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The classifying topos of a group scheme and invariants of symmetric bundles

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Abstract

Let Y be a scheme in which 2 is invertible and let V be a rank n vector bundle on Y endowed with a non-degenerate symmetric bilinear form q. The orthogonal group $\mathbf{O}(q)$ of the form q is a group scheme over Y whose cohomology ring $H^*(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z}) \simeq A_Y[HW_1(q), \ldots, HW_n(q)]$ is a polynomial algebra over the étale cohomology ring $A_Y := H^*(Y_{\text{et}}, \mathbf{Z}/2\mathbf{Z})$ of the scheme Y. Here, the $HW_i(q)$'s are Jardine's universal Hasse–Witt invariants and $B_{\mathbf{O}(q)}$ is the classifying topos of $\mathbf{O}(q)$ as defined by Grothendieck and Giraud. The cohomology ring $H^*(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z})$ contains canonical classes $\det[q]$ and $[C_q]$ of degree 1 and 2, respectively, which are obtained from the determinant map and the Clifford group of q. The classical Hasse–Witt invariants $w_i(q)$ live in the ring A_Y.

Our main theorem provides a computation of $\det[q]$ and $[C_q]$ as polynomials in $HW_1(q)$ and $HW_2(q)$ with coefficients in A_Y written in terms of $w_1(q), w_2(q) \in A_Y$. This result is the source of numerous standard comparison formulas for classical Hasse–Witt invariants of quadratic forms. Our proof is based on computations with (abelian and non-abelian) Cech cocycles in the topos $B_{\mathbf{O}(q)}$. This requires a general study of the cohomology of the classifying topos of a group scheme, which we carry out in the first part of this paper.

1. *Introduction*

In [**6**, **18**], Fröhlich and Serre proved some beautiful formulas that compared invariants associated to various kinds of Galois representations and quadratic forms defined over a field K of characteristic different from 2. Their work has inspired numerous generalizations (see, for example, [**2**, **4**, **17**]). The basic underlying idea may be summarized as follows. Let (V, q) be a symmetric bundle, defined over a scheme Y in which 2 is invertible and let $\mathbf{O}(q)$ be the orthogonal group of (V, q) considered as a group scheme over Y. We may associate to any orthogonal representation $\rho : G \to \mathbf{O}(q)$ of a finite discrete group G and any G-torsor X on Y a cocycle in the cohomology set $H^1(Y_{\text{et}}, \mathbf{O}(q))$. Since this set classifies the isometry classes of symmetric bundles with the same rank of q, we may attach to (ρ, X) a new symmetric bundle (V_X, q_X), known as the Fröhlich twist of (V, q). The results consist of various comparison formulas, in the étale cohomology ring $H^*(Y_{\text{et}}, \mathbf{Z}/2\mathbf{Z})$, which relate the Hasse–Witt invariants of (V, q) to those of its twisted form (V_X, q_X). One of the principal aims of this paper is to show that all these comparison formulas, together with a number of new results, can be immediately deduced by pulling back from a single equation which sits in the cohomology ring $H^*(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z})$ of the classifying topos $B_{\mathbf{O}(q)}$ and which is independent of any choice of particular orthogonal representation and particular torsor.

In [**7**, **10**], Grothendieck and Giraud introduced the notion of the classifying topos of a group object in given topos, and they suggested that it could be used in the theory of characteristic classes in algebraic geometry. Building on their insight, we will prove our main theorem using

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both abelian and non-abelian Cech cohomology of the classifying topos $B_{\mathbf{O}(q)}$ of the group scheme $\mathbf{O}(q)$. To this end, the first part of this paper, namely Sections 2 and 3, is devoted to the study of some basic properties of the classifying topos B_G of a Y-group scheme G which is defined as follows. Let Y_{fl} denote the category of sheaves of sets on the big fppf-site of Y and let yG denote the sheaf of groups of Y_{fl} represented by G. Then B_G is simply the category of objects $\mathcal{F}$ of Y_{fl} endowed with a left action of yG. We may view a Y-scheme as an object of Y_{fl} and write G for yG.

In Section 2, we recall the fact that there is a canonical equivalence

$$\mathbf{Homtop}_{Y_{fl}}(\mathcal{E}, B_G)^{\mathrm{op}} \xrightarrow{\sim} \mathbf{Tors}(\mathcal{E}, f^*G), \tag{1}$$

where $f : \mathcal{E} \to Y_{fl}$ is any topos over Y_{fl}, $\mathbf{Homtop}_{Y_{fl}}(\mathcal{E}, B_G)$ is the category of morphisms of Y_{fl}-topoi from $\mathcal{E}$ to B_G and $\mathbf{Tors}(\mathcal{E}, f^*G)$ is the groupoid of f^*G-torsors in $\mathcal{E}$.

Section 3 is devoted to the study of the cohomology of B_G. First, we show that there is a canonical isomorphism

$$H^*(B_G, \mathcal{A}) \simeq H^*(\mathbf{B}G_{\mathrm{et}}, \mathcal{A}), \tag{2}$$

where the right-hand side denotes the étale cohomology of the simplicial scheme $\mathbf{B}G$ (as defined in [**5**]) and $\mathcal{A}$ is an abelian object of B_G that is representable by a smooth Y-scheme supporting a G-action. It follows that, for a constant group G, the cohomology of B_G (or more generally the cohomology of B_G/X for any Y-scheme X with a G-action) computes Grothendieck's mixed cohomology (see [**8**, 2.1]). There are several interesting spectral sequences and exact sequences that relate the cohomology of B_G to other kinds of cohomology. For example, for any commutative group scheme A endowed with a left action of G, there is an exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0(B_G, \mathcal{A}) \longrightarrow H^0(Y_{fl}, \mathcal{A}) \longrightarrow \mathrm{Crois}_Y(G, A) \longrightarrow H^1(B_G, \mathcal{A}) \\ &\longrightarrow H^1(Y_{fl}, \mathcal{A}) \longrightarrow \mathrm{Ext}_Y(G, A) \longrightarrow H^2(B_G, \mathcal{A}) \longrightarrow H^2(Y_{fl}, \mathcal{A}), \end{aligned} \tag{3}$$

where $\mathcal{A} = y(A)$, $\mathrm{Crois}_Y(G, A)$ is the group of crossed homomorphisms from G to A (which is just $\mathrm{Hom}_Y(G, A)$ if G acts trivially on A) and $\mathrm{Ext}_Y(G, A)$ is the group of extensions $1 \to A \to \tilde{G} \to G \to 1$ inducing the given G-action on A.

We shall also establish the existence of a Hochschild–Serre spectral sequence in this context. Let $1 \to N \to G \to G/N \to 1$ be an exact sequence of S-group schemes (with respect to the fppf-topology). Then, for any abelian object $\mathcal{A}$ of B_G, there is a natural G/N-action on the cohomology $H^j_S(B_N, \mathcal{A})$ of $\mathcal{A}$ with values in S_{fl} (see Notation 3.9) and we have the spectral sequence

$$H^i(B_{G/N}, H^j_S(B_N, \mathcal{A})) \Longrightarrow H^{i+j}(B_G, \mathcal{A}).$$

The five-term exact sequence induced by this spectral sequence reads as follows:

$$\begin{aligned} 0 &\longrightarrow H^1(B_{G/N}, \mathcal{A}) \longrightarrow H^1(B_G, \mathcal{A}) \longrightarrow H^0(B_{G/N}, \underline{\mathrm{Hom}}(N, \mathcal{A})) \\ &\longrightarrow H^2(B_{G/N}, \mathcal{A}) \longrightarrow H^2(B_G, \mathcal{A}), \end{aligned} \tag{4}$$

where we assume for simplicity that $\mathcal{A}$ is given with trivial G-action.

