On the Weil-étale topos of regular arithmetic schemes

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Abstract. We define and study a Weil-étale topos for any regular, proper scheme $\mathcal{X}$ over $\mathrm{Spec}(\mathbb{Z})$ which has some of the properties suggested by Lichtenbaum for such a topos. In particular, the cohomology with $\tilde{\mathbb{R}}$-coefficients has the expected relation to $\zeta(\mathcal{X}, s)$ at $s = 0$ if the Hasse-Weil L-functions $L(h^i(\mathcal{X}_{\mathbb{Q}}), s)$ have the expected meromorphic continuation and functional equation. If $\mathcal{X}$ has characteristic p the cohomology with $\mathbb{Z}$-coefficients also has the expected relation to $\zeta(\mathcal{X}, s)$ and our cohomology groups recover those previously studied by Lichtenbaum and Geisser.

2000 Mathematics Subject Classification: Primary: 14F20, 11S40, Secondary: 11G40, 18F10

1. Introduction

In [30] Lichtenbaum suggested the existence of Weil-étale cohomology groups for arithmetic schemes $\mathcal{X}$ (i.e. separated schemes of finite type over $\mathrm{Spec}(\mathbb{Z})$) which are related to the zeta-function $\zeta(\mathcal{X}, s)$ of $\mathcal{X}$ as follows.

a) The compact support cohomology groups $H^i_c(\mathcal{X}_W, \tilde{\mathbb{R}})$ are finite dimensional vector spaces over $\mathbb{R}$, vanish for almost all i and satisfy
$$\sum_{i\in\mathbb{Z}}(-1)^i \dim_{\mathbb{R}} H^i_c(\mathcal{X}_W, \tilde{\mathbb{R}}) = 0.$$

b) The function $\zeta(\mathcal{X}, s)$ has a meromorphic continuation to $s = 0$ and
$$\mathrm{ord}_{s=0}\,\zeta(\mathcal{X}, s) = \sum_{i\in\mathbb{Z}}(-1)^i \cdot i \cdot \dim_{\mathbb{R}} H^i_c(\mathcal{X}_W, \tilde{\mathbb{R}}).$$

c) There exists a canonical class $\theta \in H^1(\mathcal{X}_W, \tilde{\mathbb{R}})$ so that the sequence
$$\cdots \xrightarrow{\cup\theta} H^i_c(\mathcal{X}_W, \tilde{\mathbb{R}}) \xrightarrow{\cup\theta} H^{i+1}_c(\mathcal{X}_W, \tilde{\mathbb{R}}) \xrightarrow{\cup\theta} \cdots$$
is exact.

d) The compact support cohomology groups $H^i_c(\mathcal{X}_W, \mathbb{Z})$ are finitely generated over $\mathbb{Z}$ and vanish for almost all i.

e) The natural map from $\mathbb{Z}$ to $\tilde{\mathbb{R}}$-coefficients induces an isomorphism
$$H^i_c(\mathcal{X}_W, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} H^i_c(\mathcal{X}_W, \tilde{\mathbb{R}}).$$

f) If $\zeta^*(\mathcal{X},0)$ denotes the leading Taylor-coefficient of $\zeta(\mathcal{X},s)$ at $s=0$ and
$$\lambda:\mathbb{R}\cong\bigotimes_{i\in\mathbb{Z}}\det{}_{\mathbb{R}}H^i_c(\mathcal{X}_W,\tilde{\mathbb{R}})^{(-1)^i}$$
the isomorphism induced by c) then
$$\mathbb{Z}\cdot\lambda(\zeta^*(\mathcal{X},0)^{-1})=\bigotimes_{i\in\mathbb{Z}}\det{}_{\mathbb{Z}}H^i_c(\mathcal{X}_W,\mathbb{Z})^{(-1)^i}$$
where the determinant is understood in the sense of [28] and $\tilde{\mathbb{R}}$ is the sheaf associated to the field of real numbers endowed with its standard topology (see Definition 14 below).

If $\mathcal{X}$ has finite characteristic these groups are well defined and well understood by work of Lichtenbaum [29] and Geisser [18, 19]. In particular all the above properties a)-f) hold for $\dim(\mathcal{X})\leq 2$ and in general under resolution of singularities. Lichtenbaum also defined such groups for $\mathcal{X}=\mathrm{Spec}(\mathcal{O}_F)$ where F is a number field and showed that a)-f) hold if one artificially redefines $H^i_c(\mathrm{Spec}(\mathcal{O}_F)_W,\mathbb{Z})$ to be zero for $i\geq 4$. In [15] it was then shown that $H^i_c(\mathrm{Spec}(\mathcal{O}_F)_W,\mathbb{Z})$ as defined by Lichtenbaum does indeed vanish for odd $i\geq 5$ but is an abelian group of infinite rank for even $i\geq 4$.
In any case, in Lichtenbaum's definition the groups $H^i_c(\mathrm{Spec}(\mathcal{O}_F)_W,\mathbb{Z})$ and $H^i_c(\mathrm{Spec}(\mathcal{O}_F)_W,\tilde{\mathbb{R}})$ are defined via an Artin-Verdier type compactification $\overline{\mathrm{Spec}(\mathcal{O}_F)}$ of $\mathrm{Spec}(\mathcal{O}_F)$ [1], where however $H^i(\overline{\mathrm{Spec}(\mathcal{O}_F)}_W,\mathcal{F})$ is not the cohomology group of a topos but rather a direct limit of such. The first purpose of this article is to give a definition of a topos $\overline{\mathrm{Spec}(\mathcal{O}_F)}_W$ which recovers Lichtenbaum's groups (see section 5 below). This definition was proposed in the second author's thesis [33] and is a natural modification of Lichtenbaum's idea which is suggested by a closer look at the étale topos $\mathrm{Spec}(\mathcal{O}_F)_{\mathrm{et}}$.
In [1] Artin and Verdier defined a topos $\overline{\mathcal{X}}_{\mathrm{et}}$ for any arithmetic scheme $\mathcal{X}\to\mathrm{Spec}(\mathbb{Z})$ so that there are complementary open and closed immersions
$$\mathcal{X}_{\mathrm{et}}\to\overline{\mathcal{X}}_{\mathrm{et}}\leftarrow Sh(\mathcal{X}_\infty)$$
the sense of topos theory [20]. Here $\mathcal{X}_\infty$ is the topological quotient space $\mathcal{X}(\mathbb{C})/G_{\mathbb{R}}$ where $\mathcal{X}(\mathbb{C})$ is the set of complex points with its standard Euclidean topology and $G_{\mathbb{R}}=\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. If $\mathcal{X}$ is an arithmetic scheme and $\mathcal{Y}$ denotes either $\mathcal{X}$ or $\overline{\mathcal{X}}$ we define the Weil-étale topos of $\mathcal{Y}$ by
$$\mathcal{Y}_W:=\mathcal{Y}_{\mathrm{et}}\times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}}\overline{\mathrm{Spec}(\mathbb{Z})}_W,$$
a fibre product in the 2-category of topoi. This definition is suggested by the fact that the Weil-étale topos defined by Lichtenbaum for varieties over finite fields is isomorphic to a similar fibre product, as was shown in the second author's thesis [33] and will be recalled in section 3 below. The work of Geisser [19] shows that Lichtenbaums's definition is only reasonable (i.e. satisfies a)-f)) for smooth, proper varieties over finite fields. Correspondingly, one can only expect our fibre product definition to be reasonable for proper *regular* arithmetic schemes.

The second purpose of this article is to show that this is indeed the case as far as $\tilde{\mathbb{R}}$-coefficients are concerned. Our main result is the following

Theorem 1.1. *Let $\mathcal{X}$ be a regular scheme, proper over* $\mathrm{Spec}(\mathbb{Z})$.

i) *For $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$ one has*

$$\overline{\mathrm{Spec}(\mathcal{O}_F)}_W \cong \overline{\mathrm{Spec}(\mathcal{O}_F)}_{\mathrm{et}} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \overline{\mathrm{Spec}(\mathbb{Z})}_W,$$

where $\overline{\mathrm{Spec}(\mathcal{O}_F)}_W$ is the topos defined in section 5 below, based on Lichtenbaum's idea of replacing Galois groups by Weil groups.

ii) *If $\mathcal{X} \to \mathrm{Spec}(\mathbb{F}_p)$ has characteristic p then our groups agree with those of Lichtenbaum and Geisser and a)-f) hold for $\mathcal{X}$.*

iii) *If $\mathcal{X}$ is flat over* $\mathrm{Spec}(\mathbb{Z})$ *and the Hasse-Weil L-functions $L(h^i(\mathcal{X}_{\mathbb{Q}}), s)$ of all motives $h^i(\mathcal{X}_{\mathbb{Q}})$ satisfy the expected meromorphic continuation and functional equation. Then a)-c) hold for $\mathcal{X}$.*

The assumptions of iii) are satisfied, for example, if $\mathcal{X}$ is a regular model of a Shimura curve, or of a self product $E \times \cdots \times E$ where E is an elliptic curve, over a totally real field F.
Unfortunately, properties d) and e) do not hold with our fibre product definition, even in low degrees, and we also do not expect them to hold with any similar definition (see the remarks in section 9.3). The right definition of Weil-étale cohomology with $\mathbb{Z}$-coefficients for schemes of characteristic zero will require a key new idea, as is already apparent for $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$.
We briefly describe the content of this article. In section 2 we recall preliminaries on sites, topoi and classifying topoi. Section 3 contains the proof that Lichtenbaum's Weil-étale topos in characteristic p is a fibre product via a method that is different from the one in the second author's thesis [33]. In section 4 we recall the definition of $\overline{\mathcal{X}}_{\mathrm{et}}$ and the corresponding compact support cohomology groups $H^i_c(\mathcal{X}_{\mathrm{et}}, \mathcal{F})$. In section 5 we define $\overline{\mathrm{Spec}(\mathcal{O}_F)}_W$ and give the proof of Theorem 1.1 i) (see Proposition 5.5). In section 6 we define $\overline{\mathcal{X}}_W$, describe its fibres above all places $p \leq \infty$ and its generic point. In section 7 we compute the cohomology of $\overline{\mathcal{X}}_W$ with $\tilde{\mathbb{R}}$-coefficients following Lichtenbaum's method of studying the Leray spectral sequence from the generic point. This section is the technical heart of this article. In section 8 we compute the compact support cohomology $H^i_c(\mathcal{X}_W, \tilde{\mathbb{R}})$ via the natural morphism $\overline{\mathcal{X}}_W \to \overline{\mathcal{X}}_{\mathrm{et}}$ and prove properties a) and c) (see Theorem 8.2). The class θ in c) is defined in subsection 8.3.
Section 9 introduces Hasse-Weil L-functions of varieties over $\mathbb{Q}$ as well as Zeta-functions of arithmetic schemes and contains the proof of Theorem 1.1 ii) (see Theorem 9.2) and of property b) (see Theorem 9.1), thereby concluding the proof Theorem 1.1 iii). In subsection 9.4 we show that property f) for $\zeta(\mathcal{X}, s)$ is compatible with the Tamagawa number conjecture of Bloch and Kato [4] (or rather of Fontaine and Perrin-Riou [16]) for $\prod_{i\in\mathbb{Z}} L(h^i(\mathcal{X}_{\mathbb{Q}}), s)^{(-1)^i}$ at $s = 0$. In order to do this we need to augment the list of properties a)-f) for Weil-étale cohomology with further natural assumptions g)-j), some of which hold

in characteristic p, and we need to assume a number of conjectures which are preliminary to the formulation of the Tamagawa number conjecture. Finally, in section 10 we prove some results related to the so called local theorem of invariant cycles in l-adic cohomology, and we formulate analogous conjectures in p-adic cohomology. These results may be of some interest independently of Weil-étale cohomology, and are necessary to establish the equality of vanishing orders

$$\mathrm{ord}_{s=0}\,\zeta(\mathcal{X},s) = \mathrm{ord}_{s=0}\prod_{i\in\mathbb{Z}} L(h^i(\mathcal{X}_{\mathbb{Q}}),s)^{(-1)^i}$$

for regular schemes $\mathcal{X}$ proper and flat over $\mathrm{Spec}(\mathbb{Z})$.

Acknowledgements: The first author is supported by grant DMS-0701029 from the National Science Foundation. He would also like to thank Spencer Bloch for a helpful discussion about the material in section 10 and the MPI Bonn for its hospitality during the final preparation of this paper.

2. Preliminaries

In this paper, a topos is a Grothendieck topos over $\underline{Set}$, and a morphism of topoi is a geometric morphism. A pseudo-commutative diagram of topoi is said to be commutative. We suppress any mention of universes.

2.1. Left exact sites.

Recall that a Grothendieck topology $\mathcal{J}$ on a category $\mathcal{C}$ is said to be *sub-canonical* if $\mathcal{J}$ is coarser than the canonical topology, i.e. if any representable presheaf on $\mathcal{C}$ is a sheaf for the topology $\mathcal{J}$. A category $\mathcal{C}$ is said to be *left exact* when finite projective limits exist in $\mathcal{C}$, i.e. when $\mathcal{C}$ has a final object and fiber products. A functor between left exact categories is said to be left exact if it commutes with finite projective limits.

DEFINITION 1. *A Grothendieck site $(\mathcal{C},\mathcal{J})$ is said to be* left exact *if $\mathcal{C}$ is a left exact category endowed with a subcanonical topology $\mathcal{J}$. A* morphism of left exact sites *$(\mathcal{C}',\mathcal{J}')\to(\mathcal{C},\mathcal{J})$ is a continuous left exact functor $\mathcal{C}'\to\mathcal{C}$.*

Note that any Grothendieck topos, i.e. any category satisfying Giraud's axioms, is equivalent to the category of sheaves of sets on a left exact site. Note also that a Grothendieck site $(\mathcal{C},\mathcal{J})$ is left exact if and only if the canonical functor (given in general by Yoneda and sheafification)

$$y:\mathcal{C}\longrightarrow\widetilde{(\mathcal{C},\mathcal{J})}$$

identifies $\mathcal{C}$ with a left exact full subcategory of $\widetilde{(\mathcal{C},\mathcal{J})}$. The following result is proven in [20] IV.4.9.

LEMMA 1. *A* morphism of left exact sites *$f^*:(\mathcal{C}',\mathcal{J}')\to(\mathcal{C},\mathcal{J})$ induces a morphism of topoi $f:\widetilde{(\mathcal{C},\mathcal{J})}\to\widetilde{(\mathcal{C}',\mathcal{J}')}$. Moreover we have a commutative* *diagram*

$$\begin{array}{ccc} (\widetilde{\mathcal{C},\mathcal{J}}) & \xleftarrow{f^*} & (\widetilde{\mathcal{C}',\mathcal{J}'}) \\ \uparrow y_{\mathcal{C}} & & \uparrow y_{\mathcal{C}'} \\ \mathcal{C} & \xleftarrow{f^*} & \mathcal{C}' \end{array}$$

where the vertical arrows are the fully faithful Yoneda functors.

2.2. The topos $\mathcal{T}$. We denote by Top^{lc} (respectively by Top^{c}) the category of locally compact topological spaces (respectively of compact spaces). A locally compact space is assumed to be Hausdorff. The category Top^{lc} is endowed with the open cover topology $\mathcal{J}_{op}$, which is subcanonical. We denote by $\mathcal{T}$ the topos of sheaves of sets on the site $(Top^{lc}, \mathcal{J}_{op})$. The Yoneda functor

$$y : Top^{lc} \longrightarrow \mathcal{T}$$

is fully faithful, and Top^{lc} is viewed as a generating full subcategory of $\mathcal{T}$. For any object T of Top^{lc}, T is locally compact hence there exist morphisms

$$\coprod yU_i \to \coprod yK_i \to yT$$

where $\{U_i \subset T\}$ is an open covering, and K_i is a compact subspace of T. It follows that $\coprod yU_i \to yT$ is an epimorphism in $\mathcal{T}$, hence so is $\coprod yK_i \to yT$. This shows that the category of compact spaces Top^c is a generating full subcategory of $\mathcal{T}$.
The unique morphism $t : \mathcal{T} \to \underline{Set}$ has a section $s : \underline{Set} \to \mathcal{T}$ such that $t_* = s^*$ hence we have three adjoint functors t^*, $t_* = s^*$, s_*. In particular t_* is exact hence we have $H^n(\mathcal{T}, \mathcal{A}) = H^n(\underline{Set}, \mathcal{A}(*)) = 0$ for any $n \geq 1$ and any abelian object $\mathcal{A}$ (see Lemma 8 for a generalization of this fact).

2.3. Classifying topoi.

2.3.1. *General case.* For any topos $\mathcal{S}$ and any group object G in $\mathcal{S}$, we denote by B_G the category of left G-object in $\mathcal{S}$. Then B_G is a topos, as it follows from Giraud's axioms, and B_G is endowed with a canonical morphism $B_G \to \mathcal{S}$, whose inverse image functor sends an object F of $\mathcal{S}$ to F with trivial G-action. If there is a risk of ambiguity, the topos B_G is denoted by $B_{\mathcal{S}}(G)$. We denote by EG the object of B_G given by the action of G on itself by left multiplication. The topos B_G is said to be the classifying topos of G since for any topos $f : \mathcal{E} \to \mathcal{S}$ over $\mathcal{S}$, the category $\underline{Homtop}_{\mathcal{S}}(\mathcal{E}, B_G)$ is equivalent to the category of f^*G-torsors in $\mathcal{E}$ (see [20] IV. Exercice 5.9).

2.3.2. *Examples.* Let G be a discrete group, i.e. a group object of the final topos $\underline{Set}$. Then $B_{\underline{Set}}G$ is the category of left G-sets, and the cohomology groups $H^*(B_{\underline{Set}}G, A)$, where A is an abelian object of B_G i.e. a G-module, is precisely the cohomology of the discrete group G. Here $B_{\underline{Set}}G$ is called the *small classifying topos of the discrete group* G and is denoted by B_G^{sm}. If G is a profinite group, the *small classifying topos* B_G^{sm} *of the profinite group* G is the category of continuous G-sets.

Let G be a locally compact topological group. Then G represents a group object of $\mathcal{T}$, where $\mathcal{T}$ is defined above. Then B_G is the classifying topos of the topological group G, and the cohomology groups $H^*(B_G, \mathcal{A})$, where $\mathcal{A}$ is an abelian object of B_G (e.g. a topological G-module) is the cohomology of the topological group G. If G is not locally compact, then we just need to replace $\mathcal{T}$ with the category of sheaves on $(Top, \mathcal{J}_{op})$.
Let S be a scheme and let G be a smooth group scheme over S. We denote by S_{Et} the big étale topos of S. Then G represents a group object of S_{Et} and B_G is the classifying topos of G. The cohomology groups $H^*(B_G, \mathcal{A})$, where $\mathcal{A}$ is an abelian object of B_G (e.g. an abelian group scheme over S endowed with a G-action) is the étale cohomology of the S-group scheme G.

2.3.3. *The local section site.* For G any locally compact topological group, we denote by $B_{Top^{lc}}G$ the category of G-equivariant locally compact topological spaces endowed with the local section topology $\mathcal{J}_{ls}$ (see [30] section 1). The Yoneda functor yields a canonical fully faithful functor

$$B_{Top^{lc}}G \longrightarrow B_G.$$

Then one can show that the local section topology $\mathcal{J}_{ls}$ on $B_{Top^{lc}}G$ is the topology induced by the canonical topology of B_G. Moreover $B_{Top^{lc}}G$ is a generating family of B_G. It follows that the morphism

$$B_G \longrightarrow (\widetilde{B_{Top^{lc}}G, \mathcal{J}_{ls}})$$

is an equivalence. In other words the site $(B_{Top^{lc}}G, \mathcal{J}_{ls})$ is a site for the classifying topos B_G (see [15] for more details).

2.3.4. *The classifying topos of a strict topological pro-group.* A *locally compact topological pro-group* $\underline{G}$ is a pro-object in the category of locally compact topological groups, i.e. a functor $I^{op} \to Gr(Top^{lc})$, where I is a filtered category and $Gr(Top^{lc})$ is the category of locally compact topological groups. A locally compact topological pro-group $\underline{G}$ is said to be *strict* if the transition maps $G_j \to G_i$ have local sections. We define the limit of $\underline{G}$ in the 2-category of topoi as follows.

Definition 2. *The classifying topos of a strict topological pro-group $\underline{G}$ is defined as*

$$B_{\underline{G}} := \varprojlim_I B_{G_i},$$

where the the projective limit is computed in the 2-category of topoi.

2.3.5. In order to ease the notations, we will simply denote by Top the category of locally compact spaces. For any locally compact group G, we denote by $B_{Top}G$ the category of locally compact spaces endowed with a continuous G-action.

2.4. Fiber products of topoi. The class of topoi forms a 2-category. In particular, $\underline{Homtop}(\mathcal{E},\mathcal{F})$ is a category for any pair of topoi $\mathcal{E}$ and $\mathcal{F}$. If $f,g:\mathcal{E}\rightrightarrows\mathcal{F}$ are two objects of $\underline{Homtop}(\mathcal{E},\mathcal{F})$, then a morphism $\sigma:f\rightarrow g$ is a natural transformation $\sigma:f_*\rightarrow g_*$. Consider now two morphisms of topoi with the same target $f:\mathcal{E}\rightarrow\mathcal{S}$ and $g:\mathcal{F}\rightarrow\mathcal{S}$. For any topos $\mathcal{G}$, we define the category

$$\underline{Homtop}(\mathcal{G},\mathcal{E})\times_{\underline{Homtop}(\mathcal{G},\mathcal{S})}\underline{Homtop}(\mathcal{G},\mathcal{F})$$

whose objects are given by triples of the form (a,b,α), where a and b are objects of $\underline{Homtop}(\mathcal{G},\mathcal{E})$ and $\underline{Homtop}(\mathcal{G},\mathcal{F})$ respectively, and

$$\alpha:f\circ a\cong g\circ b$$

is an isomorphism in the category $\underline{Homtop}(\mathcal{G},\mathcal{S})$.
A fiber product $\mathcal{E}\times_{\mathcal{S}}\mathcal{F}$ in the 2-category of topoi is a topos endowed with canonical projections $p_1:\mathcal{E}\times_{\mathcal{S}}\mathcal{F}\rightarrow\mathcal{E}$, $p_2:\mathcal{E}\times_{\mathcal{S}}\mathcal{F}\rightarrow\mathcal{F}$ and an isomorphism $\alpha:f\circ p_1\cong g\circ p_2$ satisfying the following universal condition. For any topos $\mathcal{G}$ the natural functor

$$\begin{array}{ccc}\underline{Homtop}(\mathcal{G},\mathcal{E}\times_{\mathcal{S}}\mathcal{F}) & \longrightarrow & \underline{Homtop}(\mathcal{G},\mathcal{E})\times_{\underline{Homtop}(\mathcal{G},\mathcal{S})}\underline{Homtop}(\mathcal{G},\mathcal{F})\\ d & \longmapsto & (p_1\circ d,p_2\circ d,\alpha\circ d_*)\end{array}$$

is an equivalence. It is known that fiber products of topoi always exist (see [25] for example). The universal condition implies that such a fiber product is unique up to equivalence. A product of topoi is a fiber product over the final topos

$$\mathcal{E}\times\mathcal{F}=\mathcal{E}\times_{\underline{Set}}\mathcal{F}.$$

A square of topoi

$$\begin{array}{ccc}\mathcal{E}' & \longrightarrow & \mathcal{S}'\\ \downarrow & & \downarrow\\ \mathcal{E} & \longrightarrow & \mathcal{S}\end{array}$$

is said to be a *pull-back* if it is commutative and if the morphism

$$\mathcal{E}'\longrightarrow\mathcal{E}\times_{\mathcal{S}}\mathcal{S}',$$

given by the universal condition for the fiber product, is an equivalence. The following examples will be used in this paper. Let $f:\mathcal{E}\rightarrow\mathcal{S}$ be a morphism of topoi. For any object X of $\mathcal{S}$, the commutative diagram

$$\begin{array}{ccc}\mathcal{E}/f^*X & \longrightarrow & \mathcal{S}/X\\ \downarrow & & \downarrow\\ \mathcal{E} & \xrightarrow{f} & \mathcal{S}\end{array}\qquad(1)$$

is a pull-back (see [20] IV Proposition 5.11). For any group-object G in $\mathcal{S}$, the commutative diagram

$$(2)\qquad \begin{array}{ccc} B_{\mathcal{E}}(f^*G) & \longrightarrow & B_{\mathcal{S}}(G) \\ \downarrow & & \downarrow \\ \mathcal{E} & \xrightarrow{f} & \mathcal{S} \end{array}$$

is a pull-back. This follows from the fact that $B_{\mathcal{S}}(G)$ classifies G-torsors.

3. The Weil-étale topos in characteristic p is a fiber product

For any scheme Y, we denote by Y_{et} the (small) étale topos of Y, i.e. the category of sheaves of sets on the étale site on Y. Let G be a discrete group acting on a scheme Y. An étale sheaf $\mathcal{F}$ on Y is G-equivariant if $\mathcal{F}$ is endowed with a family of morphisms $\{\varphi_g : g_*\mathcal{F} \to \mathcal{F};\ g \in G\}$ satisfying $\varphi_{1_G} = Id_{\mathcal{F}}$ and $\varphi_{gh} = \varphi_g \circ g_*(\varphi_h)$, for any $g, h \in G$. The category $\mathcal{S}(G; Y_{et})$ of G-equivariant étale sheaves on Y is a topos, as it follows from Giraud's axioms. The cohomology $H^*(\mathcal{S}(G; Y_{et}), \mathcal{A})$, for any G-equivariant abelian étale sheaf on Y, is the equivariant étale cohomology for the action (G, Y).
An equivariant map of G-schemes $u : X \to Y$ induces a morphism of topoi $\mathcal{S}(G; X_{\mathrm{et}}) \to \mathcal{S}(G; Y_{\mathrm{et}})$. Let Y be a scheme separated and of finite type over a field k, let $\overline{k}/k$ be a separable closure and let $\mathcal{F}$ be an étale sheaf on $Y \otimes_k \overline{k}$. An action of the Galois group G_k on $\mathcal{F}$ is said to be *continuous* when the induced action of the profinite group G_k on the discrete set $\mathcal{F}(U \times_k \overline{k})$ is continuous, for any U étale and quasi-compact over Y. It is well known that the étale topos Y_{et} is equivalent to the category $\mathcal{S}(G_k, \overline{Y}_{\mathrm{et}})$ of étale sheaves on $\overline{Y} := Y \otimes_k \overline{k}$ endowed with a continuous action of the Galois group G_k.
Let Y be a separated scheme of finite type over a finite field $k = \mathbb{F}_q$. Let $\overline{k}/k$ be an algebraic closure. Let W_k and G_k be the Weil group and the Galois group of k respectively. Recall that W_k is the discrete subgroup of G_k generated by the Frobenius element. The small classifying topos $B^{sm}_{W_k}$ is defined as the category of W_k-sets, while $B^{sm}_{G_k}$ is the category of continuous G_k-sets. We denote by Y^{sm}_W the Weil-étale topos of the scheme Y, which is defined as follows. We consider the scheme $\overline{Y} = Y \otimes_k \overline{k}$ endowed with the action of W_k. Then the Weil-étale topos Y^{sm}_W is the topos of W_k-equivariant sheaves of sets on $\overline{Y}$. We have a morphism

$$\gamma_Y : Y^{sm}_W := \mathcal{S}(W_k, \overline{Y}_{\mathrm{et}}) \longrightarrow \mathcal{S}(G_k, \overline{Y}_{\mathrm{et}}) \cong Y_{et}.$$

Indeed, consider the functor γ_Y^* which takes an étale sheaf $\mathcal{F}$ on $\overline{Y}$ endowed with a continuous G_k-action to the sheaf $\mathcal{F}$ endowed with the induced W_k-action via the canonical map $W_k \to G_k$. Then γ_Y^* commutes with arbitrary inductive limits and with projective limits. Hence γ_Y^* is the inverse image of a morphism of topoi γ_Y. This morphism has been defined and studied by T. Geisser in [18]. Note that the Weil-étale topos of $\mathrm{Spec}(k)$ is precisely $B^{sm}_{W_k}$ and that the étale topos $\mathrm{Spec}(k)_{\mathrm{et}}$ is equivalent to $B^{sm}_{G_k}$. In this case the morphism $\gamma_k := \alpha : B^{sm}_{W_k} \to B^{sm}_{G_k}$, from the Weil-étale topos of $\mathrm{Spec}(k)$ to its étale topos is the morphism induced by the canonical map $W_k \to G_k$. The structure map $Y \to \mathrm{Spec}(k)$ gives a W_k-equivariant morphism of schemes $\overline{Y} \to \mathrm{Spec}(\overline{k})$, inducing in turn a morphism $Y_W^{sm} \to B_{W_k}^{sm}$. This structure map also induces a morphism of étale topos $Y_{\mathrm{et}} \to B_{G_k}^{sm}$. The diagram

$$\begin{array}{ccc} Y_W^{sm} & \xrightarrow{\gamma_Y} & Y_{\mathrm{et}} \\ \downarrow & & \downarrow \\ B_{W_k} & \xrightarrow{\alpha} & B_{G_k} \end{array} \tag{3}$$

is commutative, where α is induced by the morphism $W_k \to G_k$. The aim of this section is to prove that the previous diagram is a pull-back of topoi. Our proof is based on a descent argument. We need some basic facts concerning truncated simplicial topoi. A truncated simplicial topos $\mathcal{S}_\bullet$ is given by the usual diagram

$$\mathcal{S}_2 \rightrightarrows\!\to \mathcal{S}_1 \rightrightarrows\!\leftarrow \mathcal{S}_0$$

Given such a truncated simplicial topos $\mathcal{S}_\bullet$, we define the category $Desc(\mathcal{S}_\bullet)$ of objects of S_0 endowed with a descent data. By [32], the category $Desc(\mathcal{S}_\bullet)$ is a topos. More precisely, $Desc(\mathcal{S}_\bullet)$ is the inductive limit of the diagram $\mathcal{S}_\bullet$ in the 2-category of topoi. The simplest example is the following. Let $\mathcal{S}$ be a topos and let X be an object of $\mathcal{S}$. We consider the truncated simplicial topos

$$(\mathcal{S},X)_\bullet : \ \mathcal{S}/(X\times X\times X) \rightrightarrows\!\to \mathcal{S}/(X\times X) \rightrightarrows\!\leftarrow \mathcal{S}/X$$

where these morphisms of topoi are induced by the projections maps (of the form $X\times X\times X \to X\times X$ and $X\times X \to X$) and by the diagonal map $X \to X\times X$. It is well known that, if X covers the final object of $\mathcal{S}$ (i.e. $X \to e_{\mathcal{S}}$ is epimorphic where $e_{\mathcal{S}}$ is the final object of $\mathcal{S}$), then the natural morphism

$$Desc(\mathcal{S},X)_\bullet \longrightarrow \mathcal{S}$$

is an equivalence (see [12] Chapter 4 Example 4.1). In other words $\mathcal{S}/X \to \mathcal{S}$ is an effective descent morphism for any X covering the final object of $\mathcal{S}$.

Lemma 2. *Let $f : \mathcal{E} \to \mathcal{S}$ be a morphism of topoi and let X be an object of $\mathcal{S}$ covering the final object. The morphism f is an equivalence if and only if the induced morphism*

$$f/X : \mathcal{E}/f^*X \longrightarrow \mathcal{S}/X$$

is an equivalence.

Proof. The condition is clearly necessary. Assume that f/X is an equivalence. We have $\mathcal{S}/(X\times X) = (\mathcal{S}/X)/(X\times X)$ and $\mathcal{S}/(X\times X\times X) = (\mathcal{S}/X)/(X\times X\times X)$, for any projection maps $X\times X \to X$ and $X\times X\times X \to X$. Hence the triple of morphisms $(f/X\times X\times X, f/X\times X, f/X)$ yields an equivalence of truncated simplicial topoi

$$f/ : (\mathcal{E}, f^*X)_\bullet \longrightarrow (\mathcal{S},X)_\bullet$$

This equivalence induces an equivalence of descent topoi

$$Desc(f/) : Desc(\mathcal{E}, f^*X)_\bullet \longrightarrow Desc(\mathcal{S},X)_\bullet$$

such that the following square is commutative

$$\begin{array}{ccc} Desc(\mathcal{E}, f^*X)_\bullet & \xrightarrow{Desc(f/)} & Desc(\mathcal{S}, X)_\bullet \\ \downarrow & & \downarrow \\ \mathcal{E} & \xrightarrow{f} & \mathcal{S} \end{array}$$

This shows that f is an equivalence since the vertical maps are equivalences. □

THEOREM 3.1. *Let Y be a scheme separated and of finite type over a finite field k. The canonical morphism*

$$Y_W^{sm} \longrightarrow Y_{et} \times_{B_{G_k}^{sm}} B_{W_k}^{sm}$$

is an equivalence.

Proof. The morphism

$$f : Y_W^{sm} \longrightarrow Y_{et} \times_{B_{G_k}^{sm}} B_{W_k}^{sm}$$

is defined by the commutative square (3). Let $p : Y_{et} \times_{B_{G_k}^{sm}} B_{W_k}^{sm} \to B_{W_k}^{sm}$ be the second projection. Consider the object EW_k of $B_{W_k}^{sm}$ defined by the action of W_k on itself by multiplication, and let p^*EW_k be its pull-back in $Y_{et} \times_{B_{G_k}^{sm}} B_{W_k}^{sm}$. It is enough to show that the morphism

$$f/p^*EW_k : Y_W^{sm}/f^*p^*EW_k \longrightarrow (Y_{et} \times_{B_{G_k}^{sm}} B_{W_k}^{sm})/p^*EW_k$$

is an equivalence.
Recall that $Y_W^{sm} := \mathcal{S}_{et}(W_k, \overline{Y})$ is the topos of W_k-equivariant étale sheaves on $\overline{Y}$. The object $f^*p^*EW_k$ is represented by the W_k-equivariant étale $\overline{Y}$-scheme $\coprod_{W_k} \overline{Y} \to \overline{Y}$. One has the following equivalences

$$Y_W^{sm}/f^*p^*EW_k = \mathcal{S}(W_k, \overline{Y}_{et})/\coprod_{W_k} y\overline{Y} \cong \mathcal{S}(W_k, \coprod_{W_k} \overline{Y}_{et}) \cong \overline{Y}_{et}.$$

Consider now the localization $(Y_{et} \times_{B_{G_k}^{sm}} B_{W_k}^{sm})/p^*EW_k$. We have the following canonical equivalences:

$$(Y_{et} \times_{B_{G_k}^{sm}} B_{W_k}^{sm})/p^*EW_k \cong Y_{et} \times_{B_{G_k}^{sm}} B_{W_k}^{sm} \times_{B_{W_k}^{sm}} \underline{Set} \tag{4}$$

$$\cong Y_{et} \times_{B_{G_k}^{sm}} \underline{Set} \tag{5}$$

$$\cong \varprojlim (Y_{et} \times_{B_{G_{k'/k}}^{sm}} \underline{Set}) \tag{6}$$

$$\cong \varprojlim (Y_{et} \times_{B_{G_{k'/k}}^{sm}} (B_{G_{k'/k}}^{sm}/EG_{k'/k})) \tag{7}$$

$$\cong \varprojlim (Y_{et}/Y') \tag{8}$$

$$\cong \varprojlim Y'_{et} \tag{9}$$

$$\cong (\varprojlim Y')_{et} = \overline{Y}_{et} \tag{10}$$

Indeed, (4) follows from the canonical equivalence $B^{sm}_{W_k}/EW_k \cong \underline{Set}$. The inverse limit in (6) is taken over the Galois extensions k'/k. Using the natural equivalence

$$B^{sm}_{G_k} \cong \underleftarrow{lim}\, B^{sm}_{G(k'/k)}$$

(6) follow from the universal property of limits of topoi. For (7) we use again

$$B^{sm}_{G(k'/k)}/EG(k'/k) \cong \underline{Set}.$$

Then (8) follows from the fact that the inverse image of $EG(k'/k)$ in the étale topos Y_{et} is the sheaf represented by the étale Y-scheme $Y' := Y \times_k k'$. Then (9) is given by ([20] III Proposition 5.4), and (10) is given by ([33] Lemma 8.3), since the schemes Y' are all quasi-compact and quasi-separated. We obtain a commutative square

$$\begin{array}{ccc} \overline{Y}_{et} & \xrightarrow{Id} & \overline{Y}_{et} \\ \downarrow & & \downarrow \\ Y^{sm}_W/f^*p^*EW_k & \xrightarrow{f/p^*EW_k} & (Y_{et} \times_{B^{sm}_{G_k}} B^{sm}_{W_k})/p^*EW_k \end{array} \tag{11}$$

where the vertical maps are the equivalences defined above. It follows that f/p^*EW_k is an equivalence, and so is f by Lemma 2. □

Corollary 1. *There is a canonical equivalence*

$$Y_{et} \times_{B^{sm}_{G_k}} B_{W_k} \cong Y^{sm}_W \times \mathcal{T}$$

Proof. The Weil group W_k is a group of the final topos $\underline{Set}$. If $u : \mathcal{T} \to \underline{Set}$ denotes the unique map, then u^*W_k is the group object of $\mathcal{T}$ represented by the discrete group W_k. Hence one has (see the pull-back diagram (2)):

$$B^{sm}_{W_k} \times \mathcal{T} := B_{\underline{Set}}(W_k) \times \mathcal{T} \cong B_{\mathcal{T}}(yW_k) =: B_{W_k}.$$

The previous theorem therefore yields

$$Y_{et} \times_{B^{sm}_{G_k}} B_{W_k} \cong Y_{et} \times_{B^{sm}_{G_k}} B^{sm}_{W_k} \times \mathcal{T} \cong Y^{sm}_W \times \mathcal{T}.$$

□

Definition 3. *We define the* big Weil-étale topos *of* Y *as the fiber product*

$$Y_W := Y_{et} \times_{B^{sm}_{G_k}} B_{W_k} \cong Y^{sm}_W \times \mathcal{T}.$$

Corollary 2. *Let* $p_1 : Y_W \to Y^{sm}_W$ *and* $p_2 : Y_W \to \mathcal{T}$ *be the projections. Then for any abelian object* $\mathcal{A}'$ *of* Y_W*, one has*

$$H^n(Y_W, \mathcal{A}') \cong H^n(Y^{sm}_W, p_{1*}\mathcal{A}').$$

If $\mathcal{A}$ *is an abelian object of* $\mathcal{T}$*, then*

$$H^n(Y_W, p_2^*\mathcal{A}) \cong H^n(Y^{sm}_W, \mathcal{A}(*)).$$

Proof. This follows from Corollary 12, using the equivalence $Y_W \cong Y^{sm}_W \times \mathcal{T}$. □

Define the sheaf $\tilde{\mathbb{R}}$ on Y_W as $p_2^*(y\mathbb{R})$, where $y\mathbb{R}$ is the object of $\mathcal{T}$ represented by the standard topological group $\mathbb{R}$. Then we have canonical isomorphisms

$$H^n(Y_W, \tilde{\mathbb{R}}) \cong H^n(Y_W^{sm}, \mathbb{R}) \text{ and } H^n(Y_W, \mathbb{Z}) \cong H^n(Y_W^{sm}, \mathbb{Z})$$

as it follows from the previous corollary.

COROLLARY 3. *Let $\alpha : \mathcal{G} \to \mathcal{H}$ and $\beta : \mathcal{G}' \to \mathcal{H}$ be two homomorphisms of group objects in a topos $\mathcal{S}$. If α is an epimorphism then the natural morphism*

$$f : B_{\mathcal{G}\times_{\mathcal{H}}\mathcal{G}'} \longrightarrow B_{\mathcal{G}} \times_{B_{\mathcal{H}}} B_{\mathcal{G}'}$$

is an equivalence.

Proof. Let $e_{\mathcal{S}}$ be the final object in $\mathcal{S}$. The unique map $\mathcal{G}' \to e_{\mathcal{S}}$ is epimorphic, since the unit of $\mathcal{G}'$ yields a section $e_{\mathcal{S}} \to \mathcal{G}'$. Therefore, the morphism $E\mathcal{G}' \to e_{\mathcal{G}'}$ in $B_{\mathcal{G}'}$, where $e_{\mathcal{G}'}$ is the final object of $B_{\mathcal{G}'}$, is epimorphic. We denote the second projection by

$$p : B_{\mathcal{G}} \times_{B_{\mathcal{H}}} B_{\mathcal{G}'} \longrightarrow B_{\mathcal{G}'}$$

Let $\mathcal{K}$ be the kernel of α, so that $\mathcal{G}/\mathcal{K} \cong \mathcal{H}$. On the one hand, we have the following canonical equivalences:

$$\begin{aligned}
(B_{\mathcal{G}} \times_{B_{\mathcal{H}}} B_{\mathcal{G}'})/p^*E\mathcal{G}' &\cong B_{\mathcal{G}} \times_{B_{\mathcal{H}}} (B_{\mathcal{G}'}/E\mathcal{G}') \\
&\cong B_{\mathcal{G}} \times_{B_{\mathcal{H}}} \mathcal{S} \\
&\cong B_{\mathcal{G}} \times_{B_{\mathcal{H}}} (B_{\mathcal{H}}/E\mathcal{H}) \\
&\cong B_{\mathcal{G}}/\alpha^*E\mathcal{H} \\
&\cong B_{\mathcal{G}}/(\mathcal{G}/\mathcal{K}) \\
&\cong B_{\mathcal{K}}
\end{aligned}$$

Here $\mathcal{G}/\mathcal{K}$ is endowed with its natural $\mathcal{G}$-action. The second, the third and the last equivalences are given by ([20] IV.5.8), and the fourth equivalence is given by the pull-back diagram (1).

On the other hand, we have an exact sequence of group objects in $\mathcal{S}$

$$1 \to \mathcal{K} \to \mathcal{G} \times_{\mathcal{H}} \mathcal{G}' \to \mathcal{G}' \to 1.$$

Indeed, the kernel of $\mathcal{G} \times_{\mathcal{H}} \mathcal{G}' \to \mathcal{G}'$ is given by

$$\mathcal{G} \times_{\mathcal{H}} \mathcal{G}' \times_{\mathcal{G}'} e_{\mathcal{S}} = \mathcal{G} \times_{\mathcal{H}} e_{\mathcal{S}} = \mathcal{K}.$$

Moreover, $\mathcal{G} \times_{\mathcal{H}} \mathcal{G}' \to \mathcal{G}'$ is epimorphic, since epimorphisms are universal in a topos. We obtain

$$\begin{aligned}
B_{\mathcal{G}\times_{\mathcal{H}}\mathcal{G}'}/f^*p^*E\mathcal{G}' &= B_{\mathcal{G}\times_{\mathcal{H}}\mathcal{G}'}/(\mathcal{G} \times_{\mathcal{H}} \mathcal{G}'/\mathcal{K}) \\
&= B_{\mathcal{K}}
\end{aligned}$$

and we have a commutative square

$$\begin{array}{ccc}
B_{\mathcal{K}} & \xrightarrow{Id} & B_{\mathcal{K}} \\
\downarrow & & \downarrow \\
B_{\mathcal{G}\times_{\mathcal{H}}\mathcal{G}'}/f^*p^*E\mathcal{G}' & \xrightarrow{f/p^*E\mathcal{G}'} & (B_{\mathcal{G}} \times_{B_{\mathcal{H}}} B_{\mathcal{G}'})/p^*E\mathcal{G}'
\end{array} \tag{12}$$

where the vertical maps are the equivalences defined above. Hence $f/p^*E\mathcal{G}'$ is an equivalence. By Lemma 2, f is an equivalence as well, since $E\mathcal{G}' \to e_{\mathcal{G}'}$ is epimorphic. □

Corollary 4. *Let $\alpha : G \to H$ and $\beta : G' \to H$ be two morphisms of locally compact topological groups. If α has local sections then the natural morphism*

$$f : B_{G\times_H G'} \longrightarrow B_G \times_{B_H} B_{G'}$$

is an equivalence.

