrightarrow} k$$

given by cup-product and trace maps.

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