This then concludes our description of the first part of the article, which is of a relatively general nature.

The aim of the second part of this paper, which starts from Section 4, is to apply the general results of the first part to the study of symmetric bundles and their invariants. From Section 4 on, we fix a scheme Y in which 2 is invertible and a symmetric bundle (V, q) on Y, that is, a locally free $\mathcal{O}_Y$-module V of rank n endowed with a non-degenerate bilinear form $V \otimes_{\mathcal{O}_Y} V \to \mathcal{O}_Y$. A special case is given by $(\mathcal{O}_Y^n, t_n = x_1^2 + \cdots + x_n^2)$, the standard form of rank n, and $\mathbf{O}(n)$ is defined as the orthogonal group for this form. The isomorphism (2), together with a fundamental result of Jardine (see [**14**, Theorem 2.8]), yields a canonical identification of A-algebras

$$H^*(B_{\mathbf{O}(n)}, \mathbf{Z}/2\mathbf{Z}) \simeq H^*(\mathbf{BO}(n)_{\text{et}}, \mathbf{Z}/2\mathbf{Z}) \simeq A[HW_1, \ldots, HW_n],$$

where HW_i has degree i and $A := H^*(Y_{\text{et}}, \mathbf{Z}/2\mathbf{Z})$ is the étale cohomology ring of Y. The symmetric bundle (V, q) provides us with the object $\mathbf{Isom}(t_n, q)$ of Y_{fl}, which naturally supports a right action of $\mathbf{O}(n)$ and a left action of $\mathbf{O}(q)$. It is easily seen that $\mathbf{Isom}(t_n, q)$ is in fact an $\mathbf{O}(n)$-torsor of $B_{\mathbf{O}(q)}$. It follows from (1) that this torsor may be viewed as a Y_{fl}-morphism

$$T_q : B_{\mathbf{O}(q)} \longrightarrow B_{\mathbf{O}(n)}.$$

This morphism is actually an equivalence of Y_{fl}-topoi. Indeed, T_q has a quasi-inverse

$$T_q^{-1} : B_{\mathbf{O}(n)} \xrightarrow{\sim} B_{\mathbf{O}(q)}$$

given by the $\mathbf{O}(q)$-torsor $\mathbf{Isom}(q, t_n)$ of $B_{\mathbf{O}(n)}$. Note that, however, the groups $\mathbf{O}(n)$ and $\mathbf{O}(q)$ are not isomorphic in general. For a proof of the analogous fact in the simplicial framework, we refer the reader to [**13**, Theorem 3.1] where it is shown that the simplicial sheaves associated to $\mathbf{O}(n)$ and $\mathbf{O}(q)$ are weakly equivalent).

This yields a canonical isomorphism of A-algebras

$$H^*(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z}) \simeq A[HW_1(q), \ldots, HW_n(q)],$$

where $HW_i(q) := T_q^*(HW_i)$ has degree i. The classes $HW_i(q), 1 \leqslant i \leqslant n$, will be called the universal Hasse–Witt invariants of q. We may now view the object $\mathbf{Isom}(t_n, q)$ as an $\mathbf{O}(n)$-torsor of Y_{fl}; it therefore yields a map

$$\{q\} : Y_{fl} \longrightarrow B_{\mathbf{O}(n)},$$

which, incidentally, determines q. The classical Hasse–Witt invariants of q are defined by

$$w_i(q) := \{q\}^*(HW_i) \in H^i(Y, \mathbf{Z}/2\mathbf{Z}).$$

We can attach to (V, q) both a canonical map $\det_{\mathbf{O}(q)} : \mathbf{O}(q) \to \mathbf{Z}/2\mathbf{Z}$ and also the central group extension

$$1 \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow \tilde{\mathbf{O}}(q) \longrightarrow \mathbf{O}(q) \longrightarrow 1 \tag{5}$$

derived from the Clifford algebra and the Clifford group of q. It turns out, by considering the sequence (3), that the map $\det_{\mathbf{O}(q)}$ yields a cohomology class $\det[q] \in H^1(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z})$, while the extension (5) gives us a cohomology class $[C_q] \in H^2(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z})$. The main result of the second part of the paper provides an explicit expression of $\det[q]$ and $[C_q]$ as polynomials in $HW_1(q)$ and $HW_2(q)$ with coefficients in A expressed in terms of $w_1(q), w_2(q) \in A$. To be more precise, we will prove the following theorem.

THEOREM 1.1. *Let Y be a scheme in which 2 is invertible and let (V, q) be a symmetric bundle on Y. Assume that Y is the disjoint union of its connected components. Then we have the equalities*

$$det[q] = w_1(q) + HW_1(q)$$

and

$$[C_q] = (w_1(q) \cdot w_1(q) + w_2(q)) + w_1(q) \cdot HW_1(q) + HW_2(q)$$

in the polynomial ring

$$H^*(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z}) \simeq A[HW_1(q), \ldots, HW_n(q)].$$

Section 5 is devoted to the proof of this result. The identity in degree 1 is proved using simple computations with torsors. The proof of the identity in degree 2 is more involved and is based on computations with Cech cocycles. A technical reduction makes use of the exactness of the sequence (4) derived from the Hochschild–Serre spectral sequence for group extension (5).

Theorem 1.1 is the source of numerous comparison formulas, which are either new results or generalizations of known results (see [**2**, **4**, **15**]), by using the following method: for any topos $\mathcal{E}$ given with an $\mathbf{O}(q)$-torsor, we have the canonical map $f : \mathcal{E} \to B_{\mathbf{O}(q)}$, and we derive comparison formulas in $H^*(\mathcal{E}, \mathbf{Z}/2\mathbf{Z})$ by applying functor f^* to the universal comparison formulas of Theorem 1.1. For example, given an $\mathbf{O}(q)$-torsor α on Y, we consider the map $f : \mathcal{E} = Y_{fl} \to B_{\mathbf{O}(q)}$, which classifies α, and we thereby obtain an identity in $H^i(Y_{fl}, \mathbf{Z}/2\mathbf{Z})$ for $i = 1, 2$. Our result, Corollary 6.1, generalizes a result of Serre to any base scheme Y. A second example is provided by an orthogonal representation $\rho : G \to \mathbf{O}(q)$ of a Y-group scheme G: here we consider the map $B_\rho : B_G \to B_{\mathbf{O}(q)}$ and thereby get identities in $H^*(B_G, \mathbf{Z}/2\mathbf{Z})$. It should be noted that Corollary 6.3 is new, even in the case when G is a constant (= discrete) group. A third example is provided by an orthogonal representation $\rho : G \to \mathbf{O}(q)$ and a G-torsor X on Y: in this case we may consider the map

$$Y_{fl} \xrightarrow{X} B_G \xrightarrow{B_\rho} B_{\mathbf{O}(q)}$$

in order to derive identities in $H^*(Y_{fl}, \mathbf{Z}/2\mathbf{Z})$ (see Corollary 6.5); the result that we obtain essentially generalizes the theorem of Fröhlich–Kahn–Snaith (see [**13**, Theorem 2.4]). Our result is general in the sense that Y is an arbitrary scheme (except that 2 must be invertible) and G is not assumed to be constant. However, we should remark that we do not obtain a complete analogue of [**13**, Theorem 1.6(ii)], when G is a non-constant group scheme (see the remark of Section 6). Twists of symmetric bundles by G-torsors (for a non-constant group scheme G) appear naturally in situations of arithmetic interest (for instance, the trace form of any finite and separable algebra is a twist of the standard form). The formulas in Corollary 6.5 provide us with tools to deal with the embedding problems associated to torsors; it is our intention to return to these questions in a forthcoming paper.