Proof. Since $\alpha : G \to H$ has local sections, the induced morphism $y(G) \to y(H)$ is an epimorphism in $\mathcal{T}$. Hence the result follows from Corollary 3. □

4. Artin-Verdier étale topos of an arithmetic scheme

Let $\mathcal{X}$ be a scheme separated and of finite type over $\mathrm{Spec}(\mathbb{Z})$. We denote by $\mathcal{X}^{an}$ the complex analytic variety associated to $\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{C}$, endowed with the standard complex topology. The Galois group $G_{\mathbb{R}}$ of $\mathbb{R}$ acts on $\mathcal{X}^{an}$. The quotient space $\mathcal{X}_\infty := \mathcal{X}^{an}/G_{\mathbb{R}}$ is endowed with the quotient topology. We consider the pair

$$\overline{\mathcal{X}} := (\mathcal{X}, \mathcal{X}_\infty).$$

As a set, $\overline{\mathcal{X}}$ is the disjoint union $\mathcal{X} \coprod \mathcal{X}_\infty$. The Zariski topology on $\overline{\mathcal{X}}$ is defined as follows. An open subset $(\mathcal{U}, D)$ of $\overline{\mathcal{X}}$ is given by a Zariski open subscheme $\mathcal{U} \subset \mathcal{X}$ and an open subspace $D \subset \mathcal{U}_\infty$ for the complex topology. We define the category $Et_{\overline{\mathcal{X}}}$ of étale $\overline{\mathcal{X}}$-schemes as follows. An étale $\overline{\mathcal{X}}$-scheme is an arrow $f : (\mathcal{U}, D) \to (\mathcal{X}, \mathcal{X}_\infty)$, where $\mathcal{U} \to \mathcal{X}$ is an étale morphism in the usual sense and D is an open subset of $\mathcal{U}_\infty$. The map $f_\infty : D \to \mathcal{X}_\infty$ is supposed to be unramified in the sense that $f_\infty(d) \in \mathcal{X}(\mathbb{R})$ if and only if $d \in D \cap \mathcal{U}(\mathbb{R})$. An étale $\overline{\mathcal{X}}$-scheme $\overline{\mathcal{U}}$ is said to be connected (respectively irreducible) if it is connected (respectively irreducible) as a topological space. A morphism $(\mathcal{U}, D) \to (\mathcal{U}', D')$ in the category $Et_{\overline{\mathcal{X}}}$ is given by a morphism of étale $\mathcal{X}$-schemes $\mathcal{U} \to \mathcal{U}'$ inducing a map $D \to D'$. The étale topology $\mathcal{J}_{et}$ on the category $Et_{\overline{\mathcal{X}}}$ is the topology generated by the pretopology for which a covering family is a surjective family. The Artin-Verdier étale site is left exact.

Definition 4. *The* Artin-Verdier étale topos *of $\overline{\mathcal{X}}$ is the category of sheaves of sets on the Artin-Verdier étale site:*

$$\overline{\mathcal{X}}_{et} := (\widetilde{Et_{\overline{\mathcal{X}}}, \mathcal{J}_{et}}).$$

Let $\mathcal{X}$ and $\mathcal{Y}$ be schemes which are separated and of finite type over $\mathrm{Spec}(\mathbb{Z})$. A map $f : \mathcal{X} \to \mathcal{Y}$ induces a map $\bar{f} : \overline{\mathcal{X}} \to \overline{\mathcal{Y}}$ (in the obvious sense) and a morphism of topoi $\bar{f}_{et} : \overline{\mathcal{X}}_{et} \to \overline{\mathcal{Y}}_{et}$. The object $y\mathcal{X} := y(\mathcal{X}, \emptyset)$ is a subobject of the final object $y\overline{\mathcal{X}}$ of $\overline{\mathcal{X}}_{et}$. This yields an open subtopos

$$\overline{\mathcal{X}}_{et}/y(\mathcal{X}, \emptyset) \hookrightarrow \overline{\mathcal{X}}_{et}.$$

We have the following canonical identifications (see [20] III Proposition 5.4):

$$\overline{\mathcal{X}}_{et}/y(\mathcal{X}, \emptyset) \cong (\widetilde{Et_{\overline{\mathcal{X}}}/_{(\mathcal{X},\emptyset)}, \mathcal{J}_{ind}}) \cong (\widetilde{Et_{\mathcal{X}}, \mathcal{J}_{et}}) = \mathcal{X}_{et}$$

where $\mathcal{X}_{et}$ is the usual étale topos of $\mathcal{X}$, and $\mathcal{J}_{ind}$ is the topology on $Et_{\overline{\mathcal{X}}}/(\mathcal{X},\emptyset)$ induced by $\mathcal{J}_{et}$ on $Et_{\overline{\mathcal{X}}}$ via the forgetful functor $Et_{\overline{\mathcal{X}}}/(\mathcal{X},\emptyset)\to Et_{\overline{\mathcal{X}}}$. We thus obtain an open embedding

$$\varphi:\mathcal{X}_{et}\hookrightarrow\overline{\mathcal{X}}_{et}.$$

Let $Sh(\mathcal{X}_\infty)$ be the topos of sheaves of sets on the topological space $\mathcal{X}_\infty$, i.e. the category of étalé spaces on $\mathcal{X}_\infty$. We consider $Sh(\mathcal{X}_\infty)$ as a site endowed with the canonical topology $\mathcal{J}_{can}$. There is a morphism of left exact sites

$$\begin{array}{cccc} u_\infty^*: & (Et_{\overline{\mathcal{X}}},\mathcal{J}_{et}) & \longrightarrow & (Sh(\mathcal{X}_\infty),\mathcal{J}_{can}) \\ & (\mathcal{U},D) & \longmapsto & D\to\mathcal{X}_\infty \end{array}$$

The resulting morphism of topoi

$$u_\infty:Sh(\mathcal{X}_\infty)\longrightarrow\overline{\mathcal{X}}_{et}$$

is precisely the closed complement of the open subtopos $\mathcal{X}_{et}\hookrightarrow\overline{\mathcal{X}}_{et}$ defined above, i.e. we have the following result.

Proposition 4.1. *There is an open-closed decomposition of topoi*

$$\varphi:\mathcal{X}_{et}\longrightarrow\overline{\mathcal{X}}_{et}\longleftarrow Sh(\mathcal{X}_\infty):u_\infty$$

The gluing functor $u_\infty^*\phi_*$ can be made more explicit as follows. There is a canonical morphism of topoi

$$\alpha:Sh(G_{\mathbb{R}},\mathcal{X}^{an})\longrightarrow\mathcal{X}_{et}$$

where $Sh(G_{\mathbb{R}},\mathcal{X}^{an})$ is the topos of $G_{\mathbb{R}}$-equivariant sheaves on the topological space $\mathcal{X}^{an}$, i.e. the category of $G_{\mathbb{R}}$-equivariant étalé spaces on $\mathcal{X}^{an}$. The map α is defined by the morphism of left exact sites which takes an étale $\mathcal{X}$-scheme $\mathcal{U}$ to the $G_{\mathbb{R}}$-equivariant étalé space $\mathcal{U}^{an}$ over $\mathcal{X}^{an}$ (note that $\mathcal{U}^{an}\to\mathcal{X}^{an}$ is a $G_{\mathbb{R}}$-equivariant local homeomorphism since the morphism $\mathcal{U}\otimes_{\mathbb{Z}}\mathbb{C}\to\mathcal{X}\otimes_{\mathbb{Z}}\mathbb{C}$ is étale and compatible with complex conjugation).
The quotient map $\mathcal{X}^{an}\to\mathcal{X}^{an}/G_{\mathbb{R}}$ yields another morphism of topoi

$$(\pi^*,\pi_*^{G_{\mathbb{R}}}):Sh(G_{\mathbb{R}},\mathcal{X}^{an})\longrightarrow Sh(\mathcal{X}_\infty).$$

Here $\pi:\mathcal{X}^{an}\to\mathcal{X}_\infty$ is the quotient map, π^* is the usual inverse image and $\pi_*^{G_{\mathbb{R}}}\mathcal{F}$ is the $G_{\mathbb{R}}$-invariant subsheaf of the the direct image $\pi_*\mathcal{F}$, i.e. for any open $U\subset\mathcal{X}_\infty$ one has

$$\pi_*^{G_{\mathbb{R}}}\mathcal{F}(U):=\mathcal{F}(\pi^{-1}U)^{G_{\mathbb{R}}}.$$

Then we have an identification of functors

$$u_\infty^*\varphi_*\cong\pi_*^{G_{\mathbb{R}}}\alpha^*:\mathcal{X}_{et}\longrightarrow Sh(\mathcal{X}_\infty)$$

Let us consider the category $(Sh(\mathcal{X}_\infty),\mathcal{X}_{et},\pi_*^{G_{\mathbb{R}}}\alpha^*)$ defined in ([20] IV.9.5.1) by Artin gluing. Recall that an object of this category is a triple (F,E,σ), where F is an object of $Sh(\mathcal{X}_\infty)$, E is an object of $\mathcal{X}_{et}$ and σ is a map $\sigma:F\to\pi_*^{G_{\mathbb{R}}}\alpha^*E$.

Corollary 5. *The category $\overline{\mathcal{X}}_{et}$ is canonically equivalent to $(Sh(\mathcal{X}_\infty),\mathcal{X}_{et},\pi_*^{G_{\mathbb{R}}}\alpha^*)$.*

Proof. There is a canonical functor

$$\begin{array}{rccc} \Phi: & \overline{\mathcal{X}}_{et} & \longrightarrow & (Sh(\mathcal{X}_\infty), \mathcal{X}_{et}, u_\infty^*\varphi_*) \\ & \mathcal{F} & \longmapsto & (u_\infty^*\mathcal{F}, \varphi^*\mathcal{F}, \sigma) \end{array}$$

where the morphism

$$\sigma : u_\infty^*\mathcal{F} \longrightarrow u_\infty^*\varphi_*(\varphi^*\mathcal{F})$$

is induced by the adjunction transformation $Id \to \varphi_*\varphi^*$. By ([20] IV.9.5.4.a) the functor Φ is an equivalence of categories, since $u_\infty : Sh(\mathcal{X}_\infty) \hookrightarrow \overline{\mathcal{X}}_{et}$ is the closed complement of the open embedding $\phi : \mathcal{X}_{et} \hookrightarrow \overline{\mathcal{X}}_{et}$. Hence the result follows from the isomorphism

$$u_\infty^*\varphi_* \cong \pi_*^{G_{\mathbb{R}}}\alpha^*.$$

□

COROLLARY 6. *We denote by ∞ the archimedean place of $\mathbb{Q}$. The commutative square*

$$(13)\qquad \begin{array}{ccc} Sh(\mathcal{X}_\infty) & \longrightarrow & Sh(\infty) \\ \downarrow & & \downarrow \\ \overline{\mathcal{X}}_{et} & \xrightarrow{f} & \overline{\mathrm{Spec}(\mathbb{Z})}_{et} \end{array}$$

is a pull-back, where $Sh(\infty) = \underline{Set}$ is the category of sheaves on the one point space.

Proof. The map $\overline{\mathcal{X}} \to \overline{\mathrm{Spec}(\mathbb{Z})}$ induces a morphism of étale topos f. Consider the open embedding $\mathrm{Spec}(\mathbb{Z})_{et} \hookrightarrow \overline{\mathrm{Spec}(\mathbb{Z})}_{et}$. Its inverse image under the map f is $\mathcal{X}_{et} \hookrightarrow \overline{\mathcal{X}}_{et}$. The result therefore follows from Proposition 4.1 and ([20] IV Corollaire 9.4.3). □

PROPOSITION 4.2. *For any prime number p, we have a pull-back*

$$(14)\qquad \begin{array}{ccc} (\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{F}_p)_{et} & \longrightarrow & \mathrm{Spec}(\mathbb{F}_p)_{et} \\ \downarrow & & \downarrow \\ \overline{\mathcal{X}}_{et} & \xrightarrow{f} & \overline{\mathrm{Spec}(\mathbb{Z})}_{et} \end{array}$$

Proof. The morphism $\mathrm{Spec}(\mathbb{F}_p)_{et} \to \overline{\mathrm{Spec}(\mathbb{Z})}_{et}$ factors through $\mathrm{Spec}(\mathbb{Z})_{et}$, hence one is reduced to show that

$$(\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{F}_p)_{et} \cong \mathcal{X}_{et} \times_{\mathrm{Spec}(\mathbb{Z})_{et}} \mathrm{Spec}(\mathbb{F}_p)_{et}.$$

This follows from ([20] IV Corollaire 9.4.3) since $\mathrm{Spec}(\mathbb{F}_p) \to \mathrm{Spec}(\mathbb{Z})$ is a closed embedding. □

4.1. ÉTALE COHOMOLOGY WITH COMPACT SUPPORT. It follows from Corollary 5 that we have the usual sequences of adjoint functors (see [20] IV.14)

$$\varphi_!,\ \varphi^*,\ \varphi_* \text{ and } u_\infty^*,\ u_{\infty*},\ u_\infty^!.$$

between the categories of abelian sheaves on $\overline{\mathcal{X}}_{et}$, $\mathcal{X}_{et}$ and $Sh(\mathcal{X}_\infty)$. In particular $u_{\infty*}$ is exact and φ^* preserves injective objects since $\varphi_!$ is exact. For any abelian sheaf $\mathcal{A}$ on $\overline{\mathcal{X}}_{et}$, one has the exact sequence

$$0 \to \varphi_!\varphi^*\mathcal{A} \to \mathcal{A} \to u_{\infty*}u_\infty^*\mathcal{A} \to 0, \tag{15}$$

where the morphisms are given by adjunction.

DEFINITION 5. *Assume that $\mathcal{X}$ is proper over* Spec($\mathbb{Z}$) *and let $\mathcal{A}$ be an abelian sheaf on $\mathcal{X}_{et}$. The étale cohomology with compact support is defined by*

$$H^n_c(\mathcal{X}_{et},\mathcal{A}) := H^n(\overline{\mathcal{X}}_{et},\varphi_!\mathcal{A}).$$

PROPOSITION 4.3. *Let $\mathcal{X}$ be a flat proper scheme over* Spec($\mathbb{Z}$). *Assume that $\mathcal{X}$ is normal and connected. Then the $\mathbb{R}$-vector space $H^n_c(\mathcal{X}_{et},\mathbb{R})$ is finite dimensional, zero for n large, and we have*

$$\begin{aligned} H^n_c(\mathcal{X}_{et},\mathbb{R}) &= 0 \textit{ for } n=0\\ &= H^0(\mathcal{X}_\infty,\mathbb{R})/\mathbb{R} \textit{ for } n=1\\ &= H^{n-1}(\mathcal{X}_\infty,\mathbb{R}) \textit{ for } n\geq 2 \end{aligned}$$

Proof. The exact sequence

$$0\to\varphi_!\mathbb{R}\to\mathbb{R}\to u_{\infty*}\mathbb{R}\to 0$$

and the fact that $u_{\infty*}$ is exact give a long exact sequence

$$0\to H^0_c(\mathcal{X}_{et},\mathbb{R})\to H^0(\overline{\mathcal{X}}_{et},\mathbb{R})\to H^0(\mathcal{X}_\infty,\mathbb{R})\to H^1_c(\mathcal{X}_{et},\mathbb{R})\to H^1(\overline{\mathcal{X}}_{et},\mathbb{R})\to$$

The inclusion of the generic point of $\mathcal{X}$ yields a morphism of topoi

$$\eta : (\text{Spec } K(\mathcal{X}))_{et} \longrightarrow \overline{\mathcal{X}}_{et}.$$

We have immediately $R^n\eta_*\mathbb{R}=0$ for any $n\geq 1$ since Galois cohomology is torsion and $\mathbb{R}$ is uniquely divisible. Moreover, we have $\eta_*\mathbb{R}=\mathbb{R}$. Indeed, the scheme $\mathcal{X}$ is normal hence the set of connected components of an étale $\overline{\mathcal{X}}$-scheme $\overline{\mathcal{U}}$ is in 1-1 correspondence with the set of connected components of $\mathcal{U}\times_{\mathcal{X}}\text{Spec } K(\mathcal{X})$, i.e. one has

$$\pi_0(\mathcal{U}\times_{\mathcal{X}}\text{Spec } K(\mathcal{X}))=\pi_0(\mathcal{U})=\pi_0(\overline{\mathcal{U}}).$$

Therefore the Leray spectral sequence associated to the morphism η gives

$$H^n(\overline{\mathcal{X}}_{et},\mathbb{R})=H^n(G_{K(\mathcal{X})},\mathbb{R}).$$

We obtain $H^0(\overline{\mathcal{X}}_{et},\mathbb{R})=\mathbb{R}$ and $H^n(\overline{\mathcal{X}}_{et},\mathbb{R})=0$ for $n\geq 1$, and the result follows. $\square$

5. The definition of $\overline{\mathrm{Spec}(\mathcal{O}_F)}_W$

Let F be a number field. We consider the Arakelov compactification $\bar{X} = (\mathrm{Spec}\ \mathcal{O}_F, X_\infty)$ of $X = \mathrm{Spec}\ \mathcal{O}_F$, where X_∞ is the finite set of archimedean places of F. Note that this is a special case of the previous section, since X_∞ is the quotient of $X \otimes \mathbb{C}$ by complex conjugation. We endow $\bar{X}$ with the Zariski topology described previously.
If $\bar{F}/F$ is an algebraic closure and $\bar{F}/K/F$ a finite Galois extension then the relative Weil group $W_{K/F}$ is defined by the extension of topological groups

$$1 \to C_K \to W_{K/F} \to G_{K/F} \to 1$$

corresponding to the fundamental class in $H^2(G_{K/F}, C_K)$ given by class field theory, where C_K is the idèle class group of K. A Weil group of F is then defined as the projective limit $W_F := \underleftarrow{lim}\, W_{K/F}$, computed in the category of topological groups. Alternatively, let $\bar{F}/K/F$ be a finite Galois extension and let S be a finite set of places of F containing all the places which ramify in K. Then the fundamental class in

$$H^2(G_{K/F}, C_K) \cong H^2(G_{K/F}, C_{K,S})$$

yields a group extension

$$1 \to C_{K,S} \to W_{K/F,S} \to G_{K/F} \to 1$$

where $C_{K,S}$ is the S-idèle class group of K. Then one has (see [30])

$$W_F := \underleftarrow{lim}\, W_{K/F} = \underleftarrow{lim}\, W_{K/F,S},$$

where K runs over the finite Galois extensions $\bar{F}/K/F$ and S runs over the finite sets of places of F containing all the places which ramify in K.
However, the structure of the Weil-?tale topos at the generic point suggests to consider the projective system of topological groups $W_{K/F,S}$ as a strict topological pro-group $\underline{W}_{K/F,S}$ (see Sect. 2.3.4).

5.1. The Weil-étale topos.

We choose an algebraic closure $\bar{F}/F$ and a Weil group W_F. For any place v of F, we choose an algebraic closure $\bar{F_v}/F_v$ and an embedding $\bar{F} \to \bar{F}_v$ over F. Then we choose a local Weil group W_{F_v} and a Weil map $\theta_v : W_{F_v} \to W_F$ compatible with $\bar{F} \to \bar{F}_v$.
Let $W^1_{F_v}$ be the maximal compact subgroup of W_{F_v}. For any closed point $v \in \bar{X}$ (ultrametric or archimedean), we define the Weil group of "the residue field at v" as follows

$$W_{k(v)} := W_{F_v}/W^1_{F_v},$$

while the Galois group of the residue field at v can be defined as $G_{k(v)} := G_{F_v}/I_v$. Note that $G_{k(v)}$ is the trivial group for v archimedean. For any v, the Weil map $W_{F_v} \to G_{F_v}$ chosen above induces a morphism $W_{k(v)} \to G_{k(v)}$. Finally, we denote by

$$q_v : W_{F_v} \longrightarrow W_{F_v}/W^1_{F_v} =: W_{k(v)}$$

the map from the local Weil group W_{F_v} to the Weil group of the residue field at $v \in \bar{X}$.

DEFINITION 6. *Let $T_{\bar{X}}$ be the category of objects (Z_0, Z_v, f_v) defined as follows. The topological space Z_0 is endowed with a continuous W_F-action. For any place v of F, Z_v is a topological space endowed with a continuous $W_{k(v)}$-action. The continuous map $f_v : Z_v \to Z_0$ is W_{F_v}-equivariant, when Z_v and Z_0 are seen as W_{F_v}-spaces via the maps $\theta_v : W_{F_v} \to W_F$ and $q_v : W_{F_v} \to W_{k(v)}$. Moreover, we require the following facts.*

- *The spaces Z_v are* locally compact.
- *The map f_v is an* homeomorphism for almost all places v of F and a continuous injective map for all places.
- *The action of W_F on Z_0 factors through $W_{K/F}$, for some finite Galois subextension $\bar{F}/K/F$.*

A morphism
$$\phi : (Z_0, Z_v, f_v) \to (Z'_0, Z'_v, f'_v)$$
in the category $T_{\bar{X}}$ is a continuous W_F-equivariant map $\phi : Z_0 \to Z'_0$ inducing *a continuous map $\phi_v : Z_v \to Z'_v$ for any place v. Then ϕ_v is $W_{k(v)}$-equivariant. The category $T_{\bar{X}}$ is endowed with the local section topology $\mathcal{J}_{ls}$, i.e. the topology generated by the pretopology for which a family*
$$\{\varphi_i : (Z_{i,0}, Z_{i,v}, f_{i,v}) \to (Z_0, Z_v, f_v),\ i \in I\}$$
is a covering family if $\coprod_{i\in I} Z_{i,v} \to Z_v$ has local continuous sections, for any place v.

LEMMA 3. *The site $(T_{\bar{X}}, \mathcal{J}_{ls})$ is left exact.*

Proof. The category $T_{\bar{X}}$ has fiber products and a final object, hence finite projective limits are representable in $T_{\bar{X}}$. It remains to show that $\mathcal{J}_{ls}$ is subcanonical. This follows easily from the fact that, for any topological group G, the local section topology $\mathcal{J}_{ls}$ on $B_{Top}G$ coincides with the open cover topology $\mathcal{J}_{op}$, which is subcanonical. □

DEFINITION 7. *We define the* Weil-étale topos $\bar{X}_W$ *as the topos of sheaves of sets on the site defined above:*
$$\bar{X}_W := (\widetilde{T_{\bar{X}}, \mathcal{J}_{ls}}).$$

PROPOSITION 5.1. *We have a morphism of topoi*
$$j : B_{W_F} \longrightarrow \bar{X}_W.$$

Proof. By [15] Corollary 2, the site $(B_{Top}W_F, \mathcal{J}_{ls})$ is a site for the classifying topos B_{W_F}. By [15] Corollary 2, the site $(B_{Top}W_F, \mathcal{J}_{ls})$ is a site for B_{W_F}. The morphism of left exact sites
$$\begin{array}{cccc} j^* : & (T_{\bar{X}}, \mathcal{J}_{ls}) & \longrightarrow & (B_{Top}W_F, \mathcal{J}_{ls}) \\ & (Z_0, Z_v, f_v) & \longmapsto & Z_0 \end{array}$$
induces the morphism of topoi j. □

Proposition 5.2. *The morphism of topoi* $j : B_{W_F} \to \bar{X}_W$ *factors through the classifying topos*

$$B_{\underline{W}_{K/F,S}} := \varprojlim B_{W_{K/F,S}}$$

of the strict topological pro-group $\underline{W}_{K/F,S}$ *(see Sect. 2.3.4). The induced morphism* $i_0 : B_{\underline{W}_{K/F,S}} \to \bar{X}_W$ *is an embedding.*

Proof. Let (Z_0, Z_v, f_v) be an object of $T_{\bar{X}}$. The action of W_F on Z_0 factors through $W_{K/F}$, for some finite Galois sub-extension $\bar{F}/K/F$. Since $W_{K/F}$ and Z_0 are both locally compact, this action is given by a continuous morphism

$$\rho : W_{K/F} \longrightarrow Aut(Z_0)$$

where $Aut(Z_0)$ is the homeomorphism group of Z_0 endowed with the compact-open topology. The kernel of ρ is a closed normal subgroup of $W_{K/F}$ since $Aut(Z_0)$ is Hausdorff. Moreover, there exists an open subset V of $\bar{X}$ such that $f_v : Z_v \to Z_0$ is an isomorphism of W_{F_v}-spaces for any $v \in V$. Let $\widetilde{W}^1_{F_v}$ denote the image of the continuous morphism

$$W^1_{F_v} \longrightarrow W_{F_v} \longrightarrow W_{K/F},$$

endowed with the induced topology. Then $\widetilde{W}^1_{F_v}$ is in the kernel of ρ for any $v \in V$. Let N_V be the closed normal subgroup of $W_{K/F}$ generated by the subgroups $\widetilde{W}^1_{F_v}$ for any $v \in V$. Then ρ induces a continuous morphism

$$W_{K/F}/N_V \longrightarrow Aut(Z_0).$$

We choose V small enough so that K/F is unramified above V and we set $S := \bar{X} - V$. Then we have

$$N_V = \prod_{w|v,\, v\in V} \mathcal{O}^{\times}_{K_w} \subseteq C_K \subseteq W_{K/F} \text{ and } W_{K/F}/N_V = W_{K/F,S}.$$

Hence the action of W_F on Z_0 factors through $W_{K/F,S}$, for some finite Galois sub-extension $\bar{F}/K/F$ and some finite set S of places of F containing all the places which ramify in K. The morphism of left exact sites

$$\begin{array}{rccc} j^* : & (T_{\bar{X}}, \mathcal{J}_{ls}) & \longrightarrow & (B_{Top}W_F, \mathcal{J}_{ls}) \\ & (Z_0, Z_v, f_v) & \longmapsto & Z_0 \end{array}.$$

therefore induces a morphism

$$\begin{array}{rccc} i_0^* : & (T_{\bar{X}}, \mathcal{J}_{ls}) & \longrightarrow & (\varinjlim B_{Top}W_{K/F,S}, \mathcal{J}_{ls}) \\ & (Z_0, Z_v, f_v) & \longmapsto & Z_0 \end{array}$$

where $(\varinjlim B_{Top}W_{K/F,S}, \mathcal{J}_{ls})$ is the direct limit site. More precisely, $\varinjlim B_{Top}W_{K/F,S}$ is the direct limit category endowed with the coarsest topology $\mathcal{J}$ such that the functors $B_{Top}W_{K/F,S} \to \varinjlim B_{Top}W_{K/F,S}$ are all continuous, when $B_{Top}W_{K/F,S}$ is endowed with the local section topology. One can identify $\varinjlim B_{Top}W_{K/F,S}$ with a full subcategory of $B_{Top}W_F$ and $\mathcal{J}$ with the local section topology $\mathcal{J}_{ls}$. By ([20] VI.8.2.3), the direct limit site

$(\underrightarrow{lim}\, B_{Top}W_{K/F,S}, \mathcal{J}_{ls})$ is a site for the projective limit topos $B_{\underline{W}_{K/F,S}}$. We obtain a morphism of topoi

$$i_0 : B_{\underline{W}_{K/F,S}} \longrightarrow \bar{X}_W.$$

It remains to show that this morphism is an embedding. Let $\mathcal{F}$ be an object of $B_{\underline{W}_{K/F,S}}$. Then $i_0^* i_{0*}\mathcal{F}$ is the sheaf associated with the presheaf

$$\begin{array}{rccc} i_0^p i_{0*}\mathcal{F}: & \underrightarrow{lim}\, B_{Top}W_{K/F,S} & \longrightarrow & \underline{Set} \\ & Z & \longmapsto & \varinjlim_{Z\to i_0^*(Y_0,Y_v,f_v)} i_{0*}\mathcal{F}(Y_0,Y_v,f_v) \end{array}$$

where the direct limit is taken over the category of arrows $Z \to i_0^*(Y_0, Y_v, f_v)$. For any object Z of $\underrightarrow{lim}\, B_{Top}W_{K/F,S}$, there exist a finite Galois extension K_Z/F and a finite set S_Z such that Z is an object of $B_{Top}W_{K_Z/F,S_Z}$. Consider the cofinal subcategory I_Z of the category of arrows defined above, where I_Z consists of the following objects. For any finite set S of places of F such that $S_Z \subseteq S$, we consider the map $Z \to i_0^*(Z_0, Z_v, f_v)$ with $Z_0 = Z$ as a W_F-space, $Z_v = Z$ as a $W_{k(v)}$-space for any place v not in S and $Z_v = \emptyset$ for any $v \in S$. We thus have

$$\varinjlim_{Z\to i_0^*(Y_0,Y_v,f_v)} i_{0*}\mathcal{F}(Y_0,Y_v,f_v) = \varinjlim_{I_Z} i_{0*}\mathcal{F}(Z_0,Z_v,f_v) = \mathcal{F}(Z).$$

Hence $i_0^p i_{0*}\mathcal{F}$ is already a sheaf and we have

$$i_0^* i_{0*}\mathcal{F} = i_0^p i_{0*}\mathcal{F} = \mathcal{F}.$$

This shows that i_{0*} is fully faithful, i.e. i_0 is an embedding.

□

PROPOSITION 5.3. *There is canonical morphism of topoi*

$$\mathfrak{f} : \bar{X}_W \longrightarrow B_{\mathbb{R}}.$$

Proof. We have a commutative diagram of topological groups

$$\begin{array}{ccc} W_{F_v} & \longrightarrow & W_{k(v)} \\ \downarrow & & \downarrow \\ W_F & \longrightarrow & \mathbb{R} \end{array} \tag{16}$$

where $W_F \to \mathbb{R}$ is defined as the composition

$$W_F \to W_F^{ab} \cong C_F \to \mathbb{R}_+^\times \cong \mathbb{R}.$$

Hence there is a morphism of left exact sites

$$\begin{array}{rccc} \mathfrak{f}^*: & (B_{Top}\mathbb{R}, \mathcal{J}_{ls}) & \longrightarrow & (T_{\bar{X}}, \mathcal{J}_{ls}) \\ & Z & \longmapsto & (Z, Z, Id_Z) \end{array} \tag{17}$$

where Z is seen as W_F-space (respectively a $W_{k(v)}$-space) via the canonical morphism $W_F \to \mathbb{R}$ (respectively via $W_{k(v)} \to \mathbb{R}$). The result follows. □

5.2. The morphism from the Weil-étale topos to the Artin-Verdier étale topos. Let $\bar{X}$ be the Arakelov compactification of the number ring $\mathcal{O}_F$. We consider below the Artin-Verdier étale site $(Et_{\bar{X}};\mathcal{J}_{et})$ and the Artin-Verdier étale topos $\bar{X}_{et}$ of the arithmetic curve $\bar{X}$.

Proposition 5.4. *There exists a morphism of left exact sites*

$$\begin{array}{cccc} \gamma^*: & (Et_{\bar{X}};\mathcal{J}_{et}) & \longrightarrow & (T_{\bar{X}};\mathcal{J}_{ls}) \\ & \bar{U} & \longmapsto & (U_0,U_v,f_v) \end{array}.$$

The underlying functor γ^* is fully faithful and its essential image consists exactly of objects (U_0,U_v,f_v) of $T_{\bar{X}}$ where U_0 is a finite W_F-set.

This result is a reformulation of [33] Proposition 4.61 and [33] Proposition 4.62. We give below a sketch of the proof.

Proof. For any étale $\bar{X}$-scheme $\bar{U}$, we define an object $\gamma^*(\bar{U})=(U_0,U_v,f_v)$ of $T_{\bar{X}}$ as follows. The scheme $\bar{U}\times_{\bar{X}}\mathrm{Spec}\,F$ is the spectrum of an étale F-algebra and the Grothendieck-Galois theory shows that this F-algebra is uniquely determined by the finite G_F-set

$$U_0:=Hom_{\mathrm{Spec}\,F}(\mathrm{Spec}\,\bar{F},\bar{U}\times_{\bar{X}}\mathrm{Spec}\,F)=Hom_{\bar{X}}(\mathrm{Spec}\,\overline{F},\bar{U}).$$

Let v be an ultrametric place of F. The maximal unramified sub-extension of the algebraic closure $\bar{F}_v/F_v$ yields an algebraic closure of the residue field $\overline{k(v)}/k(v)$. The scheme $\bar{U}\times_{\bar{X}}\mathrm{Spec}\,k(v)$ is the spectrum of an étale $k(v)$-algebra, corresponding to the finite $G_{k(v)}$-set

$$U_v:=Hom_{\mathrm{Spec}\,k(v)}(\mathrm{Spec}\,\overline{k(v)},\bar{U}\times_{\bar{X}}\mathrm{Spec}\,k(v))=Hom_{\bar{X}}(\mathrm{Spec}\,\overline{k(v)},\bar{U})$$

The chosen F-embedding $\bar{F}\to\bar{F}_v$ induces a G_{F_v}-equivariant map

$$f_v:U_v\longrightarrow U_0.$$

Consider now an archimedean place v of F. Define

$$U_v:=Hom_{\bar{X}}(v,\bar{U})=\bar{U}\times_{\bar{X}}v$$

where the map $v\to\bar{X}$ is the closed embedding corresponding to the archimedean place v of F. As above, the F-embedding $\bar{F}\to\bar{F}_v$ induces a G_{F_v}-equivariant map

$$f_v:U_v\longrightarrow U_0.$$

For any place v of F, the set U_v is viewed as a $W_{k(v)}$-topological space via the morphism $W_{k(v)}\to G_{k(v)}$. Respectively, U_0 is viewed as a W_F-topological space via $W_F\to G_F$. Then the map f_v defined above is W_{F_v}-equivariant. We check that the map f_v is bijective for almost all valuations and injective for all valuations (see [33] Proposition 4.62). We obtain a functor

$$\gamma^*:Et_{\bar{X}}\longrightarrow T_{\bar{X}}.$$

This functor is left exact by construction (i.e. it preserves the final objects and fiber product) and continuous (i.e. it preserves covering families) since a surjective map of discrete sets is a local section cover. The last claim of the proposition follows from Galois theory. □

COROLLARY 7. *There is a morphism of topoi* $\gamma : \bar{X}_W \to \bar{X}_{et}$.

Proof. This follows from the fact that a morphism of left exact sites induces a morphism of topoi. □

DEFINITION 8. *Let* X' *be an open subscheme of* $\bar{X}$. *We define*

$$X'_W := \bar{X}_W / \gamma^*(yX').$$

5.3. STRUCTURE OF $\bar{X}_W$ AT THE CLOSED POINTS. Let v be a place of F. We consider the Weil group $W_{k(v)}$ and the Galois group $G_{k(v)}$ of the residue field $k(v)$ at $v \in \bar{X}$. Note that for v archimedean one has $W_{k(v)} \cong \mathbb{R}$ and $G_{k(v)} = \{1\}$. Consider the big classifying topos $B_{W_{k(v)}}$, i.e. the category of $y(W_{k(v)})$-objects in $\mathcal{T}$. We consider also the small classifying topos $B^{sm}_{G_{k(v)}}$, which is defined as the category of continuous $G_{k(v)}$-sets. The category of locally compact $W_{k(v)}$-spaces $B_{Top}W_{k(v)}$ is endowed with the local section topology $\mathcal{J}_{ls}$. Recall that the site $(B_{Top}W_{k(v)}, \mathcal{J}_{ls})$ is a site for the classifying topos $B_{W_{k(v)}}$. We denote by $B_{fSets}G_{k(v)}$ the category of finite $G_{k(v)}$-sets endowed with the canonical topology $\mathcal{J}_{can}$. The site $(B_{fSets}G_{k(v)}, \mathcal{J}_{can})$ is a site for the small classifying topos $B^{sm}_{G_{k(v)}}$.
For any place v of F, we have a morphism of left exact sites

$$\begin{array}{rccc} i_v^* : & (T_{\bar{X}}, \mathcal{J}_{ls}) & \longrightarrow & (B_{Top}W_{k(v)}, \mathcal{J}_{ls}) \\ & (Z_0, Z_v, f_v) & \longmapsto & Z_v \end{array}$$

hence a morphism of topoi

$$i_v : B_{W_{k(v)}} \longrightarrow \bar{X}_W.$$

On the other hand one has morphism of topoi

$$u_v : B^{sm}_{G_{k(v)}} \longrightarrow \bar{X}_{et}$$

for any closed point v of $\bar{X}$. For v ultrametric, this morphism is induced by the closed embedding of schemes

$$\mathrm{Spec}\ k(v) \longrightarrow \bar{X}$$

since the étale topos of Spec $k(v)$ is equivalent to the category $B^{sm}_{G_{k(v)}}$ of continuous $G_{k(v)}$-sets. Note that this equivalence is induced by the choice of an algebraic closure of $k(v)$ made at the beginning of section 5.1. By Corollary 4.1, there is a closed embedding

$$Sh(X_\infty) = \coprod_{X_\infty} \underline{Set} \longrightarrow \bar{X}_{et}$$

which yields the closed embedding

$$u_v : B^{sm}_{G_{k(v)}} = \underline{Set} \longrightarrow \bar{X}_{et}$$

for any archimedean valuation v of F. In both cases, we have a commutative diagram of left exact sites

$$\begin{tikzcd}
(B_{Top}W_{k(v)}, \mathcal{J}_{ls}) & (B_{fSets}G_{k(v)}, \mathcal{J}_{can}) \arrow[l, "\alpha_v^*"'] \\
(T_{\bar{X}}, \mathcal{J}_{ls}) \arrow[u, "i_v^*"] & (Et_{\bar{X}}, \mathcal{J}_{et}) \arrow[l, "\gamma^*"'] \arrow[u, "u_v^*"']
\end{tikzcd}$$

where $u_v^*(\bar{U})$ is the finite $G_{k(v)}$-set $Hom_{\bar{X}}(\mathrm{Spec}\,\overline{k(v)}, \bar{U})$ (respectively $Hom_{\bar{X}}(v, \bar{U})$) for v ultrametric (respectively archimedean). Moreover, α_v^* maps a finite $G_{k(v)}$-set E to the discrete topological space E on which $W_{k(v)}$ acts via $W_{k(v)} \to G_{k(v)}$. This commutative diagram of sites induces a commutative diagram of topoi.

Theorem 5.1. *For any closed point v of $\bar{X}$, the following diagram is a pull-back of topoi.*

$$\begin{tikzcd}
B_{W_{k(v)}} \arrow[r, "\alpha_v"] \arrow[d, "i_v"] & B^{sm}_{G_{k(v)}} \arrow[d, "u_v"] \\
\bar{X}_W \arrow[r, "\gamma"] & \bar{X}_{et}
\end{tikzcd}$$

In particular, the morphism i_v is a closed embedding.

Proof. We first prove a partial result.

Lemma 4. *The morphism i_v is an embedding, i.e. i_{v*} is fully faithful.*

Proof. We use below the fact that the *full* subcategory

$$W_{k(v)} \times Top \hookrightarrow B_{Top}W_{k(v)}$$

is a topologically generating subcategory of the site $(B_{Top}W_{k(v)}, \mathcal{J}_{ls})$. Here $W_{k(v)} \times Top$ consists in locally compact topological spaces of the form $Z = W_{k(v)} \times T$ on which $W_{k(v)}$ acts by left multiplication on the first factor. In particular, a sheaf $\mathcal{F}$ of

$$B_{W_{k(v)}} = (\widetilde{B_{Top}W_{k(v)}, \mathcal{J}_{ls}})$$

is completely determined by its values $\mathcal{F}(W_{k(v)} \times T)$ on objects of $W_{k(v)} \times Top$. Let $\mathcal{F}$ be an object of $B_{W_{k(v)}}$. Consider the adjunction map

$$i_v^* \circ i_{v*}\mathcal{F} \longrightarrow \mathcal{F}. \tag{18}$$

The sheaf $i_v^* \circ i_{v*}\mathcal{F}$ is the sheaf on $(B_{Top}W_{k(v)}, \mathcal{J}_{ls})$ associated to the presheaf

$$Z \to \lim_{Z \to i_v^*(Y_0, Y_w, f_w)} i_{v*}\mathcal{F}(Y_0, Y_w, f_w) = \lim_{Z \to i_v^*(Y_0, Y_w, f_w)} \mathcal{F}(Y_v)$$

where the direct limit is taken over the category of arrows $Z \to i_v^*(Y_0, Y_w, f_w)$ with (Y_0, Y_w, f_w) an object of $T_{\bar{X}}$.