2. *The classifying topos of a group scheme*

2.1. *The definition of* B_G

Let S be a scheme. We consider the category of S-schemes $\mathbf{Sch}/S$ endowed with the étale topology or fppf-topology. Recall that a fundamental system of covering families for the fppf-topology is given by the surjective families $(f_i : X_i \to X)$ consisting of flat morphisms that are locally finitely presented. The corresponding sites are denoted by $(\mathbf{Sch}/S)_{\mathrm{et}}$ and $(\mathbf{Sch}/S)_{fppf}$. The big flat topos and the big étale topos of S are defined as the categories of sheaves of sets on these sites:

$$S_{fl} := (\widetilde{\mathbf{Sch}/S})_{fppf} \quad \text{and} \quad S_{\mathrm{Et}} := (\widetilde{\mathbf{Sch}/S})_{\mathrm{et}}.$$

Here, $\tilde{C}$ denotes the category of sheaves on a site C. The identity $(\mathbf{Sch}/S)_{\mathrm{et}} \to (\mathbf{Sch}/S)_{fppf}$ is a continuous functor. It yields a canonical morphism of topoi

$$i : S_{fl} \longrightarrow S_{\mathrm{Et}}.$$

This map is an embedding, that is, i_* is fully faithful; hence S_{fl} can be identified with the full subcategory of S_{Et} consisting of big étale sheaves on S which are sheaves for the fppf-topology. The fppf-topology on the category $\mathbf{Sch}/S$ is subcanonical (hence so is the étale topology). In other words, any representable presheaf is a sheaf. It follows that the Yoneda functor yields a fully faithful functor

$$y : \mathbf{Sch}/S \longrightarrow S_{fl}. \tag{6}$$

For any S-scheme Y, we consider the slice topos S_{fl}/yY (that is, the category of maps $\mathcal{F} \to yY$ in S_{fl}). We have a canonical equivalence [**10**, IV, Section 5.10]

$$S_{fl}/yY := (\widetilde{\mathbf{Sch}/S})_{fl}/yY \simeq (\widetilde{\mathbf{Sch}/Y})_{fl} =: Y_{fl}.$$

The Yoneda functor (6) commutes with projective limits. In particular, it preserves products and the final object; hence a group scheme G over S represents a group object yG in S_{fl}, that is, a sheaf of groups on the site $(\mathbf{Sch}/S)_{fl}$.

DEFINITION 2.1. The *classifying topos* B_G of the S-group scheme G is defined as the category of objects in S_{fl} given with a left action of yG. The *étale classifying topos* B_G^{et} of the S-group scheme G is defined as the category of objects in S_{Et} given with a left action of yG.

More explicitly, an object of B_G (respectively, of B_G^{et}) is a sheaf $\mathcal{F}$ on $\mathbf{Sch}/S$ for the fppf-topology (respectively, for the étale topology) such that, for any S-scheme Y, the set $\mathcal{F}(Y)$ is endowed with a $G(Y)$-action

$$\mathrm{Hom}_S(Y, G) \times \mathcal{F}(Y) \longrightarrow \mathcal{F}(Y),$$

which is functorial in Y. We have a commutative diagram (in fact a pull-back) of topoi

$$\begin{array}{ccc} B_G & \longrightarrow & B_G^{\mathrm{et}} \\ \downarrow & & \downarrow \\ S_{fl} & \xrightarrow{\ i\ } & S_{\mathrm{Et}} \end{array}$$

where the vertical morphisms are defined as in (7).

2.2. *Classifying torsors*

More generally, let $\mathcal{S}$ be any topos and let G be any group in $\mathcal{S}$. We denote by $\mathbf{Tors}(\mathcal{S}, G)$ the category of G-torsors in $\mathcal{S}$. Recall that a (right) G-torsor in $\mathcal{S}$ is an object T endowed with a right action $\mu : T \times G \to T$ of G such that:

(i) the map $T \to e_{\mathcal{S}}$ is an epimorphism, where $e_{\mathcal{S}}$ is the final object of $\mathcal{S}$;
(ii) the map $(p_1, \mu) : T \times G \to T \times T$ is an isomorphism, where p_1 is the projection on the first component.

An object T in $\mathcal{S}$, endowed with a right G-action, is a G-torsor if and only if there exists an epimorphic family $\{U_i \to e_{\mathcal{S}}\}$ such that the base change $U_i \times T$ is isomorphic to the trivial $(U_i \times G)$-torsor in $\mathcal{S}/U_i$, that is, if there is a $(U_i \times G)$-equivariant isomorphism

$$U_i \times T \simeq U_i \times G$$

defined over U_i, where $U_i \times G$ acts on itself by right multiplication.

The classifying topos

$$B_G := B_G(\mathcal{S})$$

is the category of left G-objects in $\mathcal{S}$. The fact that B_G is a topos follows easily from Giraud's axioms; the fact that B_G classifies G-torsors is recalled below. We denote by

$$\pi : B_G \longrightarrow \mathcal{S} \tag{7}$$

the canonical map: the inverse image functor π^* sends an object $\mathcal{F}$ in $\mathcal{S}$ to $\mathcal{F}$ with trivial G-action. Indeed, π^* commutes with arbitrary inductive and projective limits; hence π^* is the inverse image of a morphism of topoi π. In particular, the group π^*G is given by the trivial action of G on itself. Let E_G denote the object of B_G defined by the action of G on itself by left multiplication. Then the map

$$E_G \times \pi^* G \longrightarrow E_G,$$

given by right multiplication is a morphism of B_G (that is, it is G-equivariant). This action provides E_G with the structure of a right π^*G-torsor in B_G. We shall also use the following notation.

NOTATION 2.2. Let G be a group in a topos $\mathcal{S}$ and let X (respectively, Y) be an object in $\mathcal{S}$ endowed with a right action of G (respectively, with a left action of G). Then the contracted product

$$X \wedge^G Y := (X \times Y)/G$$

is the quotient of the diagonal G-action on $X \times Y$.

If $f : \mathcal{E} \to \mathcal{S}$ and $f' : \mathcal{E}' \to \mathcal{S}$ are topoi over the base topos $\mathcal{S}$, then we denote by $\mathbf{Homtop}_{\mathcal{S}}(\mathcal{E}, \mathcal{E}')$ the category of $\mathcal{S}$-morphisms from $\mathcal{E}$ to $\mathcal{E}'$. An object of this category is a pair (a, α) where $a : \mathcal{E} \to \mathcal{E}'$ is a morphism and $\alpha : f' \circ a \simeq f$ is an isomorphism, that is, an isomorphism of functors $\alpha : f'_* \circ a_* \simeq f_*$, or equivalently, an isomorphism of functors $\alpha : f^* \simeq a^* \circ f'^*$. A map $\tau : (a, \alpha) \to (b, \beta)$ in the category $\mathbf{Homtop}_{\mathcal{S}}(\mathcal{E}, \mathcal{E}')$ is a morphism (of morphism of topoi) $\tau : a \to b$ compatible (in the obvious sense) with α and β. The following result is well known; see [**7**, VIII.4.3].