Let $\bar{F}/K/F$ be a finite Galois sub-extension, let S be a finite set of closed points of $\bar{X}$ such that $v \in S$ and let $Z = W_{k(v)} \times T$ be an object of $W_{k(v)} \times Top$. Consider the object of $T_{\bar{X}}$

$$\mathcal{Y}(K,S,Z) = (T_0, T_w, f_w)$$

defined as follows. We first define the topological space

$$T_0 = W_{K/F,S} \times^{W_{F_v}} Z := (W_{K/F,S} \times W_{k(v)} \times T)/W_{F_v} \cong (W_{K/F,S}/W^1_{F_v}) \times T$$

endowed with its natural W_F-action. For any w not in S, we consider $T_w = W_{K/F,S} \times^{W_{F_v}} Z$ on which $W_{k(w)}$ acts via the map

$$W_{k(w)} = W_{F_w}/W^1_{F_w} \longrightarrow W_{K/F,S}.$$

For any $w \in S$ such that $w \neq v$, we set $T_w = \emptyset$, and we define $T_v = Z$. The map f_w is the identity for any w not in S and

$$f_v : Z \longrightarrow W_{K/F,S} \times^{W_{F_v}} Z$$

is the canonical map. This map f_v is continuous and injective. The image of $W^1_{F_v}$ in $W_{K/F,S}$ is compact, and the spaces T_0 and T_w are locally compact for any place w so that $\mathcal{Y}(K,S,Z)$ is an object of $T_{\bar{X}}$.
On the one hand, the functor $Z \mapsto W_{K/F,S} \times^{W_{F_v}} Z$ is left adjoint to the forgetful functor $B_{Top}W_{K/F,S} \to B_{Top}W_{F_v}$. On the other hand, for any object (Y_0, Y_w, f_w) of $T_{\bar{X}}$, the action of W_F on Z_0 factors through $W_{K/F,S}$ for some finite Galois extension K/F and some finite set S of places of F. It follows that

$$\{\mathcal{Y}(K,S,Z), \text{ for } K/F \text{ Galois}, S \text{ finite }\}$$

yields a cofinal system in the category of arrows $Z \to i_v^*(Y_0, Y_w, f_w)$ considered above, for any fixed object Z of $W_{k(v)} \times Top$. Hence $i_v^* \circ i_{v*}\mathcal{F}$ is the sheaf on $(B_{Top}W_{k(v)}, \mathcal{J}_{ls})$ associated to the presheaf

$$\begin{array}{ccc} W_{k(v)} \times Top & \longrightarrow & \underline{Set} \\ Z & \longmapsto & \varinjlim_{Z \to i_v^*(Y_0,Y_w,f_w)} \mathcal{F}(Y_v) = \varinjlim_{Z \to i_v^*\mathcal{Y}(K,S,Z)} \mathcal{F}(Z) = \mathcal{F}(Z) \end{array}$$

Since $W_{k(v)} \times Top$ is a topologically generating subcategory of $(B_{Top}W_{k(v)}, \mathcal{J}_{ls})$, the sheaf on $B_{W_{k(v)}}$ associated to this presheaf is $\mathcal{F}$, and the adjunction morphism (18) is an isomorphism. This shows that i_{v*} is fully faithful, i.e. i_v is an embedding. □

Definition 9. *Let v be a closed point of $\bar{X}$. We consider the morphism $p_v : \mathcal{T} \to B_{W_{k(v)}}$ whose inverse image p_v^* is the forgetful functor, and we denote by $i_{\overline{v}}$ the composite morphism*

$$i_{\overline{v}} := i_v \circ p_v : \mathcal{T} \longrightarrow B_{W_{k(v)}} \longrightarrow \bar{X}_W.$$

For any object $Z = W_{k(v)} \times T$ of the full subcategory $W_{k(v)} \times Top \hookrightarrow B_{Top}W_{k(v)}$ and for any sheaf $\mathcal{F}$ of $\bar{X}_W$, we have

$$i_v^p\mathcal{F}(Z) = \varinjlim_{Z \to i_v^*(Y_0,Y_w,f_w)} \mathcal{F}(Y_0, Y_w, f_w) = \varinjlim_{Z \to i_v^*\mathcal{Y}(K,S,Z)} \mathcal{F}(\mathcal{Y}(K,S,Z)) \tag{19}$$

where we consider the pull-back presheaf $i_v^p\mathcal{F}$ on $B_{Top}W_{k(v)}$. The morphism $p_v : \mathcal{T} \to B_{W_{k(v)}}$ is induced by the morphism of left exact sites given by the forgetful functor $B_{Top}W_{k(v)} \to Top$. By adjunction, for any space T of Top and any presheaf $\mathcal{P}$ on $B_{Top}W_{k(v)}$ we have

$$p_v^p\mathcal{P}(T) = \mathcal{P}(W_{k(v)} \times T).$$

Hence the isomorphism $i_{\overline{v}}^p \cong p_v^p \circ i_v^p$ gives

$$i_{\overline{v}}^p\mathcal{F}(T) = i_v^p\mathcal{F}(Z) = \lim_{Z \to i_v^*\mathcal{Y}(K,S,Z)} \mathcal{F}(\mathcal{Y}(K,S,Z)) \tag{20}$$

where $Z := W_{k(v)} \times T$. We consider the category of compact spaces Top^c. The morphism of sites $(Top^c, \mathcal{J}_{op}) \to (Top, \mathcal{J}_{op})$ induces an equivalence of topoi, hence one can restrict our attention to compact spaces. Let us show that $i_{\overline{v}}^p\mathcal{F}$ restricts to a sheaf on $(Top^c, \mathcal{J}_{op})$. Let $\{T_i \to T,\, i \in I\}$ be a covering family of $(Top^c, \mathcal{J}_{op})$, i.e. a local section cover of compact spaces. One can assume that I is finite, since any covering family of $(Top^c, \mathcal{J}_{op})$ can be refined by a finite covering family. For any K/F and any S,

$$\{\mathcal{Y}(K,S,W_{k(v)} \times T_i) \to \mathcal{Y}(K,S,W_{k(v)} \times T)\}$$

is a covering family of $(T_{\bar{X}}, \mathcal{J}_{ls})$. Moreover the fiber product

$$\mathcal{Y}(K,S,W_{k(v)} \times T_i) \times_{\mathcal{Y}(K,S,W_{k(v)} \times T)} \mathcal{Y}(K,S,W_{k(v)} \times T_j)$$

computed in the category $T_{\bar{X}}$, is isomorphic to $\mathcal{Y}(K,S,W_{k(v)} \times T_{ij})$, where T_{ij} denotes $T_i \times_T T_i$. It follows that the diagram of sets

$$\mathcal{F}(\mathcal{Y}(K,S,W_{k(v)} \times T)) \to \prod_i \mathcal{F}(\mathcal{Y}(K,S,W_{k(v)} \times T_i))$$
$$\rightrightarrows \prod_{i,j} \mathcal{F}(\mathcal{Y}(K,S,W_{k(v)} \times T_{ij}))$$

is exact. Passing to the inductive limit over K and S, and using left exactness of filtered inductive limits (i.e. using the fact that filtered inductive limits commute with finite products and equalizers), we obtain an exact diagram of sets

$$i_{\overline{v}}^p\mathcal{F}(T) \to \prod_i i_{\overline{v}}^p\mathcal{F}(T_i) \rightrightarrows \prod_{i,j} i_{\overline{v}}^p\mathcal{F}(T_{ij}),$$

as it follows from (20). Hence $i_{\overline{v}}^p\mathcal{F}$ is a sheaf on $(Top^c, \mathcal{J}_{op})$. Therefore, for any compact space T, one has

$$i_{\overline{v}}^*\mathcal{F}(T) = i_{\overline{v}}^p\mathcal{F}(T) = \lim_{Z \to i_v^*\mathcal{Y}(K,S,Z)} \mathcal{F}(\mathcal{Y}(K,S,Z)) \tag{21}$$

where $Z = W_{k(v)} \times T$.

Lemma 5. *The family of functors*

$$\{i_{\overline{v}}^* : \bar{X}_W \to \mathcal{T},\, v \in \bar{X}^0\}$$

is conservative, where $\bar{X}^0$ is the set of closed points of $\bar{X}$.

Proof. Let $\mathcal{F}$ be an object of $\bar{X}_W$. We need to show that the adjunction map

$$\mathcal{F} \longrightarrow \prod_{v \in \bar{X}^0} i_{\overline{v}*} i_{\overline{v}}^* \mathcal{F}. \tag{22}$$

is injective. For any (Z_0, Z_w, f_w) of $T_{\bar{X}}$, we have

$$\prod_{v \in \bar{X}^0} (i_{\overline{v}*} i_{\overline{v}}^* \mathcal{F})(Z_0, Z_w, f_w) = \prod_{v \in \bar{X}^0} i_{\overline{v}}^* \mathcal{F}(Z_v).$$

Note that, in the term on the right hand side of the equality above, Z_v is considered as a topological space without any action. For any v, we choose a local section cover of the space Z_v:

$$\{T_{v,l} \hookrightarrow Z_v,\, l \in \Lambda_v\}$$

such that $T_{v,l}$ is a compact subspace of Z_v for any index l. Such a local section cover exists since Z_v is locally compact. The map

$$i_{\overline{v}}^* \mathcal{F}(Z_v) \longrightarrow \prod_{l \in \Lambda_v} i_{\overline{v}}^* \mathcal{F}(T_{v,l}).$$

is injective since $i_{\overline{v}}^* \mathcal{F}$ is a sheaf. It is therefore enough to show that the composite map

$$\kappa : \mathcal{F}(Z_0, Z_w, f_w) \longrightarrow \prod_{v \in \bar{X}^0} i_{\overline{v}}^* \mathcal{F}(Z_v) \longrightarrow \prod_{v \in \bar{X}^0,\, l \in \Lambda_v} i_{\overline{v}}^* \mathcal{F}(T_{v,l})$$

is injective. Let $\alpha, \beta \in \mathcal{F}(Z_0, Z_w, f_w)$ be two sections such that $\kappa(\alpha) = \kappa(\beta)$. For any pair (v, l), we consider

$$\kappa_{v,l} : \mathcal{F}(Z_0, Z_w, f_w) \longrightarrow \prod_{v \in \bar{X}^0,\, l \in \Lambda_v} i_{\overline{v}}^* \mathcal{F}(T_{v,l}) \longrightarrow i_{\overline{v}}^* \mathcal{F}(T_{v,l}).$$

For any (v, l), we have $\kappa_{v,l}(\alpha) = \kappa_{v,l}(\beta)$ and by (21)

$$i_{\overline{v}}^* \mathcal{F}(T_{v,l}) = \varinjlim \mathcal{F}(\mathcal{Y}(K, S, W_{k(v)} \times T_{v,l}))$$

where the direct limit is taken over the category of arrows

$$W_{k(v)} \times T_{v,l} \longrightarrow i_v^* \mathcal{Y}(K, S, W_{k(v)} \times T_{v,l}).$$

The inclusion $T_{v,l} \subseteq Z_v$ gives a $W_{k(v)}$-equivariant continuous map

$$W_{k(v)} \times T_{v,l} \longrightarrow i_v^*(Z_0, Z_w, f_w) = Z_v.$$

Thus for any pair (v, l), there is an object $\mathcal{Y}(K, S, W_{k(v)} \times T_{v,l})$ and a morphism

$$\mathcal{Y}(K, S, W_{k(v)} \times T_{v,l}) \longrightarrow (Z_0, Z_w, f_w)$$

in the category $T_{\bar{X}}$ inducing the previous map

$$W_{k(v)} \times T_{v,l} = i_v^* \mathcal{Y}(K, S, W_{k(v)} \times T_{v,l}) \longrightarrow i_v^*(Z_0, Z_w, f_w) = Z_v$$

and such that $\alpha_{|(v,l)} = \beta_{|(v,l)}$, where $\alpha_{|(v,l)}$ (respectively $\beta_{|(v,l)}$) denotes the restriction of α (respectively of β) to $\mathcal{Y}(K, S, W_{k(v)} \times T_{v,l})$. We obtain a local section cover

$$\{\mathcal{Y}(K, S, W_{k(v)} \times T_{v,l}) \rightarrow (Z_0, Z_w, f_w),\, v \in \bar{X}^0,\, l \in \Lambda_v)\}$$

in the site $(T_{\bar{X}}, \mathcal{J}_{ls})$ such that $\alpha_{|(v,l)} = \beta_{|(v,l)}$ for any (v, l). It follows that $\alpha = \beta$ since $\mathcal{F}$ is a sheaf. Hence κ is injective and so is the adjunction map (22).

□

A morphism of topoi f is said to be surjective if its inverse image functor f^* is faithful.

Corollary 8. *The following morphism is surjective:*

$$(i_v)_{v\in\bar{X}^0} : \coprod_{v\in\bar{X}^0} B_{W_{k(v)}} \longrightarrow \bar{X}_W.$$

Proof. The morphism of topoi

$$(i_{\overline{v}})_{v\in\bar{X}^0} : \coprod_{v\in\bar{X}^0} \mathcal{T} \longrightarrow \bar{X}_W$$

is surjective since its inverse image is faithful by the previous result. But $(i_{\overline{v}})_{v\in\bar{X}^0}$ factors through $(i_v)_{v\in\bar{X}^0}$, hence $(i_v)_{v\in\bar{X}^0}$ is surjective as well. □

Proof of Theorem 5.1. Since the morphism i_v is an embedding, we have in fact two embeddings of topoi

$$B_{W_{k(v)}} \longrightarrow \bar{X}_W \times_{\bar{X}_{et}} B^{sm}_{G_{k(v)}} \longrightarrow \bar{X}_W$$

where the fiber product $\bar{X}_W \times_{\bar{X}_{et}} B^{sm}_{G_{k(v)}}$ is defined as the inverse image $\gamma^{-1}(B^{sm}_{G_{k(v)}})$ of the closed sub-topos $B^{sm}_{G_{k(v)}} \hookrightarrow \bar{X}_{et}$ under the morphism γ (see [20] IV. Corollaire 9.4.3). Therefore $B_{W_{k(v)}}$ is equivalent to a full subcategory of $\bar{X}_W \times_{\bar{X}_{et}} B^{sm}_{G_{k(v)}}$. This fiber product is the closed complement of the open subtopos $\bar{Y}_W \hookrightarrow \bar{X}_W$ where $\bar{Y} := \bar{X} - v$ (see the next section for the definition of $\bar{Y}_W$). In other words, the strictly full subcategory $\bar{X}_W \times_{\bar{X}_{et}} B^{sm}_{G_{k(v)}}$ of $\bar{X}_W$ consists in objects $\mathcal{G}$ such that $\mathcal{G} \times \gamma^*\bar{Y}$ is the final object of $\bar{Y}_W$. It follows that

$$i_{v*}\mathcal{F} \times \gamma^*\bar{Y}$$

is the final object of $\bar{Y}_W$, for any object $\mathcal{F}$ of $B_{W_{k(v)}}$.
We have to prove that $B_{W_{k(v)}}$ is in fact equivalent to $\bar{X}_W \times_{\bar{X}_{et}} B^{sm}_{G_{k(v)}}$. Let $\mathcal{G}$ be an object of this fiber product, i.e. an object of $\bar{X}_W$ such that $\mathcal{G} \times \gamma^*\bar{Y}$ is the final object. Consider the adjunction map

$$\mathcal{G} \longrightarrow i_{v*}i_v^*\mathcal{G}.$$

If w is a closed point of $\bar{X}$ such that $w \neq v$, then the morphism i_w factors through $\bar{Y}_W$:

$$i_w : B_{W_{k(w)}} \longrightarrow \bar{Y}_W \longrightarrow \bar{X}_W.$$

We denote by $i_{\bar{Y},w} : B_{W_{k(v)}} \to \bar{Y}_W$ the induced map. Hence

$$i_w^*\mathcal{G} = i_{\bar{Y},w}^*(\mathcal{G} \times \bar{Y})$$

is the final object of $B_{W_{k(w)}}$, since $\mathcal{G}\times\bar{Y}$ is the final object of $\bar{Y}_W$ and $i^*_{\bar{Y},w}$ is left exact. On the other hand

$$i_w^* i_{v*} i_v^* \mathcal{G} = i^*_{\bar{Y},w}(i_{v*} i_v^* \mathcal{G}\times\bar{Y})$$

is the final object of $B_{W_{k(w)}}$, since $i_{v*}i_v^*\mathcal{G}\times\bar{Y}$ is the final object of $\bar{Y}_W$. Hence the map

$$i_w^*(\mathcal{G}) \longrightarrow i_w^*(i_{v*}i_v^*\mathcal{G})$$

is an isomorphism for any closed point $w\neq v$ of $\bar{X}$. Suppose now that $w=v$. Then the map

$$i_v^*(\mathcal{G}) \longrightarrow i_v^*(i_{v*}i_v^*\mathcal{G}) = (i_v^* i_{v*})i_v^*\mathcal{G} = i_v^*\mathcal{G}$$

is an isomorphism by Lemma 4. Hence the morphism

$$i_w^*(\mathcal{G}) \longrightarrow i_w^*(i_{v*}i_v^*\mathcal{G})$$

induced by the adjunction map $\mathcal{G}\to i_{v*}i_v^*\mathcal{G}$ is an isomorphism for any closed point w of $\bar{X}$. Since the family of functors

$$\{i_w^* : \bar{X}_W \to B_{W_{k(w)}},\, w\in\bar{X}\}$$

is conservative, the adjunction map $\mathcal{G}\to i_{v*}i_v^*\mathcal{G}$ is an isomorphism for any object $\mathcal{G}$ of $\gamma^{-1}(B^{sm}_{G_{k(v)}})$. Hence any object of $\gamma^{-1}(B^{sm}_{G_{k(v)}})$ is in the essential image of i_{v*}. This shows that the morphism

$$B_{W_{k(v)}} \longrightarrow \bar{X}_W \times_{\bar{X}_{et}} B^{sm}_{G_{k(v)}}$$

is an equivalence (this is a connected embedding). Theorem 5.1 follows. □

We consider the morphism

$$\bar{X}_W = \overline{\mathrm{Spec}(\mathcal{O}_F)}_W \longrightarrow \overline{\mathrm{Spec}(\mathbb{Z})}_W$$

induced by the map $\overline{\mathrm{Spec}(\mathcal{O}_F)}\to\overline{\mathrm{Spec}(\mathbb{Z})}$.

PROPOSITION 5.5. *The canonical morphism*

$$\delta_{\bar{X}} : \bar{X}_W \longrightarrow \bar{X}_{et}\times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \overline{\mathrm{Spec}(\mathbb{Z})}_W$$

is an equivalence.

Proof. Let $\bar{X}'$ be the open subscheme of $\bar{X}$ consisting of the points of $\bar{X}$ where the map $\bar{X}\to\overline{\mathrm{Spec}(\mathbb{Z})}$ is étale. Let $Y\to\bar{X}$ be the complementary reduced closed subscheme.

(I) THE MORPHISM $\delta_{\bar{X}}$ IS AN EQUIVALENCE OVER $\bar{X}'$ AND OVER Y. Recall from Definition 8 that $\bar{X}'_W$ is defined as an open subtopos of $\bar{X}_W$. The canonical morphism

$$\bar{X}'_W \to \bar{X}'_{et}\times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \overline{\mathrm{Spec}(\mathbb{Z})}_W$$

is an equivalence. Indeed, the morphism $\bar{X}' \to \overline{\mathrm{Spec}(\mathbb{Z})}$ is étale hence we have

$$\begin{aligned}\bar{X}'_{et} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \overline{\mathrm{Spec}(\mathbb{Z})}_W &\cong (\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}/\bar{X}') \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \overline{\mathrm{Spec}(\mathbb{Z})}_W \\ &\cong \overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} (\overline{\mathrm{Spec}(\mathbb{Z})}_W/\gamma^*\bar{X}') \\ &\cong \overline{\mathrm{Spec}(\mathbb{Z})}_W/\gamma^*\bar{X}'. \\ &\cong \bar{X}'_W\end{aligned}$$

Let Y' be the image of Y in $\overline{\mathrm{Spec}(\mathbb{Z})}$, such that $Y' \times_{\overline{\mathrm{Spec}(\mathbb{Z})}} \mathrm{Spec}(\mathbb{Z})$ is given with a structure of reduced closed subscheme of $\mathrm{Spec}(\mathbb{Z})$. The morphism of étale topoi $Y_{et} \to \overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}$ factors through Y'_{et}. It follows from Theorem 5.1 that one has

$$\begin{aligned}Y_{et} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \overline{\mathrm{Spec}(\mathbb{Z})}_W &\cong Y_{et} \times_{Y'_{\mathrm{et}}} Y'_{et} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \overline{\mathrm{Spec}(\mathbb{Z})}_W \\ &\cong Y_{et} \times_{Y'_{\mathrm{et}}} Y'_W.\end{aligned}$$

We have the following equivalences $Y_{et} \cong \coprod_{v\in Y} B^{sm}_{G_{k(v)}}$, $Y'_{\mathrm{et}} \cong \coprod_{p\in Y'} B^{sm}_{G_{k(p)}}$ and $Y'_W := \coprod_{p\in Y'} B_{W_{k(p)}}$. We obtain

$$\begin{aligned}Y_{et} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \overline{\mathrm{Spec}(\mathbb{Z})}_W &\cong Y_{et} \times_{Y'_{\mathrm{et}}} Y'_W \\ &\cong \coprod_{v\in Y} B^{sm}_{G_{k(v)}} \times_{(\coprod_{p\in Y'} B^{sm}_{G_{k(p)}})} \coprod_{p\in Y'} B_{W_{k(p)}} \\ &\cong \coprod_{v\in Y} (B^{sm}_{G_{k(v)}} \times_{B^{sm}_{G_{k(p)}}} B_{W_{k(p)}}) \\ &\cong \coprod_{v\in Y} B_{W_{k(v)}} = Y_W\end{aligned}$$

In view of the pull-back square (1), the last equivalence above follows from the fact that

$$B^{sm}_{G_{k(v)}} \cong B^{sm}_{G_{k(p)}}/(G_{k(p)}/G_{k(v)}) \longrightarrow B^{sm}_{G_{k(p)}}$$

is a localization morphism.

(ii) The natural transformation t between the glueing functors. The previous step (i) shows that there is an open-closed decomposition of topoi

$$\mathrm{j} : \bar{X}'_W \to \bar{X}_{et} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \overline{\mathrm{Spec}(\mathbb{Z})}_W \leftarrow Y_W : \mathrm{i}$$

By Theorem 5.1, we have another open-closed decomposition

$$j : \bar{X}'_W \to \bar{X}_W \leftarrow Y_W : i$$

The glueing functors associated to these open-closed decompositions are given by $\mathrm{i}^*\mathrm{j}_*$ and i^*j_*. The map $\bar{X}_W \to \bar{X}_{et} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \overline{\mathrm{Spec}(\mathbb{Z})}_W$ induces a natural transformation

$$t : \mathrm{i}^*\mathrm{j}_* \longrightarrow i^*j_*. \tag{23}$$

Indeed, the following commutative diagram

$$\begin{tikzcd} \bar{X}'_W \arrow[r, "j"] \arrow[d, "Id"] & \bar{X}_W \arrow[d, "\delta_{\bar{X}}"] & Y_W \arrow[l, "i"'] \arrow[d, "Id"] \\ \bar{X}'_W \arrow[r, "\mathrm{j}"] & \bar{X}_{et} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \overline{\mathrm{Spec}(\mathbb{Z})}_W & Y_W \arrow[l, "\mathrm{i}"'] \end{tikzcd} \tag{24}$$

gives $\delta_{\bar{X}} \circ i = \mathrm{i}$ and $\delta_{\bar{X}} \circ j = \mathrm{j}$. Then the natural transformation (23) is induced by the adjunction transformation $\delta^*_{\bar{X}}\delta_{\bar{X}*} \to Id$ as follows :

$$\mathrm{i}^*\mathrm{j}_* \cong i^*\delta^*_{\bar{X}}\delta_{\bar{X}*}j_* \longrightarrow i^*j_*.$$

(III) THE GLUEING FUNCTORS ARE NATURALLY ISOMORPHIC. Since the disjoint sum topos $Y_W = \coprod_{v\in S} B_{W_{k(v)}}$ is given by the direct product of the categories $B_{W_{k(v)}}$, it is enough to show that the natural transformation

$$\mathrm{i}^*_v\mathrm{j}_* \longrightarrow i^*_v j_* \tag{25}$$

is an isomorphism for any $v \in Y$.

Let $\mathcal{F}$ be an object of $\bar{X}'_W$. The sheaf $i^*_v \circ j_*\mathcal{F}$ (respectively $\mathrm{i}^*_v \circ \mathrm{j}_*\mathcal{F}$) is the sheaf on $(W_{k(v)} \times Top, \mathcal{J}_{ls})$ associated to the presheaf $i^p_v \circ j_*\mathcal{F}$ (respectively to the presheaf $\mathrm{i}^p_v \circ \mathrm{j}_*\mathcal{F}$). Recall that $W_{k(v)} \times Top$ is a topologically generating subcategory of $(B_{Top}W_{k(v)}, \mathcal{J}_{ls})$. It is therefore enough to show that the natural map

$$\mathrm{i}^p_v \circ \mathrm{j}_*\mathcal{F} \to i^p_v \circ j_*\mathcal{F}, \tag{26}$$

of presheaves on $W_{k(v)} \times Top$, is an isomorphism.

On the one hand, for any object $\mathcal{F}$ of $\bar{X}'_W$ we have

$$i^p_v j_*\mathcal{F}(W_{k(v)} \times T) = \lim_{W_{k(v)}\times T \to i^*_v(Y_0,Y_w,f_w)} \mathcal{F}((Y_0, Y_w, f_w) \times \bar{X}') \tag{27}$$

$$= \underrightarrow{lim}_{K/F,S}\mathcal{F}(\mathcal{Y}(K/F, S, W_{k(v)} \times T) \times \bar{X}) \tag{28}$$

where (28) is given by (19). See the proof of Lemma 4 for the definition of $\mathcal{Y}(K/F, S, W_{k(v)} \times T)$. On the other hand, for any object $\mathcal{F}$ of $\bar{X}'_W$ we have

$$\begin{aligned} \mathrm{i}^p_v\mathrm{j}_*\mathcal{F}(W_{k(v)} \times T) &= \underrightarrow{lim}\, \mathrm{j}_*\mathcal{F}((Z_0, Z_w, f_w) \to V \leftarrow U) \\ &= \underrightarrow{lim}\, \mathcal{F}((Z_0, Z_w, f_w) \times_V U \times \bar{X}') \end{aligned}$$

where the direct limit is taken over the category of arrows

$$W_{k(v)} \times T \to \mathrm{i}^*_v((Z_0, Z_w, f_w) \to V \leftarrow U) = Z_p \times_{V_p} U_v. \tag{29}$$

Here, $((Z_0, Z_w, f_w) \to V \leftarrow U)$ is an object of the fiber product site $\mathcal{C}_{\bar{X}}$, i.e. (Z_0, Z_w, f_w), V and U are objects of the sites $T_{\overline{\mathrm{Spec}(\mathbb{Z})}}$, $Et_{\overline{\mathrm{Spec}(\mathbb{Z})}}$ and $Et_{\bar{X}}$ respectively. Then $(Z_0, Z_w, f_w) \times_V U$ is seen as an object of $T_{\bar{X}}$. Finally, the place p is defined as the image of $v \in \bar{X}$ in $\overline{\mathrm{Spec}(\mathbb{Z})}$. We refer to [25] and section 7 for the definition of the site $\mathcal{C}_{\bar{X}}$.

There is a natural functor from the category of arrows of the form (29) to the category of arrows $(W_{k(v)} \times T) \to i_v^*(Y_0, Y_w, f_w)$ sending $((Z_0, Z_w, f_w) \to V \leftarrow V')$ to $(Z_0, Z_w, f_w) \times_V V'$. This provides us with the natural map

$$\mathrm{i}_v^p \mathrm{j}_* \mathcal{F}(W_{k(v)} \times T) \longrightarrow i_v^p j_* \mathcal{F}(W_{k(v)} \times T). \tag{30}$$

In order to show that (30) is an isomorphism, we have to show that the system

$$W_{k(v)} \times T \longrightarrow i_v^*((Z_0, Z_w, f_w) \times_V U),$$

where $(Z_0, Z_w, f_w) \to V \leftarrow U)$ runs over the class of objects in $C_{\bar{X}}$, is cofinal in the category of arrows $\mathcal{A}_{v,T}$:

$$W_{k(v)} \times T \to i_v^*(Y_0, Y_w, f_w).$$

We know that the system given by the $\mathcal{Y}(K, S, W_{k(v)} \times T)$'s is cofinal in $\mathcal{A}_{v,T}$. Here $v \in S$ and K/F is unramified outside S. One can choose S large enough so that S contains Y. Let S' be the image of S in $\overline{\mathrm{Spec}(\mathbb{Z})}$. Then $K/\mathbb{Q}$ is unramified outside S'. If we denote by $L/\mathbb{Q}$ the Galois closure of $K/\mathbb{Q}$ (in the fixed algebraic closure $\overline{\mathbb{Q}}/\mathbb{Q}$), then $L/\mathbb{Q}$ remains unramified outside S', and L/F is Galois and unramified outside S. Moreover, we have a morphism

$$\mathcal{Y}(L/F, S, W_{k(v)} \times T) \to \mathcal{Y}(K/F, S, W_{k(v)} \times T) \text{ in } \mathcal{A}_{v,T}.$$

Hence one can restrict our attention to the objects of $\mathcal{A}_{v,T}$ of the form

$$W_{k(v)} \times T \to i_v^* \mathcal{Y}(K/F, S, W_{k(v)} \times T)$$

where $K/\mathbb{Q}$ is a Galois extension unramified outside S'. We denote again by p the image of v in $\overline{\mathrm{Spec}(\mathbb{Z})}$ and we consider the object

$$\mathcal{Y}(K/\mathbb{Q}, S', W_{k(p)} \times T) \to V \leftarrow U \text{ in } \mathcal{C}_{\bar{X}},$$

where V and U are defined as follows. Using Proposition 5.4, the étale $\overline{\mathrm{Spec}(\mathbb{Z})}$-scheme V is given by the $G_{\mathbb{Q}}$-set $G_{K/\mathbb{Q}}/I_p$, with no point over $S' - \{p\}$, and exactly one point over the place p corresponding to the distinguished $G_{\mathbb{Q}_p}$-orbit of $G_{K/\mathbb{Q}}/I_p$ on which the inertia group I_p acts trivially. The étale $\bar{X}$-scheme U is given by the G_F-set $G_{K/F}/I_v$, with no point over $S - \{v\}$, and exactly one point over the place v corresponding to the distinguished G_{F_v}-orbit of $G_{K/F}/I_v$ on which the inertia group I_v acts trivially. Finally, we enlarge S so that S is the inverse image of S' (which is the image of S) along the map $\bar{X} \to \overline{\mathrm{Spec}(\mathbb{Z})}$. Then the map $U \to V$ is well defined.
Assume that one has an identification

$$\mathcal{Y}(K/F, S, W_{k(v)} \times T) = \mathcal{Y}(K/\mathbb{Q}, S', W_{k(p)} \times T) \times_V U \tag{31}$$

in the category $T_{\bar{X}}$. It would follow that the system of objects

$$W_{k(v)} \times T \to i_v^*((Z_0, Z_w, f_w) \times_V U)$$

is cofinal in the category $\mathcal{A}_{v,T}$. The map (30) would be an isomorphism for any T and any $\mathcal{F}$, hence (26) would be an isomorphism of presheaves for any $\mathcal{F}$. This would show that the transformation (25) is an isomorphism. Hence the transformation (23) would be an isomorphism as well.

It is therefore enough to show (31). One has

$$\mathcal{Y}(K/F, S, W_{k(v)} \times T) = \mathcal{Y}(K/F, S, W_{k(v)}) \times (T, T, Id_T)$$

and

$$\mathcal{Y}(K/\mathbb{Q}, S', W_{k(p)} \times T) = \mathcal{Y}(K/\mathbb{Q}, S', W_{k(p)}) \times (T, T, Id_T)$$

in the category $T_{\bar{X}}$, hence one can assume that $T = *$ is the point. We have a map in $T_{\bar{X}}$

$$\mathcal{Y}(K/F, S, W_{k(v)}) \longrightarrow \mathcal{Y}(K/\mathbb{Q}, S', W_{k(p)}) \times_V U. \tag{32}$$

and we need to show that it is an isomorphism. Let w be a point of $\bar{X}$. If $w \in S$ and $w \neq v$, then the w-component of both the right hand side and the left hand side in (32) are empty. Assume that w is not in S. Then the w-components of $\mathcal{Y}(K/F, S, W_{k(v)})$, $\mathcal{Y}(K/\mathbb{Q}, S', W_{k(p)})$, V and U are the $W_{k(w)}$-spaces $W_{K/F,S}/W^1_{F_v}$, $W_{K/\mathbb{Q},S'}/W^1_{\mathbb{Q}_p}$, $G_{K/\mathbb{Q}}/I_p$ and $G_{K/F}/I_v$ respectively. But we have an $W_{k(w)}$-equivariant homeomorphism

$$W_{K/F,S}/W^1_{F_v} \cong (W_{K/\mathbb{Q},S'}/W^1_{\mathbb{Q}_p}) \times_{(G_{K/\mathbb{Q}}/I_p)} (G_{K/F}/I_v).$$

Moreover, the v-component of $\mathcal{Y}(K/F, S, W_{k(v)})$, $\mathcal{Y}(K/\mathbb{Q}, S', W_{k(p)})$, V and U are the $W_{k(v)}$-spaces $W_{k(v)}$, $W_{k(p)}$, $G_{k(p)}/G_{k(u)}$ and $G_{k(v)}/G_{k(u)}$, where u the unique point of U lying over v. But we have an $W_{k(v)}$-equivariant homeomorphism

$$W_{k(v)} \cong W_{k(p)} \times_{(G_{k(p)}/G_{k(u)})} (G_{k(v)}/G_{k(u)}).$$

This shows that (32) is an isomorphism in $T_{\bar{X}}$, and (31) follows.

(v) THE MORPHISM $\delta_{\bar{X}}$ IS AN EQUIVALENCE. We consider the glued topoi $(Y_W, \bar{X}'_W, i^*j_*)$ and $(Y_W, \bar{X}'_W, \mathrm{i}^*\mathrm{j}_*)$. Recall that an object of $(Y_W, \bar{X}'_W, i^*j_*)$ is a triple (E, F, σ) with $E \in Y_W$, $F \in \bar{X}'_W$ and $\sigma : E \to i^*j_*F$ (see [20] IV.9.5.3). There is a canonical functor

$$\begin{array}{ccc} \bar{X}_W & \longrightarrow & (Y_W, \bar{X}'_W, i^*j_*) \\ \mathcal{F} & \longmapsto & (i^*\mathcal{F}, j^*\mathcal{F}, i^*\mathcal{F} \to i^*j_*j^*\mathcal{F}) \end{array}$$

where $i^*\mathcal{F} \to i^*j_*j^*\mathcal{F}$ is given by adjunction. By ([20] IV Theorem 9.5.4), this functor is an equivalence, and the same is true for the canonical functor

$$\bar{X}_{et} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \overline{\mathrm{Spec}(\mathbb{Z})}_W \longrightarrow (Y_W, \bar{X}'_W, \mathrm{i}^*\mathrm{j}_*)$$

Under these identifications, the inverse image functor $\delta^*_{\bar{X}}$ is given by (see diagram (24))

$$\begin{array}{cccc} \delta^*_{\bar{X}} : & (Y_W, \bar{X}'_W, \mathrm{i}^*\mathrm{j}_*) & \longrightarrow & (Y_W, \bar{X}'_W, i^*j_*) \\ & (E, F, \tau) & \longmapsto & (E, F, t_F \circ \tau) \end{array}$$

Here t is the transformation defined in step (ii), and $t_F \circ \tau$ denotes the following composition :

$$t_F \circ \tau : E \to \mathrm{i}^*\mathrm{j}_*F \to i^*j_*F.$$

Since t is an isomorphism of functors, the inverse image functor $\delta^*_{\bar{X}}$ is an equivalence, hence so is the morphism $\delta_{\bar{X}}$. $\square$

6. The definition of $\overline{\mathcal{X}}_W$

6.1. Let $\mathcal{X}$ be a scheme separated and of finite type over $\mathrm{Spec}(\mathbb{Z})$. Recall the defining site $Et_{\overline{\mathcal{X}}}$ of the Artin-Verdier étale topos $\overline{\mathcal{X}}_{\mathrm{et}}$ from section 4. For any object $\overline{\mathcal{U}}$ of $Et_{\overline{\mathcal{X}}}$ one has the induced topos

$$\overline{\mathcal{U}}_{\mathrm{et}} = \overline{\mathcal{X}}_{\mathrm{et}}/\overline{\mathcal{U}} \cong (\widetilde{Et_{\overline{\mathcal{X}}}/\overline{\mathcal{U}}, \mathcal{J}_{ind}}).$$

Definition 10. *For any object $\overline{\mathcal{U}}$ of $Et_{\overline{\mathcal{X}}}$ we define the Weil-étale topos of $\overline{\mathcal{U}}$ as the fiber product*

$$\overline{\mathcal{U}}_W := \overline{\mathcal{U}}_{\mathrm{et}} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \overline{\mathrm{Spec}(\mathbb{Z})}_W.$$

This topos is defined by a universal property in the 2-category of topoi. As a consequence, it is well defined up to a canonical equivalence. We point out two special cases. If $\overline{\mathcal{U}} = (\mathcal{X}, \mathcal{X}_\infty) = \overline{\mathcal{X}}$ is the final object we obtain the definition of $\overline{\mathcal{X}}_W$ and if $\overline{\mathcal{U}} = (\mathcal{X}, \emptyset)$ we obtain the definition of $\mathcal{X}_W$. The topos $\mathcal{X}_W$ will play no role in this paper but $\overline{\mathcal{X}}_W$ is our central object of study in case $\mathcal{X}$ is proper and regular.
Note also that for $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$ Definition 10 is consistent with Definition 7 by Proposition 5.5.

Proposition 6.1. *The first projection yields a canonical morphism*

$$\gamma_{\overline{\mathcal{X}}} : \overline{\mathcal{X}}_W \longrightarrow \overline{\mathcal{X}}_{\mathrm{et}}.$$

Proposition 6.2. *There is a canonical morphism*

$$\mathfrak{f}_{\overline{\mathcal{X}}} : \overline{\mathcal{X}}_W \longrightarrow B_{\mathbb{R}}.$$

Proof. The morphism $\mathfrak{f}_{\overline{\mathcal{X}}}$ is defined as the composition

$$\overline{\mathcal{X}}_W \longrightarrow \overline{\mathrm{Spec}(\mathbb{Z})}_W \longrightarrow B_{\mathbb{R}}$$

where the first arrow is the projection and the second is the morphism of Proposition 5.3. □

The structure of the topos $\overline{\mathcal{X}}_W$ over any closed point of $\overline{\mathrm{Spec}(\mathbb{Z})}$ is made explicit below. Note that $\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{F}_p$ is not assumed to be regular.

Proposition 6.3. *Let* $\mathrm{Spec}(\mathbb{F}_p)$ *be a closed point of* $\mathrm{Spec}(\mathbb{Z})$*. Then*

$$\overline{\mathcal{X}}_W \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \mathrm{Spec}(\mathbb{F}_p)_{\mathrm{et}} \cong (\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{F}_p)_W \cong (\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{F}_p)^{sm}_W \times \mathcal{T}$$

where $(\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{F}_p)_W$ denotes the big Weil-étale topos of $\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{F}_p$.

Proof. The result follows from the following equivalences.

$$\begin{aligned}
\overline{\mathcal{X}}_W \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \mathrm{Spec}(\mathbb{F}_p)_{\mathrm{et}} &\cong \overline{\mathcal{X}}_{\mathrm{et}} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \overline{\mathrm{Spec}(\mathbb{Z})}_W \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \mathrm{Spec}(\mathbb{F}_p)_{\mathrm{et}} \\
&\cong \overline{\mathcal{X}}_{\mathrm{et}} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} B_{W_{\mathbb{F}_p}} \\
&\cong \overline{\mathcal{X}}_{\mathrm{et}} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \mathrm{Spec}(\mathbb{F}_p)_{et} \times_{\mathrm{Spec}(\mathbb{F}_p)_{et}} B_{W_{\mathbb{F}_p}} \\
&\cong (\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{F}_p)_{\mathrm{et}} \times_{B^{sm}_{G_{\mathbb{F}_p}}} B_{W_{\mathbb{F}_p}} \\
&\cong (\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{F}_p)_W
\end{aligned}$$

The second equivalence, the fourth and the last one are given by Theorem 5.1, Proposition 4.2 and Corollary 1 respectively. □

COROLLARY 9. *The closed immersion of schemes $(\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{F}_p) \to \mathcal{X}$ induces a closed embedding of topoi*

$$(\mathcal{X} \otimes_{\mathbb{Z}} \mathbb{F}_p)_W \longrightarrow \mathcal{X}_W.$$

We denote by ∞ the closed point of $\overline{\mathrm{Spec}(\mathbb{Z})}$ corresponding to the archimedean place of $\mathbb{Q}$. This point yields a closed embedding of topoi

$$\underline{Set} = Sh(\infty) \longrightarrow \overline{\mathrm{Spec}(\mathbb{Z})}_{et}.$$

This paper suggests the following definition.

DEFINITION 11. *We define the Weil-étale topos of $\mathcal{X}_\infty$ as follows:*

$$\mathcal{X}_{\infty,W} := Sh(\mathcal{X}_\infty) \times B_{\mathbb{R}}.$$

The argument of Proposition 6.3 is also valid for the archimedean fiber.

PROPOSITION 6.4. *We have a pull-back square of topoi:*

$$\begin{array}{ccc}
\mathcal{X}_{\infty,W} & \longrightarrow & \underline{Set} \\
\Big\downarrow{\scriptstyle i_\infty} & & \Big\downarrow \\
\overline{\mathcal{X}}_W & \longrightarrow & \overline{\mathrm{Spec}(\mathbb{Z})}_{et}
\end{array} \tag{33}$$

In particular i_∞ is a closed embedding.

Proof. The result follows from the following equivalences.

$$\begin{aligned}
\overline{\mathcal{X}}_W \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \underline{Set} &\cong \overline{\mathcal{X}}_{\mathrm{et}} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \overline{\mathrm{Spec}(\mathbb{Z})}_W \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \underline{Set} \\
&\cong \overline{\mathcal{X}}_{\mathrm{et}} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} B_{\mathbb{R}} \\
&\cong \overline{\mathcal{X}}_{\mathrm{et}} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \underline{Set} \times_{\underline{Set}} B_{\mathbb{R}} \\
&\cong Sh(\mathcal{X}_\infty) \times_{\underline{Set}} B_{\mathbb{R}} \\
&\cong Sh(\mathcal{X}_\infty) \times B_{\mathbb{R}} \\
&= \mathcal{X}_{\infty,W}
\end{aligned}$$

Indeed, the second (respectively the fourth) equivalence above is given by Theorem 5.1 (respectively by Corollary 6). □

6.2. We assume here that $\mathcal{X}$ is irreducible and flat over $Spec(\mathbb{Z})$. Let us study the structure of $\overline{\mathcal{X}}_W$ at the generic point of $\mathcal{X}$. We denote by $K(\mathcal{X})$ the function field of the irreducible scheme $\mathcal{X}$. Let $\overline{K(\mathcal{X})}/K(\mathcal{X})$ be an algebraic closure. The algebraic closure $\overline{\mathbb{Q}}/\mathbb{Q}$ is taken as a sub-extension of $\overline{K(\mathcal{X})}/\mathbb{Q}$. Then we have a continuous morphism $G_{K(\mathcal{X})} \to G_{\mathbb{Q}}$.

Definition 12. *Let $\mathcal{X}$ be an irreducible scheme which is flat, separated and of finite type over $Spec(\mathbb{Z})$. We consider the locally compact topological group*

$$W_{K(\mathcal{X})} := G_{K(\mathcal{X})} \times_{G_{\mathbb{Q}}} W_{\mathbb{Q}}$$

defined as a fiber product in the category of topological groups.

If $K(\mathcal{X})/\mathbb{Q}$ is a number field, then $W_{K(\mathcal{X})} = G_{K(\mathcal{X})} \times_{G_{\mathbb{Q}}} W_{\mathbb{Q}}$ coincides with the Weil group of $K(\mathcal{X})$ defined in Sect. 5.