THEOREM 2.3. *Let $f : \mathcal{E} \to \mathcal{S}$ be a morphism of topoi and let B_G be the classifying topos of a group G in $\mathcal{S}$. The functor*

$$\begin{array}{rccc} \Psi: & \mathbf{Homtop}_{\mathcal{S}}(\mathcal{E}, B_G)^{\mathrm{op}} & \longrightarrow & \mathbf{Tors}(\mathcal{E}, f^*G), \\ & (a, \alpha) & \longmapsto & a^* E_G \end{array}$$

is an equivalence of categories. A quasi-inverse for Ψ is given by

$$\begin{array}{rccc} \Psi^{-1}: & \mathbf{Tors}(\mathcal{E}, f^*G) & \longrightarrow & \mathbf{Homtop}_{\mathcal{S}}(\mathcal{E}, B_G)^{\mathrm{op}}, \\ & T & \longmapsto & (a_T, \alpha_T), \end{array}$$

where

$$\begin{array}{rccc} a_T^*: & B_G & \longrightarrow & \mathcal{E}, \\ & X & \longmapsto & f^*X \wedge^{f^*G} T \end{array}$$

and $\alpha_T : f^* \simeq a_T^* \circ \pi^*$ is the obvious isomorphism.

An immediate corollary is the following result; see [**7**, VIII. Corollaire 4.3].

COROLLARY 2.4. *Let $f : \mathcal{E} \to \mathcal{S}$ be a morphism of topoi and let G be a group in $\mathcal{S}$. Then the following square*:

$$\begin{array}{ccc} B_{f^*G}(\mathcal{E}) & \longrightarrow & B_G(\mathcal{S}) \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{E} & \xrightarrow{f} & \mathcal{S} \end{array}$$

is a pull-back.

2.3. *Torsors under group scheme actions*

Corollary 2.5. *Let G be a group scheme over S and let Y be an S-scheme. There are canonical equivalences*

$$\begin{aligned}\mathbf{Tors}(Y_{fl}, G_Y)^{\mathrm{op}} &\simeq \mathbf{Homtop}_{Y_{fl}}(Y_{fl}, B_{G_Y})\\ &\simeq \mathbf{Homtop}_{S_{fl}}(Y_{fl}, B_G)\\ &\simeq \mathbf{Homtop}_{S_{\mathrm{Et}}}(Y_{fl}, B_G^{\mathrm{et}}).\end{aligned}$$

Proof. The first equivalence follows directly from the previous theorem, and so does the second equivalence, since the inverse image of $y(G)$ along the morphism $Y_{fl} \to S_{fl}$ is the sheaf on Y represented by $G_Y = G \times_S Y$. The third equivalence follows from the canonical equivalence

$$B_G := B_G(S_{fl}) \simeq S_{fl} \times_{S_{\mathrm{Et}}} B_G(S_{\mathrm{Et}})$$

given by Corollary 2.4. □

The key case of interest is provided by an S-group scheme G which is flat and locally of finite presentation over S. For an S-scheme Y, denote by $\mathbf{Tors}(Y, G_Y)$ the category of G_Y-torsors of the scheme Y; that is, the category of maps $T \to Y$ which are faithfully flat and locally of finite presentation, supporting a right action $T \times_Y G_Y \to T$ such that the morphism $T \times_Y G_Y \to T \times T$ is an isomorphism of T-schemes. The Yoneda embedding yields a fully faithful functor

$$y : \mathbf{Tors}(Y, G_Y) \longrightarrow \mathbf{Tors}(Y_{fl}, G_Y).$$

This functor is not an equivalence (that is, it is not essentially surjective) in general. However, it is an equivalence in certain special cases; see [**16**, III, Theorem 4.3]. In particular, this is the case when G is affine over S.

Corollary 2.6. *Let Y be an S-scheme. Let G be a flat group scheme over S that is locally of finite type. Assume that G is affine over S. Then we have an equivalence of categories*

$$\mathbf{Tors}(Y, G_Y)^{\mathrm{op}} \simeq \mathbf{Homtop}_{S_{fl}}(Y_{fl}, B_G).$$

Notation 2.7. Let G be a flat affine group scheme over S that is of finite type and let Y be an S-scheme. We have canonical equivalences

$$\begin{aligned}\mathbf{Tors}(Y, G_Y)^{\mathrm{op}} &\simeq \mathbf{Tors}(Y_{fl}, G_Y)^{\mathrm{op}}\\ &\simeq \mathbf{Homtop}_{S_{fl}}(Y_{fl}, B_G(S_{fl}))\\ &\simeq \mathbf{Homtop}_{Y_{fl}}(Y_{fl}, B_{G_Y}(Y_{fl})).\end{aligned}$$

If a Y-scheme T is a G_Y-torsor over Y, then we again denote by T the object of $\mathbf{Tors}(Y_{fl}, G_Y)^{\mathrm{op}}$, and also denote by T,

$$T : Y_{fl} \longrightarrow B_G(S_{fl})$$

the corresponding object of $\mathbf{Homtop}_{S_{fl}}(Y_{fl}, B_G(S_{fl}))$; and similarly we denote by T,

$$T : Y_{fl} \longrightarrow B_{G_Y}(Y_{fl}),$$

the corresponding object of $\mathbf{Homtop}_{Y_{fl}}(Y_{fl}, B_{G_Y}(Y_{fl}))$.

2.4. *The big topos B_G/X of G-equivariant sheaves*

Let X be an S-scheme endowed with a left action (over S) of G. Then yX is a sheaf on $(\mathbf{Sch}/S)_{fppf}$ with a left action of yG (since y commutes with finite projective limits). The resulting object of B_G will be denoted by $y(G, X)$, or just by X if there is no risk of ambiguity. The slice category B_G/X is a topos, which we refer to as the topos of G-equivariant sheaves on X. This terminology is justified by the following observation: an object of B_G/X is given by an object $\mathcal{F} \to X$ of $S_{fl}/X \simeq X_{fl}$ (that is, a sheaf on the fppf-site of X), endowed with an action of yG such that the structure map $\mathcal{F} \to X$ is G-equivariant. We have a (localization) morphism

$$f : B_G/X \longrightarrow B_G$$

whose inverse image maps an object $\mathcal{F}$ of B_G to the (G-equivariant) projection $\mathcal{F} \times X \to X$, where yG acts diagonally on $\mathcal{F} \times X$.

Let Y be an S-scheme with trivial G-action, and consider the topos B_G/Y. We denote by $G_Y := G \times_S Y$ the base change of the S-group scheme G to Y and we consider the classifying topos B_{G_Y} of the Y-group scheme G_Y. Recall that B_{G_Y} is the category of $y(G_Y)$-equivariant sheaves on $(\mathbf{Sch}/Y)_{fl}$. The following result shows that the classifying topos B_G behaves well with respect to base change.

PROPOSITION 2.8. *If G acts trivially on an S-scheme Y, then there is a canonical equivalence*

$$B_{G_Y} \simeq B_G/Y.$$

Proof. Let $\pi : B_G \to S_{fl}$ denote the canonical map. On the one hand by [**10**, IV, Section 5.10], the square

$$\begin{array}{ccc} B_G/\pi^*(yY) & \longrightarrow & S_{fl}/yY \\ \downarrow & & \downarrow \\ B_G & \xrightarrow{\pi} & S_{fl} \end{array}$$

is a pull-back. Note that $\pi^*(yY)$ is given by the trivial action of G on Y so that $B_G/\pi^*(yY) = B_G/Y$. On the other hand, the square

$$\begin{array}{ccc} B_{g^*(yG)}(S_{fl}/yY) & \longrightarrow & B_G \\ \downarrow & & \downarrow \\ S_{fl}/yY & \xrightarrow{g} & S_{fl} \end{array}$$

is also a pull-back by Corollary 2.4. Hence, we have canonical equivalences

$$B_G/Y \simeq B_G \times_{S_{fl}} S_{fl}/yY \simeq B_{S_{fl}/yY}(g^*(yG)).$$

Here, the first equivalence (respectively, the second) is induced by the first (respectively the second) pull-back square above. Finally, we have $g^*(yG) = y(G \times_S Y) = y(G_Y)$ in the topos $S_{fl}/yY \simeq Y_{fl}$; hence, we obtain

$$B_{S_{fl}/yY}(g^*(yG)) \simeq B_{Y_{fl}}(yG_Y) = B_{G_Y}. \qquad \square$$

3. *Cohomology of group schemes*

The cohomology of a Lie group can be defined as the cohomology of its classifying space. Analogously, Grothendieck and Giraud defined the cohomology of a group object G in a topos as the cohomology of its classifying topos B_G.

DEFINITION 3.1. Let G be an S-group scheme and let $\mathcal{A}$ be an abelian object of $B_G = B_{yG}(S_{fl})$. The cohomology of the S-group scheme G with coefficients in $\mathcal{A}$ is defined as

$$H^i(G, \mathcal{A}) := H^i(B_G, \mathcal{A}).$$

Note that any commutative group scheme $\mathcal{A}$ over S, endowed with an action of G, gives rise to an abelian object in B_G. Note also that, in the case where the S-group scheme G is trivial (that is, $G = S$), the cohomology of G is reduced to the flat cohomology of S.

In this section, we show that the cohomology of the classifying topos B_G of a group scheme with coefficients in a smooth commutative group scheme coincides with the étale cohomology of the simplicial classifying scheme $\mathbf{B}G$. This fact holds in the more general situation given by the action of G on a scheme X over S.

DEFINITION 3.2. Let X be an S-scheme endowed with a left G-action. We define the equivariant cohomology of the pair (G, X) with coefficients in an abelian object $\mathcal{A}$ of B_G/X by

$$H^i(G, X, \mathcal{A}) := H^i(B_G/X, \mathcal{A}).$$

Note that if $X = S$ is trivial, then the equivariant cohomology of the pair (G, X) is just the cohomology of the S-group scheme G as defined before. If the group scheme G is trivial, then the equivariant cohomology of the pair (G, X) is the flat cohomology of the scheme X.

3.1. *Etale cohomology of simplicial schemes*

After recalling the notions of simplicial schemes and simplicial topoi, we observe that the big and the small étale sites of a simplicial scheme have the same cohomology. References for this section are [**5**, I,II, **11**, VI, Section 5].

The category Δ of standard simplices is the category whose objects are the finite ordered sets $[0, n] = \{0 < 1 < \cdots < n\}$ and whose morphisms are non-decreasing functions. Any morphism $[0, n] \to [0, m]$, other than identity, can be written as a composite of degeneracy maps s^i and face maps d^i. Here, recall that $s^i : [0, n+1] \to [0, n]$ is the unique surjective map with two elements mapping to i and that $d^i : [0, n-1] \to [0, n]$ is the unique injective map avoiding i. A simplicial scheme is a functor $X_\bullet : \Delta^{\mathrm{op}} \to \mathbf{Sch}$. As usual, we write $X_n := X_\bullet([0, n])$, $d_i = X_\bullet(d^i)$ for the face map and $s_i = X_\bullet(s^i)$ for the degeneracy map. From the functor $X_\bullet$, we deduce a simplicial topos

$$\begin{array}{rccc} X_{\bullet,\mathrm{et}} : & \Delta^{\mathrm{op}} & \longrightarrow & \mathbf{Top}, \\ & [0, n] & \longmapsto & X_{n,\mathrm{et}}, \end{array}$$

where $X_{\bullet,\mathrm{et}}([0, n]) = X_{n,\mathrm{et}}$ is the small étale topos of the scheme X_n, that is, the category of sheaves on the category of étale X_n-schemes endowed with the étale topology. Strictly speaking, $X_{\bullet,\mathrm{et}}$ is a pseudo-functor from Δ^{op} to the 2-category of topoi.

Finally, we consider the total topos $\mathrm{TOP}(X_{\bullet,\mathrm{et}})$ associated to this simplicial topos (see [**11**, VI. 5.2]). Recall that an object of $\mathrm{TOP}(X_{\bullet,\mathrm{et}})$ consists of the data of objects F_n of $X_{n,\mathrm{et}}$ together with maps $\alpha^* F_m \to F_n$ in $X_{n,\mathrm{et}}$ for each $\alpha : [0, m] \to [0, n]$ in Δ satisfying the natural transitivity condition for a composite map in Δ. The arrows in $\mathrm{TOP}(X_{\bullet,\mathrm{et}})$ are defined in the obvious way. We observe that this category is equivalent to the category of sheaves on the etale site $\mathrm{Et}(X_\bullet)$ as defined in [**5**, I, Definition 1.4].

In a similar way, we define the big étale simplicial topos associated to $X_\bullet$ as follows:

$$X_{\bullet,\,\mathrm{Et}}: \quad \begin{array}{ccc} \Delta^{\mathrm{op}} & \longrightarrow & \mathbf{Top}, \\ [0,n] & \longmapsto & X_{n,\,\mathrm{Et}}, \end{array}$$

here, $X_{n,\,\mathrm{Et}}$ is the big étale topos of the scheme X_n, that is, the category of sheaves on the category $\mathbf{Sch}/X_n$ endowed with the étale topology. Then we denote by $\mathrm{T{\small OP}}(X_{\bullet,\mathrm{Et}})$ the total topos associated to $X_{\bullet,\,\mathrm{Et}}$.

LEMMA 3.3. *For any simplicial scheme $X_\bullet$, there is a canonical morphism of topoi*

$$\iota : \mathrm{T{\small OP}}(X_{\bullet,\,\mathrm{Et}}) \longrightarrow \mathrm{T{\small OP}}(X_{\bullet,\,\mathrm{et}})$$

such that the map

$$H^i(\mathrm{T{\small OP}}(X_{\bullet,\,\mathrm{et}}), \mathcal{A}) \longrightarrow H^i(\mathrm{T{\small OP}}(X_{\bullet,\,\mathrm{Et}}), \iota^*\mathcal{A})$$

is an isomorphism for any $i \geqslant 0$ and for any abelian sheaf $\mathcal{A}$ of $\mathrm{T{\small OP}}(X_{\bullet,\,\mathrm{et}})$.

Proof. The canonical morphism $Y_{\mathrm{Et}} \to Y_{\mathrm{et}}$, from the big étale topos of a scheme Y to its small étale topos, is pseudo-functorial in Y; this follows immediately from the description of this morphism in terms of morphism of sites. Hence, we have a morphism of simplicial topoi

$$\iota_\bullet : X_{\bullet,\,\mathrm{Et}} \longrightarrow X_{\bullet,\,\mathrm{et}}$$

inducing a morphism between total topoi:

$$\iota : \mathrm{T{\small OP}}(X_{\bullet,\,\mathrm{Et}}) \longrightarrow \mathrm{T{\small OP}}(X_{\bullet,\,\mathrm{et}}).$$