Proposition 6.5. *Let $\mathcal{X}$ be an irreducible scheme which is flat, separated and of finite type over $Spec(\mathbb{Z})$. There is a canonical morphism $j_{\overline{\mathcal{X}}} : B_{W_{K(\mathcal{X})}} \to \overline{\mathcal{X}}_W$.*

Proof. The continuous morphism $W_{K(\mathcal{X})} \to G_{K(\mathcal{X})}$ induces a morphism

$$B_{W_{K(\mathcal{X})}} \to B_{G_{K(\mathcal{X})}} \to B^{sm}_{G_{K(\mathcal{X})}}.$$

Here the second map is the canonical morphism from the big classifying topos of $G_{K(\mathcal{X})}$ to its small classifying topos, whose inverse image sends a continuous $G_{K(\mathcal{X})}$-set E to the sheaf represented by the discrete $G_{K(\mathcal{X})}$-space E (see [15] Section 7). The generic point of the irreducible scheme $\mathcal{X}$ and the previous choice of the algebraic closure $\overline{K(\mathcal{X})}/K(\mathcal{X})$ yield an embedding $B^{sm}_{G_{K(\mathcal{X})}} \hookrightarrow \overline{\mathcal{X}}_{\mathrm{et}}$. We obtain a morphism

$$B_{W_{K(\mathcal{X})}} \longrightarrow \overline{\mathcal{X}}_{\mathrm{et}}.$$

On the other hand we have maps

$$B_{W_{K(\mathcal{X})}} \longrightarrow B_{W_{\mathbb{Q}}} \longrightarrow \overline{\mathrm{Spec}(\mathbb{Z})}_W$$

and a commutative diagram

$$\begin{tikzcd} B_{W_{K(\mathcal{X})}} \arrow[r] \arrow[d] & \overline{\mathrm{Spec}(\mathbb{Z})}_W \arrow[d] \\ \overline{\mathcal{X}}_{\mathrm{et}} \arrow[r] & \overline{\mathrm{Spec}(\mathbb{Z})}_{et} \end{tikzcd}$$

The result therefore follows from the very definition of $\overline{\mathcal{X}}_W$. $\square$

Unfortunately the morphism $j_{\overline{\mathcal{X}}}$ is not an embedding. The structure of $\overline{\mathcal{X}}_W$ at the generic point is more subtle, as it is shown below. We assume again that $\mathcal{X}$ is irreducible, flat, separated and of finite type over $Spec(\mathbb{Z})$. The generic point $\mathrm{Spec}(\mathbb{Q}) \to \overline{\mathrm{Spec}(\mathbb{Z})}$ and $\overline{\mathbb{Q}}/\mathbb{Q}$ induce an embedding

$$B^{sm}_{G_{\mathbb{Q}}} \cong \mathrm{Spec}(\mathbb{Q})_{\mathrm{et}} \hookrightarrow \overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}.$$

The corresponding subtopos of $\overline{\mathrm{Spec}(\mathbb{Z})}_W$ is the classifying topos of the topological pro-group (see Proposition 5.2)

$$\underline{W}_{K/\mathbb{Q},S} := \{W_{K/\mathbb{Q},S}, \text{ for } \overline{\mathbb{Q}}/K/\mathbb{Q} \text{ finite Galois and } S \text{ finite}\}.$$

Recall that we have

$$B_{\underline{W}_{K/\mathbb{Q},S}} := \varprojlim B_{W_{K/\mathbb{Q},S}}$$

where the projective limit is understood in the 2-category of topoi. In other words, there is a pull-back

$$\begin{tikzcd} B_{\underline{W}_{K/\mathbb{Q},S}} \arrow[r] \arrow[d, "i_0"] & B^{sm}_{G_{\mathbb{Q}}} \arrow[d, "u_0"] \\ \overline{\mathrm{Spec}(\mathbb{Z})}_W \arrow[r] & \overline{\mathrm{Spec}(\mathbb{Z})}_{et} \end{tikzcd}$$

The generic point of the irreducible scheme $\mathcal{X}$ and an algebraic closure $\overline{K(\mathcal{X})}/K(\mathcal{X})$ yield an embedding $B^{sm}_{G_{K(\mathcal{X})}} \hookrightarrow \overline{\mathcal{X}}_{\mathrm{et}}$. We obtain

$$\begin{aligned}
\overline{\mathcal{X}}_W \times_{\overline{\mathcal{X}}_{\mathrm{et}}} B^{sm}_{G_{K(\mathcal{X})}} &= \overline{\mathrm{Spec}(\mathbb{Z})}_W \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \overline{\mathcal{X}}_{\mathrm{et}} \times_{\overline{\mathcal{X}}_{\mathrm{et}}} B^{sm}_{G_{K(\mathcal{X})}} \\
&\cong \overline{\mathrm{Spec}(\mathbb{Z})}_W \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} B^{sm}_{G_{K(\mathcal{X})}} \\
&\cong \overline{\mathrm{Spec}(\mathbb{Z})}_W \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} B^{sm}_{G_{\mathbb{Q}}} \times_{B^{sm}_{G_{\mathbb{Q}}}} B^{sm}_{G_{K(\mathcal{X})}} \\
&\cong B_{\underline{W}_{K/\mathbb{Q},S}} \times_{B^{sm}_{G_{\mathbb{Q}}}} B^{sm}_{G_{K(\mathcal{X})}}
\end{aligned}$$

The small classifying topos $B^{sm}_{G_{K(\mathcal{X})}}$ is the projective limit $\varprojlim B^{sm}_{G_{L/K(\mathcal{X})}}$ where $L/K(\mathcal{X})$ runs over the finite Galois sub-extension of $\overline{K(\mathcal{X})}/K(\mathcal{X})$. For such L we set $L' := L \cap \overline{\mathbb{Q}}$. Then the same is true for $B^{sm}_{G_{\mathbb{Q}}}$, i.e. we have $B^{sm}_{G_{\mathbb{Q}}} = \varprojlim B^{sm}_{G_{L'/\mathbb{Q}}}$. Since projective limits commute between themselves, we have

$$\begin{aligned}
B^{sm}_{G_{K(\mathcal{X})}} \times_{B^{sm}_{G_{\mathbb{Q}}}} B_{\underline{W}_{K/\mathbb{Q},S}} &= \varprojlim B^{sm}_{G_{L/K(\mathcal{X})}} \times_{\varprojlim B^{sm}_{G_{L'/\mathbb{Q}}}} \varprojlim B_{W_{L'/\mathbb{Q},S}} \\
&= \varprojlim_{L,S} \left(B^{sm}_{G_{L/K(\mathcal{X})}} \times_{B^{sm}_{G_{L'/\mathbb{Q}}}} B_{W_{L'/\mathbb{Q},S}}\right)
\end{aligned}$$

By Corollary 4, the fiber product $B^{sm}_{G_{L/K(\mathcal{X})}} \times_{B^{sm}_{G_{L'/\mathbb{Q}}}} B_{W_{L'/\mathbb{Q},S}}$ is equivalent to the classifying topos of the topological group $G_{L/K(\mathcal{X})} \times_{G_{L'/\mathbb{Q}}} W_{L'/\mathbb{Q},S}$ where the fiber product is in turn computed in the category of topological groups. Note that $W_{L'/\mathbb{Q},S} \to G_{L'/\mathbb{Q}}$ has local sections since $G_{L'/\mathbb{Q}}$ is profinite (see [15] Proposition 2.1).

DEFINITION 13. *Let $\overline{K(\mathcal{X})}/L/K(\mathcal{X})$ be a finite Galois sub-extension and let S be a finite set of places of $\mathbb{Q}$ containing all the places which ramify in $L' = L\cap\overline{\mathbb{Q}}$. We consider the locally compact topological group*

$$W_{L/K(\mathcal{X}),S} := G_{L/K(\mathcal{X})} \times_{G_{L'/\mathbb{Q}}} W_{L'/\mathbb{Q},S}$$

defined as a fiber product in the category of topological groups.

We have obtained the following result.

Proposition 6.6. *Let $\mathcal{X}$ be an irreducible scheme which is flat, separated and of finite type over Spec($\mathbb{Z}$). We have a pull-back square of topoi*

$$\begin{array}{ccc} \varprojlim_{L,S} B_{W_{L/K(\mathcal{X}),S}} & \longrightarrow & B^{sm}_{G_{K(\mathcal{X})}} \\ \downarrow & & \downarrow \\ \overline{\mathcal{X}}_W & \longrightarrow & \overline{\mathcal{X}}_{et} \end{array}$$

where the vertical arrows are embedding.

7. Cohomology of $\overline{\mathcal{X}}_W$ with $\tilde{\mathbb{R}}$-coefficients

The fiber product topos $\overline{\mathcal{X}}_W$, as defined in section 6, is equivalent to the category of sheaves on a site $(\mathcal{C}_{\overline{\mathcal{X}}}, \mathcal{J}_{\overline{\mathcal{X}}})$ lying in a *non-commutative* diagram of sites

$$\begin{array}{ccc} (\mathcal{C}_{\overline{\mathcal{X}}}, \mathcal{J}_{\overline{\mathcal{X}}}) & \longleftarrow & (T_{\overline{\mathrm{Spec}(\mathbb{Z})}}, \mathcal{J}_{ls}) \\ \uparrow & & \uparrow \gamma^* \\ (Et_{\overline{\mathcal{X}}}, \mathcal{J}_{\mathrm{et}}) & \xleftarrow{f^*} & (Et_{\overline{\mathrm{Spec}(\mathbb{Z})}}, \mathcal{J}_{\mathrm{et}}) \end{array}$$

The site $(\mathcal{C}_{\overline{\mathcal{X}}}, \mathcal{J}_{\overline{\mathcal{X}}})$ is defined as follows (see [25]). The category $\mathcal{C}_{\overline{\mathcal{X}}}$ is the category of pairs of morphisms $\mathcal{U} \to V \leftarrow \mathcal{Z}$, where $\mathcal{U}$ is an object of $Et_{\overline{\mathcal{X}}}$, V is an object of $Et_{\overline{\mathrm{Spec}(\mathbb{Z})}}$ and $\mathcal{Z}$ is an object of $T_{\overline{\mathrm{Spec}(\mathbb{Z})}}$. The map $\mathcal{U} \to V$ (respectively $\mathcal{Z} \to V$) is understood as a morphism $\mathcal{U} \to f^*V$ in $Et_{\overline{\mathcal{X}}}$ (respectively as a morphism $\mathcal{Z} \to \gamma^*V$ in $T_{\overline{\mathcal{X}}}$).
The topology $\mathcal{J}_{\overline{\mathcal{X}}}$ is generated by the covering families

$$\{(\mathcal{U}_i \to V_i \leftarrow \mathcal{Z}_i) \to (\mathcal{U} \to V \leftarrow \mathcal{Z}),\ i \in I\}$$

of the following types:
(a) $\mathcal{U}_i = \mathcal{U}$, $V_i = V$ and $\{\mathcal{Z}_i \to \mathcal{Z}\}$ is a covering family.
(b) $\mathcal{Z}_i = \mathcal{Z}$, $V_i = V$ and $\{\mathcal{U}_i \to \mathcal{U}\}$ is a covering family.
(c) $\{(\mathcal{U}' \to V' \leftarrow \mathcal{Z}') \to (\mathcal{U} \to V \leftarrow \mathcal{Z})\}$ with $\mathcal{U}' = \mathcal{U}$, and $\mathcal{Z}' \to \mathcal{Z}$ is obtained by base change from the map $V' \to V$ of $Et_{\overline{\mathrm{Spec}(\mathbb{Z})}}$.
(d) $\{(\mathcal{U}' \to V' \leftarrow \mathcal{Z}') \to (\mathcal{U} \to V \leftarrow \mathcal{Z})\}$ with $\mathcal{Z}' = \mathcal{Z}$, and $\mathcal{U}' \to \mathcal{U}$ is obtained by base change from the map $V' \to V$ of $Et_{\overline{\mathrm{Spec}(\mathbb{Z})}}$.
Then $(\mathcal{C}_{\overline{\mathcal{X}}}, \mathcal{J}_{\overline{\mathcal{X}}})$ is a defining site for the fiber product topos $\overline{\mathcal{X}}_W$. The topology $\mathcal{J}_{\overline{\mathcal{X}}}$ *is not subcanonical.*

Definition 14. *For any $\mathcal{T}$-topos $t : \mathcal{E} \to \mathcal{T}$, we define the sheaf of* continuous real valued functions on $\mathcal{E}$ *as follows:*

$$\tilde{\mathbb{R}} := t^*(y\mathbb{R})$$

Here $y\mathbb{R}$ is the abelian object of $\mathcal{T}$ represented by the standard topological group $\mathbb{R}$.

For an irreducible scheme $\mathcal{X}$ which is flat, separated and of finite type over Spec($\mathbb{Z}$), we consider the morphism $j_{\overline{\mathcal{X}}} : B_{W_{K(\mathcal{X})}} \to \overline{\mathcal{X}}_W$ defined in Proposition 6.5.

PROPOSITION 7.1. *Let $\mathcal{X}$ be an irreducible scheme which is flat, separated and of finite type over Spec($\mathbb{Z}$). We have $R^n j_{\overline{\mathcal{X}},*}\tilde{\mathbb{R}} = 0$ for any $n \geq 1$.*

Proof. Recall that the morphism $j_{\overline{\mathcal{X}}}$ is defined by the following commutative diagram of topoi.

$$\begin{array}{ccc} B_{W_{K(\mathcal{X})}} & \xrightarrow{b} & \overline{\mathrm{Spec}(\mathbb{Z})}_W \\ {\scriptstyle a}\downarrow & & {\scriptstyle \gamma}\downarrow \\ \overline{\mathcal{X}}_{\mathrm{et}} & \xrightarrow{f} & \overline{\mathrm{Spec}(\mathbb{Z})}_{et} \end{array}$$

The site $(B_{Top}W_{K(\mathcal{X})}, \mathcal{J}_{ls})$ is a defining site for $B_{W_{K(\mathcal{X})}}$, and we denote by a^*, b^*, γ^* and f^* the morphism of sites inducing the morphism of topos a, b, γ and f. The morphism $j_{\overline{\mathcal{X}}} : B_{W_{K(\mathcal{X})}} \to \overline{\mathcal{X}}_W$ is induced by the morphism of sites :

$$\begin{array}{ccc} \mathcal{C}_{\overline{\mathcal{X}}} & \longrightarrow & B_{Top}W_{K(\mathcal{X})} \\ (\mathcal{U} \to V \leftarrow \mathcal{Z}) & \longmapsto & a^*\mathcal{U} \times_{a^*f^*V} b^*\mathcal{Z} \end{array}$$

Note that one has an identification $a^*\mathcal{U} \times_{a^*f^*V} b^*\mathcal{Z} = a^*\mathcal{U} \times_{b^*\gamma^*V} b^*\mathcal{Z}$. Consider the object of $T_{\overline{\mathrm{Spec}(\mathbb{Z})}}$ whose components are all given by the action of $W_{\mathbb{Q}}$ on $W_{\mathbb{Q}}/W^1_{\mathbb{Q}} \cong \mathbb{R}$:

$$(\mathbb{R}, \mathbb{R}, Id_{\mathbb{R}}) = \mathfrak{f}^*E\mathbb{R}.$$

Recall that $W^1_{\mathbb{Q}}$ denotes the maximal compact subgroup of $W_{\mathbb{Q}}$. Then $\mathfrak{f}^*E\mathbb{R}$ is a covering of the final object in $T_{\overline{\mathrm{Spec}(\mathbb{Z})}}$ for the local section topology, hence

$$\mathfrak{f}^*_{\overline{\mathcal{X}}}E\mathbb{R} = (* \to * \leftarrow \mathfrak{f}^*E\mathbb{R}) \longrightarrow (* \to * \leftarrow *)$$

is a covering of the final object of $\mathcal{C}_{\overline{\mathcal{X}}}$ for the topology $\mathcal{J}_{\overline{\mathcal{X}}}$.
The sheaf $R^n j_{\overline{\mathcal{X}},*}\tilde{\mathbb{R}}$ is the sheaf on $(\mathcal{C}_{\overline{\mathcal{X}}}, \mathcal{J}_{\overline{\mathcal{X}}})$ associated to the presheaf

$$\begin{array}{cccc} P^n j_{\overline{\mathcal{X}},*}\tilde{\mathbb{R}} : & \mathcal{C}_{\overline{\mathcal{X}}} & \longrightarrow & Ab \\ & (\mathcal{U} \to V \leftarrow \mathcal{Z}) & \longmapsto & H^n(B_{W_{K(\mathcal{X})}}, a^*\mathcal{U} \times_{a^*f^*V} b^*\mathcal{Z}, \tilde{\mathbb{R}}) \end{array}$$

Since the object $\mathfrak{f}^*_{\overline{\mathcal{X}}}E\mathbb{R}$ defined above covers the final object of $\mathcal{C}_{\overline{\mathcal{X}}}$, we can restrict our attention to the slice category $\mathcal{C}_{\overline{\mathcal{X}}}/\mathfrak{f}^*_{\overline{\mathcal{X}}}E\mathbb{R}$. Let $(\mathcal{U} \to V \leftarrow \mathcal{Z})$ be an object of $\mathcal{C}_{\overline{\mathcal{X}}}/\mathfrak{f}^*_{\overline{\mathcal{X}}}E\mathbb{R}$, i.e. $(\mathcal{U} \to V \leftarrow \mathcal{Z})$ is given with a map $\mathcal{Z} \to \mathfrak{f}^*_{\overline{\mathcal{X}}}E\mathbb{R}$ in $T_{\overline{\mathrm{Spec}(\mathbb{Z})}}$. We obtain a morphism

$$a^*\mathcal{U} \times_{a^*f^*V} b^*\mathcal{Z} \longrightarrow W_{K(\mathcal{X})}/W^1_{K(\mathcal{X})}$$

in the category $B_{Top}W_{K(\mathcal{X})}$, where the homogeneous space $(W_{K(\mathcal{X})}/W^1_{K(\mathcal{X})}) \cong \mathbb{R}$ is seen as an object of $B_{Top}W_{K(\mathcal{X})}$. Here $W^1_{K(\mathcal{X})}$ denotes the maximal compact subgroup of $W_{K(\mathcal{X})}$.
On the other hand the continuous morphism

$$W_{K(\mathcal{X})} \longrightarrow W_{K(\mathcal{X})}/W^1_{K(\mathcal{X})} = \mathbb{R}$$

has a global continuous section. This gives an isomorphism in $\mathcal{T}$

$$y(W_{K(\mathcal{X})}/W^1_{K(\mathcal{X})}) = yW_{K(\mathcal{X})}/yW^1_{K(\mathcal{X})}$$

and a canonical equivalence

$$B_{W_{K(\mathcal{X})}}/y(W_{K(\mathcal{X})}/W^1_{K(\mathcal{X})}) \cong B_{W_{K(\mathcal{X})}}/(yW_{K(\mathcal{X})}/yW^1_{K(\mathcal{X})}) \cong B_{W^1_{K(\mathcal{X})}}.$$

Under this equivalence the object represented by

$$\alpha : a^*\mathcal{U} \times_{a^*f^*V} b^*\mathcal{Z} \longrightarrow (W_{K(\mathcal{X})}/W^1_{K(\mathcal{X})}) \cong \mathbb{R}$$

corresponds to the object of $B_{W^1_{K(\mathcal{X})}}$ represented by the subspace

$$\alpha^{-1}(0) \subset a^*\mathcal{U} \times_{a^*f^*V} b^*\mathcal{Z}$$

endowed with the induced continuous action of $W^1_{K(\mathcal{X})}$. Thus one has

$$H^n(B_{W_{K(\mathcal{X})}}, a^*\mathcal{U} \times_{a^*f^*V} b^*\mathcal{Z}, \tilde{\mathbb{R}}) = H^n(B_{W^1_{K(\mathcal{X})}}, \alpha^{-1}(0), \tilde{\mathbb{R}}).$$

Therefore it is enough to prove

$$(34) \qquad H^n(B_{W^1_{K(\mathcal{X})}}, Z, \tilde{\mathbb{R}}) := H^n(B_{W^1_{K(\mathcal{X})}}/yZ, \tilde{\mathbb{R}} \times yZ) = 0,$$

for any object Z of $B_{Top}W^1_{K(\mathcal{X})}$ and any $n \geq 1$. We have two canonical equivalences

$$B_{W^1_{K(\mathcal{X})}}/EW^1_{K(\mathcal{X})} \cong \mathcal{T} \text{ and } (B_{W^1_{K(\mathcal{X})}}/yZ)/(EW^1_{K(\mathcal{X})} \times yZ) \cong \mathcal{T}/yZ.$$

We obtain a pull-back square

$$\begin{array}{ccc} \mathcal{T}/yZ & \xrightarrow{l'} & \mathcal{T} \\ {\scriptstyle h'}\downarrow & & {\scriptstyle h}\downarrow \\ B_{W^1_{K(\mathcal{X})}}/yZ & \xrightarrow{l} & B_{W^1_{K(\mathcal{X})}} \end{array}$$

where all the maps are localization morphisms (local homeomorphisms of topoi in the modern language). It follows easily that this pull-back square satisfies the Beck-Chevalley condition

$$h^*l_* \cong l'_*h'^*.$$

Moreover the functor h'^*, being a localization functor, preserves injective abelian objects. We obtain

$$(35) \qquad h^*R^n(l_*)\mathcal{A} \cong R^n(l'_*)h'^*\mathcal{A}$$

for any abelian object $\mathcal{A}$ of $B_{W^1_{K(\mathcal{X})}}/yZ$ and any $n \geq 0$. The forgetful functor h^* takes an object $\mathcal{F}$ of $\mathcal{T}$ endowed with an action of $yW^1_{K(\mathcal{X})}$ to $\mathcal{F}$. Hence $R^n(l_*)\mathcal{A}$ is the object $R^n(l'_*)\mathcal{A}$ endowed with the induced $yW^1_{K(\mathcal{X})}$-action.

Lemma 6. *We have $R^n(l'_*)(\tilde{\mathbb{R}} \times yZ) = 0$ for any $n \geq 1$.*

Proof. We consider the morphism $l' : \mathcal{T}/yZ \to \mathcal{T}$. The sheaf $R^n(l'_*)(\tilde{\mathbb{R}} \times yZ)$ on $\mathcal{T} = (Top^{lc}, \mathcal{J}_{ls})$ is the sheaf associated to the presheaf

$$\begin{array}{rccc} P^n(l'_*)(\tilde{\mathbb{R}} \times yZ) : & Top^{lc} & \longrightarrow & Ab \\ & T & \longmapsto & H^n(\mathcal{T}/y(Z \times T), \tilde{\mathbb{R}} \times y(Z \times T)) \end{array}$$

It is enough to show that $H^n(\mathcal{T}/yT', \tilde{\mathbb{R}} \times yT') = 0$ for any locally compact topological space $T' = Z \times T$. By ([20] IV.4.10.5) we have a canonical isomorphism

$$H^n(\mathcal{T}/yT', \tilde{\mathbb{R}} \times yT') = H^n(T', \mathcal{C}^0(T', \mathbb{R}))$$

where the right hand side is the usual sheaf cohomology of the *paracompact* space T' with values in the sheaf $\mathcal{C}^0(T', \mathbb{R})$ of continuous real valued functions on T'. It is well known that the sheaf $\mathcal{C}^0(T', \mathbb{R})$ is fine, hence acyclic for the global section functor. The Lemma follows. $\square$

Therefore the sheaf

$$h^* R^n(l_*)(\tilde{\mathbb{R}} \times yZ) \cong R^n(l'_*)h'^*(\tilde{\mathbb{R}} \times yZ)$$

vanishes for any $n \geq 1$, hence so does $R^n(l_*)(\tilde{\mathbb{R}} \times yZ)$. The spectral sequence

$$H^p(B_{W^1_{K(\mathcal{X})}}, R^q(l_*)(\tilde{\mathbb{R}} \times yZ)) \Rightarrow H^{p+q}(B_{W^1_{K(\mathcal{X})}}/yZ, \tilde{\mathbb{R}} \times yZ)$$

degenerates and yields an isomorphism

$$H^n(B_{W^1_{K(\mathcal{X})}}, l_*(\tilde{\mathbb{R}} \times yZ)) \cong H^n(B_{W^1_{K(\mathcal{X})}}/yZ, \tilde{\mathbb{R}} \times yZ)$$

for any $n \geq 0$. The sheaf $l_*(\tilde{\mathbb{R}} \times yZ)$ is given by the object $l'_*(\tilde{\mathbb{R}} \times yZ)$ of $\mathcal{T}$ endowed with the induced action of $yW^1_{K(\mathcal{X})}$, as it follows from (35). Furthermore, one has

$$l'_*(\tilde{\mathbb{R}} \times yZ) = l'_* l'^*(\tilde{\mathbb{R}}) = \underline{Hom}_{\mathcal{T}}(yZ, \tilde{\mathbb{R}})$$

where the right hand side is the internal Hom-object in $\mathcal{T}$ (see [20] IV Corollaire 10.8). The sheaf $\underline{Hom}_{\mathcal{T}}(yZ, \tilde{\mathbb{R}})$ is represented by the abelian topological group $\underline{Hom}_{Top}(Z, \mathbb{R})$ of continuous maps from Z to $\mathbb{R}$ endowed with the compact-open topology, since Z is locally compact. The compact-open topology on $\underline{Hom}_{Top}(Z, \mathbb{R})$ is the topology of uniform convergence on compact sets, since $\mathbb{R}$ is a metric space. The real vector space $\underline{Hom}_{Top}(Z, \mathbb{R})$ is locally convex, Hausdorff and complete (see [6] X.16. Corollaire 3). Note that the action of $W^1_{K(\mathcal{X})}$ on $\underline{Hom}_{Top}(Z, \mathbb{R})$ is induced by the action on Z, and that the group $W^1_{K(\mathcal{X})}$ is compact. By [15] Corollary 8, one has

$$H^n(B_{W^1_{K(\mathcal{X})}}, \underline{Hom}_{Top}(Z, \mathbb{R})) = 0.$$

In summary, for any locally compact topological space Z with a continuous action of $W^1_{K(\mathcal{X})}$ and any $n \geq 1$, one has

$$\begin{aligned} H^n(B_{W^1_{K(\mathcal{X})}}/yZ, \tilde{\mathbb{R}} \times yZ) &= H^n(B_{W^1_{K(\mathcal{X})}}, l_*(\tilde{\mathbb{R}} \times yZ)) \\ &= H^n(B_{W^1_{K(\mathcal{X})}}, \underline{Hom}_{Top}(Z, \mathbb{R})) \\ &= 0 \end{aligned}$$

Hence (34) holds and $R^n(j_{\overline{\mathcal{X}},*})\tilde{\mathbb{R}} = 0$ for any $n \geq 1$. $\square$

Lemma 7. *Let $\mathcal{X}$ be an irreducible scheme which is flat, separated and of finite type over Spec$(\mathbb{Z})$. If $\mathcal{X}$ is normal, then the adjunction map*

$$\mathfrak{f}^*_{\overline{\mathcal{X}}}\tilde{\mathbb{R}} \longrightarrow j_{\overline{\mathcal{X}}*}\, j^*_{\overline{\mathcal{X}}}\, \mathfrak{f}^*_{\overline{\mathcal{X}}}\tilde{\mathbb{R}} \cong j_{\overline{\mathcal{X}}*}\tilde{\mathbb{R}}$$

is an isomorphism.

Proof. Firstly, we need to restrict the site $\mathcal{C}_{\overline{\mathcal{X}}}$. The class of connected étale $\overline{\mathcal{X}}$-schemes (respectively of connected étale $\overline{\mathrm{Spec}(\mathbb{Z})}$-schemes) is a topologically generating family for the étale site of $\overline{\mathcal{X}}$ (respectively of $\overline{\mathrm{Spec}(\mathbb{Z})}$). It follows easily that the subcategory $\mathcal{C}'_{\overline{\mathcal{X}}} \subset \mathcal{C}_{\overline{\mathcal{X}}}$, consisting in objects $(\mathcal{U} \to V \leftarrow \mathcal{Z})$ of $\mathcal{C}_{\overline{\mathcal{X}}}$ such that $\mathcal{U}$ and V are both connected, is a topologically generating family. Then we endow the full subcategory $\mathcal{C}'_{\overline{\mathcal{X}}}$ with the induced topology via the natural fully faithful functor

$$\mathcal{C}'_{\overline{\mathcal{X}}} \hookrightarrow \mathcal{C}_{\overline{\mathcal{X}}}.$$

Then $\mathcal{C}'_{\overline{\mathcal{X}}}$ is a defining site for the topos $\overline{\mathcal{X}}_W$.

The composite map $\mathfrak{f}_{\overline{\mathcal{X}}}\circ j_{\overline{\mathcal{X}}} : B_{W_{K(\mathcal{X})}} \to \overline{\mathcal{X}}_W \to B_{\mathbb{R}}$ is induced by the morphism of topological groups $W_{K(\mathcal{X})} \to \mathbb{R}$. The canonical isomorphism $j^*_{\overline{\mathcal{X}}}\mathfrak{f}^*_{\overline{\mathcal{X}}}\tilde{\mathbb{R}} \cong \tilde{\mathbb{R}}$ induces

$$j_{\overline{\mathcal{X}}*}\, j^*_{\overline{\mathcal{X}}}\, \mathfrak{f}^*_{\overline{\mathcal{X}}}\tilde{\mathbb{R}} \cong j_{\overline{\mathcal{X}}*}\tilde{\mathbb{R}}.$$

On the one hand $\mathfrak{f}^*_{\overline{\mathcal{X}}}\tilde{\mathbb{R}}$ is the sheaf associated to the abelian presheaf

$$\begin{array}{llll} \mathfrak{f}^p_{\overline{\mathcal{X}}}\tilde{\mathbb{R}} : & \mathcal{C}'_{\overline{\mathcal{X}}} & \longrightarrow & Ab \\ & (\mathcal{U} \to V \leftarrow \mathcal{Z}) & \longmapsto & Hom_{\mathcal{C}'_{\overline{\mathcal{X}}}}((\mathcal{U} \to V \leftarrow \mathcal{Z}), (* \to * \leftarrow \mathfrak{f}^*\tilde{\mathbb{R}})) \end{array}$$

where $\mathfrak{f}^*\tilde{\mathbb{R}}$ denotes the object $(\mathbb{R}, \mathbb{R}, Id)$ of $T_{\overline{\mathrm{Spec}(\mathbb{Z})}}$ (with trivial action of the Weil groups on $\mathbb{R}$). For any object $(\mathcal{U} \to V \leftarrow \mathcal{Z})$ of $\mathcal{C}'_{\overline{\mathcal{X}}}$ with $\mathcal{Z} = (Z_0, Z_v, f_v)$, one has

$$\begin{aligned} \mathfrak{f}^p_{\overline{\mathcal{X}}}\tilde{\mathbb{R}}(\mathcal{U} \to V \leftarrow \mathcal{Z}) &= Hom_{\mathcal{C}'_{\overline{\mathcal{X}}}}((\mathcal{U} \to V \leftarrow \mathcal{Z}), (* \to * \leftarrow \mathfrak{f}^*\tilde{\mathbb{R}})) \\ &= Hom_{T_{\overline{\mathrm{Spec}(\mathbb{Z})}}}(\mathcal{Z}, \mathfrak{f}^*\tilde{\mathbb{R}}) \\ &= Hom_{B_{Top}W_{\mathbb{Q}}}(Z_0, \mathbb{R}) \\ &= Hom_{Top}(Z_0/W_{\mathbb{Q}}, \mathbb{R}). \end{aligned}$$

One the other hand, the morphism $j_{\mathcal{X}}$ is induced by the continuous functor:

$$\begin{array}{ccc} \mathcal{C}'_{\overline{\mathcal{X}}} & \longrightarrow & B_{Top}W_{K(\mathcal{X})} \\ (\mathcal{U} \to V \leftarrow \mathcal{Z}) & \longmapsto & a^*\mathcal{U} \times_{a^*f^*V} b^*\mathcal{Z} \end{array}$$

Hence the direct image $j_{\overline{\mathcal{X}}*}$ is given by

$$j_{\overline{\mathcal{X}}*}\tilde{\mathbb{R}}(\mathcal{U} \to V \leftarrow \mathcal{Z}) = Hom_{B_{Top}W_{K(\mathcal{X})}}(a^*\mathcal{U} \times_{a^*f^*V} b^*\mathcal{Z}, \mathbb{R}).$$

Here the topological group $\mathbb{R}$ is given with the trivial action of $W_{K(\mathcal{X})}$. We set

$$\mathcal{U}_0 := a^*\mathcal{U} \text{ and } V_0 := a^*f^*V.$$

Note that $\mathcal{U}_0$ (respectively V_0) is given by the finite $G_{K(\mathcal{X})}$-set (respectively the finite $G_{\mathbb{Q}}$-set) corresponding, via Galois theory, to the étale $K(\mathcal{X})$-scheme $\mathcal{U} \times_{\mathcal{X}}$

$\operatorname{Spec} K(\mathcal{X})$ (respectively to the étale $\mathbb{Q}$-scheme $V\otimes\mathbb{Q}$). Here, $\mathcal{U}_0$ (respectively V_0) is considered as a finite set on which $W_{K(\mathcal{X})}$ acts via $W_{K(\mathcal{X})}\to G_{K(\mathcal{X})}$ (respectively via $W_{K(\mathcal{X})}\to G_{\mathbb{Q}}$). Finally, $W_{K(\mathcal{X})}$ acts on the space $Z_0:=b^*\mathcal{Z}$ via $W_{K(\mathcal{X})}\to W_{\mathbb{Q}}$.
Since $W_{K(\mathcal{X})}$ acts trivially on $\mathbb{R}$, one has

$$\begin{aligned} j_{\overline{\mathcal{X}}*}\tilde{\mathbb{R}}(\mathcal{U}\to V\leftarrow\mathcal{Z}) &= Hom_{B_{Top}W_{K(\mathcal{X})}}(a^*\mathcal{U}\times_{a^*f^*V}b^*\mathcal{Z},\mathbb{R})\\ &= Hom_{Top}((\mathcal{U}_0\times_{V_0}Z_0)/W_{K(\mathcal{X})},\mathbb{R}).\end{aligned}$$

(i) The map $\mathfrak{f}^*_{\overline{\mathcal{X}}}\tilde{\mathbb{R}}\to j_{\overline{\mathcal{X}}*}\tilde{\mathbb{R}}$ is a monomorphism.
The morphism $\mathfrak{f}^*_{\overline{\mathcal{X}}}\tilde{\mathbb{R}}\to j_{\overline{\mathcal{X}}*}\tilde{\mathbb{R}}$ is given by adjunction. It is induced by the morphism of presheaves on $\mathcal{C}'_{\overline{\mathcal{X}}}$ given by the functorial map

$$Hom_{Top}(Z_0/W_{\mathbb{Q}},\mathbb{R})\longrightarrow Hom_{Top}((\mathcal{U}_0\times_{V_0}Z_0)/W_{K(\mathcal{X})},\mathbb{R}) \tag{36}$$

which is in turn induced by the continuous map

$$(\mathcal{U}_0\times_{V_0}Z_0)/W_{K(\mathcal{X})}\longrightarrow Z_0/W_{\mathbb{Q}}. \tag{37}$$

Let $(\mathcal{U}\to V\leftarrow\mathcal{Z})$ be an object of $\mathcal{C}'_{\overline{\mathcal{X}}}$. Hence $\mathcal{U}$ and V are both connected. Since $\mathcal{U}$ and V are both normal, the schemes $\mathcal{U}\times_{\mathcal{X}}\operatorname{Spec}(K(\mathcal{X}))$ and $V\times_{\overline{\operatorname{Spec}(\mathbb{Z})}}\operatorname{Spec}(\mathbb{Q})$ are connected as well. By Galois theory, the Galois groups $G_{K(\mathcal{X})}$ and $G_{\mathbb{Q}}$ act transitively on $\mathcal{U}_0$ and V_0 respectively. Hence the Weil groups $W_{K(\mathcal{X})}$ and $W_{\mathbb{Q}}$ act transitively on $\mathcal{U}_0$ and V_0 respectively.
We have maps of compactified schemes

$$\mathcal{U}\to\overline{\mathcal{X}}\to\overline{\operatorname{Spec}(\mathbb{Z})}\text{ and }\mathcal{U}\to V\to\overline{\operatorname{Spec}(\mathbb{Z})}.$$

We consider the subfield $L(\mathcal{U})$ of $K(\mathcal{U})$ consisting in elements of $K(\mathcal{U})$ that are algebraic over $\mathbb{Q}$, i.e. we set

$$L(\mathcal{U}):=K(\mathcal{U})\cap\overline{\mathbb{Q}}.$$

Note that $\mathcal{U}$ is normal and connected, hence irreducible, so that its function field $K(\mathcal{U})$ is well defined. We consider the arithmetic curve $\overline{\operatorname{Spec}(\mathcal{O}_{L(\mathcal{U})})}$. Since $\mathcal{U}$ is normal, we have a canonical map

$$\mathcal{U}\longrightarrow\overline{\operatorname{Spec}(\mathcal{O}_{L(\mathcal{U})})}.$$

We denote by U' the (open) image of $\mathcal{U}$ in $\overline{\operatorname{Spec}(\mathcal{O}_{L(\mathcal{U})})}$. Then we have a factorization

$$\mathcal{U}\to U'\to V\to\overline{\operatorname{Spec}(\mathbb{Z})}$$

since $V\to\overline{\operatorname{Spec}(\mathbb{Z})}$ is étale.
The group $W_{K(\mathcal{X})}$ acts transitively on $\mathcal{U}_0$, hence the choice of a base point $u_0\in\mathcal{U}_0$ induces an isomorphism of $W_{K(\mathcal{X})}$-sets :

$$\alpha:\mathcal{U}_0\cong W_{K(\mathcal{X})}/W_{K(\mathcal{U})}=G_{K(\mathcal{X})}/G_{K(\mathcal{U})}.$$

We fix such a base point $u_0 \in \mathcal{U}_0$. Then one has a 1-1 correspondence

$$\begin{array}{ccc} (\mathcal{U}_0 \times Z_0)/W_{K(\mathcal{X})} & \longrightarrow & Z_0/W_{K(\mathcal{U})} = Z_0/W_{L(\mathcal{U})} \\ (u_0, z) & \longmapsto & z \end{array}$$

where the equality $Z_0/W_{K(\mathcal{U})} = Z_0/W_{L(\mathcal{U})}$ follows from the fact that the image of $W_{K(\mathcal{U})}$ in $W_{\mathbb{Q}}$ is precisely $W_{L(\mathcal{U})}$.
We consider now the commutative diagram of topological spaces

$$\begin{array}{ccc} (\mathcal{U}_0 \times_{V_0} Z_0)/W_{K(\mathcal{X})} & \xrightarrow{(37)} & Z_0/W_{\mathbb{Q}} \\ {\scriptstyle i}\downarrow & & \uparrow{\scriptstyle s} \\ (\mathcal{U}_0 \times Z_0)/W_{K(\mathcal{X})} & \underset{\alpha}{\xrightarrow{\cong}} & Z_0/W_{L(\mathcal{U})} \end{array}$$

where i is injective and s is surjective. We denote by v_0 the image of $u_0 \in \mathcal{U}_0$ in V_0 (note that $\mathcal{U} \to V$ induces a map $\mathcal{U}_0 \to V_0$).
Let $\overline{z} \in Z_0/W_{\mathbb{Q}}$. There exists $w \in W_{\mathbb{Q}}$ such that $w.z$ goes to v_0 under the $W_{\mathbb{Q}}$-equivariant map $Z_0 \to V_0$, since $W_{\mathbb{Q}}$ acts transitively on V_0. Then one has $\overline{w.z} = \overline{z}$, $(u_0, w.z) \in \mathcal{U}_0 \times_{V_0} Z_0$, and

$$s \circ \alpha \circ i\overline{(u_0, w.z)} = s \circ \alpha\overline{(u_0, w.z)} = s(\overline{w.z}) = \overline{z}$$

where $\overline{*}$ stands for the orbit of some point $*$ under some group action. Using the previous commutative diagram, this shows that the map (37) is surjective whenever $\mathcal{U}$ and V are both connected. Hence the map (36) is injective for any object $(\mathcal{U} \to V \leftarrow \mathcal{Z})$ of $\mathcal{C}'_{\overline{\mathcal{X}}}$. In other words the morphism of presheaves on $\mathcal{C}'_{\overline{\mathcal{X}}}$

$$\mathfrak{f}^{p}_{\overline{\mathcal{X}}}\tilde{\mathbb{R}} \longrightarrow j_{\overline{\mathcal{X}}*}\tilde{\mathbb{R}}$$

is injective. Since the associated sheaf functor is exact, the morphism of sheaves

$$\mathfrak{f}^{*}_{\overline{\mathcal{X}}}\tilde{\mathbb{R}} \longrightarrow j_{\overline{\mathcal{X}}*}\tilde{\mathbb{R}}$$

is injective.
(II) THE MAP $\mathfrak{f}^{*}_{\overline{\mathcal{X}}}\tilde{\mathbb{R}} \to j_{\overline{\mathcal{X}}*}\tilde{\mathbb{R}}$ IS AN EPIMORPHISM.
One has

$$\mathfrak{f}^{*}_{\overline{\mathcal{X}}}\tilde{\mathbb{R}}(\mathcal{U} \to * \leftarrow \mathcal{Z}) = Hom_{\overline{\mathcal{X}}_W}(\varepsilon\mathcal{U} \times \varepsilon\mathcal{Z}; \mathfrak{f}^{*}_{\overline{\mathcal{X}}}\tilde{\mathbb{R}}).$$

Here we denote by $*$ the final object $Et_{\overline{\mathrm{Spec}(\mathbb{Z})}}$. Moreover, we denote by $\varepsilon\mathcal{U}$ (respectively by $\varepsilon\mathcal{Z}$) the sheaf on the topos $\overline{\mathcal{X}}_W$ associated to the presheaf represented by $(\mathcal{U} \to * \leftarrow *)$ (respectively by $(* \to * \leftarrow \mathcal{Z})$), where $*$ stands for the final object of the corresponding site. Finally, the product $\varepsilon\mathcal{U} \times \varepsilon\mathcal{Z}$ is computed in the topos $\overline{\mathcal{X}}_W$. By adjunction, we have

$$\begin{aligned} \mathfrak{f}^{*}_{\overline{\mathcal{X}}}\tilde{\mathbb{R}}(\mathcal{U} \to * \leftarrow \mathcal{Z}) &= Hom_{\overline{\mathcal{X}}_W}(\varepsilon\mathcal{U} \times \varepsilon\mathcal{Z}; \mathfrak{f}^{*}_{\overline{\mathcal{X}}}\tilde{\mathbb{R}}) \\ &= Hom_{\overline{\mathcal{X}}_W/\varepsilon\mathcal{U}}(\varepsilon\mathcal{U} \times \varepsilon\mathcal{Z}, \varepsilon\mathcal{U} \times \mathfrak{f}^{*}_{\overline{\mathcal{X}}}\tilde{\mathbb{R}}) \end{aligned}$$

where we consider the slice topos $\overline{\mathcal{X}}_W/\varepsilon\mathcal{U}$. On the other hand we have

$$\begin{aligned}\overline{\mathcal{X}}_W/\varepsilon\mathcal{U} &= (\overline{\mathcal{X}}_{\mathrm{et}} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \overline{\mathrm{Spec}(\mathbb{Z})}_W)/\varepsilon\mathcal{U}\\ &\cong (\overline{\mathcal{X}}_{\mathrm{et}}/y\mathcal{U}) \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \overline{\mathrm{Spec}(\mathbb{Z})}_W\\ &\cong \mathcal{U}_{\mathrm{et}} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \overline{\mathrm{Spec}(\mathbb{Z})}_W\\ &=: \mathcal{U}_W\\ &\cong \mathcal{U}_{\mathrm{et}} \times_{U'_{\mathrm{et}}} U'_{\mathrm{et}} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{\mathrm{et}}} \overline{\mathrm{Spec}(\mathbb{Z})}_W\\ &\cong \mathcal{U}_{\mathrm{et}} \times_{U'_{\mathrm{et}}} U'_W\end{aligned}$$

where $U' \subseteq \overline{\mathrm{Spec}(\mathcal{O}_{L(\mathcal{U})})}$ is defined as in the proof of step (i). The last equivalence above is given by Proposition 5.5. Hence the fiber product site $\mathcal{C}_{\mathcal{U}}$ for $\mathcal{U}_{\mathrm{et}} \times_{U'_{\mathrm{et}}} U'_W$ given by the sites $Et_{\mathcal{U}}$, $Et_{U'}$ and $T_{U'}$, is a defining site for the topos $\mathcal{U}_W$. Then the object $\varepsilon\mathcal{U} \times \varepsilon\mathcal{Z}$ of

$$\overline{\mathcal{X}}_W/\varepsilon\mathcal{U} = \mathcal{U}_W \cong \mathcal{U}_{\mathrm{et}} \times_{U'_{\mathrm{et}}} U'_W$$

is the sheaf associated to the presheaf on $\mathcal{C}_{\mathcal{U}}$ represented by $(* \to * \leftarrow \mathcal{Z})$ where $\mathcal{Z} = (Z_0, Z_v, f_v)$ is seen as an object of $T_{U'}$ by restricting the group of operators on Z_0 and Z_v for any place v of $\mathbb{Q}$. Moreover, the object $\varepsilon\mathcal{U} \times \mathfrak{f}^*_{\overline{\mathcal{X}}}\tilde{\mathbb{R}}$ of $\overline{\mathcal{X}}_W/\varepsilon\mathcal{U} = \mathcal{U}_W$ is precisely $\mathfrak{f}^*_{\mathcal{U}}\tilde{\mathbb{R}}$. It is the sheaf associated to the presheaf $\mathfrak{f}^p_{\mathcal{U}}\tilde{\mathbb{R}}$ on $\mathcal{C}_{\mathcal{U}}$ represented by $(* \to * \leftarrow \mathfrak{f}^*_{U'}\tilde{\mathbb{R}})$. Therefore, we have

$$\begin{aligned}\mathfrak{f}^p_{\mathcal{U}}\tilde{\mathbb{R}}(* \to * \leftarrow \mathcal{Z}) &= Hom_{\mathcal{C}_{\overline{\mathcal{X}}}}((* \to * \leftarrow \mathcal{Z}), (* \to * \leftarrow \mathfrak{f}^*_{U'}\tilde{\mathbb{R}}))\\ &= Hom_{T_{U'}}(\mathcal{Z}, \mathfrak{f}^*_{U'}\tilde{\mathbb{R}})\\ &= Hom_{B_{Top}W_{L(\mathcal{U})}}(Z_0, \mathbb{R})\\ &= Hom_{Top}(Z_0/W_{L(\mathcal{U})}, \mathbb{R})\end{aligned}$$

By the universal property of the associated sheaf functor, we obtain a map from

$$\mathfrak{f}^p_{\mathcal{U}}\tilde{\mathbb{R}}(* \to * \leftarrow \mathcal{Z}) = Hom_{Top}(Z_0/W_{L(\mathcal{U})}, \mathbb{R})$$

to the set

$$\begin{aligned}\mathfrak{f}^*_{\mathcal{U}}\tilde{\mathbb{R}}(* \to * \leftarrow \mathcal{Z}) &= Hom_{\mathcal{U}_W}(\varepsilon\mathcal{U} \times \varepsilon\mathcal{Z}, \mathfrak{f}^*_{\mathcal{U}}\tilde{\mathbb{R}})\\ &= Hom_{\overline{\mathcal{X}}_W/\varepsilon\mathcal{U}}(\varepsilon\mathcal{U} \times \varepsilon\mathcal{Z}, \varepsilon\mathcal{U} \times \mathfrak{f}^*_{\overline{\mathcal{X}}}\tilde{\mathbb{R}})\\ &= Hom_{\overline{\mathcal{X}}_W}(\varepsilon\mathcal{U} \times \varepsilon\mathcal{Z}, \mathfrak{f}^*_{\overline{\mathcal{X}}}\tilde{\mathbb{R}})\\ &= \mathfrak{f}^*_{\overline{\mathcal{X}}}\tilde{\mathbb{R}}(\mathcal{U} \to * \leftarrow \mathcal{Z})\end{aligned}$$

Composing this map $\mathfrak{f}^p_{\mathcal{U}}\tilde{\mathbb{R}}(* \to * \leftarrow \mathcal{Z}) \to \mathfrak{f}^*_{\overline{\mathcal{X}}}\tilde{\mathbb{R}}(\mathcal{U} \to * \leftarrow \mathcal{Z})$ with

$$\mathfrak{f}^*_{\overline{\mathcal{X}}}\tilde{\mathbb{R}}(\mathcal{U} \to * \leftarrow \mathcal{Z}) \longrightarrow j_{\overline{\mathcal{X}}*}\tilde{\mathbb{R}}(\mathcal{U} \to * \leftarrow \mathcal{Z}), \tag{38}$$

we obtain the natural bijective map from

$$\mathfrak{f}^p_{\mathcal{U}}\tilde{\mathbb{R}}(* \to * \leftarrow \mathcal{Z}) = Hom_{Top}(Z_0/W_{L(\mathcal{U})}, \mathbb{R})$$

to the set

$$j_{\overline{\mathcal{X}}*}\tilde{\mathbb{R}}(\mathcal{U} \to * \leftarrow \mathcal{Z}) = Hom_{Top}(Z_0/W_{L(\mathcal{U})}, \mathbb{R}).$$

It follows that the map (38) is surjective.
It remains to show that the map

$$\mathfrak{f}^*_{\overline{\mathcal{X}}}\tilde{\mathbb{R}}(\mathcal{U} \to V \leftarrow \mathcal{Z}) \longrightarrow j_{\overline{\mathcal{X}}*}\tilde{\mathbb{R}}(\mathcal{U} \to V \leftarrow \mathcal{Z}) \tag{39}$$

is surjective when V is not necessarily the final object. For any object $(\mathcal{U} \to V \leftarrow \mathcal{Z})$ of $\mathcal{C}'_{\overline{\mathcal{X}}}$, we consider the following commutative diagram

$$\begin{array}{ccc}
\mathfrak{f}^*_{\overline{\mathcal{X}}}\tilde{\mathbb{R}}(\mathcal{U} \to * \leftarrow \mathcal{Z}) & \xrightarrow{(38)} & j_{\overline{\mathcal{X}}*}\tilde{\mathbb{R}}(\mathcal{U} \to * \leftarrow \mathcal{Z}) \\
\downarrow & & p\downarrow \\
\mathfrak{f}^*_{\overline{\mathcal{X}}}\tilde{\mathbb{R}}(\mathcal{U} \to V \leftarrow \mathcal{Z}) & \xrightarrow{(39)} & j_{\overline{\mathcal{X}}*}\tilde{\mathbb{R}}(\mathcal{U} \to V \leftarrow \mathcal{Z})
\end{array}$$

We have proven above that the map (38) is surjective. The vertical arrow p is the natural map from

$$j_{\overline{\mathcal{X}}*}\tilde{\mathbb{R}}(\mathcal{U} \to * \leftarrow \mathcal{Z}) = Hom_{B_{Top}W_{K(\mathcal{X})}}(\mathcal{U}_0 \times Z_0, \mathbb{R})$$

to the set

$$j_{\overline{\mathcal{X}}*}\tilde{\mathbb{R}}(\mathcal{U} \to V \leftarrow \mathcal{Z}) = Hom_{B_{Top}W_{K(\mathcal{X})}}(\mathcal{U}_0 \times_{V_0} Z_0, \mathbb{R}),$$

which is surjective as well. Indeed, $\mathcal{U}_0 \times_{V_0} Z_0$ is an open and closed $W_{K(\mathcal{X})}$-equivariant subspace of $\mathcal{U}_0 \times Z_0$, hence any equivariant continuous map $\mathcal{U}_0 \times_{V_0} Z_0 \to \mathbb{R}$ extends to an equivariant continuous map $\mathcal{U}_0 \times Z_0 \to \mathbb{R}$. It follows immediately from the previous commutative diagram that the map (39) is surjective, for any object $(\mathcal{U} \to V \leftarrow \mathcal{Z})$ of $\mathcal{C}'_{\overline{\mathcal{X}}}$. Therefore the morphism of sheaves $\mathfrak{f}^*_{\overline{\mathcal{X}}}\tilde{\mathbb{R}} \to j_{\overline{\mathcal{X}}*}\tilde{\mathbb{R}}$ is surjective. ☐

Theorem 7.1. *Let $\mathcal{X}$ be an irreducible scheme which is flat, separated and of finite type over $Spec(\mathbb{Z})$. If $\mathcal{X}$ is normal then the morphism*

$$\mathfrak{f}^*_{\overline{\mathcal{X}}} : H^n(B_{\mathbb{R}}, \tilde{\mathbb{R}}) \longrightarrow H^n(\overline{\mathcal{X}}_W, \tilde{\mathbb{R}})$$

is an isomorphism for any $n \geq 0$.

Proof. The Leray spectral sequence

$$H^p(\overline{\mathcal{X}}_W, R^q j_{\overline{\mathcal{X}}*}\tilde{\mathbb{R}}) \Rightarrow H^{p+q}(B_{W_{K(\mathcal{X})}}, \tilde{\mathbb{R}})$$

degenerates by Proposition 7.1. This shows that the canonical morphism

$$H^n(\overline{\mathcal{X}}_W, \tilde{\mathbb{R}}) \cong H^n(\overline{\mathcal{X}}_W, j_{\overline{\mathcal{X}}*}\tilde{\mathbb{R}}) \longrightarrow H^n(B_{W_{K(\mathcal{X})}}, \tilde{\mathbb{R}})$$

is an isomorphism, where the first identification is given by Lemma 7. These cohomology groups can be computed using the spectral sequence associated with the extension

$$1 \to W^1_{K(\mathcal{X})} \to W_{K(\mathcal{X})} \to \mathbb{R} \to 1.$$

Indeed, localizing along $E\mathbb{R}$, we obtain a pull-back square

$$\begin{tikzcd} B_{W^1_{K(\mathcal{X})}} \arrow[r, "q'"] \arrow[d, "p'"'] & \mathcal{T} \cong B_{\mathbb{R}}/E\mathbb{R} \arrow[d, "p"] \\ B_{W_{K(\mathcal{X})}} \arrow[r, "q"] & B_{\mathbb{R}} \end{tikzcd} \tag{40}$$

This gives an isomorphism

$$p^*R^n(q_*) \cong R^n(q'_*)p'^*$$

for any $n \geq 0$. The argument of the proof of Proposition 7.1 shows that $R^n(q'_*)\tilde{\mathbb{R}} = 0$ for any $n \geq 1$. Hence $R^n(q_*)\tilde{\mathbb{R}} = 0$ for any $n \geq 1$. Moreover q is connected, i.e. q^* is fully faithful, hence we have

$$q_*\tilde{\mathbb{R}} = q_*q^*\tilde{\mathbb{R}} = \tilde{\mathbb{R}}.$$

Therefore, the Leray spectral sequence given by the morphism q

$$H^i(B_{\mathbb{R}}, R^j(q_*)\tilde{\mathbb{R}}) \Rightarrow H^{i+j}(B_{W_{K(\mathcal{X})}}, \tilde{\mathbb{R}})$$

degenerates and yields

$$H^n(B_{\mathbb{R}}, \tilde{\mathbb{R}}) = H^n(B_{\mathbb{R}}, q_*\tilde{\mathbb{R}}) \cong H^n(B_{W_{K(\mathcal{X})}}, \tilde{\mathbb{R}})$$

for any $n \geq 0$. The result follows. □

8. Compact support Cohomology of $\mathcal{X}_W$ with $\tilde{\mathbb{R}}$-coefficients

Throughout this section, the arithmetic scheme $\mathcal{X}$ is supposed to be irreducible, normal, flat, and proper over Spec($\mathbb{Z}$).

8.1. The morphism $\gamma_{\overline{\mathcal{X}}} : \overline{\mathcal{X}}_W \to \overline{\mathcal{X}}_{et}$. Recall the notion of étale $\overline{\mathcal{X}}$-scheme defined in section 4. An étale $\overline{\mathcal{X}}$-scheme is in particular a topological space so that it makes sense to speak of connected étale $\overline{\mathcal{X}}$-schemes. Theorem 7.1 yields the following result.

Corollary 10. *For any connected étale $\overline{\mathcal{X}}$-scheme $\overline{\mathcal{U}}$, the morphism*

$$\mathfrak{f}^*_{\overline{\mathcal{U}}} : H^n(B_{\mathbb{R}}, \tilde{\mathbb{R}}) \longrightarrow H^n(\overline{\mathcal{U}}_W, \tilde{\mathbb{R}})$$

is an isomorphism for any $n \geq 0$. In particular, one has $H^n(\overline{\mathcal{U}}_W, \tilde{\mathbb{R}}) = \mathbb{R}$ for $n = 0, 1$ and $H^n(\overline{\mathcal{U}}_W, \tilde{\mathbb{R}}) = 0$ for $n \geq 2$.

Proof. This is clear from the fact that an étale $\overline{\mathcal{X}}$-scheme $\overline{\mathcal{U}} = (\mathcal{U}, D)$ is connected if and only if the scheme $\mathcal{U}$ is connected, and Theorem 7.1 applies to $\mathcal{U}$. □

Recall that $\gamma_{\overline{\mathcal{X}}} : \overline{\mathcal{X}}_W \to \overline{\mathcal{X}}_{et}$ is the projection induced by Definition 10.

Proposition 8.1. *The sheaf $R^n\gamma_{\overline{\mathcal{X}}*}(\tilde{\mathbb{R}})$ is the constant étale sheaf on $\overline{\mathcal{X}}$ associated to the discrete abelian group $\mathbb{R}$ for $n = 0, 1$ and $R^n\gamma_{\overline{\mathcal{X}}*}(\tilde{\mathbb{R}}) = 0$ for $n \geq 2$.*

Proof. For any $n \geq 0$, $R^n\gamma_{\overline{\mathcal{X}}*}(\tilde{\mathbb{R}})$ is the sheaf associated to the presheaf

$$\begin{array}{ccc} Et_{\overline{\mathcal{X}}} & \longrightarrow & Ab \\ \overline{\mathcal{U}} & \longmapsto & H^n(\overline{\mathcal{X}}_W/\gamma^*_{\overline{\mathcal{X}}}\overline{\mathcal{U}}, \tilde{\mathbb{R}}) \end{array}$$

Hence the result follows immediately from Corollary 10, since $\overline{\mathcal{X}}_W/\gamma^*_{\overline{\mathcal{X}}}\overline{\mathcal{U}} \cong \overline{\mathcal{U}}_W$. □

8.2. THE MORPHISM $\gamma_\infty : \mathcal{X}_{\infty,W} \to \mathcal{X}_\infty$.

8.2.1. If T is a locally compact space (or any space), one can define the big topos $TOP(T)$ of T as the category of sheaves on the site $(Top/T, \mathcal{J}_{op})$ where $\mathcal{J}_{op}$ is the open cover topology. It is well known that the natural morphism $TOP(T) \to Sh(T)$ is a cohomological equivalence. The following lemma gives a slight generalization of this result.

LEMMA 8. *Let T be an object of Top. Let $\mathcal{T}/_T$ be the big topos of T and let $Sh(T)$ be its small topos. For any topos $\mathcal{S}$, the canonical morphism*

$$f : \mathcal{T}/_T \times \mathcal{S} \longrightarrow Sh(T) \times \mathcal{S}$$

has a section s such that $s^* \cong f_*$.

Proof. We first observe that one has a canonical equivalence

$$TOP(T) := (\widetilde{Top/T, \mathcal{J}_{op}}) \cong \mathcal{T}/_T,$$

where $TOP(T)$ is the big topos of the topological space T. In what follows, we shall identify $\mathcal{T}/_T$ with $TOP(T)$. The morphism $f' : \mathcal{T}/_T \to Sh(T)$ has a canonical section $s' : Sh(T) \to \mathcal{T}/_T$, hence the map

$$f = (f', Id_{\mathcal{S}}) : \mathcal{T}/_T \times \mathcal{S} \longrightarrow Sh(T) \times \mathcal{S}$$

has a section

$$s := (s', Id_{\mathcal{S}}) : Sh(T) \times \mathcal{S} \longrightarrow \mathcal{T}/_T \times \mathcal{S}.$$

Moreover, we have $s'^* = f'_*$ hence a sequence of three adjoint functors

$$f'^*, \;\; f'_* = s'^*, \;\; s'_*.$$

The functor $f'_* = s'^*$ is called *restriction* and is denoted by Res. The functor f'^* is called *prolongement* and is denoted by $Prol$.
The category $Op(*)$ of open sets of the one point space is a defining site for the final topos $\underline{Set}$. The site $(\mathcal{S}, \mathcal{J}_{can})$ and $(\mathcal{T}/_T, \mathcal{J}_{can})$ can be seen as sites for the topoi $\mathcal{S}$ and $\mathcal{T}/_T$ respectively, where $\mathcal{J}_{can}$ denotes the canonical topology. Then the morphisms f' and s' are induced by the left exact continuous functors f'^* and s'^* respectively. A site for $\mathcal{T}/_T \times \mathcal{S}$ (respectively for $Sh(T) \times \mathcal{S}$) is given by the category $\mathcal{C}$ of objects of the form $(\mathcal{F} \to * \leftarrow S)$ (respectively by the category C of objects $(F \to * \leftarrow S)$). Here S is an object of $\mathcal{S}$, F is an étalé space on T, $\mathcal{F}$ is a big sheaf on T and $*$ is the set with one element. The

categories C and $\mathcal{C}$ both have an initial object $(\emptyset \to \emptyset \leftarrow \emptyset)$. The morphism of topoi f is induced by the morphism of sites

$$\begin{array}{cccc} f^{-1}: & C & \longrightarrow & \mathcal{C} \\ & (F \to * \leftarrow S) & \longmapsto & (Prol(F) \to * \leftarrow S) \end{array}$$

and the morphism s is induced by the morphism of sites

$$\begin{array}{cccc} s^{-1}: & \mathcal{C} & \longrightarrow & C \\ & (\mathcal{F} \to * \leftarrow S) & \longmapsto & (Res(\mathcal{F}) \to * \leftarrow S) \end{array}$$

Let $\mathcal{L}$ be a sheaf of $\mathcal{T}/_T \times \mathcal{S}$. Then one has

$$f_*\mathcal{L}(F \to * \leftarrow S) = \mathcal{L}(Prol(F) \to * \leftarrow S)$$

for any object $(F \to * \leftarrow S)$ of C. On the other hand, $s^*\mathcal{L}$ is the sheaf associated with the presheaf

$$\begin{array}{cccc} s^p\mathcal{L}: & C & \longrightarrow & \underline{Set} \\ & (F \to * \leftarrow S) & \longmapsto & \varinjlim \mathcal{L}(\mathcal{F} \to * \leftarrow S') \end{array}$$

where the direct limit is taken over the category of arrows

$$(F \to * \leftarrow S) \longrightarrow (Res(\mathcal{F}) \to * \leftarrow S').$$

But this category has an initial object given by $(Prol(F) \to * \leftarrow S)$, since $Prol$ is left adjoint to Res. We obtain

$$s^{-1}\mathcal{L}(F \to * \leftarrow S) = \mathcal{L}(Prol(F) \to * \leftarrow S).$$

Hence $s^{-1}\mathcal{L}$ is already a sheaf isomorphic to $f_*\mathcal{L}$. This identification is functorial hence gives an isomorphism of functors

$$f_* \cong s^*.$$

□

COROLLARY 11. *Let $\mathcal{A}$ be an abelian object of $\mathcal{T}/_T \times \mathcal{S}$ and let $\mathcal{A}'$ be an abelian object of $Sh(T) \times \mathcal{S}$. We have the following canonical isomorphisms:*

$$H^n(\mathcal{T}/_T \times \mathcal{S}, \mathcal{A}) \cong H^n(Sh(T) \times \mathcal{S}, f_*\mathcal{A})$$
$$H^n(Sh(T) \times \mathcal{S}, \mathcal{A}') \cong H^n(\mathcal{T}/_T \times \mathcal{S}, f^*\mathcal{A}').$$

Proof. By Lemma 8, the Leray spectral sequence associated with the morphism $f : \mathcal{T}/_T \times \mathcal{S} \to Sh(T) \times \mathcal{S}$ degenerates since $f_* \cong s^*$ is exact. This yields the first isomorphism

$$H^n(\mathcal{T}/_T \times \mathcal{S}, \mathcal{A}) \cong H^n(Sh(T) \times \mathcal{S}, f_*\mathcal{A}) = H^n(Sh(T) \times \mathcal{S}, s^*\mathcal{A}).$$

Applying this identification to the sheaf $f^*\mathcal{A}'$, we obtain

$$H^n(\mathcal{T}/_T \times \mathcal{S}, f^*\mathcal{A}') \cong H^n(Sh(T) \times \mathcal{S}, \mathcal{A}')$$

Indeed, we have $f \circ s \cong Id$ hence

$$f_*f^*\mathcal{A}' \cong s^*f^*\mathcal{A}' \cong (f \circ s)^*\mathcal{A}' \cong \mathcal{A}'.$$

□

Corollary 12. *Let $\mathcal{S}$ be any topos. We denote by $p_1 : \mathcal{T} \times \mathcal{S} \to \mathcal{T}$ and $p_2 : \mathcal{T} \times \mathcal{S} \to \mathcal{S}$ the projections. For any abelian object $\mathcal{A}'$ of $\mathcal{T} \times \mathcal{S}$, one has*

$$H^n(\mathcal{T} \times \mathcal{S}, \mathcal{A}') \cong H^n(\mathcal{S}, p_{2*}\mathcal{A}').$$

For any abelian object $\mathcal{A}$ of $\mathcal{T}$, one has

$$H^n(\mathcal{T} \times \mathcal{S}, p_1^*\mathcal{A}) \cong H^n(\mathcal{S}, \mathcal{A}(*)).$$

Proof. The topos $\mathcal{T}$ is the big topos of the one point space $\{*\}$ while $Sh(*) = \underline{Set}$ is the final topos. The map

$$f : \mathcal{T} \times \mathcal{S} \to Sh(*) \times \mathcal{S} = \underline{Set} \times \mathcal{S} = \mathcal{S}$$

is the second projection p_2. Hence one has

$$H^n(\mathcal{T} \times \mathcal{S}, \mathcal{A}') \cong H^n(\mathcal{S}, p_{2*}\mathcal{A}') \tag{41}$$

as it follows from Corollary 11.
There is a pull-back square :

$$\begin{array}{ccc} \mathcal{T} \times \mathcal{S} & \xrightarrow{p_2} & \mathcal{S} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle e_{\mathcal{S}}} \\ \mathcal{T} & \xrightarrow{e_{\mathcal{T}}} & \underline{Set} \end{array} \tag{42}$$

The functor $e_{\mathcal{T}*}$ has a right adjoint, so that $e_{\mathcal{T}*}$ commutes with arbitrary inductive limits and in particular with filtered inductive limits. Hence the morphism $e_{\mathcal{T}}$ is tidy (see [26] C.3.4.2). It follows that the Beck-Chevalley natural transformation

$$e_{\mathcal{S}}^* \circ e_{\mathcal{T}*} \cong p_{2*} \circ p_1^*.$$

is an isomorphism (see [26] C.3.4.10). But the sheaf

$$p_{2*}p_1^*\mathcal{A} = e_{\mathcal{S}}^* e_{\mathcal{T}*}\mathcal{A}$$

is the constant sheaf on $\mathcal{S}$ associated with the abelian group $\mathcal{A}(*)$, since $e_{\mathcal{T}*}$ is the global section functor and $e_{\mathcal{S}}^*$ is the constant sheaf functor. Applying (41) to the sheaf $p_1^*\mathcal{A}$, we obtain

$$H^n(\mathcal{T} \times \mathcal{S}, p_1^*\mathcal{A}) \cong H^n(\mathcal{S}, p_{2*}p_1^*\mathcal{A}) \cong H^n(\mathcal{S}, e_{\mathcal{S}}^* e_{\mathcal{T}*}\mathcal{A}) = H^n(\mathcal{S}, \mathcal{A}(*))$$

for any $n \geq 0$. □

Lemma 9. *Let U be a contractible topological space and let*

$$q : \mathcal{T} \times Sh(U) \longrightarrow \mathcal{T}$$

be the first projection. Then one has

$$R^n(q_*)q^*\tilde{\mathbb{R}} = 0 \text{ for } n \geq 1.$$

Proof. The sheaf $R^n(q_*)q^*\tilde{\mathbb{R}}$ is the sheaf associated to the presheaf

$$\begin{array}{rccc} P^n(q_*)q^*\tilde{\mathbb{R}} : & Top & \longrightarrow & Ab \\ & K & \longmapsto & H^n(\mathcal{T} \times Sh(U), q^*yK, q^*\tilde{\mathbb{R}}) \end{array}$$

Recall that Top denotes the category of locally compact topological spaces. The category Top^c of compact spaces is a topologically generating family of the site $(Top, \mathcal{J}_{ls})$. It is therefore enough to show

$$H^n(\mathcal{T} \times Sh(U), q^*yK, q^*\tilde{\mathbb{R}}) := H^n((\mathcal{T} \times Sh(U))/_{q^*yK}, q^*\tilde{\mathbb{R}} \times q^*yK) = 0 \tag{43}$$

for any compact space K and any $n \geq 1$. We have immediately

$$(\mathcal{T} \times Sh(U))/_{q^*yK} = \mathcal{T}/yK \times Sh(U).$$

We denote by $q_K : \mathcal{T}/yK \times Sh(U) \to \mathcal{T}/yK \to \mathcal{T}$ the morphism obtained by projection and localization. Equivalently q_K is the composition

$$\mathcal{T}/yK \times Sh(U) \cong (\mathcal{T} \times Sh(U))/_{q^*yK} \longrightarrow (\mathcal{T} \times Sh(U)) \longrightarrow \mathcal{T}.$$

We consider also the map

$$s : Sh(K) \times Sh(U) \longrightarrow \mathcal{T}/yK \times Sh(U)$$

defined in Lemma 8. Then the following identifications

$$\begin{aligned} H^n(\mathcal{T} \times Sh(U), q^*yK, \tilde{\mathbb{R}}) &\cong H^n(\mathcal{T}/yK \times Sh(U), q_K^*\tilde{\mathbb{R}}) \\ &\cong H^n(Sh(K) \times Sh(U), s^*q_K^*\tilde{\mathbb{R}}) \\ &\cong H^n(Sh(K \times U), \tilde{s}^*q_K^*\tilde{\mathbb{R}}) \end{aligned}$$

are induced by the following composite morphism of topoi

$$\tilde{s} : Sh(K \times U) \longrightarrow Sh(K) \times Sh(U) \longrightarrow \mathcal{T}/yK \times Sh(U).$$

Indeed, the first map $Sh(K \times U) \to Sh(K) \times Sh(U)$ is an equivalence since K is compact, and the second map induces an isomorphism on cohomology by Corollary 11. The commutative diagram

$$\begin{tikzcd} Sh(K \times U) \arrow[r] \arrow[d, "p_1"] & Sh(K) \times Sh(U) \arrow[r, "s"] \arrow[d] & \mathcal{T}/yK \times Sh(U) \arrow[d, "q_K"] \\ Sh(K) \arrow[r] & \mathcal{T}/_K \arrow[r] & \mathcal{T} \end{tikzcd} \tag{44}$$

shows that the sheaf $\tilde{s}^*q_K^*\tilde{\mathbb{R}}$ on the product space $K \times U$ is the inverse image of the sheaf $\mathcal{C}^0(K, \mathbb{R})$, of continuous real functions on K, along the continuous projection $p_1 : K \times U \to K$. In other words, one has

$$\tilde{s}^*q_K^*\tilde{\mathbb{R}} = p_1^*\mathcal{C}^0(K, \mathbb{R}).$$

Consider the proper map

$$p_2 : K \times U \longrightarrow U.$$

By proper base change, the stalk of the sheaf $R^n(p_{2*})p_1^*\mathcal{C}^0(K, \mathbb{R})$ on U at some point $u \in U$ is given by

$$(R^n(p_{2*})p_1^*\mathcal{C}^0(K, \mathbb{R}))_u = H^n(p_2^{-1}(u), p_1^*\mathcal{C}^0(K, \mathbb{R}) \mid_{p_2^{-1}(u)}) = H^n(K, \mathcal{C}^0(K, \mathbb{R})).$$

This group is trivial for any $n \geq 1$. Indeed K is compact, in particular paracompact, hence $\mathcal{C}^0(K, \mathbb{R})$ is fine on K. Thus we have

$$R^n(p_{2*})p_1^*\mathcal{C}^0(K, \mathbb{R}) = 0 \text{ for any } n \geq 1.$$

Applying again proper base change to the proper map $K \to *$, we see that $p_{2*}p_1^*\mathcal{C}^0(K,\mathbb{R})$ is the constant sheaf on U associated with the discrete abelian group

$$C^0(K,\mathbb{R}) := H^0(K,\mathcal{C}^0(K,\mathbb{R})).$$

The Leray spectral sequence associated with the continuous map $K\times U \to U$ therefore yields

$$H^n(Sh(K\times U), p_1^*\mathcal{C}^0(K,\mathbb{R})) \cong H^n(U, p_{2*}p_1^*\mathcal{C}^0(K,\mathbb{R})) = H^n(U, C^0(K,\mathbb{R}))$$

for any $n\geq 0$. But U is contractible hence $H^n(U,C^0(K,\mathbb{R}))=0$ for $n\geq 1$, since sheaf cohomology with constant coefficients of locally contractible spaces coincides with singular cohomology, which is in turn homotopy invariant. We obtain

$$H^n(K\times U, p_1^*\mathcal{C}^0(K,\mathbb{R})) = H^n(U,C^0(K,\mathbb{R}))=0$$

for any $n\geq 1$. The result follows since we have

$$\begin{aligned} H^n(\mathcal{T}\times Sh(U), q^*yK,\tilde{\mathbb{R}}) &\cong H^n(Sh(K\times U), \tilde{s}^*q_K^*\tilde{\mathbb{R}}) \\ &\cong H^n(Sh(K\times U), p_1^*\mathcal{C}^0(K,\mathbb{R})) \\ &= 0 \end{aligned}$$

for any compact space K and any $n\geq 1$. □

8.2.2. We still denote by $\mathcal{X}$ an irreducible normal scheme which is flat and proper over Spec($\mathbb{Z}$). Recall that $\mathcal{X}_\infty$ is the topological space $\mathcal{X}^{an}/G_{\mathbb{R}}$, and that the Weil-étale topos of $\mathcal{X}_\infty$ is defined as follows (see Definition 11):

$$\mathcal{X}_{\infty,W} := B_{\mathbb{R}} \times Sh(\mathcal{X}_\infty)$$

Proposition 8.2. *Consider the projection morphism*

$$\gamma_\infty : \mathcal{X}_{\infty,W} = B_{\mathbb{R}}\times Sh(\mathcal{X}_\infty) \longrightarrow Sh(\mathcal{X}_\infty).$$

If $\mathbb{R}$ denotes the constant sheaf on $\mathcal{X}_\infty$ associated to the discrete abelian group $\mathbb{R}$ we have

$$R^n\gamma_{\infty*}(\tilde{\mathbb{R}}) \cong \begin{cases} \mathbb{R} & n=0,1 \\ 0 & n\geq 2 \end{cases}$$

and

$$R^n\gamma_{\infty*}(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n=0 \\ 0 & n\geq 1. \end{cases}$$

Proof. The sheaf $R^n\gamma_{\infty*}(\tilde{\mathbb{R}})$ is the sheaf on the topological space $\mathcal{X}_\infty$ associated to the presheaf

$$\begin{array}{cccc} P^n(\gamma_{\infty*})(\tilde{\mathbb{R}}): & Op(\mathcal{X}_\infty) & \longrightarrow & Ab \\ & U & \longmapsto & H^n(B_{\mathbb{R}}\times Sh(\mathcal{X}_\infty), U, \tilde{\mathbb{R}}) \end{array}$$

where $Op(\mathcal{X}_\infty)$ is the category of open sets of $\mathcal{X}_\infty$. One has

$$H^n(B_{\mathbb{R}}\times Sh(\mathcal{X}_\infty), U,\tilde{\mathbb{R}}) := H^n(B_{\mathbb{R}}\times Sh(\mathcal{X}_\infty)/U,\tilde{\mathbb{R}}) = H^n(B_{\mathbb{R}}\times Sh(U),\tilde{\mathbb{R}}).$$

The family of contractible open subsets $U \subset \mathcal{X}_\infty$ forms a topologically generating family of the site $(Op(\mathcal{X}_\infty), \mathcal{J}_{op})$, since $\mathcal{X}_\infty$ is locally contractible. It is therefore enough to compute the groups $H^n(B_{\mathbb{R}} \times Sh(U), \tilde{\mathbb{R}})$ for U contractible. For any contractible open subset $U \subset \mathcal{X}_\infty$, we consider the following pull-back square :

$$\begin{array}{ccc} \mathcal{T} \times Sh(U) & \xrightarrow{q} & \mathcal{T} \\ {\scriptstyle l'}\downarrow & & \downarrow{\scriptstyle l} \\ B_{\mathbb{R}} \times Sh(U) & \xrightarrow{p} & B_{\mathbb{R}} \end{array} \tag{45}$$

Here the vertical arrows l and l' are both localization maps (recall that $B_{\mathbb{R}}/E\mathbb{R} \cong \mathcal{T}$), while p and and q are the projections. This yields a canonical isomorphism

$$l^*(Rp_*) \cong (Rq_*)l'^*.$$

By Lemma 9, we obtain

$$l^*(R^n p_*)\tilde{\mathbb{R}} \cong (R^n q_*)l'^*\tilde{\mathbb{R}} = (R^n q_*)q^*\tilde{\mathbb{R}} = 0$$

for any $n \geq 1$. It follows immediately that $(R^n p_*)\tilde{\mathbb{R}} = 0$ for $n \geq 1$, since $l^* : B_{\mathbb{R}} \to \mathcal{T}$ is the forgetful functor (forget the $y\mathbb{R}$-action).
The contractible topological space U is connected and locally connected, hence so is the morphism of topoi $Sh(U) \to \underline{Set}$. Since connected and locally connected morphisms are stable under base change (see [26] C.3.3.15), the first projection $p : B_{\mathbb{R}} \times Sh(U) \to B_{\mathbb{R}}$ is also connected and locally connected. In particular, p^* is fully faithful hence we have

$$p_*\tilde{\mathbb{R}} := p_*p^*\tilde{\mathbb{R}} = \tilde{\mathbb{R}}$$

The Leray spectral sequence associated to the morphism p therefore yields

$$H^n(B_{\mathbb{R}} \times Sh(U), p^*\tilde{\mathbb{R}}) \cong H^n(B_{\mathbb{R}}, p_*p^*\tilde{\mathbb{R}}) = H^n(B_{\mathbb{R}}, \tilde{\mathbb{R}}). \tag{46}$$

But one has $H^n(B_{\mathbb{R}}, \tilde{\mathbb{R}}) = \mathbb{R}$ for $n = 0, 1$ and $H^n(B_{\mathbb{R}}, \tilde{\mathbb{R}}) = 0$ for $n \geq 2$. Hence the sheaf $R^n\gamma_{\infty *}(\tilde{\mathbb{R}})$ is the constant sheaf on $\mathcal{X}_\infty$ associated to the discrete abelian group $\mathbb{R}$ for $n = 0, 1$ and $R^n(\gamma_{\infty *})\tilde{\mathbb{R}} = 0$ for $n \geq 2$.
To compute $R^n\gamma_{\infty *}(\mathbb{Z})$ recall that for any group object $\mathcal{G}$ in a topos $\mathcal{E}$ and any abelian $\mathcal{G}$-object $\mathcal{A}$ there is a spectral sequence

$$H^p(H^q(\mathcal{E}/\mathcal{G}^\bullet, \mathcal{A})) \Rightarrow H^{p+q}(B_{\mathcal{G}}, \mathcal{A}).$$

Applying this to $\mathcal{G} = \mathbb{R}$ in $\mathcal{E} = \mathcal{T} \times Sh(U)$ we note that the classifying topos of $\mathcal{G}$ is just $B_{\mathbb{R}} \times Sh(U)$ by [10]. Hence for $\mathcal{A} = \mathbb{Z}$ we obtain a spectral sequence

$$H^p(H^q(\mathcal{T}/\mathbb{R}^\bullet \times Sh(U), \mathbb{Z})) \cong H^p(H^q(Sh(\mathbb{R}^\bullet \times U), \mathbb{Z})) \Rightarrow H^{p+q}(B_{\mathbb{R}} \times Sh(U), \mathbb{Z})$$

where we have again used Corollary 11 and the fact that the spaces $\mathbb{R}^q$ are locally compact. Now if U is contractible so is $\mathbb{R}^q \times U$ and $H^q(Sh(\mathbb{R}^\bullet \times U), \mathbb{Z}) =$

$\mathbb{Z}$ (resp. 0) for $q = 0$ (resp. $q > 0$). The spectral sequence degenerates to an isomorphism

$$H^p(B_{\mathbb{R}} \times Sh(U), \mathbb{Z}) \cong H^p(C(\mathbb{Z})) = \begin{cases} \mathbb{Z} & p = 0 \\ 0 & p > 0 \end{cases}$$

where $C(\mathbb{Z})$ is the complex associated to the constant simplicial abelian group $\mathbb{Z}$ which is quasi-isomorphic to $\mathbb{Z}[0]$. The sheaf $R^n\gamma_{\infty *}(\mathbb{Z})$ is associated to the presheaf $U \mapsto H^p(B_{\mathbb{R}} \times Sh(U), \mathbb{Z})$ and hence takes the values in the statement of Proposition 8.2. □

By Proposition 8.2 the Leray spectral sequence for γ_∞ induces a long exact sequence

$$\cdots \to H^i(\mathcal{X}_\infty, \mathbb{R}) \to H^i(\mathcal{X}_{\infty,W}, \tilde{\mathbb{R}}) \to H^{i-1}(\mathcal{X}_\infty, \mathbb{R}) \to \cdots$$

which decomposes into a collection of canonical isomorphisms

$$H^i(\mathcal{X}_{\infty,W}, \tilde{\mathbb{R}}) \cong H^i(\mathcal{X}_\infty, \mathbb{R}) \oplus H^{i-1}(\mathcal{X}_\infty, \mathbb{R}) \tag{47}$$

since γ_∞ is canonically split by the morphism of topoi $\sigma : Sh(\mathcal{X}_\infty) \to B_{\mathbb{R}} \times Sh(\mathcal{X}_\infty)$ which is the product with $Sh(\mathcal{X}_\infty)$ of the canonical splitting $\underline{Set} \to \mathcal{T} \to B_{\mathbb{R}}$ of the canonical projection $B_{\mathbb{R}} \to \mathcal{T} \to \underline{Set}$. Note here that σ^* applied to the adjunction map $\mathbb{R} = \gamma_\infty^* \gamma_{\infty,*}\tilde{\mathbb{R}} \to \tilde{\mathbb{R}}$ is an isomorphism $\mathbb{R} = \sigma^*\mathbb{R} \cong \sigma^*\tilde{\mathbb{R}} \cong \mathbb{R}$.