Note that we have a commutative diagram of topoi

$$\begin{array}{ccc} X_{n,Et} & \xrightarrow{\ \iota_n\ } & X_{n,et} \\ {\scriptstyle f_n}\big\downarrow & & \big\downarrow{\scriptstyle g_n} \\ \mathrm{T{\small OP}}(X_{\bullet,\,\mathrm{Et}}) & \xrightarrow{\ \iota\ } & \mathrm{T{\small OP}}(X_{\bullet,\,\mathrm{et}}) \end{array}$$

for any object $[0,n]$ of Δ. Here, the inverse image g_n^* (respectively, f_n^*) of the vertical morphism $g_n : X_{n,et} \to \mathrm{T{\small OP}}(X_{\bullet,\,\mathrm{et}})$ (respectively, $f_n : X_{n,Et} \to \mathrm{T{\small OP}}(X_{\bullet,\,\mathrm{Et}})$) maps an object $F = (F_n;\, \alpha^* F_m \to F_n)$ of the total topos $\mathrm{T{\small OP}}(X_{\bullet,\,\mathrm{et}})$ (respectively, of $\mathrm{T{\small OP}}(X_{\bullet,\,\mathrm{Et}})$) to $F_n \in X_{n,\,\mathrm{et}}$ (respectively, to $F_n \in X_{n,\,\mathrm{Et}}$). Recall that the functors g_n^* and f_n^* preserve injective objects. This leads to spectral sequences (see [**10**, VI Exercice 7.4.15])

$$E_1^{i,j} = H^j(X_{i,\mathrm{et}}, \mathcal{A}_i) \Rightarrow H^{i+j}(\mathrm{T{\small OP}}(X_{\bullet,\,\mathrm{et}}), \mathcal{A}) \tag{8}$$

and

$$'E_1^{i,j} = H^j(X_{i,Et}, (\iota^*\mathcal{A})_i) \Rightarrow H^{i+j}(\mathrm{T{\small OP}}(X_{\bullet,\,\mathrm{Et}}), \iota^*\mathcal{A}) \tag{9}$$

for any abelian object $\mathcal{A}$ of $\mathrm{T{\small OP}}(X_{\bullet,\,\mathrm{et}})$. The morphism $\iota_\bullet$ of simplicial topoi induces a morphism of spectral sequences from (8) to (9). This morphism of spectral sequences is an isomorphism since the natural map

$$H^j(X_{i,\mathrm{et}}, \mathcal{A}_i) \longrightarrow H^j(X_{i,Et}, (\iota^*\mathcal{A})_i) = H^j(X_{i,Et}, \iota_i^*(\mathcal{A}_i)) \tag{10}$$

is an isomorphism, where the equality on the right-hand side follows from the previous commutative square. Then the map (10) is the natural morphism from the cohomology of the small étale site of X_i to the cohomology of its big étale site, which is well known to be an isomorphism. Therefore, the induced morphism on abutments

$$H^i(\mathrm{Top}(X_{\bullet,\,\mathrm{et}}),\mathcal{A}) \longrightarrow H^i(\mathrm{Top}(X_{\bullet,\,\mathrm{Et}}),\iota^*\mathcal{A})$$

is an isomorphism. □

3.2. *Classifying topoi and classifying simplicial schemes*

Let S be a scheme, let G be an S-group scheme and let X be an S-scheme that supports a left G-action $G\times_S X\to X$. We consider the classifying simplicial scheme $\mathbf{B}(G,X)$ as defined in [5, Example 1.2]. Recall that

$$\mathbf{B}(G,X)_n = G^n\times X,$$

where G^n is the n-fold fiber product of G with itself over S and the product $G^n\times X$ is taken over S, with structure maps given in the usual way by using the multiplication in G, the action of G on X and the unit section $S\to G$. We consider the big étale simplicial topos

$$\begin{array}{rccc}\mathbf{B}(G,X)_{\mathrm{Et}}: & \Delta^{\mathrm{op}} & \longrightarrow & \mathbf{Top},\\ & [0,n] & \longmapsto & (G^n\times X)_{\mathrm{Et}}\end{array}$$

and the total topos $\mathrm{Top}(\mathbf{B}(G,X)_{\mathrm{Et}})$ as defined in the previous subsection.

Lemma 3.4. *There is a canonical morphism of topoi*

$$\kappa:\mathrm{Top}(\mathbf{B}(G,X)_{\mathrm{Et}}) \longrightarrow B_G^{\mathrm{et}}/X.$$

Proof. We let $\mathrm{Desc}(\mathbf{B}(G,X)_{\mathrm{Et}})$ be the descent topos. It is defined as the category of objects L of $X_{\mathrm{Et}}=\mathbf{B}(G,X)_{0,Et}$ endowed with descent data, that is, an isomorphism $a: d_1^*L\to d_0^*L$ such that

(i) $s_0^*(a)=\mathrm{Id}_L$;
(ii) $d_0^*(a)\circ d_2^*(a)=d_1^*(a)$ (neglecting the transitivity isomorphisms).

Then there is an equivalence of categories

$$\mathrm{Desc}(\mathbf{B}(G,X)_{\mathrm{Et}}) \longrightarrow B_G^{\mathrm{et}}/X. \tag{11}$$

Indeed, for any object L of $S_{\mathrm{Et}}/X\simeq X_{\mathrm{Et}}$, descent data on L are equivalent to a left action of G on L such that the structure map $L\to X$ is G-equivariant (see [11, VI, Section 8]). We define the functor

$$\begin{array}{rccc}\mathrm{Ner}: & \mathrm{Desc}(\mathbf{B}(G,X)_{\mathrm{Et}}) & \longrightarrow & \mathrm{Top}(\mathbf{B}(G,X)_{\mathrm{Et}}),\\ & (L,a) & \longmapsto & \mathrm{Ner}(L,a)\end{array}$$

as follows. Let (L,a) be an object of $\mathrm{Desc}(\mathbf{B}(G,X)_{\mathrm{Et}})$. We consider

$$\mathrm{Ner}_n(L,a)=(d_0\cdots d_0)^*L$$

in the topos $S_{\mathrm{Et}}/(G^n\times X)\simeq (G^n\times X)_{\mathrm{Et}}$. The map

$$d_i^*\,\mathrm{Ner}_{n-1}(L,a)\longrightarrow \mathrm{Ner}_n(L,a)$$

is Id for $i<n$ and $(d_0\cdots d_0)^*(a)$ for $i=n$. Finally, the map

$$s_i^*\,\mathrm{Ner}_n(L,a)\longrightarrow \mathrm{Ner}_{n-1}(L,a)$$

is the identity for any i. The functor Ner commutes with inductive limits and finite projective limits, since the inverse image of a morphism of topoi commutes with such limits and since these limits are computed component-wise in both $\mathrm{Desc}(\mathbf{B}(G,X)_{\mathrm{Et}})$ and $\mathrm{Top}(\mathbf{B}(G,X)_{\mathrm{Et}})$.

Hence, Ner is the inverse image of a morphism of topoi

$$\mathrm{Top}(\mathbf{B}(G,X)_{\mathrm{Et}}) \longrightarrow \mathrm{Desc}(\mathbf{B}(G,X)_{\mathrm{Et}}).$$

Composing this map with equivalence (11), we obtain the desired morphism

$$\kappa : \mathrm{Top}(\mathbf{B}(G,X)_{\mathrm{Et}}) \longrightarrow \mathrm{Desc}(\mathbf{B}(G,X)_{\mathrm{Et}}) \simeq B_G^{\mathrm{et}}/X. \qquad \square$$

Lemma 3.5. *The canonical map*

$$H^i(B_G^{\mathrm{et}}/X, \mathcal{A}) \longrightarrow H^i(\mathrm{Top}(\mathbf{B}(G,X)_{\mathrm{Et}}), \kappa^*\mathcal{A})$$

is an isomorphism for any i and any abelian sheaf $\mathcal{A}$ on B_G^{et}/X.

Proof. We will prove this lemma as follows: we describe spectral sequences converging to $H^*(B_G^{\mathrm{et}}/X, \mathcal{A})$ and $H^*(\mathrm{Top}(\mathbf{B}(G,X)_{\mathrm{Et}}), \kappa^*\mathcal{A})$, respectively; then we show that these spectral sequences are isomorphic at E_1. Let e_G be the final object of B_G^{et}. Since the map $E_G \to e_G$ has a section, it is an epimorphism, hence so is $E_G \times X \to X$, since epimorphisms are universal in a topos. We obtain a covering $\mathcal{U} = (E_G \times X \to X)$ of the final object in B_G^{et}/X. This covering gives rise to the Cartan–Leray spectral sequence (see [10, V, Corollary 3.3])

$$\check{H}^i(\mathcal{U}, \underline{H}^j(\mathcal{A})) \Rightarrow H^{i+j}(B_G^{\mathrm{et}}/X, \mathcal{A}),$$

where $\underline{H}^j(\mathcal{A})$ denotes the presheaf on B_G^{et}/X

$$\underline{H}^j(\mathcal{A}) : (\mathcal{F} \longrightarrow X) \longrightarrow H^j((B_G^{\mathrm{et}}/X)/\mathcal{F}, \mathcal{F} \times_X \mathcal{A})$$

and $\check{H}^i(\mathcal{U}, -)$ denotes Cech cohomology. By [10, IV, 5.8.3], we have a canonical equivalence

$$(B_G^{\mathrm{et}}/X)/(E_G \times X) \simeq S_{\mathrm{Et}}/X.$$

Consider more generally the n-fold product of $(E_G \times X)$ with itself over the final object in B_G^{et}/X

$$(E_G \times X)^n = (E_G \times X) \times_X \cdots \times_X (E_G \times X) = (E_G^n \times X).$$

Here, E_G^n is the object of B_G^{et} represented by the scheme $G^n = G \times_S \cdots \times_S G$ on which G acts diagonally. Then we have an equivalence

$$(B_G^{\mathrm{et}}/X)/(E_G^{n+1} \times X) = (B_G^{\mathrm{et}}/X)/(E_G \times (E_G^n \times X)) \simeq S_{\mathrm{Et}}/(G^n \times X)$$

for any $n \geqslant 0$. Therefore, the term $E_1^{i,j}$ of the Cartan–Leray spectral sequence takes the following form:

$$\begin{aligned} E_1^{i,j} &= H^j((B_G^{\mathrm{et}}/X)/(E_G^{i+1} \times X), (E_G^{i+1} \times X) \times_X \mathcal{A}) \\ &= H^j((S_{\mathrm{Et}}/(G^i \times X), G^i \times \mathcal{A}) \\ &= H^j((G^i \times X)_{\mathrm{Et}}, G^i \times \mathcal{A}) \end{aligned}$$

for any abelian object $\mathcal{A} \to X$ of B_G^{et}/X. We conclude that the spectral sequence can be written as follows:

$$E_1^{i,j} = H^j((G^i \times X)_{\mathrm{Et}}, G^i \times \mathcal{A}) \Rightarrow H^{i+j}(B_G^{\mathrm{et}}/X, \mathcal{A}). \tag{12}$$

We also have a spectral sequence (see (9))

$$'E_1^{i,j} = H^j(\mathbf{B}(G,X)_{i,Et}, (\kappa^*\mathcal{A})_i) \Rightarrow H^{i+j}(\mathrm{Top}(\mathbf{B}(G,X)_{\mathrm{Et}}), \kappa^*\mathcal{A}), \tag{13}$$

where $\kappa : \mathrm{Top}(\mathbf{B}(G,X)_{\mathrm{Et}}) \to B_G^{\mathrm{et}}/X$ is the map of Lemma 3.4. For any $i \geqslant 0$, the following square:

$$\begin{array}{ccc} \mathbf{B}(G,X)_{i,Et} & \longrightarrow & (B_G^{\mathrm{et}}/X)/(E_G^{i+1} \times X) \\ \downarrow & & \downarrow \\ \mathrm{Top}(\mathbf{B}(G,X)_{\mathrm{Et}}) & \xrightarrow{\ \kappa\ } & B_G^{\mathrm{et}}/X \end{array}$$

is commutative, where the top horizontal map is the canonical equivalence

$$\mathbf{B}(G,X)_{i,Et} = (G^i \times X)_{\mathrm{Et}} \simeq S_{\mathrm{Et}}/(G^i \times X) \simeq (B_G^{\mathrm{et}}/X)/(E_G^{i+1} \times X).$$

Note that this last equivalence is precisely the equivalence from which we have deduced the isomorphism $E_1^{i,j} = H^j((G^i \times X)_{\mathrm{Et}}, G^i \times A)$. We obtain a morphism of spectral sequences from (12) to (13). This morphism of spectral sequences is an isomorphism since $(\kappa^*\mathcal{A})_i = G^i \times \mathcal{A}$, which in turn follows from the fact that the square above commutes. The result follows. □

We now consider the flat topos S_{fl}, the big étale topos S_{Et} and their classifying topoi

$$B_G^{\mathrm{et}} = B_{yG}(S_{\mathrm{Et}}) \quad \text{and} \quad B_G^{fl} = B_{yG}(S_{fl}).$$

It follows from Corollary 2.4 that the canonical morphism $i : S_{fl} \to S_{\mathrm{et}}$ induces a morphism $B_G \to B_G^{\mathrm{et}}$ such that the following square:

$$\begin{array}{ccc} B_G & \longrightarrow & B_G^{\mathrm{et}} \\ \downarrow & & \downarrow \\ S_{fl} & \xrightarrow{\ i\ } & S_{\mathrm{Et}} \end{array}$$

is a pull-back. This morphism induces a morphism [**10**, IV, Section 5.10]:

$$\gamma : B_G/X \longrightarrow B_G^{\mathrm{et}}/X.$$

The following is an equivariant refinement of the classical comparison theorem [**9**, Theorem 11.7] between étale and flat cohomology.

Lemma 3.6. *Let $\mathcal{A} = yA$ be an abelian object of B_G/X represented by a smooth X-scheme A. Then the canonical morphism*

$$\gamma^* : H^i(B_G^{\mathrm{et}}/X, yA) \longrightarrow H^i(B_G/X, yA)$$

is an isomorphism for any $i \geqslant 0$.

Proof. Consider the spectral sequence (a special case of (12))

$$H^j((G^i \times X)_{\mathrm{Et}}, G^i \times \mathcal{A}) \Rightarrow H^{i+j}(B_G^{\mathrm{et}}/X, \mathcal{A}) \tag{14}$$

associated to the covering $(E_G \times X \to X)$ in B_G^{et}/X. Applying the functor γ^* to $\mathcal{U} = (E_G \times X \to X)$, we obtain the covering $\gamma^*\mathcal{U} = (E_G \times X \to X)$ in B_G/X, and we get a morphism of spectral sequences from (14) to

$$H^j((G^i \times X)_{fl}, G^i \times \mathcal{A}) \Rightarrow H^{i+j}(B_G/X, \mathcal{A}).$$

But the canonical maps

$$H^j((G^i \times X)_{\mathrm{Et}}, G^i \times \mathcal{A}) \longrightarrow H^j((G^i \times X)_{fl}, G^i \times \mathcal{A})$$

are isomorphisms [**9**, Theorem 11.7]. It therefore follows that the maps

$$H^i(B_G^{\mathrm{et}}/X, \mathcal{A}) \longrightarrow H^i(B_G/X, \mathcal{A})$$

are also isomorphisms. □

Theorem 3.7. *Let $\mathcal{A} = yA$ be an abelian object of B_G/X represented by a smooth X-scheme A. Then there is a canonical isomorphism*

$$H^i(B_G/X, yA) \simeq H^i(\mathrm{Et}(\mathbf{B}(G, X)), yA)$$

for any $i \geqslant 0$, where B_G denotes the classifying topos of G and $\mathrm{Et}(\mathbf{B}(G, X))$ denotes the (small) étale site of the simplicial scheme $\mathbf{B}(G, X)$ as defined in [**5**].