8.3. The fundamental class. The map $\mathfrak{f}_{\overline{\mathcal{X}}} : \overline{\mathcal{X}}_W \to B_{\mathbb{R}}$ induces an isomorphism

$$\mathfrak{f}_{\overline{\mathcal{X}}}^* : Hom_c(\mathbb{R}, \mathbb{R}) = H^1(B_{\mathbb{R}}, \tilde{\mathbb{R}}) \to H^1(\overline{\mathcal{X}}_W, \tilde{\mathbb{R}}).$$

Definition 15. *The* fundamental class *is defined as follows:*

$$\theta := \mathfrak{f}_{\overline{\mathcal{X}}}^*(Id_{\mathbb{R}}) \in H^1(\overline{\mathcal{X}}_W, \tilde{\mathbb{R}}).$$

We consider the sheaf $\tilde{\mathbb{R}}$ as a ring object on the topos $\overline{\mathcal{X}}_W$ and we denote by $\mathcal{D}^+(\tilde{\mathbb{R}})$ the derived category of bounded below complexes of $\tilde{\mathbb{R}}$-modules on $\overline{\mathcal{X}}_W$. For any such complex of $\tilde{\mathbb{R}}$-modules M one has (see [20] V.3.5)

$$H^n(\overline{\mathcal{X}}_W, M) = Ext^n_{\mathbb{Z}}(\overline{\mathcal{X}}_W, \mathbb{Z}, M) = Ext^n_{\tilde{\mathbb{R}}}(\overline{\mathcal{X}}_W, \tilde{\mathbb{R}}, M) = Hom_{\mathcal{D}^+(\tilde{\mathbb{R}})}(\tilde{\mathbb{R}}, M[n])$$

and we consider $\theta : \tilde{\mathbb{R}} \to \tilde{\mathbb{R}}[1]$ as a map in $\mathcal{D}^+(\tilde{\mathbb{R}})$. Applying $M \otimes^L_{\tilde{\mathbb{R}}} -$ we get

$$\cup\theta : M = M \otimes^L_{\tilde{\mathbb{R}}} \tilde{\mathbb{R}} \longrightarrow M \otimes^L_{\tilde{\mathbb{R}}} \tilde{\mathbb{R}}[1] = M[1].$$

Applying $R\gamma_{\overline{\mathcal{X}},*}$ we obtain

$$\cup\theta : R\gamma_{\overline{\mathcal{X}},*}(M) \longrightarrow R\gamma_{\overline{\mathcal{X}},*}(M)[1]. \tag{48}$$

Applying in turn $R\Gamma_{\overline{\mathcal{X}}_{et}}$ we get

$$\cup\theta : R\Gamma_{\overline{\mathcal{X}}_W}(M) \longrightarrow R\Gamma_{\overline{\mathcal{X}}_W}(M)[1] \tag{49}$$

Similarly, for any $\overline{\mathcal{U}}$ étale over $\overline{\mathcal{X}}$ we have

$$\cup\theta_{\overline{\mathcal{U}}} : R\Gamma_{\overline{\mathcal{U}}_W}(M) \longrightarrow R\Gamma_{\overline{\mathcal{U}}_W}(M)[1]$$

Taking cohomology we get

$$\cup\theta : R^n\gamma_{\overline{\mathcal{X}},*}(M) \longrightarrow R^{n+1}\gamma_{\overline{\mathcal{X}},*}(M), \tag{50}$$

$$\cup\theta : H^n(\overline{\mathcal{X}}_W, M) \longrightarrow H^{n+1}(\overline{\mathcal{X}}_W, M) \tag{51}$$

and

$$\cup\theta_{\overline{\mathcal{U}}} : H^n(\overline{\mathcal{U}}_W, M) \longrightarrow H^{n+1}(\overline{\mathcal{U}}_W, M). \tag{52}$$

Note that the collection of maps (52), when $\overline{\mathcal{U}}$ runs over the category of étale $\overline{\mathcal{X}}$-schemes, gives a map of presheaves inducing the morphism of sheaves (50). Consider now the open-closed decomposition

$$\varphi : \mathcal{X}_{et} \longrightarrow \overline{\mathcal{X}}_{et} \longleftarrow Sh(\mathcal{X}_\infty) : u_\infty$$

given by Corollary 4.1. The morphism $\gamma_{\overline{\mathcal{X}}} : \overline{\mathcal{X}}_W \to \overline{\mathcal{X}}_{et}$ gives pull-back squares

$$\begin{array}{ccc}
\mathcal{X}_W & \xrightarrow{\gamma_{\mathcal{X}}} & \mathcal{X}_{\mathrm{et}} \\
\phi\downarrow & & \varphi\downarrow \\
\overline{\mathcal{X}}_W & \xrightarrow{\gamma_{\overline{\mathcal{X}}}} & \overline{\mathcal{X}}_{\mathrm{et}}
\end{array}$$

and

$$\begin{array}{ccc}
\mathcal{X}_{\infty,W} & \xrightarrow{\gamma_{\infty}} & Sh(\mathcal{X}_\infty) \\
i_\infty\downarrow & & u_\infty\downarrow \\
\overline{\mathcal{X}}_W & \xrightarrow{\gamma_{\overline{\mathcal{X}}}} & \overline{\mathcal{X}}_{\mathrm{et}}
\end{array}$$

The second square is indeed a pull-back, as can be seen from the following commutative diagram:

$$\begin{array}{ccccc}
\mathcal{X}_{\infty,W} & \xrightarrow{\gamma_{\infty}} & Sh(\mathcal{X}_\infty) & \longrightarrow & Sh(\infty) = \underline{Set} \\
i_\infty\downarrow & & u_\infty\downarrow & & \downarrow \\
\overline{\mathcal{X}}_W & \xrightarrow{\gamma_{\overline{\mathcal{X}}}} & \overline{\mathcal{X}}_{et} & \longrightarrow & \overline{\mathrm{Spec}(\mathbb{Z})}_{et}
\end{array}$$

The right hand side square and the total square are both pull-backs by Corollary 6 and Proposition 6.4 respectively. It follows that the left hand side square is a pull-back as well.

THEOREM 8.1. *There is an isomorphism $R^n\gamma_{\overline{\mathcal{X}}*}(\phi_!\tilde{\mathbb{R}}) \cong \varphi_!\tilde{\mathbb{R}}$ for $n = 0, 1$, and $R^n\gamma_{\overline{\mathcal{X}}*}(\phi_!\tilde{\mathbb{R}}) = 0$ for $n \geq 2$. Under these identifications, the morphism*

$$\cup\theta : R^0\gamma_{\overline{\mathcal{X}}*}(\phi_!\tilde{\mathbb{R}}) \longrightarrow R^1\gamma_{\overline{\mathcal{X}}*}(\phi_!\tilde{\mathbb{R}})$$

given by cup product with the fundamental class, is the identity of the sheaf $\varphi_!\tilde{\mathbb{R}}$.

Proof. We have an exact sequence of abelian sheaves on $\overline{\mathcal{X}}_W$:

$$0 \to \phi_!\tilde{\mathbb{R}} \to \tilde{\mathbb{R}} \to i_{\infty*}\tilde{\mathbb{R}} \to 0$$

Applying the functor $R\gamma_{\overline{\mathcal{X}}*}$, we obtain an exact sequence of étale sheaves

$$0 \to \gamma_{\overline{\mathcal{X}}*}\phi_!\tilde{\mathbb{R}} \to \gamma_{\overline{\mathcal{X}}*}\tilde{\mathbb{R}} \to \gamma_{\overline{\mathcal{X}}*}i_{\infty*}\tilde{\mathbb{R}} \to R^1\gamma_{\overline{\mathcal{X}}*}(\phi_!\tilde{\mathbb{R}}) \to R^1\gamma_{\overline{\mathcal{X}}*}(\tilde{\mathbb{R}}) \to R^1\gamma_{\overline{\mathcal{X}}*}(i_{\infty*}\tilde{\mathbb{R}}) \to ...$$

But we have canonical isomorphisms

$$R^n\gamma_{\overline{\mathcal{X}}*}(i_{\infty*}\tilde{\mathbb{R}}) \cong R^n(\gamma_{\overline{\mathcal{X}}*}i_{\infty*})\tilde{\mathbb{R}} \cong R^n(u_{\infty*}\gamma_{\infty*})\tilde{\mathbb{R}} \cong u_{\infty*}R^n\gamma_{\infty*}(\tilde{\mathbb{R}}) \tag{53}$$

for any $n \geq 0$, since the direct image of a closed embedding of topoi is exact. Therefore, by Proposition 8.1 and Proposition 8.2, we obtain an exact sequence

$$0 \to \gamma_{\overline{\mathcal{X}}*}\phi_!\tilde{\mathbb{R}} \to \mathbb{R} \to u_{\infty*}\mathbb{R} \to R^1\gamma_{\overline{\mathcal{X}}*}(\phi_!\tilde{\mathbb{R}}) \to \mathbb{R} \to u_{\infty*}\mathbb{R} \to R^2\gamma_{\overline{\mathcal{X}}*}(\phi_!\tilde{\mathbb{R}}) \to 0$$

and $R^n\gamma_{\overline{\mathcal{X}}*}(\phi_!\tilde{\mathbb{R}}) = 0$ for $n \geq 3$. The map $\mathbb{R} \to u_{\infty*}\mathbb{R}$ is surjective since u_∞ is a closed embedding. Hence we have an exact sequence

$$0 \to R^n\gamma_{\overline{\mathcal{X}}*}(\phi_!\tilde{\mathbb{R}}) \to \mathbb{R} \to u_{\infty*}\mathbb{R} \to 0$$

for $n = 0, 1$ and $R^n\gamma_{\overline{\mathcal{X}}*}(\phi_!\tilde{\mathbb{R}}) = 0$ for $n \geq 2$. The first claim of the theorem follows.
For any connected étale $\overline{\mathcal{X}}$-scheme $\overline{\mathcal{U}}$, we have a commutative square of $\mathbb{R}$-vector spaces

$$\begin{array}{ccc} H^0(\overline{\mathcal{U}}_W, \tilde{\mathbb{R}}) & \xrightarrow{\cup\theta_{\overline{\mathcal{U}}}} & H^1(\overline{\mathcal{U}}_W, \tilde{\mathbb{R}}) \\ \uparrow & & \uparrow \\ H^0(B_{\mathbb{R}}, \tilde{\mathbb{R}}) = \mathbb{R} & \xrightarrow{\cup Id_{\mathbb{R}}} & H^1(B_{\mathbb{R}}, \tilde{\mathbb{R}}) \cong \mathbb{R} \end{array}$$

where the vertical maps are isomorphisms by Corollary 10. The $\mathbb{R}$-linear map

$$\cup Id_{\mathbb{R}} : H^0(B_{\mathbb{R}}, \tilde{\mathbb{R}}) = \mathbb{R} \longrightarrow H^1(B_{\mathbb{R}}, \tilde{\mathbb{R}}) = Hom_{cont}(\mathbb{R}, \mathbb{R}) \tag{54}$$

sends $1 \in \mathbb{R}$ to $Id_{\mathbb{R}}$. Under the identification

$$H^1(B_{\mathbb{R}}, \tilde{\mathbb{R}}) = Hom_{cont}(\mathbb{R}, \mathbb{R}) \cong \mathbb{R}$$

which maps $f : \mathbb{R} \to \mathbb{R}$ to $f(1)$, the morphism (54) is the identity of $\mathbb{R}$. Hence the morphism

$$\cup\theta_{\overline{\mathcal{U}}} : H^0(\overline{\mathcal{U}}_W, \tilde{\mathbb{R}}) = \mathbb{R} \longrightarrow H^1(\overline{\mathcal{U}}_W, \tilde{\mathbb{R}}) \cong \mathbb{R}$$

is just the identity, for any connected étale $\overline{\mathcal{X}}$-scheme $\overline{\mathcal{U}}$. It follows that the morphism of sheaves defined in (50)

$$\cup\theta : R^0\gamma_{\overline{\mathcal{X}}*}(\tilde{\mathbb{R}}) = \tilde{\mathbb{R}} \longrightarrow R^1\gamma_{\overline{\mathcal{X}}*}(\tilde{\mathbb{R}}) \cong \tilde{\mathbb{R}} \tag{55}$$

is the identity of the sheaf $\tilde{\mathbb{R}}$.
The same argument is valid for the sheaf $i_{\infty*}\tilde{\mathbb{R}}$. The composite morphism

$$p : \mathcal{X}_{\infty,W} = B_{\mathbb{R}} \times Sh(\mathcal{X}_\infty) \longrightarrow \overline{\mathcal{X}} \longrightarrow B_{\mathbb{R}}$$

is the first projection. We consider the fundamental class

$$\theta_\infty := p^*(Id_{\mathbb{R}}) = i^*_\infty(\theta) \in H^1(\mathcal{X}_{\infty,W}, \tilde{\mathbb{R}}).$$

Then the morphism

$$\cup\theta : R^0\gamma_{\overline{\mathcal{X}}*}(i_{\infty*}\tilde{\mathbb{R}}) \longrightarrow R^1\gamma_{\overline{\mathcal{X}}*}(i_{\infty*}\tilde{\mathbb{R}})$$

coincides, via the canonical isomorphism (53), with the morphism $u_{\infty*}R^0\gamma_{\infty*}(\tilde{\mathbb{R}}) \to u_{\infty*}R^1\gamma_{\infty*}(\tilde{\mathbb{R}})$ induced by

$$\cup\theta_\infty : R^0\gamma_{\infty*}(\tilde{\mathbb{R}}) \longrightarrow R^1\gamma_{\infty*}(\tilde{\mathbb{R}}).$$

But for any contractible open subset $U \subset \mathcal{X}_\infty$, one has a commutative square

$$\begin{array}{ccc} H^0(\mathcal{X}_{\infty,W}, U, \tilde{\mathbb{R}}) & \xrightarrow{\cup\theta_\infty} & H^1(\mathcal{X}_{\infty,W}, U, \tilde{\mathbb{R}}) \\ \uparrow & & \uparrow \\ H^0(B_{\mathbb{R}}, \tilde{\mathbb{R}}) = \mathbb{R} & \xrightarrow{\cup Id_{\mathbb{R}}} & H^1(B_{\mathbb{R}}, \tilde{\mathbb{R}}) \cong \mathbb{R} \end{array}$$

where all the maps are isomorphisms, as it follows from (46). Hence the map

$$\cup\theta_\infty : \tilde{\mathbb{R}} = \gamma_{\infty*}(\tilde{\mathbb{R}}) \longrightarrow R^1\gamma_{\infty*}(\tilde{\mathbb{R}}) \cong \tilde{\mathbb{R}}$$

is the identity, and so is the morphism

$$\cup\theta : R^0\gamma_{\overline{\mathcal{X}}*}(i_{\infty*}\tilde{\mathbb{R}}) = u_{\infty*}\tilde{\mathbb{R}} \longrightarrow R^1\gamma_{\overline{\mathcal{X}}*}(i_{\infty*}\tilde{\mathbb{R}}) \cong u_{\infty*}\tilde{\mathbb{R}}. \tag{56}$$

The morphism (50) is functorial hence $\cup\theta$ gives a morphism of exact sequences from

$$0 \to R^0\gamma_{\overline{\mathcal{X}}*}(\phi_!\tilde{\mathbb{R}}) \to R^0\gamma_{\overline{\mathcal{X}}*}(\tilde{\mathbb{R}}) \to R^0\gamma_{\overline{\mathcal{X}}*}(i_{\infty*}\tilde{\mathbb{R}}) \to 0$$

to

$$0 \to R^1\gamma_{\overline{\mathcal{X}}*}(\phi_!\tilde{\mathbb{R}}) \to R^1\gamma_{\overline{\mathcal{X}}*}(\tilde{\mathbb{R}}) \to R^1\gamma_{\overline{\mathcal{X}}*}(i_{\infty*}\tilde{\mathbb{R}}) \to 0$$

But the morphisms (55) and (56) are both given by the identity map, hence so is the morphism

$$\cup\theta : R^0\gamma_{\overline{\mathcal{X}}*}(\phi_!\tilde{\mathbb{R}}) = \varphi_!\tilde{\mathbb{R}} \longrightarrow R^1\gamma_{\overline{\mathcal{X}}*}(\phi_!\tilde{\mathbb{R}}) \cong \varphi_!\tilde{\mathbb{R}}.$$

$\square$

DEFINITION 16. *For any abelian sheaf $\mathcal{A}$ on $\mathcal{X}_W$, the compact support cohomology groups $H^i_c(\mathcal{X}_W, \mathcal{A})$ are defined as follows:*

$$H^i_c(\mathcal{X}_W, \mathcal{A}) := H^i(\overline{\mathcal{X}}_W, \phi_!\mathcal{A})$$

THEOREM 8.2. *Assume that $\mathcal{X}$ is irreducible, normal, flat and proper over* Spec($\mathbb{Z}$). *The compact support cohomology groups $H^i_c(\mathcal{X}_W, \tilde{\mathbb{R}})$ are finite dimensional vector spaces over $\mathbb{R}$, vanish for almost all i and satisfy*

$$\sum_{i\in\mathbb{Z}}(-1)^i \dim_{\mathbb{R}} H^i_c(\mathcal{X}_W, \tilde{\mathbb{R}}) = 0.$$

Moreover, the complex of $\mathbb{R}$-vector spaces

$$\cdots \xrightarrow{\cup\theta} H^i_c(\mathcal{X}_W, \tilde{\mathbb{R}}) \xrightarrow{\cup\theta} H^{i+1}_c(\mathcal{X}_W, \tilde{\mathbb{R}}) \xrightarrow{\cup\theta} \cdots$$

is acyclic.

Proof. Consider the Leray spectral sequence

$$H^p(\overline{\mathcal{X}}_{et}, R^q\gamma_{\overline{\mathcal{X}}*}(\phi_!\tilde{\mathbb{R}})) \Longrightarrow H^{p+q}(\overline{\mathcal{X}}_W, \phi_!\tilde{\mathbb{R}})$$

given by the morphism $\gamma_{\overline{\mathcal{X}}}$. This spectral sequence yields

$$H^0(\overline{\mathcal{X}}_W, \phi_!\tilde{\mathbb{R}}) = H^0(\overline{\mathcal{X}}_{et}, \varphi_!\mathbb{R}) = 0$$

and a long exact sequence

$$\begin{array}{ccccccc}
0 & \to & H^1(\overline{\mathcal{X}}_{et}, R^0\gamma_{\overline{\mathcal{X}}*}(\phi_!\mathbb{R})) & \to & H^1(\overline{\mathcal{X}}_W, \phi_!\tilde{\mathbb{R}}) & \to & H^0(\overline{\mathcal{X}}_{et}, R^1\gamma_{\overline{\mathcal{X}}*}(\phi_!\mathbb{R})) \\
 & & & & \big\downarrow{\scriptstyle \cup\theta} & & \\
... & \to & H^2(\overline{\mathcal{X}}_{et}, R^0\gamma_{\overline{\mathcal{X}}*}(\phi_!\mathbb{R})) & \to & H^2(\overline{\mathcal{X}}_W, \phi_!\tilde{\mathbb{R}}) & \to & H^1(\overline{\mathcal{X}}_{et}, R^1\gamma_{\overline{\mathcal{X}}*}(\phi_!\mathbb{R})) \\
 & & & & \big\downarrow{\scriptstyle \cup\theta} & & \\
... & \to & H^3(\overline{\mathcal{X}}_{et}, R^0\gamma_{\overline{\mathcal{X}}*}(\phi_!\mathbb{R})) & \to & H^3(\overline{\mathcal{X}}_W, \phi_!\tilde{\mathbb{R}}) & \to & H^2(\overline{\mathcal{X}}_{et}, R^1\gamma_{\overline{\mathcal{X}}*}(\phi_!\mathbb{R})) \\
 & & & & & & \\
... & \to & H^4(\overline{\mathcal{X}}_{et}, R^0(\gamma_{\overline{\mathcal{X}}*})\phi_!\mathbb{R}) & \to & ... & &
\end{array}$$

Here the vertical maps $\cup\theta$ are given by cup product with the fundamental class. More precisely, the morphism (48)

$$R\gamma_{\overline{\mathcal{X}},*}(\phi_!\mathbb{R}) \longrightarrow R\gamma_{\overline{\mathcal{X}},*}(\phi_!\mathbb{R})[1]$$

induces a morphism of spectral sequences. This morphism of spectral sequences induces in turn these vertical maps $\cup\theta$. It follows that the composite map

$$H^i(\overline{\mathcal{X}}_{et}, R^0\gamma_{\overline{\mathcal{X}}*}(\phi_!\mathbb{R})) \to H^i(\overline{\mathcal{X}}_W, \phi_!\tilde{\mathbb{R}}) \xrightarrow{\cup\theta} H^{i+1}(\overline{\mathcal{X}}_W, \phi_!\tilde{\mathbb{R}}) \to H^i(\overline{\mathcal{X}}_{et}, R^1\gamma_{\overline{\mathcal{X}}*}(\phi_!\mathbb{R})) \tag{57}$$

is induced by the isomorphism of sheaves

$$R^0\gamma_{\overline{\mathcal{X}}*}(\phi_!\tilde{\mathbb{R}}) = \varphi_!\mathbb{R} \xrightarrow{\cup\theta} R^1\gamma_{\overline{\mathcal{X}}*}(\phi_!\tilde{\mathbb{R}}) \cong \varphi_!\mathbb{R}.$$

Hence the map (57) is an isomorphism for any $i \geq 0$, by Theorem 8.1. This yields a section to the map

$$H^{i+1}(\overline{\mathcal{X}}_W, \phi_!\tilde{\mathbb{R}}) \longrightarrow H^i(\overline{\mathcal{X}}_{et}, R^1\gamma_{\overline{\mathcal{X}}*}(\phi_!\mathbb{R})).$$

It follows that the long exact sequence above decomposes into a collection of canonical isomorphisms

$$H^i(\overline{\mathcal{X}}_W, \phi_!\tilde{\mathbb{R}}) \cong H^i(\overline{\mathcal{X}}_{et}, R^0\gamma_{\overline{\mathcal{X}}*}(\phi_!\tilde{\mathbb{R}})) \oplus H^{i-1}(\overline{\mathcal{X}}_{et}, R^1\gamma_{\overline{\mathcal{X}}*}(\phi_!\tilde{\mathbb{R}})) \tag{58}$$

$$\cong H^i(\overline{\mathcal{X}}_{et}, \varphi_!\mathbb{R}) \oplus H^{i-1}(\overline{\mathcal{X}}_{et}, \varphi_!\mathbb{R}) \tag{59}$$

$$\cong H^i_c(\mathcal{X}_{et}, \mathbb{R}) \oplus H^{i-1}_c(\mathcal{X}_{et}, \mathbb{R}) \tag{60}$$

for any $i \geq 1$. By Proposition 4.3, the $\mathbb{R}$-vector space $H^i_c(\mathcal{X}_{et}, \mathbb{R})$ is finite dimensional and zero for i large. Hence we have

$$\dim_{\mathbb{R}} H^i_c(\mathcal{X}_W, \tilde{\mathbb{R}}) = \dim_{\mathbb{R}} H^i_c(\mathcal{X}_{et}, \mathbb{R}) + \dim_{\mathbb{R}} H^{i-1}_c(\mathcal{X}_{et}, \mathbb{R})$$

and

$$\sum_{i\in\mathbb{Z}}(-1)^i \dim_{\mathbb{R}} H^i_c(\mathcal{X}_W,\tilde{\mathbb{R}})=0.$$

Under the identification (60), the morphism given by cup product with the fundamental class

$$H^i_c(\mathcal{X}_W,\tilde{\mathbb{R}})\xrightarrow{\cup\theta}H^{i+1}_c(\mathcal{X}_W,\tilde{\mathbb{R}})$$

is obtained by composing the projection with the inclusion as follows:

$$H^i_c(\mathcal{X}_W,\tilde{\mathbb{R}})\twoheadrightarrow H^i_c(\mathcal{X}_{et},\mathbb{R})\hookrightarrow H^{i+1}_c(\mathcal{X}_W,\tilde{\mathbb{R}}). \tag{61}$$

It follows immediately from (60) and (61) that the complex of $\mathbb{R}$-vector spaces

$$\cdots\xrightarrow{\cup\theta}H^i_c(\mathcal{X}_W,\tilde{\mathbb{R}})\xrightarrow{\cup\theta}H^{i+1}_c(\mathcal{X}_W,\tilde{\mathbb{R}})\xrightarrow{\cup\theta}\cdots$$

is acyclic. □

PROPOSITION 8.3. *Assume that $\mathcal{X}$ is irreducible, normal, flat and proper over* Spec($\mathbb{Z}$). *Then one has*

$$\begin{aligned}\sum_{i\in\mathbb{Z}}(-1)^i i\dim_{\mathbb{R}} H^i_c(\mathcal{X}_W,\tilde{\mathbb{R}}) &= \sum_{i\in\mathbb{Z}}(-1)^{i+1}\dim_{\mathbb{R}} H^i_c(\mathcal{X}_{\mathrm{et}},\mathbb{R})\\ &= -1+\sum_{i\in\mathbb{Z}}(-1)^i\dim_{\mathbb{R}} H^i(\mathcal{X}_\infty,\mathbb{R})\\ &= -1+\sum_{i\in\mathbb{Z}}(-1)^i\dim_{\mathbb{R}} H^i(\mathcal{X}^{an},\mathbb{R})^+\end{aligned}$$

Proof. The first equality (respectively the second) follows from (60) (respectively from Proposition 4.3). To prove the third, we consider the morphism of topoi

$$(\pi^*,\pi^{G_{\mathbb{R}}}_*):Sh(G_{\mathbb{R}},\mathcal{X}^{an})\to Sh(\mathcal{X}_\infty)$$

given by the quotient map $\pi:\mathcal{X}^{an}\to\mathcal{X}^{an}/G_{\mathbb{R}}$, where $Sh(G_{\mathbb{R}},\mathcal{X}^{an})$ is the topos of $G_{\mathbb{R}}$-equivariant sheaves on the space $\mathcal{X}^{an}$. The constant sheaf $\mathbb{R}$ on $\mathcal{X}^{an}$ is endowed with its $G_{\mathbb{R}}$-equivariant structure. For any $n\geq 1$, the stalk of $R^n(\pi^{G_{\mathbb{R}}}_*)\mathbb{R}$ at some fixed point $x\in\mathcal{X}(\mathbb{R})\subset\mathcal{X}^\infty$ is the abelian group $H^n(G_{\mathbb{R}},\mathbb{R})$, which is zero since $\mathbb{R}$ is uniquely divisible. This gives

$$R^n(\pi^{G_{\mathbb{R}}}_*)\mathbb{R}=0 \text{ for } n\geq 1$$

and a canonical isomorphism

$$H^n(\mathcal{X}_\infty,\mathbb{R})\cong H^n(Sh(G_{\mathbb{R}},\mathcal{X}^{an}),\mathbb{R})$$

for any $n\geq 0$. But the spectral sequence

$$H^p(G_{\mathbb{R}},H^q(\mathcal{X}^{an},\mathbb{R}))\Longrightarrow H^{p+q}(Sh(G_{\mathbb{R}},\mathcal{X}^{an}),\mathbb{R})$$

degenerates and gives an isomorphism

$$H^n(Sh(G_{\mathbb{R}},\mathcal{X}^{an}),\mathbb{R})\cong H^0(G_{\mathbb{R}},H^n(\mathcal{X}^{an},\mathbb{R}))=:H^n(\mathcal{X}^{an},\mathbb{R})^+$$

for any n. The result follows. $\square$

9. Relationship to the Zeta-function

9.1. Motivic L-functions. We first recall the expected properties of motivic L-functions [38]. For any smooth proper scheme $X/\mathbb{Q}$ of pure dimension d and $0 \le i \le 2d$ one defines the L-function

$$L(h^i(X), s) = \prod_p L_p(h^i(X), s)$$

as an Euler product over all primes p where

$$L_p(h^i(X), s) = P_p(h^i(X), p^{-s})^{-1}$$

and

$$P_p(h^i(X), T) = \det_{\mathbb{Q}_l}\left(1 - \mathrm{Frob}_p^{-1} \cdot T | H^i(X_{\bar{\mathbb{Q}},et}, \mathbb{Q}_l)^{I_p}\right)$$

is a polynomial (conjecturally) with rational coefficients independent of the prime $l \neq p$. By [9] this product converges for $\Re(s) > \frac{i}{2} + 1$. Set

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(\frac{s}{2}); \quad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$$

and

$$L_\infty(h^i(X), s) = \prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \cdot \prod_{p=\frac{i}{2}} \Gamma_{\mathbb{R}}(s-p)^{h^{p,+}} \Gamma_{\mathbb{R}}(s-p+1)^{h^{p,-}}$$

where $H^i(X(\mathbb{C}), \mathbb{C}) \cong \bigoplus_{p+q=i} H^{p,q}$ is the Hodge decomposition,

$$h^{p,q} = \dim_{\mathbb{C}} H^{p,q}; \quad h^{p,\pm} = \dim_{\mathbb{C}} (H^{p,p})^{F_\infty = \pm(-1)^p}$$

and F_∞ is the map induced by complex conjugation on the manifold $X(\mathbb{C})$. Here the product over $p = \frac{i}{2}$ is understood to be empty for odd i. The completed L-function

$$\Lambda(h^i(X), s) = L_\infty(h^i(X), s) L(h^i(X), s)$$

is expected to meromorphically continue to all s and satisfy a functional equation

$$\Lambda(h^i(X), s) = \epsilon(h^i(X), s)\Lambda(h^{2d-i}(X), d+1-s). \tag{62}$$

Here $\epsilon(h^i(X), s)$ is the product of a constant and an exponential function in s, in particular nowhere vanishing.

Lemma 10. *Assuming meromorphic continuation and the functional equation we have*

$$\mathrm{ord}_{s=0} L(h^i(X), s) = \begin{cases} -t + \dim_{\mathbb{C}} H^0(X(\mathbb{C}), \mathbb{C})^{F_\infty = 1} & i = 0 \\ \dim_{\mathbb{C}} H^i(X(\mathbb{C}), \mathbb{C})^{F_\infty = 1} & i > 0. \end{cases}$$

where t is the number of connected components of the scheme X.

Proof. For $i > 0$ the point $d+1 > \frac{2d-i}{2}+1$ lies in the region of absolute convergence of $L(h^{2d-i}(X),s)$ so that $L(h^{2d-i}(X),d+1) \neq 0$. The Gamma-function has no zeros and has simple poles precisely at the non-positive integers. For $p+q = 2d-i$ and $p<q$ we have $p < d-\frac{i}{2}$, hence $\Gamma_{\mathbb{C}}(d+1-p) \neq 0$. For $p = d-\frac{i}{2}$ we likewise have $\Gamma_{\mathbb{R}}(d+1-p) \neq 0$ and $\Gamma_{\mathbb{R}}(d+1+1-p) \neq 0$. Hence $L_\infty(h^{2d-i}(X),d+1) \neq 0$ and the functional equation shows $\Lambda(h^i(X),0) \neq 0$, i.e.

$$\begin{aligned}\operatorname{ord}_{s=0} L(h^i(X),s) &= -\operatorname{ord}_{s=0} L_\infty(h^i(X),s) \\ &= \sum_{p<q} h^{p,q} + \sum_{p=\frac{i}{2}} h^{p,\pm} = \dim_{\mathbb{C}} H^i(X(\mathbb{C}),\mathbb{C})^{F_\infty=1}\end{aligned} \tag{63}$$

where this last identity follows from $F_\infty(H^{p,q}) = H^{q,p}$ and the sign $\pm$ in $h^{p,\pm}$ is the one for which $\pm(-1)^p = 1$. Indeed, $\Gamma_{\mathbb{R}}(s-p)$ (resp. $\Gamma_{\mathbb{R}}(s-p+1)$) has a simple pole at $s=0$ precisely for even (resp. odd) p.
For $i=0$ the function

$$L(h^0(X),s) = \zeta_{K_1}(s)\cdots\zeta_{K_t}(s)$$

is a product of Dedekind Zeta-functions where $H^0(X,\mathcal{O}_X) = K_1 \times \cdots \times K_t$ is the ring of global regular functions on X and the K_i are number fields. It is classical that $\operatorname{ord}_{s=1}\zeta_{K_j}(s) = -1$ and therefore

$$\operatorname{ord}_{s=0}\Lambda(h^0(X),s) = \operatorname{ord}_{s=1}\Lambda(h^0(X),s) = \sum_{j=1}^{t}\operatorname{ord}_{s=1}\zeta_{K_j}(s) = -t.$$

Hence (63) holds for $i=0$ with $-t$ added to the right hand side.

□

9.2. ZETA-FUNCTIONS. For any separated scheme $\mathcal{X}$ of finite type over $\operatorname{Spec}(\mathbb{Z})$ one defines a Zeta-function

$$\zeta(\mathcal{X},s) := \prod_{x\in X^{cl}} \frac{1}{1-N(x)^{-s}} = \prod_p \zeta(\mathcal{X}_{\mathbb{F}_p},s)$$

as an Euler product over all closed points. By Grothendieck's formula [31][Thm. 13.1]

$$\zeta(\mathcal{X}_{\mathbb{F}_p},s) = \prod_{i=0}^{2\dim(\mathcal{X}_{\mathbb{F}_p})} \det_{\mathbb{Q}_l}\left(1-\mathrm{Frob}_p^{-1}\cdot p^{-s}|H^i_c(\mathcal{X}_{\bar{\mathbb{F}}_p,et},\mathbb{Q}_l)\right)^{(-1)^{i+1}}.$$

If $\mathcal{X}_{\mathbb{Q}} \to \operatorname{Spec}(\mathbb{Q})$ is smooth and proper of relative dimension d, there will be an open subscheme $U \subseteq \operatorname{Spec}(\mathbb{Z})$ on which $\mathcal{X}_U \to U$ is smooth and proper. By smooth and proper base change we have for $p \in U$

$$H^i_c(\mathcal{X}_{\bar{\mathbb{F}}_p,et},\mathbb{Q}_l) \cong H^i(\mathcal{X}_{\bar{\mathbb{F}}_p,et},\mathbb{Q}_l) \cong H^i(\mathcal{X}_{\bar{\mathbb{Q}},et},\mathbb{Q}_l) \cong H^i(\mathcal{X}_{\bar{\mathbb{Q}},et},\mathbb{Q}_l)^{I_p}$$

and therefore

$$\zeta(\mathcal{X},s)=\prod_{p\notin U}E_p(s)\prod_{i=0}^{2d}L(h^i(\mathcal{X}_{\mathbb{Q}}),s)^{(-1)^i} \tag{64}$$

where

$$E_p(s)=\prod_{i=0}^{\infty}\left(\frac{\det_{\mathbb{Q}_l}\left(1-\mathrm{Frob}_p^{-1}\cdot p^{-s}|H^i(\mathcal{X}_{\bar{\mathbb{Q}},et},\mathbb{Q}_l)^{I_p}\right)}{\det_{\mathbb{Q}_l}\left(1-\mathrm{Frob}_p^{-1}\cdot p^{-s}|H^i_c(\mathcal{X}_{\bar{\mathbb{F}}_p,et},\mathbb{Q}_l)\right)}\right)^{(-1)^i}$$

is a rational function in p^{-s}.

Theorem 9.1. *Let $\mathcal{X}$ be a regular scheme, proper and flat over* $\mathrm{Spec}(\mathbb{Z})$*. Assume that the L-functions $L(h^i(\mathcal{X}_{\mathbb{Q}}),s)$ can be meromorphically continued and satisfy the functional equation (62). Then*

$$\mathrm{ord}_{s=0}\,\zeta(\mathcal{X},s)=\sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot\dim_{\mathbb{R}}H^i_c(\mathcal{X}_W,\tilde{\mathbb{R}}).$$

Proof. Note that regularity of $\mathcal{X}$ implies that $\mathcal{X}_{\mathbb{Q}}\to\mathrm{Spec}(\mathbb{Q})$ is smooth. By Lemma 10 and Proposition 8.3 we have

$$\begin{aligned}\mathrm{ord}_{s=0}\prod_{i\in\mathbb{Z}}L(h^i(\mathcal{X}_{\mathbb{Q}}),s)^{(-1)^i}&=-t+\sum_{i\in\mathbb{Z}}(-1)^i\dim_{\mathbb{C}}H^i(\mathcal{X}_{\mathbb{Q}}(\mathbb{C}),\mathbb{C})^{F_\infty=1}\\&=-t+\sum_{i\in\mathbb{Z}}(-1)^i\dim_{\mathbb{R}}H^i(\mathcal{X}^{an},\mathbb{R})^{F_\infty=1}\\&=\sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot\dim_{\mathbb{R}}H^i_c(\mathcal{X}_W,\tilde{\mathbb{R}})\end{aligned}$$

and in view of (64) it remains to show that $\mathrm{ord}_{s=0}E_p(s)=0$ for all p (or just $p\notin U$). This follows from the fact that the Frob_p^{-1} eigenvalue 1 (of weight 0) has the same multiplicity on $H^i_c(\mathcal{X}_{\bar{\mathbb{F}}_p,et},\mathbb{Q}_l)=H^i(\mathcal{X}_{\bar{\mathbb{F}}_p,et},\mathbb{Q}_l)$ and on $H^i(\mathcal{X}_{\bar{\mathbb{Q}},et},\mathbb{Q}_l)^{I_p}$ by part b) of Theorem 10.1 in the next section. $\square$

Corollary 13. *Let F be a totally real number field and $\mathcal{X}$ a proper, regular model of a Shimura curve over F, or of $E\times E\times\cdots\times E$ where E is an elliptic curve over F. Then*

$$\mathrm{ord}_{s=0}\,\zeta(\mathcal{X},s)=\sum_{i\in\mathbb{Z}}(-1)^i\cdot i\cdot\dim_{\mathbb{R}}H^i_c(\mathcal{X}_W,\tilde{\mathbb{R}}).$$

Proof. For any Shimura curve X, by the now classical results of Eichler, Shimura, Deligne, Carayol and others, $L(h^1(X),s)$ is a product of L-functions associated to weight 2 cusp forms for a suitable arithmetic subgroup of $\mathrm{PSL}_2(\mathbb{R})$ associated to X, hence satisfies (62). It is moreover well known that any curve always has a proper regular model.
By the Kuenneth formula we have

$$h^i(E^d)\cong\bigoplus_{\substack{i_0+i_1+i_2=d\\ i_1+2i_2=i}}h^0(E)^{\otimes i_0}\otimes h^1(E)^{\otimes i_1}\otimes h^2(E)^{\otimes i_2}\cong\bigoplus_{\substack{i_0+i_1+i_2=d\\ i_1+2i_2=i}}h^1(E)^{\otimes i_1}(-i_2)$$

and each tensor power $h^1(E)^{\otimes i_1}$ is a direct sum of Tate twists of symmetric powers $\mathrm{Sym}^k h^1(E)$. But for elliptic curves E over totally real fields F the meromorphic continuation and functional equation of $L(\mathrm{Sym}^k h^1(E)/F, s)$ follows from recent deep results of Harris, Taylor, Shin et al (see [5][Cor. 8.8]). We remark that a proper regular model $\mathcal{X}$ of E^d certainly exists if E has semistable reduction at all primes since then the product singularities of $\mathcal{E}^d$, where $\mathcal{E}$ is a proper regular model of E, can be resolved [37]. □

Theorem 9.2. *Let $\mathcal{X}$ be a smooth proper variety over a finite field. Then a)-f) in the introduction hold for $\mathcal{X}$.*

Proof. This was proved for $\mathcal{X}_W^{sm}$ in [18][Thm. 9.1] since one clearly has

$$H^i(\mathcal{X}_W^{sm}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong H^i(\mathcal{X}_W^{sm}, \mathbb{R}).$$

But in view of Corollary 12 (see also Corollary 2 and the remark after it) we have

$$H^i(\mathcal{X}_W, \mathbb{Z}) \cong H^i(\mathcal{X}_W^{sm}, \mathbb{Z}); \quad H^i(\mathcal{X}_W, \tilde{\mathbb{R}}) \cong H^i(\mathcal{X}_W^{sm}, \mathbb{R})$$

when $\mathcal{X}_W$ is defined by Definition 10. Note here that our fundamental class θ defined in Definition 15 is different from the class $e \in H^1(\mathcal{X}_W, \tilde{\mathbb{R}})$ used in [18]. The class e lies in the image of $H^1(\mathcal{X}_W, \mathbb{Z})$ and is the pullback of the identity map in

$$H^1(\mathrm{Spec}(\mathbb{F}_p)_W, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(W_{\mathbb{F}_p}, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}).$$

Since the natural map $W_{\mathbb{F}_p} \to \mathbb{R}$ sends the Frobenius to $\log(p)$, the elements θ and e differ by a factor of $\log(p)$. This is consistent with the fact that

$$\zeta^*(\mathcal{X}, 0) = \log(p)^r Z^*(\mathcal{X}, 1) \tag{65}$$

where $Z(\mathcal{X}, T) \in \mathbb{Q}(T)$ is the rational function so that $\zeta(\mathcal{X}, s) = Z(\mathcal{X}, p^{-s})$ and

$$Z(\mathcal{X}, T) = (1 - T)^r Z^*(\mathcal{X}, 1) + ...$$

with $r \in \mathbb{Z}$ and $Z^*(\mathcal{X}, 1) \neq 0, \infty$. □

9.3. Remarks. We finish this section with some remarks to put our results in perspective.

9.3.1. *Cohomology with $\mathbb{Z}$-coefficients.* If $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ is a (proper, flat, regular) arithmetic scheme with a section then $R\Gamma(\overline{\mathrm{Spec}(\mathbb{Z})}_W, \mathbb{Z})$ is a direct summand of $R\Gamma(\overline{\mathcal{X}}_W, \mathbb{Z})$. Hence by [14] $H^4(\mathcal{X}_W, \mathbb{Z})$ will not be a finitely generated abelian group and d) does not hold. Even if one could find a definition of $\overline{\mathrm{Spec}(\mathbb{Z})}_W$ with the expected $\mathbb{Z}$-cohomology the definition of $\mathcal{X}_W$ as a fibre product (Definition 10) will not be the right one. Heuristically this is because one should view the fibre product of topoi as a "homotopy pullback", and the "homotopy fibre" of $\gamma : \mathcal{X}_W \to \mathcal{X}_{et}$ is not independent of $\mathcal{X}$, unlike in the situation over finite fields. Indeed, viewing $R\gamma_*\mathbb{Z}$ as the cohomology of the fibre, Geisser has shown [18] that this complex has cohomology $\mathbb{Z}$, $\mathbb{Q}$, 0 in degrees 0, 1, ≥ 2, respectively, for any $\mathcal{X}$ over $\mathrm{Spec}(\mathbb{F}_p)$. So for any $\mathcal{X}$ over $\mathrm{Spec}(\mathbb{F}_p)$ one can view the fibre as the pro-homotopy type of a solenoid.

For $\mathcal{X} = \overline{\mathrm{Spec}(\mathcal{O}_F)}$ where F is a number field, one expects $R\gamma_*\mathbb{Z}$ to be concentrated in degrees 0 and 2 (see [34][Sec.9]). On the other hand, if

$$\mathcal{X} = \mathbb{P}^1_{\mathrm{Spec}(\mathcal{O}_F)}$$

has the correct $\mathbb{Z}$-cohomology, compatible with the computations of $\tilde{\mathbb{R}}$-cohomology in this paper, then $H^4(\bar{\mathcal{X}}_W, \mathbb{Z})$ must be a finitely generated group of rank r_2, the rank of $K_3(\mathcal{O}_F)$ (see j) in section 9.4.2 below). This can only happen if $R^i\gamma_*\mathbb{Z}$ is nonzero for $i = 3$ or $i = 4$, the most likely scenario being that $R^4\gamma_*\mathbb{Z}$ is nonzero with global sections $H^0(\bar{\mathcal{X}}_{et}, R^4\gamma_*\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(K_3(\mathcal{O}_F), \mathbb{Q})$. Again, this is only a heuristic argument since we have not rigorously defined the homotopy fibre, let alone established any relation between its $\mathbb{Z}$-cohomology and $R\gamma_*\mathbb{Z}$.

9.3.2. *Weil-groups of finitely generated fields.* The definition of the Weil-étale topos as a fibre product is closely related to the idea, briefly mentioned by Lichtenbaum in the introduction of [30], of defining the Weil-étale topos via Weil-groups for all scheme points $x \in X$, and then gluing into a global topos in the spirit of [30]. This is because the Weil-group of a field $k(x)$ of finite transcendence degree over its prime subfield F would be defined as the fibre product $G_{k(x)} \times_{G_F} W_F$ (as in Definition 12) and the classifying topos of this group is the fibre product of the classifying topoi of the factors by Corollary 4. The remarks in the previous paragraph would then apply to such a definition as well.

9.3.3. *Properties a)-f) for $\overline{\mathcal{X}}$.* If $\mathcal{X}$ is regular, proper and flat over $\mathrm{Spec}(\mathbb{Z})$ with generic fibre X of dimension d it follows easily from our results that properties a)-c) hold for $\overline{\mathcal{X}}$ where of course

$$R\Gamma_c(\overline{\mathcal{X}}_W, \tilde{\mathbb{R}}) = R\Gamma(\overline{\mathcal{X}}_W, \tilde{\mathbb{R}})$$

and

$$\zeta(\overline{\mathcal{X}}, s) = \zeta(\mathcal{X}, s) \prod_{i=0}^{2d} L_\infty(h^i(X), s)^{(-1)^i}.$$

Property d) must also hold for any reasonable definition of $R\Gamma(\overline{\mathcal{X}}_W, \mathbb{Z})$ as will become clear from our discussion in section 9.4.2 below. This discussion will also show, however, that properties e) and f) will definitely not hold for any definition of $R\Gamma(\overline{\mathcal{X}}_W, \mathbb{Z})$. This is consistent with the fact that there are no special value conjectures for the completed L-functions $\Lambda(h^i(X), s)$ in the literature.

9.3.4. *Non-regular/non-proper schemes.* For varieties over finite fields which are not smooth and proper the work of Geisser [19] shows that one has to replace the étale topology by the eh-topology (which allows abstract blow-ups as coverings) in order to define groups $H^i_c(X_{Wh}, \mathbb{Z})$ and $H^i_c(X_{Wh}, \mathbb{R})$ which are independent of a choice of compactification of X and which satisfy a)-f) in the introduction (where the index W is replaced by Wh). For arithmetic schemes over $\mathrm{Spec}(\mathbb{Z})$ a similar modification will be necessary, and one also has to assume some strong form of resolution of singularities for arithmetic schemes. We have refrained from trying to incorporate the idea of the eh-topology in this paper since our results (based on the fibre product definition of $\mathcal{X}_W$) are only very partial in any case.

9.4. RELATION TO THE TAMAGAWA NUMBER CONJECTURE. In this section we establish the compatibility of the conjectural properties of Weil-étale cohomology, as outlined in the introduction and augmented with some further assumptions below, with the Tamagawa number conjecture of Bloch and Kato.

9.4.1. *Statement of the Tamagawa number conjecture.* Let $\mathcal{X}$ be a proper, flat, regular $\mathbb{Z}$-scheme with generic fibre X of dimension d. The original Tamagawa number conjecture of Bloch and Kato [4] concerned the leading Taylor coefficient of $L(h^i(X),s)$ at integers $s \geq \frac{i+1}{2}$. This was then generalized by Fontaine and Perrin-Riou [16] to a conjecture about the vanishing order and leading coefficient at any integer s. In this paper we are only concerned with $s=0$.
One defines "integral motivic cohomology" groups $H^p_M(X_{/\mathbb{Z}},\mathbb{Q}(q))$ for example, as

$$H^p_M(X_{/\mathbb{Z}},\mathbb{Q}(q)) := \mathrm{im}\big(K_{2q-p}(\mathcal{X})^{(q)}_{\mathbb{Q}} \to K_{2q-p}(X)^{(q)}_{\mathbb{Q}}\big),$$

with $K_j(X)^{(q)}_{\mathbb{Q}}$ the q-th Adams eigenspace of the algebraic K-groups $K_j(X)\otimes_{\mathbb{Z}}\mathbb{Q}$. Denote by $W^*=\mathrm{Hom}_{\mathbb{Q}}(W,\mathbb{Q})$ the dual $\mathbb{Q}$-space and set $W_{\mathbb{R}} := W\otimes_{\mathbb{Q}}\mathbb{R}$.

CONJECTURE 1. *(Vanishing order) The space $H^{2d-i+1}_M(X_{/\mathbb{Z}},\mathbb{Q}(d+1))$ is finite dimensional and*

$$\begin{aligned}\mathrm{ord}_{s=0} L(h^i(X),s) &= \dim_{\mathbb{Q}} H^1_f(h^i(X)^*(1))^* - \dim_{\mathbb{Q}} H^0_f(h^i(X)^*(1))^* \\ &= \dim_{\mathbb{Q}} H^1_f(h^{2d-i}(X)(d+1))^* - \dim_{\mathbb{Q}} H^0_f(h^{2d-i}(X)(d+1))^* \\ &= \dim_{\mathbb{Q}} H^{2d-i+1}_M(X_{/\mathbb{Z}},\mathbb{Q}(d+1))^*\end{aligned}$$

Here we refer to [13] for the notation $H^i_f(M)$ (not needed in the sequel).
Let $H^p_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(q))$ denote (real) Deligne cohomology and let

$$\rho^i_\infty : H^{2d-i+1}_M(X_{/\mathbb{Z}},\mathbb{Q}(d+1))_{\mathbb{R}} \to H^{2d-i+1}_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(d+1)) \tag{66}$$

be the Beilinson regulator.

CONJECTURE 2. *(Beilinson) The map ρ^i_∞ is an isomorphism for $i\geq 1$ and there is an exact sequence*

$$0 \to H^{2d+1}_M(X_{/\mathbb{Z}},\mathbb{Q}(d+1))_{\mathbb{R}} \xrightarrow{\rho^0_\infty} H^{2d+1}_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(d+1)) \to CH^0(X)^*_{\mathbb{R}} \to 0 \tag{67}$$

for $i=0$.

We remark that Deligne cohomology satisfies a duality

$$H^{2d-i+1}_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(d+1))^* \cong H^i_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}) = H^i(X(\mathbb{C}),\mathbb{R})^+ \tag{68}$$

for $i \geq 0$ and deduce the well known fact that the vanishing order of $L(h^i(X), s)$ predicted by Conjectures 1 and 2 is in accordance with Lemma 10. Another consequence of conjecture 2 is

$$H_M^{2d-i+1}(X_{/\mathbb{Z}}, \mathbb{Q}(d+1)) = 0 \tag{69}$$

for $i \geq 2d+1$, a particular case of the Beilinson-Soule conjecture.
Define the fundamental line

$$\Delta_f(h^i(X)) = \det\nolimits_{\mathbb{Q}}^{-1}(H^i(X(\mathbb{C}), \mathbb{Q})^+) \otimes_{\mathbb{Q}} \det\nolimits_{\mathbb{Q}} H_M^{2d-i+1}(X_{/\mathbb{Z}}, \mathbb{Q}(d+1))^*$$

for $i > 0$ and

$$\begin{aligned}&\Delta_f(h^0(X))\\ &= \det\nolimits_{\mathbb{Q}} CH^0(X)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \det\nolimits_{\mathbb{Q}}^{-1}(H^0(X(\mathbb{C}), \mathbb{Q})^+) \otimes_{\mathbb{Q}} \det\nolimits_{\mathbb{Q}} H_M^{2d+1}(X_{/\mathbb{Z}}, \mathbb{Q}(d+1))^*\end{aligned}$$

for $i = 0$. There is an isomorphism

$$\vartheta_\infty^i : \mathbb{R} \cong \Delta_f(h^i(X))_{\mathbb{R}}$$

induced by (68) and the dual of (66) (resp. (67)) for $i > 0$ (resp. $i = 0$).
Now fix a prime number l and let $U \subseteq \mathrm{Spec}(\mathbb{Z})$ an open subscheme on which l is invertible. For any smooth l-adic sheaf V on U and prime $p \neq l$ define a complex concentrated in degrees 0 and 1

$$R\Gamma_f(\mathbb{Q}_p, V) = R\Gamma(\mathbb{F}_p, i_p^* j_{p,*} V) = V^{I_p} \xrightarrow{1-\mathrm{Frob}_p^{-1}} V^{I_p}$$

where I_p is the inertia subgroup at p and $i_p : \mathrm{Spec}(\mathbb{F}_p) \to \mathrm{Spec}(\mathbb{Z})$ and $j_p : U \to \mathrm{Spec}(\mathbb{Z})$ are the natural immersions. For $p = l$ define

$$R\Gamma_f(\mathbb{Q}_p, V) = D_{cris}(V) \xrightarrow{(1-\phi,\iota)} D_{cris}(V) \oplus D_{dR}(V)/F^0 D_{dR}(V)$$

where D_{cris} and D_{dR} are Fontaine's functors [16]. In both cases there is a map of complexes

$$R\Gamma_f(\mathbb{Q}_p, V) \to R\Gamma(\mathbb{Q}_p, V)$$

and one defines $R\Gamma_{/f}(\mathbb{Q}_p, V)$ as the mapping cone. The next Lemma shows that the complex $R\Gamma_f(\mathbb{Q}_p, V)$ has a uniform description for $p = l$ and $p \neq l$ in the case that interests us.

Lemma 11. *Let V be finite dimensional $\mathbb{Q}_p$-vector space with a continuous $G_p := \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$-action and such that $D_{dR}(V)/F^0 D_{dR}(V) = 0$. Then there is a commutative diagram in the derived category of $\mathbb{Q}_p$-vector spaces*

$$\begin{array}{ccc} R\Gamma_f(\mathbb{Q}_p, V) & \longrightarrow & R\Gamma(\mathbb{Q}_p, V) \\ {\scriptstyle\kappa}\big\downarrow & & \big\| \\ R\Gamma(\mathbb{F}_p, V^{I_p}) & \longrightarrow & R\Gamma(\mathbb{Q}_p, V) \end{array}$$

where κ is a quasi-isomorphism.

Proof. For a profinite group G and continuous G-module M we denote by $C^*(G,M)$ the standard complex of continuous cochains. There is an exact sequence of continuous G_p-modules

$$0\to V\to B^0(V)\xrightarrow{d^0}B^1(V)\to 0$$

where

$$B^0(V)=B_{cris}\otimes_{\mathbb{Q}_p}V$$

(with diagonal G_p-action),

$$B^1(V)=B_{cris}\otimes_{\mathbb{Q}_p}V\oplus(B_{dR}/F^0B_{dR})\otimes_{\mathbb{Q}_p}V$$

and $d^0(x)=((1-\phi)(x),\iota(x))$ where ι is induced by the canonical inclusion $B_{cris}\to B_{dR}$ (see [16] for more on Fontaine's rings B_{cris} and B_{dR}). Viewing this sequence as a quasi-isomorphism between V and a two term complex we obtain a quasi-isomorphism

$$\begin{aligned}R\Gamma(\mathbb{Q}_p,V)&=R\Gamma(G_p,V)=C^*(G_p,V)\\&\cong\mathrm{Tot}\left(C^*(G_p,B^0(V))\xrightarrow{d^{0*}}C^*(G_p,B^1(V))\right)\end{aligned}$$

where Tot denotes the simple complex associated to a double complex. By definition $R\Gamma_f(\mathbb{Q}_p,V)$ is the subcomplex

$$H^0(G_p,B^0(V))\xrightarrow{d^0}H^0(G_p,B^1(V))$$

of this double complex, i.e. the complex

$$D_{cris}(V)\xrightarrow{d^0}D_{cris}(V)\oplus D_{dR}(V)/F^0D_{dR}(V).$$

For any continuous G_p-module M there is moreover a quasi-isomorphism

$$R\Gamma(G_p,M)\cong R\Gamma(\mathbb{F}_p,R\Gamma(I_p,M))\cong\mathrm{Tot}\left(C^*(I_p,M)\xrightarrow{1-\mathrm{Frob}_p^{-1}}C^*(I_p,M)\right)$$

where $\mathrm{Frob}_p\in G_p$ is any lift of the Frobenius automorphism in G_p/I_p, acting simultaneously on I_p (by conjugation) and on M. The complex $R\Gamma(\mathbb{F}_p,H^0(I_p,M))$ is the subcomplex

$$H^0(I_p,M)\xrightarrow{1-\mathrm{Frob}_p^{-1}}H^0(I_p,M)$$

of this double complex. Combining these two constructions, we deduce that $R\Gamma(\mathbb{Q}_p,V)$ is canonically isomorphic to the total complex of the triple complex

$$\begin{tikzcd} C^*(I_p,B^0(V)) \arrow[r, "d^{0*}"] \arrow[d, "1-\mathrm{Frob}_p^{-1}"'] & C^*(I_p,B^1(V)) \arrow[d, "1-\mathrm{Frob}_p^{-1}"'] \\ C^*(I_p,B^0(V)) \arrow[r, "d^{0*}"] & C^*(I_p,B^1(V)). \end{tikzcd} \tag{70}$$

We have an isomorphism

$$\hat{\mathbb{Q}}_p^{ur}\otimes_{\mathbb{Q}_p}D_{cris}(V)\cong H^0(I_p,B^0(V))$$

where $\hat{\mathbb{Q}}_p^{ur}$ is the p-adic completion of the maximal unramified extension of $\mathbb{Q}_p$. The map ϕ on this space (induced by ϕ on B_{cris}) is semilinear with respect to the automorphism ϕ of $\hat{\mathbb{Q}}_p^{ur}$. The classical Dieudonné-Manin structure theorem for such semilinear ϕ-modules [27] shows that $H^0(I_p, B^0(V))$ is a direct sum of simple modules $E(\frac{N}{M}) \cong \hat{\mathbb{Q}}_p^{ur}[\phi]/(\phi^M - p^N)$, parametrized by non-negative rational numbers $\frac{N}{M}$, and an explicit computation shows that $1-\phi$ is surjective on $E(\frac{N}{M})$. Hence $1-\phi$ is surjective on $H^0(I_p, B^0(V))$. By our assumption on V we have

$$\begin{aligned} H^0(I_p, B^1(V)) &\cong H^0(I_p, B^0(V)) \oplus \hat{\mathbb{Q}}_p^{ur} \otimes_{\mathbb{Q}_p} (D_{dr}(V)/F^0 D_{dR}(V)) \\ &\cong H^0(I_p, B^0(V)) \end{aligned}$$

and d^0 is simply the map $1-\phi$ on $H^0(I_p, B^0(V))$. Hence $H^0(I_p, V)$ is quasi-isomorphic to the complex

$$H^0(I_p, B^0(V)) \xrightarrow{d^0} H^0(I_p, B^1(V))$$

and $R\Gamma(\mathbb{F}_p, H^0(I_p, V))$ is canonically isomorphic to the total complex of the double subcomplex

$$\begin{array}{ccc} H^0(I_p, B^0(V)) & \xrightarrow{d^0} & H^0(I_p, B^1(V)) \\ {\scriptstyle 1-\mathrm{Frob}_p^{-1}}\downarrow & & {\scriptstyle 1-\mathrm{Frob}_p^{-1}}\downarrow \\ H^0(I_p, B^0(V)) & \xrightarrow{d^0} & H^0(I_p, B^1(V)) \end{array}$$

of (70). Now, again using our assumption $D_{dR}(V)/F^0 D_{dR}(V) = 0$, this double complex is naturally quasi-isomorphic to $R\Gamma_f(\mathbb{Q}_p, V)$ via the natural map κ (given by the inclusion $H^0(G_p, -) \to H^0(I_p, -)$) in the following diagram

$$\begin{array}{ccc} D_{cris}(V) & \xrightarrow{1-\phi} & D_{cris}(V) \\ {\scriptstyle \kappa^0}\downarrow & & {\scriptstyle \kappa^1}\downarrow \\ H^0(I_p, B^0(V)) & \xrightarrow{1-\phi} & H^0(I_p, B^0(V)) \\ {\scriptstyle 1-\mathrm{Frob}_p^{-1}}\downarrow & & {\scriptstyle 1-\mathrm{Frob}_p^{-1}}\downarrow \\ H^0(I_p, B^0(V)) & \xrightarrow{1-\phi} & H^0(I_p, B^0(V)). \end{array}$$

Indeed, the vertical sequences in this diagram are short exact sequences. The space $D_{cris}(V) = H^0(G_p, B^0(V))$ is the kernel of $1-\mathrm{Frob}_p^{-1}$ on $H^0(I_p, B^0(V))$ by definition, and $1-\mathrm{Frob}_p^{-1}$ is surjective on $H^0(I_p, B^0(V))$. This is because there is an isomorphism of Frob_p-modules

$$H^0(I_p, B^0(V)) = \hat{\mathbb{Q}}_p^{ur} \otimes_{\mathbb{Q}_p} D_{cris}(V) \cong (\hat{\mathbb{Q}}_p^{ur})^d$$

where $d = \dim_{\mathbb{Q}_p} D_{cris}(V)$, using again the fact that Frob_p acts trivially on $D_{cris}(V)$. Finally, it is well known that $1-\mathrm{Frob}_p : \hat{\mathbb{Q}}_p^{ur} \to \hat{\mathbb{Q}}_p^{ur}$ is surjective. This concludes the proof of the Lemma. $\square$

We continue with the notations introduced before Lemma 11. One defines a global complex $R\Gamma_f(\mathbb{Q}, V)$ as the mapping fibre of

$$R\Gamma(U_{\mathrm{et}}, V) \to \bigoplus_{p \notin U} R\Gamma_{/f}(\mathbb{Q}_p, V).$$

Then there is an exact triangle in the derived category of $\mathbb{Q}_l$-vector spaces

$$R\Gamma_c(U_{\mathrm{et}}, V) \to R\Gamma_f(\mathbb{Q}, V) \to \bigoplus_{p \notin U} R\Gamma_f(\mathbb{Q}_p, V) \tag{71}$$

where the primes $p \notin U$ include $p = \infty$ with the convention $R\Gamma_f(\mathbb{R}, V) = R\Gamma(\mathbb{R}, V)$. One can further show that Artin-Verdier duality induces a duality

$$H^i_f(\mathbb{Q}, V) \cong H^{3-i}_f(\mathbb{Q}, V^*(1))^*.$$

The index "f" stands for "finite" which in this context is synonymous for "unramified" or "coming from an integral model". The following proposition justifies this interpretation of the complex $R\Gamma_f$ in the case of interest in this paper.

PROPOSITION 9.1. *Let $\pi : \mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ be a regular, proper, flat $\mathbb{Z}$-scheme and $\bar{\mathcal{X}}_{\mathrm{et}}$ its Artin-Verdier étale topos. Let $U \subseteq \mathrm{Spec}(\mathbb{Z})$ be an open subscheme so that $\pi_U : \mathcal{X}_U \to U$ is proper and smooth, let l be a prime number invertible on U and set $\mathcal{X}_p = \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{F}_p$. For brevity we write $\mathcal{X}_{\infty,\mathrm{et}}$ for $Sh(\mathcal{X}_\infty)$ (see Prop. 4.1). Assume Conjecture 9 in the next section. Then there is an isomorphism of exact triangles in the derived category of $\mathbb{Q}_l$-vector spaces*

$$\begin{array}{ccccccc}
R\Gamma_c(\mathcal{X}_{U,\mathrm{et}}, \mathbb{Q}_l) & \to & R\Gamma(\bar{\mathcal{X}}_{\mathrm{et}}, \mathbb{Q}_l) & \to & \bigoplus_{p \notin U} R\Gamma(\mathcal{X}_{p,\mathrm{et}}, \mathbb{Q}_l) & \to \\
\downarrow & & \downarrow & & \downarrow & \\
\bigoplus_{i=0}^{2d} R\Gamma_c(U_{\mathrm{et}}, V_l^i)[-i] & \to & \bigoplus_{i=0}^{2d} R\Gamma_f(\mathbb{Q}, V_l^i)[-i] & \to & \bigoplus_{p \notin U} \bigoplus_{i=0}^{2d} R\Gamma_f(\mathbb{Q}_p, V_l^i)[-i] & \to
\end{array}$$

where $V_l^i := H^i(X_{\bar{\mathbb{Q}},\mathrm{et}}, \mathbb{Q}_l)$ and the bottom exact triangle is a sum over triangles (71).

Proof. For all p and l (including $p = \infty$ with a suitable interpretation of the terms) we shall first show that there is a commutative diagram

$$\begin{array}{ccccccc}
R\Gamma(\mathcal{X}_{p,\mathrm{et}}, \mathbb{Q}_l) & \to & R\Gamma(\mathcal{X}_{\mathbb{Q}_p,\mathrm{et}}, \mathbb{Q}_l) & \to & R\Gamma_{\mathcal{X}_p}(\mathcal{X}_{\mathbb{Z}_p,\mathrm{et}}, \mathbb{Q}_l)[1] & \to \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow & \\
\bigoplus_{i=0}^{2d} R\Gamma_f(\mathbb{Q}_p, V_l^i)[-i] & \to & \bigoplus_{i=0}^{2d} R\Gamma(\mathbb{Q}_p, V_l^i)[-i] & \to & \bigoplus_{i=0}^{2d} R\Gamma_{/f}(\mathbb{Q}_p, V_l^i)[-i] & \to
\end{array} \tag{72}$$

where the rows are exact and the vertical maps are quasi-isomorphisms. This then induces a commutative diagram where the vertical maps are quasi-isomorphisms

(73)
$$\begin{CD}
R\Gamma(\mathcal{X}_{U,\mathrm{et}},\mathbb{Q}_l) @>>> \bigoplus_{p\notin U} R\Gamma(\mathcal{X}_{\mathbb{Q}_p,\mathrm{et}},\mathbb{Q}_l) @>>> \bigoplus_{p\notin U} R\Gamma_{\mathcal{X}_p}(\mathcal{X}_{\mathbb{Z}_p,\mathrm{et}},\mathbb{Q}_l)[1]\\
@VVV @VVV @VVV\\
\bigoplus_{i=0}^{2d} R\Gamma(U_{\mathrm{et}},V_l^i)[-i] @>>> \bigoplus_{p\notin U}\bigoplus_{i=0}^{2d} R\Gamma(\mathbb{Q}_p,V_l^i)[-i] @>>> \bigoplus_{p\notin U}\bigoplus_{i=0}^{2d} R\Gamma_{/f}(\mathbb{Q}_p,V_l^i)[-i].
\end{CD}$$

Indeed, the first commutative square is induced by the commutative diagram

$$\begin{CD}
\mathcal{X}_U @<<< \mathcal{X}_{\mathbb{Q}_p}\\
@VVV @VVV\\
U @<<< \mathrm{Spec}(\mathbb{Q}_p)
\end{CD}$$

and a decomposition $R\pi_{U,*}\mathbb{Q}_l \cong \bigoplus_{i=0}^{2d} V_l^i[-i]$ in the derived category of l-adic sheaves on U, and the second is a sum over $p \notin U$ of the right hand square in (72). Taking mapping fibres of the composite horizontal maps in (73) we obtain an isomorphism of exact triangles

$$\begin{CD}
R\Gamma(\bar{\mathcal{X}}_{\mathrm{et}},\mathbb{Q}_l) @>>> R\Gamma(\mathcal{X}_{U,\mathrm{et}},\mathbb{Q}_l) @>>> \bigoplus_{p\notin U} R\Gamma_{\mathcal{X}_p}(\mathcal{X}_{\mathbb{Z}_p,\mathrm{et}},\mathbb{Q}_l)[1] @>>>\\
@VVV @VVV @VVV\\
\bigoplus_{i=0}^{2d} R\Gamma_f(\mathbb{Q},V_l^i)[-i] @>>> \bigoplus_{i=0}^{2d} R\Gamma(U_{\mathrm{et}},V_l^i)[-i] @>>> \bigoplus_{p\notin U}\bigoplus_{i=0}^{2d} R\Gamma_{/f}(\mathbb{Q}_p,V_l^i)[-i] @>>>
\end{CD}$$

where we use excision to identify the first fibre with $R\Gamma(\bar{\mathcal{X}}_{\mathrm{et}},\mathbb{Q}_l)$. The octahedral axiom then gives the isomorphism of exact triangles in Proposition 9.1, using the fact that the mapping fibre of the top left (resp. bottom left) horizontal map in (73) is $R\Gamma_c(\mathcal{X}_{U,\mathrm{et}},\mathbb{Q}_l)$ (resp. $\bigoplus_{i=0}^{2d} R\Gamma_c(U_{\mathrm{et}},V_l^i)[-i]$).

Concerning (72), for $p=\infty$ we declare $R\Gamma(\mathcal{X}_{p,\mathrm{et}},\mathbb{Q}_l) = R\Gamma(\mathcal{X}_{\mathbb{Q}_p,\mathrm{et}},\mathbb{Q}_l)$ and $R\Gamma_{\mathcal{X}_p}(\mathcal{X}_{\mathbb{Z}_p,\mathrm{et}},\mathbb{Q}_l)=0$. This agrees with the convention $R\Gamma_f(\mathbb{R},-)=R\Gamma(\mathbb{R},-)$ introduced above. For $p\neq\infty$ the top exact triangle is simply a localization triangle in étale cohomology since we have $R\Gamma(\mathcal{X}_{p,\mathrm{et}},\mathbb{Q}_l)\cong R\Gamma(\mathcal{X}_{\mathbb{Z}_p,\mathrm{et}},\mathbb{Q}_l)$ by proper base change. It suffices to construct quasi-isomorphisms α and β so that the left hand square in (72) commutes. For brevity we now omit the index et when referring to (continuous l-adic) étale cohomology.

The quasi-isomorphism β is induced by the Leray spectral sequence for $\pi_{\mathbb{Q}_p}$ and a decomposition

$$R\pi_{\mathbb{Q}_p,*}\mathbb{Q}_l \cong \bigoplus_{i=0}^{2d} V_l^i[-i] \tag{74}$$

in the derived category of l-adic sheaves on $\mathrm{Spec}(\mathbb{Q}_p)$. The existence of α follows if the composite map

$$H^i(\mathcal{X}_p,\mathbb{Q}_l)\to H^i(\mathcal{X}_{\mathbb{Q}_p},\mathbb{Q}_l)\xrightarrow{H^i(\beta)} H^0(\mathbb{Q}_p,V_l^i)\oplus H^1(\mathbb{Q}_p,V_l^{i-1})\oplus H^2(\mathbb{Q}_p,V_l^{i-2})$$

induces an isomorphism

$$H^i(\mathcal{X}_p,\mathbb{Q}_l)\cong H^0_f(\mathbb{Q}_p,V_l^i)\oplus H^1_f(\mathbb{Q}_p,V_l^{i-1}).$$

We shall show this only referring to the filtration F^* on $H^i(\mathcal{X}_{\mathbb{Q}_p},\mathbb{Q}_l)$ induced by the Leray spectral sequence for $\pi_{\mathbb{Q}_p}$, not any particular decomposition (74). The Hochschild-Serre spectral sequence for the covering $\mathcal{X}_{\hat{\mathbb{Z}}_p^{ur}}\to\mathcal{X}_{\mathbb{Z}_p}$, whose group we identify with $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$, induces a commutative diagram with exact rows
(75)

$$\begin{array}{ccccccc}
0\to & H^1(\mathbb{F}_p,H^{i-1}(\mathcal{X}_{\bar{\mathbb{F}}_p},\mathbb{Q}_l)) & \to & H^i(\mathcal{X}_p,\mathbb{Q}_l) & \to & H^0(\mathbb{F}_p,H^i(\mathcal{X}_{\bar{\mathbb{F}}_p},\mathbb{Q}_l)) & \to 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & H^1(\mathbb{F}_p,H^{i-1}(\mathcal{X}_{\hat{\mathbb{Q}}_p^{ur}},\mathbb{Q}_l)) & \to & H^i(\mathcal{X}_{\mathbb{Q}_p},\mathbb{Q}_l) & \to & H^0(\mathbb{F}_p,H^i(\mathcal{X}_{\hat{\mathbb{Q}}_p^{ur}},\mathbb{Q}_l)) & \to 0\\
 & \downarrow & & \downarrow & & \downarrow{\scriptstyle\gamma} & \\
 & H^1(\mathbb{F}_p,H^0(I_p,V_l^{i-1})) & & H^0(\mathbb{Q}_p,V_l^i) & = & H^0(\mathbb{F}_p,H^0(I_p,V_l^i)). &
\end{array}$$

The left and right composite vertical maps are isomorphisms by Theorem 10.1 b) for $l\neq p$ (resp. Conjecture 9 for $l=p$) and the fact that

$$R\Gamma(\mathbb{F}_p,V)\cong R\Gamma(\mathbb{F}_p,W_0V)$$

for any l-adic sheaf V on $\mathrm{Spec}(\mathbb{F}_p)$ where $W_0V\subseteq V$ is the generalized Frobenius eigenspace for eigenvalues which are roots of unity (or just for the eigenvalue 1). Note also that

$$H^k_f(\mathbb{Q}_p,V_l^i)=H^k(\mathbb{F}_p,H^0(I_p,V_l^i))$$

for $k=0,1$ and all l and i by Lemma 11 since

$$D_{dR}(V_p^i)\cong H^i_{dR}(\mathcal{X}_{\mathbb{Q}_p}/\mathbb{Q}_p)=F^0H^i_{dR}(\mathcal{X}_{\mathbb{Q}_p}/\mathbb{Q}_p)\cong F^0D_{dR}(V_p^i).$$

The kernel of the map γ in (75) is $H^0(\mathbb{F}_p,H^1(I_p,V_l^{i-1}))$, hence there is a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0\to & H^1(\mathbb{F}_p,H^{i-1}(\mathcal{X}_{\hat{\mathbb{Q}}_p^{ur}},\mathbb{Q}_l)) & \to & F^1H^i(\mathcal{X}_{\mathbb{Q}_p},\mathbb{Q}_l) & \to & H^0(\mathbb{F}_p,H^1(I_p,V_l^{i-1})) & \to 0\\
 & \downarrow & & \downarrow & & \| & \\
0\to & H^1(\mathbb{F}_p,H^0(I_p,V_l^{i-1})) & \to & H^1(\mathbb{Q}_p,V_l^{i-1}) & \to & H^0(\mathbb{F}_p,H^1(I_p,V_l^{i-1})) & \to 0
\end{array}$$

which implies that the left vertical isomorphism in (75) fits into a commutative diagram with the natural map $F^1H^i(\mathcal{X}_{\mathbb{Q}_p},\mathbb{Q}_l)\to H^1(\mathbb{Q}_p,V_l^{i-1})$. This finishes the proof of the existence of α and of Proposition 9.1.

We remark that for $l \neq p$ we have $W_0 H^1(I_p, V_l^i) = W_0((V_l^i)_{I_p}(-1)) = 0$ and hence isomorphisms

$$W_0 H^i(\mathcal{X}_{\bar{\mathbb{F}}_p}, \mathbb{Q}_l) \cong W_0 H^i(\mathcal{X}_{\hat{\mathbb{Q}}_p^{ur}}, \mathbb{Q}_l) \cong W_0 H^0(I_p, V_l^i)$$

which implies that the top left and right, and therefore the top middle vertical maps in (75) are isomorphisms. We conclude that

$$R\Gamma(\mathcal{X}_p, \mathbb{Q}_l) \cong R\Gamma(\mathcal{X}_{\mathbb{Q}_p}, \mathbb{Q}_l)$$

for $l \neq p$ like for $p = \infty$. □

We continue with the statement of the Tamagawa number conjecture. One might view the following conjecture as an l-adic analogue of Beilinson's conjecture, or as a generalization of Tate's conjecture.

Conjecture 3. *(Bloch-Kato) There are isomorphisms*

$$\rho_l^i : H_f^2(\mathbb{Q}, V_l^i) \cong H_M^{2d-i+1}(X_{/\mathbb{Z}}, \mathbb{Q}(d+1))^*_{\mathbb{Q}_l}$$

and $H_f^1(\mathbb{Q}, V_l^i) = 0$ *for any* i.

One can show easily that $H_f^0(\mathbb{Q}, V_l^0) \cong Ch^0(X)_{\mathbb{Q}_l}$, $H_f^0(\mathbb{Q}, V_l^i) = 0$ for $i > 0$ and $H_f^3(\mathbb{Q}, V_l^i) = 0$ so that Conjecture 3 computes the entire cohomology of $R\Gamma_f(\mathbb{Q}, V_l^i)$. Together with Artin's comparison isomorphism

$$V_l^i = H^i(X_{\bar{\mathbb{Q}},\mathrm{et}}, \mathbb{Q}_l) \cong H^i(X(\mathbb{C}), \mathbb{Q})_{\mathbb{Q}_l}$$

as well as the isomorphisms

$$\iota_p : \det_{\mathbb{Q}_l} R\Gamma_f(\mathbb{Q}_p, V) \cong \mathbb{Q}_l$$

induced by the identity map on $(V_l^i)^{I_p}$ and $D_{cris}(V_l^i)$, Conjecture 3 induces an isomorphism

$$\vartheta_l^i : \Delta_f(h^i(X))_{\mathbb{Q}_l} \cong \det_{\mathbb{Q}_l} R\Gamma_f(\mathbb{Q}, V_l^i) \otimes \det_{\mathbb{Q}_l} R\Gamma(\mathbb{R}, V_l^i) \cong \det_{\mathbb{Q}_l} R\Gamma_c(U_{\mathrm{et}}, V_l^i).$$

Conjecture 4. *(l-part of the Tamagawa number conjecture) There is an identity of free rank one $\mathbb{Z}_l$-submodules of* $\det_{\mathbb{Q}_l} R\Gamma_c(U_{\mathrm{et}}, V_l^i)$

$$\mathbb{Z}_l \cdot \vartheta_l^i \circ \vartheta_\infty^i(L^*(h^i(X), 0)^{-1}) = \det_{\mathbb{Z}_l} R\Gamma_c(U_{\mathrm{et}}, T_l^i)$$

for any Galois stable $\mathbb{Z}_l$-lattice $T_l^i \subseteq V_l^i$.

This conjecture is independent of the choice of the lattice T_l^i since

$$\prod_{i \in \mathbb{Z}} |H_c^i(U_{\mathrm{et}}, M)|^{(-1)^i} = 1 \tag{76}$$

for any finite locally constant sheaf M whose cardinality is invertible on U. The following conjecture allows a reformulation of the Tamagawa number conjecture in terms of the L-function

$$L_U(h^i(X), s) = \prod_{p \in U} L_p(h^i(X), s)$$

associated to the smooth l-adic sheaf V_l^i over U. Recall that a two term complex $C = \left(W \xrightarrow{\lambda} W\right)$ is called *semisimple at* 0 if the composite map

$$H^0(C) = \ker(\lambda) \subseteq W \to \mathrm{coker}(\lambda) = H^1(C)$$

is an isomorphism. This is always the case, for example, if the complex C is acyclic.

CONJECTURE 5. *(Frobenius-Semisimplicity at the eigenvalue 1) For any prime number p the complex $R\Gamma_f(\mathbb{Q}_p, V_l^i)$ is semisimple at zero.*

Under this conjecture one has a *second* isomorphism

$$\tilde{\iota}_p : \det_{\mathbb{Q}_l} R\Gamma_f(\mathbb{Q}_p, V_l^i) \cong \mathbb{Q}_l$$

which satisfies

$$\iota_p = P_p^*(h^i(X), 1)^{-1}\tilde{\iota}_p = L_p^*(h^i(X), 0)\log(p)^{r_{i,p}}\tilde{\iota}_p$$

where $r_{i,p} = \mathrm{ord}_{T=1} P_p(h^i(X), T) = -\mathrm{ord}_{s=0} L_p(h^i(X), s)$ (see [7][Lemma 2]). If $R\Gamma_f(\mathbb{Q}_p, V_l^i)$ is acyclic then $\tilde{\iota}_p$ is the canonical trivialization of the determinant of an acyclic complex. Using this second isomorphism the Tamagawa number conjecture becomes

$$\mathbb{Z}_l \cdot \tilde{\vartheta}_l^i \circ \tilde{\vartheta}_\infty^i(L_U^*(h^i(X), 0)^{-1}) = \det_{\mathbb{Z}_l} R\Gamma_c(U_{\mathrm{et}}, T_l^i) \tag{77}$$

where

$$\tilde{\vartheta}_l^i = \prod_{p \notin U} P_p^*(h^i(X), 1)\vartheta_l^i, \quad \tilde{\vartheta}_\infty^i = \prod_{p \notin U} \log(p)^{r_{i,p}}\vartheta_\infty^i.$$

9.4.2. *Further assumptions on Weil-étale cohomology.* In order to establish the compatibility of the conjectural picture a)-f) outlined in the introduction with the Tamagawa number conjecture, we need to augment it with a number of further assumptions. Even though a)-f) only refer to cohomology groups we now assume that these groups do indeed arise from a topos $\mathcal{X}_W$ - different from the one defined in Definition 10 - and that compact support cohomology is defined via an embedding into a proper scheme followed by an Artin-Verdier type compactification $\bar{\mathcal{X}}_W$ (and is independent of a choice of compactification).

g) For an open subscheme U of an arithmetic scheme $\mathcal{X}$ with closed complement Z there is an exact triangle of perfect complexes in the derived category of abelian groups

$$R\Gamma_c(U_W, \mathbb{Z}) \to R\Gamma_c(\mathcal{X}_W, \mathbb{Z}) \to R\Gamma_c(Z_W, \mathbb{Z}) \to .$$

h) There is a morphism of topoi $\gamma : \mathcal{X}_W \to \mathcal{X}_{\mathrm{et}}$ for any arithmetic scheme $\mathcal{X}$ (or the Artin-Verdier compactification of such a scheme). Moreover, for any constructible sheaf $\mathcal{F}$ on $\mathcal{X}_{\mathrm{et}}$ the adjunction $\mathcal{F} \to R\gamma_*\gamma^*\mathcal{F}$ is an isomorphism.

If $\mathcal{X}$ has finite characteristic then h) holds if one understands the index W as denoting the Weil-eh cohomology of Geisser (see [19][Thm. 5.2b), Thm. 3.6]). In addition, the exact triangle in g) exists for Weil-eh cohomology by [19][Def. 5.4, eq. (4)] but perfectness is only known under resolution of singularities

(more precisely, under the assumption R(d) of [19][Def. 2.4], see loc. cit. Lemma 2.7).
The following property is a natural extension of property g) to the Artin-Verdier compactification.

i) If $\mathcal{X}$ is regular, proper, flat over $\mathrm{Spec}(\mathbb{Z})$ then there is an exact triangle in the derived category of abelian groups

$$R\Gamma_c(\mathcal{X}_W,\mathbb{Z}) \to R\Gamma(\bar{\mathcal{X}}_W,\mathbb{Z}) \to R\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z}) \to$$

and there is an exact triangle

$$R\Gamma_c(\mathcal{X}_W,\tilde{\mathbb{R}}) \to R\Gamma(\bar{\mathcal{X}}_W,\tilde{\mathbb{R}}) \to R\Gamma(\mathcal{X}_{\infty,W},\tilde{\mathbb{R}}) \to$$

in the derived category of $\mathbb{R}$-vector spaces, where $\mathcal{X}_{\infty,W}$ was defined in Definition 11.

Note that

$$R\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z}) \cong R\Gamma(\mathcal{X}_\infty,\gamma_{\infty*}(\mathbb{Z})) \cong R\Gamma(\mathcal{X}_\infty,\mathbb{Z})$$

by Proposition 8.2 and this last complex is isomorphic to the singular complex of the (locally contractible) compact space $\mathcal{X}_\infty$ and is therefore a perfect complex of abelian groups. Since the complex $R\Gamma_c(\mathcal{X}_W,\mathbb{Z})$ is perfect by d) the triangle in i) then implies that $R\Gamma(\bar{\mathcal{X}}_W,\mathbb{Z})$ is also a perfect complex of abelian groups. Note also that, unlike in the situation g), the triangle for $\tilde{\mathbb{R}}$-coefficients is not the scalar extension of the triangle for $\mathbb{Z}$-coefficients since neither $R\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z})$ nor $R\Gamma(\bar{\mathcal{X}}_W,\mathbb{Z})$ satisfies property e). One rather has a commutative diagram of long exact sequences

$$\begin{tikzcd}
{} \arrow[r] & H^{i+1}(\mathcal{X}_{\infty,W},\tilde{\mathbb{R}}) \arrow[r] & H^{i+2}_c(\mathcal{X}_W,\tilde{\mathbb{R}}) \arrow[r] & 0 \arrow[r] & H^{i+2}(\mathcal{X}_{\infty,W},\tilde{\mathbb{R}}) \\
{} \arrow[r] & H^{i+1}(\mathcal{X}_{\infty,W},\mathbb{Z}) \arrow[r] \arrow[u, "\alpha_\infty"] & H^{i+2}_c(\mathcal{X}_W,\mathbb{Z}) \arrow[r] \arrow[u, "\alpha"] & H^{i+2}(\bar{\mathcal{X}}_W,\mathbb{Z}) \arrow[r] \arrow[u] & H^{i+2}(\mathcal{X}_{\infty,W},\mathbb{Z}) \arrow[u]
\end{tikzcd}$$

where only $\alpha_{\mathbb{R}}$ is an isomorphism by e). Here we assume $i \geq 0$ so that $H^{i+2}(\bar{\mathcal{X}}_W,\tilde{\mathbb{R}}) = 0$ by Theorem 7.1. There is a direct sum decomposition

$$H^{i+1}(\mathcal{X}_{\infty,W},\tilde{\mathbb{R}}) \cong H^{i+1}(\mathcal{X}_\infty,\mathbb{R}) \oplus H^{i}(\mathcal{X}_\infty,\mathbb{R})$$

by (47) and an isomorphism

$$H^{i+1}(\mathcal{X}_{\infty,W},\mathbb{Z})_{\mathbb{R}} \cong H^{i+1}(\mathcal{X}_\infty,\mathbb{Z})_{\mathbb{R}} \cong H^{i+1}(\mathcal{X}_\infty,\mathbb{R}).$$

One therefore obtains a map for $i \geq 0$

$$r^i_\infty : H^i(\mathcal{X}_\infty,\mathbb{R}) \to H^{i+2}(\bar{\mathcal{X}}_W,\mathbb{Z})_{\mathbb{R}}$$

which is an isomorphism for $i > 0$.
Proposition 9.1 and assumption h) yield an isomorphism for $i \geq 0$

$$r^i_l : H^2_f(\mathbb{Q},V^i_l) \cong H^{i+2}(\bar{\mathcal{X}}_{\mathrm{et}},\mathbb{Q}_l) \cong H^{i+2}(\bar{\mathcal{X}}_W,\mathbb{Q}_l) \cong H^{i+2}(\bar{\mathcal{X}}_W,\mathbb{Z})_{\mathbb{Q}_l}.$$

The following is the key requirement on a definition of a Weil-étale topos.

j) If $\mathcal{X}$ is regular, proper, flat over $\mathrm{Spec}(\mathbb{Z})$ with generic fibre X of dimension d then there are isomorphisms

$$\lambda^i : H^{i+2}(\overline{\mathcal{X}}_W, \mathbb{Z})_{\mathbb{Q}} \cong H_M^{2d-i+1}(X_{/\mathbb{Z}}, \mathbb{Q}(d+1))^*$$

for $i \geq 0$ such that $\lambda^i_{\mathbb{R}} \circ r^i_\infty = (\rho^i_\infty)^*$ and $\lambda^i_{\mathbb{Q}_l} \circ r^i_l = \rho^i_l$.

This is true for $d = i = 0$ with Lichtenbaum's current definition where

$$H^2(\overline{\mathrm{Spec}(\mathcal{O}_F)}_W, \mathbb{Z})_{\mathbb{Q}} \cong H^1_M(\mathrm{Spec}(F)_{/\mathbb{Z}}, \mathbb{Q}(1))^* = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F^\times, \mathbb{Q}).$$

Note that j) together with (69) and h) also implies

$$H^{i+2}(\overline{\mathcal{X}}_W, \mathbb{Z}) = 0$$

for $i > 2d+1$, which is not satisfied by the current definition of $\overline{\mathrm{Spec}(\mathcal{O}_F)}_W$.

PROPOSITION 9.2. *Suppose there is a definition of Weil-étale cohomology groups for arithmetic schemes satisfying a)-j) except perhaps f) for schemes of characteristic* 0*. Let X be a proper, smooth variety over $\mathbb{Q}$ of dimension d which has a proper, regular model over* $\mathrm{Spec}(\mathbb{Z})$ *such that Conjectures 1,2,3,5,9 are satisfied. Assume $L(h^i(X), s)$ has a meromorphic continuation to $s = 0$ for all i. Then the Tamagawa number conjecture (Conjecture 4) for the motive*

$$h(X) = \bigoplus_{i=0}^{2d} h^i(X)[-i]$$

is equivalent to statement f) for any arithmetic scheme $\mathcal{X}$ with generic fibre X.

Proof. If $\mathcal{X}$ is any arithmetic scheme with generic fibre X then there exists an open subscheme $U \subseteq \mathrm{Spec}(\mathbb{Z})$ so that $\pi : \mathcal{X}_U \to U$ is proper and smooth. Let Z be the closed complement of U. Then by g) we have an isomorphism

$$\det{}_{\mathbb{Z}} R\Gamma_c(\mathcal{X}_W, \mathbb{Z}) \cong \det{}_{\mathbb{Z}} R\Gamma_c(\mathcal{X}_{U,W}, \mathbb{Z}) \otimes_{\mathbb{Z}} \det{}_{\mathbb{Z}} R\Gamma_c(X_{Z,W}, \mathbb{Z})$$

as well as factorizations

$$\zeta(\mathcal{X}, s) = \zeta(\mathcal{X}_U, s)\zeta(\mathcal{X}_Z, s); \quad \zeta^*(\mathcal{X}, 0) = \zeta^*(\mathcal{X}_U, 0)\zeta^*(\mathcal{X}_Z, 0).$$

Since we assume f) for $\mathcal{X}_Z = \coprod_{p \in Z} \mathcal{X}_p$, statement f) for $\mathcal{X}$ is equivalent to statement f) for $\mathcal{X}_U$. We now assume that $\mathcal{X}$ is the proper regular model of X. For any prime $p \in Z$, the exact sequence

$$\cdots \xrightarrow{\cup e} H^i(\mathcal{X}_{p,W}, \mathbb{Z})_{\mathbb{Q}} \xrightarrow{\cup e} H^{i+1}(\mathcal{X}_{p,W}, \mathbb{Z})_{\mathbb{Q}} \xrightarrow{\cup e} \cdots,$$

where $e \in H^1(\mathcal{X}_{p,W}, \mathbb{Z})$ is Lichtenbaum's canonical class, yields a trivialization

$$\mathbb{Q} \cong \det{}_{\mathbb{Q}} R\Gamma(\mathcal{X}_{p,W}, \mathbb{Z})_{\mathbb{Q}}. \tag{78}$$

The exact triangle

$$R\Gamma_c(\mathcal{X}_{U,W}, \mathbb{Z}) \to R\Gamma(\bar{\mathcal{X}}_W, \mathbb{Z}) \to \bigoplus_{p \in Z \cup \{\infty\}} R\Gamma(\mathcal{X}_{p,W}, \mathbb{Z})$$

together with assumptions i), j), (78) and the isomorphisms

$$H^i(\mathcal{X}_{\infty,W}, \mathbb{Z})_{\mathbb{Q}} \cong H^i(\mathcal{X}(\mathbb{C}), \mathbb{Q})^+, \quad CH^0(X)_{\mathbb{Q}} \cong H^0(\bar{\mathcal{X}}_W, \mathbb{Z})_{\mathbb{Q}}$$

induce an isomorphism

$$\vartheta_W : \det_{\mathbb{Q}} R\Gamma_c(\mathcal{X}_{U,W},\mathbb{Z})_{\mathbb{Q}} \cong \det_{\mathbb{Q}} R\Gamma(\bar{\mathcal{X}}_W,\mathbb{Z})_{\mathbb{Q}} \otimes \bigotimes_{p\in Z\cup\{\infty\}} \det_{\mathbb{Q}}^{-1} R\Gamma(\mathcal{X}_{p,W},\mathbb{Z})_{\mathbb{Q}}$$
$$\cong \bigotimes_{i=0}^{2d} \Delta_f(h^i(X))^{(-1)^i}.$$

By assumption j) there is a commutative diagram of isomorphisms

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\gamma"] \arrow[d, equal] & \det_{\mathbb{R}} R\Gamma_c(\mathcal{X}_{U,W},\mathbb{Z})_{\mathbb{R}} \arrow[d, "\vartheta_{W,\mathbb{R}}"] \\ \mathbb{R} \arrow[r, "\otimes_i(\tilde{\vartheta}^i_\infty)^{(-1)^i}"] & \bigotimes_{i=0}^{2d} \Delta_f(h^i(X))_{\mathbb{R}}^{(-1)^i} \end{tikzcd}$$

where γ is induced by c). The power of $\log(p)$ in $\tilde{\vartheta}$ appears for the same reason as in the proof of Theorem 9.2.
Similarly, j) implies that for any prime $l \in Z$ we have a commutative diagram of isomorphisms

$$\begin{tikzcd} \det_{\mathbb{Q}_l} R\Gamma_c(\mathcal{X}_{U,W},\mathbb{Z})_{\mathbb{Q}_l} \arrow[r] \arrow[d, "\vartheta_{W,\mathbb{Q}_l}"] & \det_{\mathbb{Q}_l} R\Gamma_c(\mathcal{X}_{U,\mathrm{et}},\mathbb{Q}_l) \arrow[d] \\ \bigotimes_{i=0}^{2d} \Delta_f(h^i(X))_{\mathbb{Q}_l}^{(-1)^i} \arrow[r, "\otimes_i(\tilde{\vartheta}^i_l)^{(-1)^i}"] & \bigotimes_{i=0}^{2d} \det_{\mathbb{Q}_l}^{(-1)^i} R\Gamma_c(U_{\mathrm{et}},V_l^i) \end{tikzcd} \tag{79}$$

where the top isomorphism is induced by an isomorphism

$$R\Gamma_c(\mathcal{X}_{U,W},\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}_l \cong R\Gamma_c(\mathcal{X}_{U,\mathrm{et}},\mathbb{Z}_l)$$

(coming from assumption h) together with the finite generation of the cohomology of $R\Gamma_c(\mathcal{X}_{U,W},\mathbb{Z})$ claimed in g)), while the right vertical isomorphism is induced by the isomorphism

$$R\Gamma_c(\mathcal{X}_{U,\mathrm{et}},\mathbb{Z}_l) \cong R\Gamma_c(U_{\mathrm{et}},R\pi_*\mathbb{Z}_l)$$

and

$$\det_{\mathbb{Z}_l} R\Gamma_c(U_{\mathrm{et}},R\pi_*\mathbb{Z}_l) \cong \bigotimes_{i=0}^{2d} \det_{\mathbb{Z}_l}^{(-1)^i} R\Gamma_c(U_{\mathrm{et}},L_l^i) = \bigotimes_{i=0}^{2d} \det_{\mathbb{Z}_l}^{(-1)^i} R\Gamma_c(U_{\mathrm{et}},T_l^i)$$

where $L_l^i := R^i\pi_*\mathbb{Z}_l$ and $T_l^i \subseteq V_l^i$ is the torsion free part of L_l^i. Note that we have an exact sequence of locally constant $\mathbb{Z}_l$-sheaves on U

$$0 \to L^i_{l,tor} \to L_l^i \to T_l^i \to 0$$

and an identity $\det_{\mathbb{Z}_l} R\Gamma_c(U_{\mathrm{et}},T_l^i) = \det_{\mathbb{Z}_l} R\Gamma_c(U_{\mathrm{et}},L_l^i)$ of invertible $\mathbb{Z}_l$-submodules of $\det_{\mathbb{Q}_l} R\Gamma_c(U_{\mathrm{et}},V_l^i)$ by (76).
As discussed above statement f) for $\mathcal{X}_U$ is equivalent to statement f) for $\mathcal{X}_{U'}$ for $U' \subset U$, hence we can always assume that a given prime l is not in U. If

we know statement f) for $\mathcal{X}_U$ then the image under γ of

$$\zeta^*(\mathcal{X}_U, 0)^{-1} = \prod_{i=0}^{2d} L_U^*(h^i(X), 0)^{(-1)^{i+1}}$$

generates the natural invertible $\mathbb{Z}_l$-submodule

$$R\Gamma_c(\mathcal{X}_{U,W}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_l \cong \bigotimes_{i=0}^{2d} \det_{\mathbb{Z}_l}^{(-1)^i} R\Gamma_c(U_{\mathrm{et}}, T_l^i)$$

in (79) (see the discussion in the previous paragraph). Hence we obtain the Tamagawa number conjecture in the form (77) for $h(X)$. Conversely, knowing the Tamagawa number conjecture for $h(X)$, we obtain the l-primary part of statement f) for $\mathcal{X}_{U\setminus\{l\}}$ which is equivalent to the l-primary part of statement f) for $\mathcal{X}_U$. Varying l we obtain f) for $\mathcal{X}_U$. Here by l-primary parts, we mean that for any perfect complex of abelian groups C, such as $R\Gamma_c(\mathcal{X}_{U,W}, \mathbb{Z})$, an element $b \in \det_{\mathbb{Z}}(C) \otimes \mathbb{Q}$ is a generator of $\det_{\mathbb{Z}}(C)$ if and only if the image of b in $\det_{\mathbb{Z}}(C) \otimes \mathbb{Q}_l$ is a generator of $\det_{\mathbb{Z}}(C) \otimes \mathbb{Z}_l$ for all primes l.

□

10. On the local theorem of invariant cycles

Let R be a complete discrete valuation ring with quotient field K and finite residue field k of characteristic p. Set $S = \mathrm{Spec}(R)$, $\eta = \mathrm{Spec}(K)$, $s = \mathrm{Spec}(k)$. Let $\bar{S} = (\bar{S}, \bar{s}, \bar{\eta})$ be the normalization of S in a separable closure $\bar{K}$ of K and denote by $I \subseteq G := \mathrm{Gal}(\bar{K}/K)$ the inertia subgroup.

10.1. l-adic cohomology for $p \neq l$.

In this section l is a prime different from p. The following lemma might be well known as a consequence of de Jong's theorem on alterations [11], and also of Deligne's work [9] in case $\mathrm{char}(K) = p$. We shall only need it for $X_\eta \to \mathrm{Spec}(K)$ proper and smooth.

LEMMA 12. *Let $X_\eta \to \mathrm{Spec}(K)$ be separated and of finite type. Then the G-representation $H^i(X_{\bar{\eta}}, \mathbb{Q}_l)$ has a (unique) G-invariant weight filtration*

$$\cdots \subseteq W_j H^i(X_{\bar{\eta}}, \mathbb{Q}_l) \subseteq W_{j+1} H^i(X_{\bar{\eta}}, \mathbb{Q}_l) \subseteq \cdots$$

in the sense of [9]*[Prop.-Def. 1.7.5], i.e. if $F \in G$ is any lift of a geometric Frobenius element in* $\mathrm{Gal}(\bar{k}/k)$ *then the eigenvalues of F on* $\mathrm{gr}_j^W H^i(X_{\bar{\eta}}, \mathbb{Q}_l)$ *are Weil numbers of weight $j \in \mathbb{Z}$ with respect to $|k|$. The same is true for the G-representation $H_c^i(X_{\bar{\eta}}, \mathbb{Q}_l)$. One has $W_{-1} H^i(X_{\bar{\eta}}, \mathbb{Q}_l) = W_{-1} H_c^i(X_{\bar{\eta}}, \mathbb{Q}_l) = 0$.*

Proof. By [9][Prop.-Def. 1.7.5] it suffices to show that all eigenvalues α of F on $H^i(X_{\bar{\eta}}, \mathbb{Q}_l)$ are Weil numbers of some weight $j = j(\alpha) \in \mathbb{Z}$. In doing so, one may pass to an open subgroup $G' \subseteq G$, i.e. replace X_η by its base change to a finite extension K'/K, since an algebraic number α is a Weil number with respect to $|k|$ if and only if $\alpha^{[k':k]}$ is a Weil number with respect to $|k'|$. One can now argue exactly as in the proof of [2][Prop. 6.3.2] to which we refer for more details. If X_η is the generic fibre of a proper, strictly semistable scheme, then the vanishing cycle spectral sequence computed by Rapoport and Zink [36][Satz 2.10]

$$E_1^{-r,i+r} = \bigoplus_{q\geq 0, r+q\geq 0} H^{i-r-2q}(Y^{(r+2q)}, \mathbb{Q}_l)(-r-q) \Rightarrow H^i(X_{\bar\eta}, \mathbb{Q}_l) \tag{80}$$

together with the Weil conjectures for the smooth proper schemes $Y^{(i)}$ give the statement (and moreover the weight filtration on $H^i(X_{\bar\eta}, \mathbb{Q}_l)$ is the filtration induced by the spectral sequence). If X_η is only smooth and proper then by de Jong's theorem [2][Thm. 1.4.1] there is a generically finite, flat $X'_{\eta'} \to X_\eta$ where $X'_{\eta'}$ is strictly semistable. Hence $H^i(X_{\bar\eta}, \mathbb{Q}_l)$ is a direct summand of the G'-representation $H^i(X'_{\bar\eta}, \mathbb{Q}_l)$ for which the statement holds. Then one can use induction on the dimension together with the long exact localization sequence to prove the statement for $H^i_c(X_{\bar\eta}, \mathbb{Q}_l)$ for any separated X_η of finite type. Another application of de Jong's theorem is necessary here to assure that a regular open subscheme $U \subseteq X_\eta$ has a finite cover $U' \to U$ which is open in a proper regular K-scheme. For X_η smooth over K, Poincare duality then implies the statement for $H^i(X_{\bar\eta}, \mathbb{Q}_l)$ and for general X one uses a hypercovering argument. In this proof, starting with (80), all occurring F-eigenvalues have non-negative weight, i.e. we have $W_{-1}H^i(X_{\bar\eta}, \mathbb{Q}_l) = W_{-1}H^i_c(X_{\bar\eta}, \mathbb{Q}_l) = 0$. □

Let $f: X \to S$ be a proper, flat, generically smooth morphism of relative dimension d. For $0 \leq i \leq 2d$ one defines the specialization morphism

$$sp: H^i(X_{\bar s}, \mathbb{Q}_l) \to H^i(X_{\bar\eta}, \mathbb{Q}_l)^I \tag{81}$$

as the composite

$$H^i(X_{\bar s}, \mathbb{Q}_l) \cong H^i(X', \mathbb{Q}_l) \to H^i(X'_\eta, \mathbb{Q}_l) \to H^i(X_{\bar\eta}, \mathbb{Q}_l) \tag{82}$$

where X' is the base change of X to a strict Henselization of S at $\bar s$ and the first isomorphism is proper base change. The map sp is G-equivariant and respects the weight filtration.

THEOREM 10.1. *If X is* regular *then the following hold.*

a) *The map*

$$H^i(X_{\bar s}, \mathbb{Q}_l) = W_i H^i(X_{\bar s}, \mathbb{Q}_l) \to W_i H^i(X_{\bar\eta}, \mathbb{Q}_l)^I$$

induced by sp *is surjective for all i.*

b) *The map*

$$W_1 H^i(X_{\bar s}, \mathbb{Q}_l) \to W_1 H^i(X_{\bar\eta}, \mathbb{Q}_l)^I \tag{83}$$

induced by sp *is an isomorphism for all i, and the zero map for $i > d$.*

c) *The map* sp *is an isomorphism for $i = 0, 1$.*

d) *If $W_i H^i(X_{\bar\eta}, \mathbb{Q}_l)^I = H^i(X_{\bar\eta}, \mathbb{Q}_l)^I$ for all i then the map*

$$W_{i-1} H^i(X_{\bar s}, \mathbb{Q}_l) \to W_{i-1} H^i(X_{\bar\eta}, \mathbb{Q}_l)^I$$

induced by sp *is an isomorphism for all i.*

Remarks. a) By part a) of the theorem, the assumption of part d) is equivalent to the surjectivity of the map sp for all i, a statement which is called the local theorem on invariant cycles. It is known to hold if R is the local ring of a smooth curve over k by [9][Lemma 3.6.2], see also [9][Thm. 3.6.1], but it is only conjectured in mixed characteristic. Unconditionally, we were only able to prove the weak statement in b) rather than the full conclusion of d). Part c) is probably well known and follows, for example, from b) and results of [22][Exposé IX] on Neron models which assure that $W_1H^1(X_{\bar{\eta}},\mathbb{Q}_l)^I = H^1(X_{\bar{\eta}},\mathbb{Q}_l)^I$.
b) It is easy to construct examples where sp is not injective for $i \geq 2$. For example if X is the blowup of a proper smooth relative curve over S in a closed point, then $H^2(X_{\bar{s}},\mathbb{Q}_l)$ will have an extra summand $\mathbb{Q}_l(-1)$ corresponding to the exceptional divisor which gives a new irreducible component of $X_{\bar{s}}$.
c) If X arises by base change from a regular, proper, flat scheme $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ part b) of the theorem implies

$$\mathrm{ord}_{s=n}\,\zeta(\mathcal{X},s) = \mathrm{ord}_{s=n}\prod_{i=0}^{2d} L(h^i(\mathcal{X}_{\mathbb{Q}}),s)^{(-1)^i}$$

for integers $n \leq 0$ (and for $n = \frac{1}{2}$) where $\zeta(\mathcal{X},s)$ is the Zeta-function of the arithmetic scheme $\mathcal{X}$ and $L(h^i(\mathcal{X}_{\mathbb{Q}}),s)$ is the L-function of the motive $h^i(\mathcal{X}_{\mathbb{Q}})$ defined by Serre [38]. Indeed the former (resp. latter) is an Euler product of characteristic polynomials of Frobenius on $H^i(\mathcal{X}\otimes\bar{\mathbb{F}}_p,\mathbb{Q}_l)$ (resp. $H^i(\mathcal{X}\otimes\bar{\mathbb{Q}}_p,\mathbb{Q}_l)^{I_p}$). These are equal for almost all primes and at the finitely many (bad reduction) primes where they might differ, part b) assures that the vanishing order at $s \leq 0$ of both factors, which equals $(-1)^{i+1}$ times the multiplicity of the Frobenius-eigenvalue p^n (of weight 2n), is the same.
d) Regularity of X is a key assumption in the theorem. The map sp will be an isomorphism for $i = 0$ if X is only normal but for $i = 1$ normality is not even sufficient for surjectivity of sp on W_0, as the following example of de Jeu [8] shows. If E is an elliptic curve over $\mathbb{Q}$ given by a projective Weierstrass equation

$$Y^2Z = X^3 + AXZ^2 + BZ^3$$

with $A, B \in \mathbb{Q}$ then for any $u \in \mathbb{Q}^\times$ the curve

$$Y^2Z = X^3 + u^4AXZ^2 + u^6BZ^3$$

is isomorphic to E, and if $u^4A, u^6B \in \mathbb{Z}$ this equation defines a normal scheme $\mathcal{E}$, proper and flat over $\mathrm{Spec}(\mathbb{Z})$, inside $\mathbb{P}^2_{\mathbb{Z}}$. Indeed, the affine coordinate ring of the complement of the zero section $(X : Y : Z) = (0 : 1 : 0)$ is $R = \mathbb{Z}[x,y]/(y^2 - x^3 - u^4Ax - u^6B)$ and hence a complete intersection. So R is normal if and only if all local rings $A_{\mathfrak{p}}$ for primes $\mathfrak{p}$ of height ≤ 1 are regular. If $\mathfrak{p}$ maps to the generic point of $\mathrm{Spec}(\mathbb{Z})$ this is clear because E is a smooth curve over $\mathbb{Q}$. If $\mathfrak{p}$ maps to (p) for some prime number p, then $\mathfrak{p} = R\cdot p$ since $R\cdot p$ is already a prime ideal as the equation $y^2 - x^3 - u^4Ax - u^6B$ remains irreducible modulo p. Hence $\mathfrak{p}$ is principal and $A_{\mathfrak{p}}$ is a DVR. The generic point of the zero section maps to the generic point of $\mathrm{Spec}(\mathbb{Z})$, hence $\mathcal{E}$ is normal.

If we now pick u in addition to be a multiple of some prime p where E has split multiplicative reduction, then $\mathcal{E}_{\bar{s}}$ is a cuspidal cubic curve and therefore

$$0 = H^1(\mathcal{E}_{\bar{s}}, \mathbb{Q}_l) \to H^1(\mathcal{E}_{\bar{\eta}}, \mathbb{Q}_l)^I = W_0 H^1(\mathcal{E}_{\bar{\eta}}, \mathbb{Q}_l)^I \cong \mathbb{Q}_l$$

is not surjective.
However, the condition that X is locally factorial (all local rings are UFDs) lies between normality and regularity and is sufficient to ensure that our proof of c) given below goes through. Regularity is only used for the isomorphism $\mathrm{Pic}(X) \cong \mathrm{Cl}(X)$ and for [35][Thm. 6.4.1] via normality.

Proof. Since the statement of Theorem 10.1 only depends on the base change of f to the strict Henselization of S at $\bar{s}$ we may assume that S is strictly Henselian. Note that regularity is preserved by this base change by [31][I,3.17 c)].
For a) we follow Deligne's proof of [9][Thm. 3.6.1], replacing duality for the essentially smooth morphism $X \to \mathrm{Spec}(k)$ by duality for the morphism f combined with purity for the regular schemes X, proved by Thomason and Gabber (see [17]), and S, proved by Grothendieck in [21][I,Thm. 5.1]. The same arguments as in loc. cit. lead to the commutative diagram with exact rows and columns

$$\begin{tikzcd} & & H^{i+1}_{X_{\bar{s}}}(X, \mathbb{Q}_l) & & \\ 0 \arrow[r] & H^{i-1}(X_{\bar{\eta}}, \mathbb{Q}_l)_I(-1) \arrow[r] & H^i(X_\eta, \mathbb{Q}_l) \arrow[u] \arrow[r] & H^i(X_{\bar{\eta}}, \mathbb{Q}_l)^I \arrow[r] & 0 \\ & & H^i(X, \mathbb{Q}_l) \arrow[u] \arrow[r, "\sim"] & H^i(X_{\bar{s}}, \mathbb{Q}_l) \arrow[u, "sp"'] & \\ & & H^i_{X_{\bar{s}}}(X, \mathbb{Q}_l) \arrow[u] & & \end{tikzcd} \tag{84}$$

and after application of the exact functor W_i to a diagram

$$\begin{tikzcd} W_i H^{i+1}_{X_{\bar{s}}}(X, \mathbb{Q}_l) & & \\ W_i H^i(X_\eta, \mathbb{Q}_l) \arrow[u] \arrow[r] & W_i H^i(X_{\bar{\eta}}, \mathbb{Q}_l)^I \arrow[r] & 0 \\ W_i H^i(X, \mathbb{Q}_l) \arrow[u] \arrow[r, "\sim"] & W_i H^i(X_{\bar{s}}, \mathbb{Q}_l) \arrow[u, "sp"'] & \end{tikzcd}$$

so that it remains to show that

$$W_i H^{i+1}_{X_{\bar{s}}}(X, \mathbb{Q}_l) = 0 \tag{85}$$

for all i. The vertical long exact sequence in (84) arises by applying the (exact) global section functor $\Gamma(S,-)$ to the exact triangle

$$Rf_*R\underline{\mathrm{Hom}}_X(i_*\mathbb{Q}_l,\mathbb{Q}_l)\to Rf_*\mathbb{Q}_l\to Rf_*Rj_*\mathbb{Q}_l$$

where $i: X_{\bar{s}}\to X$ and $j: X_\eta\to X$ are the inclusions. By purity for X [17][§8] we have $\mathbb{Q}_l\cong Rf^!\mathbb{Q}_l(-d)[-2d]$ and the (sheafified) adjunction between $Rf^!$ and $Rf_!$ gives

$$\begin{aligned}Rf_*R\underline{\mathrm{Hom}}_X(i_*\mathbb{Q}_l,\mathbb{Q}_l)&\cong R\underline{\mathrm{Hom}}_S(Rf_!i_*\mathbb{Q}_l,\mathbb{Q}_l(-d))[-2d]\\&\cong R\underline{\mathrm{Hom}}_S(i_{s,*}Rf_{s,*}\mathbb{Q}_l,\mathbb{Q}_l(-d))[-2d]\\&\cong i_{s,*}R\underline{\mathrm{Hom}}_s(Rf_{s,*}\mathbb{Q}_l,Ri_s^!\mathbb{Q}_l(-d))[-2d]\\&\cong i_{s,*}R\underline{\mathrm{Hom}}_s(Rf_{s,*}\mathbb{Q}_l,\mathbb{Q}_l)(-d-1)[-2d-2]\end{aligned}$$

where $i_s: s\to S$ is the closed immersion and $f_s: X_s\to s$ the base change of f. Here we have also used $Rf_*=Rf_!$ (f proper) as well as the sheafified adjunction between $i_{s,!}=i_{s,*}$ and $i_s^!$, and purity for S. This last complex has cohomology in degree $i+1$ given by

$$\mathrm{Hom}_{\mathbb{Q}_l}(H^{2d+2-i-1}(X_{\bar{s}},\mathbb{Q}_l),\mathbb{Q}_l)(-d-1)$$

which has weights greater or equal to $2(d+1)-(2d+2-i-1)=i+1$ since $W_kH^k(X_{\bar{s}},\mathbb{Q}_l)=H^k(X_{\bar{s}},\mathbb{Q}_l)$ by [9][Cor. 3.3.8]. This finishes the proof of a).

Concerning b), we apply the exact functor W_1 to the diagram (84) and obtain a commutative diagram

$$\begin{array}{ccc}W_1H^i(X_\eta,\mathbb{Q}_l)&\xrightarrow{\beta}&W_1H^i(X_{\bar{\eta}},\mathbb{Q}_l)^I\\ \uparrow{\scriptstyle\alpha}&&\uparrow{\scriptstyle sp}\\ W_1H^i(X,\mathbb{Q}_l)&\xrightarrow{\sim}&W_1H^i(X_{\bar{s}},\mathbb{Q}_l).\end{array}$$

For $i\geq 2$ the map α is an isomorphism since

$$W_1H^j_{X_{\bar{s}}}(X,\mathbb{Q}_l)\subseteq W_{j-1}H^j_{X_{\bar{s}}}(X,\mathbb{Q}_l)=0$$

for $j=i,i+1$ by (85). For $i=0,1$ we already have

$$H^i_{X_{\bar{s}}}(X,\mathbb{Q}_l)\cong\mathrm{Hom}_{\mathbb{Q}_l}(H^{2d+2-i}(X_{\bar{s}},\mathbb{Q}_l),\mathbb{Q}_l)(-d-1)=0$$

before applying W_1 and the map α is also an isomorphism. For any i the map β is an isomorphism since

$$W_1\big(H^{i-1}(X_{\bar{\eta}},\mathbb{Q}_l)_I(-1)\big)=W_{-1}\big(H^{i-1}(X_{\bar{\eta}},\mathbb{Q}_l)_I\big)(-1)=0$$

by Lemma 12. Hence the map induced by sp on W_1 is also an isomorphism. For $i>d$ both sides of (83) vanish. Indeed, the weights of $H^i(X_{\bar{s}},\mathbb{Q}_l)$ are greater or equal to $2(i-d)\geq 2$ by [9][Cor. 3.3.4] and the same is true for $H^i(X_{\bar{\eta}},\mathbb{Q}_l)$ as follows from Poincare duality and the fact that the weights on $H^i(X_{\bar{\eta}},\mathbb{Q}_l)$ are $\leq 2i$ for $i<d$. This in turn can be read off from the spectral sequence (80)

in the strictly semistable case and follows in general from de Jong's theorem. Hence

$$W_1H^i(X_{\bar{s}},\mathbb{Q}_l)=W_1H^i(X_{\bar{\eta}},\mathbb{Q}_l)^I=0$$

for $i>d$ and we have finished the proof of b).
Concerning d), we apply the exact functor W_{i-1} to the diagram (84) and obtain a commutative diagram

$$\begin{array}{ccc} W_{i-1}H^i(X_\eta,\mathbb{Q}_l) & \xrightarrow{\beta} & W_{i-1}H^i(X_{\bar{\eta}},\mathbb{Q}_l)^I \\ \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle sp} \\ W_{i-1}H^i(X,\mathbb{Q}_l) & \xrightarrow{\sim} & W_{i-1}H^i(X_{\bar{s}},\mathbb{Q}_l) \end{array}$$

where α is an isomorphism for the same reason as in the proof of b) and β is an isomorphism since

$$W_{i-1}\big(H^{i-1}(X_{\bar{\eta}},\mathbb{Q}_l)_I(-1)\big)=W_{i-3}\big(H^{i-1}(X_{\bar{\eta}},\mathbb{Q}_l)_I\big)(-1)$$

is dual to

$$H^{2d-i+1}(X_{\bar{\eta}},\mathbb{Q}_l)^I(d+1)/W_{2d-i+2}\big(H^{2d-i+1}(X_{\bar{\eta}},\mathbb{Q}_l)^I\big)(d+1)$$

which vanishes by the assumption in d).
Concerning c), the case $i=0$ follows from b) since $W_1H^0(X_{\bar{s}},\mathbb{Q}_l)=H^0(X_{\bar{s}},\mathbb{Q}_l)$ and $W_1H^0(X_{\bar{\eta}},\mathbb{Q}_l)^I=H^0(X_{\bar{\eta}},\mathbb{Q}_l)^I$. The case $i=1$ can be deduced from b) and [22][Exposé IX] or from results of Raynaud on the Picard functor [35]. We give the details of this last argument because the method, essentially using motivic cohomology, might be of some interest. The short exact sequence $0\to\mu_{l^\nu}\to\mathbb{G}_m\xrightarrow{l^\nu}\mathbb{G}_m\to 0$ of sheaves on X_{et} induces an isomorphism

$$R^1f_*\mu_{l^\nu}\cong(R^1f_*\mathbb{G}_m)_{l^\nu} \tag{86}$$

of sheaves on S_{et} since $(f_*\mathbb{G}_m)/l^\nu=0$. Indeed, the stalks $H^0(Y,\mathcal{O}_Y^\times)=\prod_i R_i^\times$ and $H^0(Y_{\bar{\eta}},\mathcal{O}_{Y_{\bar{\eta}}}^\times)=\prod_i(L_i\otimes_K\bar{K})^\times$ of $f_*\mathbb{G}_m$ are l-divisible since S is strictly Henselian. Here

$$X\to Y=\coprod_i \mathrm{Spec}(R_i)\to S$$

is the Stein factorization and L_i is the fraction field of R_i. The Leray spectral sequence for f gives an exact sequence

$$0\to H^1(S,f_*\mathbb{G}_m)\to H^1(X,\mathbb{G}_m)\to H^0(S,R^1f_*\mathbb{G}_m)\to H^2(S,f_*\mathbb{G}_m)$$

and $H^i(S,f_*\mathbb{G}_m)=0$ for $i=1,2$. Indeed, $H^1(S,f_*\mathbb{G}_m)=\mathrm{Pic}(Y)=0$ (resp. $H^2(S,f_*\mathbb{G}_m)=\mathrm{Br}(Y)=0$) since Y is the disjoint union of spectra of local (resp. strictly Henselian local) rings. Hence

$$\mathrm{Pic}(X)\cong H^1(X,\mathbb{G}_m)\cong H^0(S,R^1f_*\mathbb{G}_m). \tag{87}$$

A similar argument for f_η shows

$$\mathrm{Pic}(X_\eta)\cong H^1(X_\eta,\mathbb{G}_m)\cong H^0(\eta,R^1f_*\mathbb{G}_m). \tag{88}$$

We have a commutative diagram

$$\begin{array}{ccccccc} H^1(X_{\bar s},\mu_{l^\nu}) & \xrightarrow{\sim} & H^0(S,R^1f_*\mu_{l^\nu}) & \xrightarrow{\sim} & H^0(S,R^1f_*\mathbb{G}_m)_{l^\nu} & \xrightarrow{\sim} & \mathrm{Pic}(X)_{l^\nu} \\ {\scriptstyle \mathrm{sp}}\downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(X_{\bar\eta},\mu_{l^\nu})^I & \xrightarrow{\sim} & H^0(\eta,R^1f_*\mu_{l^\nu}) & \xrightarrow{\sim} & H^0(\eta,R^1f_*\mathbb{G}_m)_{l^\nu} & \xrightarrow{\sim} & \mathrm{Pic}(X_\eta)_{l^\nu} \end{array}$$

where the isomorphisms in the top row are given by proper base change, (86) and (87) and in the bottom row by an elementary stalk computation, (86) and (88). Passing to the inverse limit over ν we are reduced to studying the map

$$\varprojlim_{\nu} \mathrm{Pic}(X)_{l^\nu} =: T_l \,\mathrm{Pic}(X) \to T_l \,\mathrm{Pic}(X_\eta) := \varprojlim_{\nu} \mathrm{Pic}(X_\eta)_{l^\nu} \tag{89}$$

and the proof of c) for $i=1$ is then finished by the following Lemma. □

LEMMA 13. *The map (89) is injective with finite cokernel.*

Proof. Since X is regular and X_η is an open subscheme the map

$$\mathrm{Pic}(X) = \mathrm{Cl}(X) \to \mathrm{Cl}(X_\eta) = \mathrm{Pic}(X_\eta)$$

is surjective and its kernel K is the subgroup of $\mathrm{Cl}(X)$ generated by divisors supported in the closed subscheme $X_{\bar s} \subset X$, hence is a finitely generated abelian group [23][II.6]. By the snake lemma we obtain an exact sequence

$$0 = T_l K \to T_l \,\mathrm{Pic}(X) \to T_l \,\mathrm{Pic}(X_\eta) \to \hat K \xrightarrow{\rho} \hat{\mathrm{Pic}}(X) \tag{90}$$

where $\hat A = \varprojlim_\nu A/l^\nu$ denotes the l-completion of an abelian group A.
Let $\mathrm{Pic}^0(X) \subseteq \mathrm{Pic}(X)$ be the subgroup defined in [35][3.2 d)], i.e. the kernel of the map

$$\mathrm{Pic}(X) = P(S) \to (P/P^0)(\bar s) \times (P/P^0)(\bar\eta) \tag{91}$$

where $P = \mathrm{Pic}_{X/S}$ is the relative Picard functor of f [35][1.2] and P^0 is the connected component of P restricted to schemes over $\bar s$ (resp. η). Note that over a field P is represented by a group scheme, locally of finite type, hence has a well defined connected component. By [35][Thm. 3.2.1] the target group in (91) - the product of the Neron-Severi groups of the geometric fibres - is finitely generated, hence so is $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$.
By [35][Thm. 6.4.1] - and this is the key fact in the proof- the group $K \cap \mathrm{Pic}^0(X)$ is finite. In the notation of loc. cit. we have $K = E(S)$ by Prop. 6.1.3 and $\mathrm{Pic}^0(X) = P^0(S) \subseteq P^\tau(S)$. Hence the kernel of $K \to \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ is finite and since both groups are finitely generated, so is the kernel on their l-completions. But this means that the map ρ in (90) has finite kernel which proves the Lemma. □

10.2. p-adic cohomology. In this section we assume that K has characteristic 0 and for simplicity also that $k = \mathbb{F}_p$. For $l = p$ one still has the specialisation map

$$\mathrm{sp} : H^i(X_{\bar{s}}, \mathbb{Q}_p) \to H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I \tag{92}$$

since proper base change holds for arbitrary torsion sheaves. However, it is well known that p-adic étale cohomology of varieties in characteristic p only captures the slope 0 part of the full p-adic cohomology, which is Berthelot's rigid cohomology $H^i_{rig}(X_s/k)$ (for proper X_s this follows from [3, Thm. 1.1] and [24, Prop. 3.28, Lemma 5.6]). Here the slope 0 part $V^{\mathrm{slope}\,0}$ of a finite dimensional $\mathbb{Q}_p$-vector space V with an endomorphism ϕ is the maximal subspace on which the eigenvalues of ϕ are p-adic units. One knows that the eigenvalues of ϕ on $H^i_{rig}(X_s/k)$ are Weil numbers, and a proof similar to that of Lemma 12 shows that the same is true for $D_{pst}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))$, and hence for

$$D_{cris}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p)) = D_{st}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))^{N=0} = D_{pst}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))^{I,N=0}.$$

Therefore one deduces weight filtrations on both spaces.

In analogy with the l-adic situation one might make the following conjecture.

Conjecture 6. *Let $X \to S$ be proper, flat and generically smooth. Then there is a ϕ-equivariant specialization map*

$$H^i_{rig}(X_s/k) \xrightarrow{\mathrm{sp}'} D_{cris}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))$$

and a commutative diagram of $\mathrm{Gal}(\bar{k}/k)$*-modules*

$$\begin{array}{ccc} H^i(X_{\bar{s}}, \mathbb{Q}_p) & \xrightarrow{\mathrm{sp}} & H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I \\ {\scriptstyle\lambda_s}\downarrow & & {\scriptstyle\lambda_\eta}\downarrow \\ H^i_{rig}(X_s/k) \otimes_{\mathbb{Q}_p} \hat{\mathbb{Q}}_p^{ur} & \xrightarrow{\mathrm{sp}'\otimes 1} & D_{cris}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p)) \otimes_{\mathbb{Q}_p} \hat{\mathbb{Q}}_p^{ur} \end{array}$$

where $\hat{\mathbb{Q}}_p^{ur}$ is the p-adic completion of the maximal unramified extension of $\mathbb{Q}_p$. Moreover, the vertical maps induce isomorphisms

$$\lambda_s : H^i(X_{\bar{s}}, \mathbb{Q}_p) \cong (H^i_{rig}(X_s/k) \otimes_{\mathbb{Q}_p} \hat{\mathbb{Q}}_p^{ur})^{\phi\otimes\phi=1} \cong H^i_{rig}(X_s/k)^{\mathrm{slope}\,0}$$

and

$$\begin{aligned} \lambda_\eta : H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I &\cong (D_{cris}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p)) \otimes_{\mathbb{Q}_p} \hat{\mathbb{Q}}_p^{ur})^{\phi\otimes\phi=1} \\ &\cong D_{cris}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))^{\mathrm{slope}\,0}. \end{aligned} \tag{93}$$

Note here that for any ϕ-module D the $\mathrm{Gal}(\bar{k}/k)$-module $(D \otimes_{\mathbb{Q}_p} \hat{\mathbb{Q}}_p^{ur})^{\phi\otimes\phi=1}$ can also be viewed as a ϕ-module (via the action of $\phi\otimes 1$) and as such is non-canonically isomorphic to $D^{\mathrm{slope}\,0}$. Moreover the action of $\mathrm{Frob}_p^{-1} \in \mathrm{Gal}(\bar{k}/k)$ coincides with that of $1 \otimes \phi^{-1} = \phi \otimes 1 = \phi$.

The p-adic analogue of Theorem 10.1 (replacing a) by the conjectural local theorem on invariant cycles) would be the following conjecture.

CONJECTURE 7. *Assume that X is moreover regular. Then the following hold.*

a) *The map* sp' *is surjective.*

b) *The map*

$$W_{i-1}H^i_{rig}(X_s/k) \xrightarrow{\mathrm{sp}'} W_{i-1}D_{cris}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))$$

induced by sp' *is an isomorphism.*

c) *The map* sp' *is an isomorphism for* $i = 0, 1$.

Combining both conjectures we deduce the following statement for p-adic étale cohomology.

CONJECTURE 8. *If X is regular then the map*

$$W_{i-1}H^i(X_{\bar{s}}, \mathbb{Q}_p) \xrightarrow{\mathrm{sp}} W_{i-1}H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I$$

induced by sp *is an isomorphism.*

Here we deduce the weight filtrations on $H^i(X_{\bar{s}}, \mathbb{Q}_p)$ and $H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I$ from Conjecture 6 via the injectivity of the maps λ_s and λ_η. For the applications in this paper we only need this isomorphism on W_0 (or in fact on the still smaller generalized eigenspace for the eigenvalue 1). For reference we record this statement separately.

CONJECTURE 9. *If X is regular then the map*

$$W_0H^i(X_{\bar{s}}, \mathbb{Q}_p) \xrightarrow{\mathrm{sp}} W_0H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I$$

induced by sp *is an isomorphism, where W_0 is the sum of generalized ϕ-eigenspaces for eigenvalues which are roots of unity.*

Again, if Conjecture 6 holds the maps λ_s and λ_η are injective and it suffices to establish an isomorphism

$$W_0H^i_{rig}(X_s/k) \cong W_0D_{cris}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p)).$$

We do not know how to establish Conjecture 6 or Conjecture 9 in general, since it seems difficult to make use of the regularity assumption. In case X has semistable reduction, however, it seems plausible that one can avoid any reference to rigid cohomology and establish a commutative diagram of $\mathrm{Gal}(\bar{k}/k)$-modules

$$\begin{array}{ccc} H^i(X_{\bar{s}}, \mathbb{Q}_p) & \xrightarrow{\mathrm{sp}} & H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I \\ \tilde{\lambda}_s \Big\downarrow & & \lambda_\eta \Big\downarrow \\ (H^i_{HK}(X_s/k)^{N=0}) \otimes_{\mathbb{Q}_p} \hat{\mathbb{Q}}_p^{ur} & \xrightarrow{c'\otimes 1} & D_{cris}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p)) \otimes_{\mathbb{Q}_p} \hat{\mathbb{Q}}_p^{ur} \end{array}$$

where $H^i_{HK}(X_s/k)$ is Hyodo-Kato cohomology. Contrary to what the notation suggests this cohomology theory not only depends on X_s/k but on the scheme X/S. Building on work of Fontaine-Messing, Bloch-Kato, Hyodo-Kato, and Kato-Messing, Tsuji [39] proved that there is an isomorphism of (ϕ, N)-modules

$$H^i_{HK}(X_s/k) \xrightarrow{c} D_{st}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))$$

and hence an isomorphism of ϕ-modules

$$H^i_{HK}(X_s/k)^{N=0} \xrightarrow{c'} D_{st}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))^{N=0} = D_{cris}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p)).$$

In addition to the commutative diagram it would then be enough to show that $\tilde{\lambda}_s$ and λ_η are injective. We refrain from giving more details since in this paper Conjecture 9 is only used in the proof of Proposition 9.2 (via Proposition 9.1) which already needs to assume a host of other, much deeper conjectures that we are unable to prove.

Remark 1. (Added in Proof) Conjectures 6, 7 c), 8 and 9 have meanwhile been proven in the Caltech thesis of Yitao Wu [40]. In fact Wu shows that

$$H^i(X_{\bar{s}}, \mathbb{Q}_p) \xrightarrow{sp} H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I$$

is an isomorphism for regular X. Conjectures 7 a) and b) however, still seem out of reach.

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