Proof. The proof follows from Lemmas 3.3–3.6. □

3.3. *Giraud's exact sequence*

Let A be a commutative S-group scheme endowed with a left action of G. We denote by $\mathrm{Ext}_S(G, A)$ the abelian group of extensions of G by A in the topos S_{fl}. More precisely, $\mathrm{Ext}_S(G, A)$ is the group of equivalence classes of exact sequences in S_{fl}

$$1 \longrightarrow yA \longrightarrow \mathcal{G} \longrightarrow yG \longrightarrow 1,$$

where yA and yG denote the sheaves in S_{fl} represented by A and G, respectively, and such that the action of $\mathcal{G}$ on yA by inner automorphisms induces the given action of G on A. Note that $\mathcal{G}$ is not a scheme in general. We denote by $\mathrm{Crois}_S(G, A)$ the abelian group of crossed morphisms $f : G \to A$. Recall that a crossed morphism is a map of S-schemes $f : G \to A$ such that

$$f(gg') = f(g) + g \cdot f(g'). \tag{15}$$

This identity makes sense on points. Equivalently (15) can be seen as a commutative diagram in $\mathbf{Sch}/S$. Note that if G acts trivially on A, then $\mathrm{Crois}_S(G, A) = \mathrm{Hom}_S(G, A)$.

More generally, for any abelian object $\mathcal{A}$ of B_G, one defines $\mathrm{Ext}_S(G, \mathcal{A})$ and $\mathrm{Crois}_S(G, \mathcal{A})$ in the very same way (of course one has $\mathrm{Ext}_S(G, A) = \mathrm{Ext}_S(G, yA)$ and $\mathrm{Crois}_S(G, A) = \mathrm{Crois}_S(G, yA)$).

Proposition 3.8. *We have an exact sequence of abelian groups*

$$\begin{aligned}0 \longrightarrow H^0(B_G, \mathcal{A}) \longrightarrow H^0(S_{fl}, \mathcal{A}) \longrightarrow \mathrm{Crois}_S(G, \mathcal{A}) \longrightarrow H^1(B_G, \mathcal{A})\\ \longrightarrow H^1(S_{fl}, \mathcal{A}) \longrightarrow \mathrm{Ext}_S(G, \mathcal{A}) \longrightarrow H^2(B_G, \mathcal{A}) \longrightarrow H^2(S_{fl}, \mathcal{A}).\end{aligned}$$

Proof. This is a special case of [**7**, VIII.7.1.5]. □

3.4. *The sheaves $H^i_S(B_G, \mathcal{A})$ for $i = 0, 1, 2$*

Recall that we denote by $\pi : B_G \to S_{fl}$ the canonical map.

Notation 3.9. For an abelian object $\mathcal{A}$ of B_G, we denote by

$$H^i_S(B_G, \mathcal{A}) := R^i(\pi_*)\mathcal{A}$$

the cohomology of B_G with values in the topos S_{fl}.

The sheaf $H^i_S(B_G,\mathcal{A})$ may be described as follows:

Proposition 3.10. *For any abelian object $\mathcal{A}$ of B_G and, for any $i\geqslant 0$, the sheaf $H^i_S(B_G,\mathcal{A})$ is the sheaf associated to the presheaf*

$$\begin{array}{ccc}\mathbf{Sch}/S & \longrightarrow & \mathbf{Ab},\\ T & \longmapsto & H^i(G_T,\mathcal{A}_T),\end{array}$$

where G_T is the T-group scheme $G\times_S T$ and $\mathcal{A}_T$ is the abelian object of B_{G_T} induced by $\mathcal{A}$.

Proof. The sheaf $H^i_S(B_G,\mathcal{A})$ is the sheaf associated to the presheaf

$$T\longrightarrow H^i(B_G/T,\mathcal{A}\times T),$$

but, by virtue of Proposition 2.8, we have

$$H^i(B_G/T,\mathcal{A}\times T)=H^i(B_{G_T},\mathcal{A}_T)=:H^i(G_T,\mathcal{A}_T). \qquad \square$$

For any abelian sheaf $\mathcal{A}$ of B_G, we denote by $\mathcal{A}^G$ the largest subobject of $\mathcal{A}$ on which G acts trivially. Then we consider the abelian presheaf

$$\begin{array}{ccc}S_{fl} & \longrightarrow & \mathbf{Ab},\\ F & \longmapsto & \mathrm{Crois}_{S_{fl}/F}(G\times F,\mathcal{A}\times F).\end{array} \tag{16}$$

This presheaf is easily seen to be a subsheaf of the sheaf of homomorphisms $\underline{\mathrm{Map}}(G,\mathcal{A})$ (here the group structure is not taken into account) in the topos S_{fl} endowed with the canonical topology. The sheaf (16) is therefore representable by an abelian object $\underline{\mathrm{Crois}}_S(G,\mathcal{A})$ of S_{fl} (recall that any sheaf on a topos endowed with the canonical topology is representable). There is a morphism:

$$\begin{array}{cccc}\tau: & \mathcal{A} & \longrightarrow & \underline{\mathrm{Crois}}_S(G,\mathcal{A}),\\ & a & \longmapsto & g\longmapsto g\cdot a-a.\end{array}$$

Finally, we consider the presheaf

$$\begin{array}{ccc}S_{fl} & \longrightarrow & \mathbf{Ab},\\ F & \longmapsto & \mathrm{Ext}_{S_{fl}/F}(G\times F,\mathcal{A}\times F).\end{array}$$

Basic descent theory in topoi shows that this is a sheaf for the canonical topology. We denote by $\underline{\mathrm{Ext}}_S(G,\mathcal{A})$ the corresponding abelian object of S_{fl}.

Corollary 3.11. *We have*

$$H^0_S(B_G,\mathcal{A})\simeq\mathcal{A}^G,\quad H^2_S(B_G,\mathcal{A})\simeq\underline{\mathrm{Ext}}_S(G,\mathcal{A})$$

and an exact sequence (of abelian objects in S_{fl})

$$0\longrightarrow\mathcal{A}^G\longrightarrow\mathcal{A}\xrightarrow{\tau}\underline{\mathrm{Crois}}_S(G,\mathcal{A})\longrightarrow H^1_S(B_G,\mathcal{A})\longrightarrow 0.$$

In particular, if G acts trivially on $\mathcal{A}$, then

$$H^1_S(B_G,\mathcal{A})=\underline{\mathrm{Hom}}(G,\mathcal{A}).$$

Proof. By Proposition 3.8, for any T over S we have the exact sequence

$$\begin{aligned}0&\longrightarrow H^0(B_{G_T},\mathcal{A}_T)\longrightarrow H^0(T_{fl},\mathcal{A}_T)\longrightarrow\mathrm{Crois}_T(G_T,\mathcal{A}_T)\longrightarrow H^1(B_{G_T},\mathcal{A}_T)\\&\longrightarrow H^1(T_{fl},\mathcal{A}_T)\longrightarrow\mathrm{Ext}_T(G_T,\mathcal{A}_T)\longrightarrow H^2(B_{G_T},\mathcal{A}_T)\longrightarrow H^2(T_{fl},\mathcal{A}_T).\end{aligned}$$

This exact sequence is functorial in T, so that it may be viewed as an exact sequence of abelian presheaves on $\mathbf{Sch}/S$. Applying the associated sheaf functor together with Proposition 3.10, we obtain the exact sequence of sheaves

$$\begin{aligned} 0 \longrightarrow \mathcal{A}^G \longrightarrow \mathcal{A} \longrightarrow \underline{\mathrm{Crois}}_S(G, \mathcal{A}) \longrightarrow H^1_S(B_G, \mathcal{A}) \\ \longrightarrow 0 \longrightarrow \underline{\mathrm{Ext}}_S(G, \mathcal{A}) \longrightarrow H^2_S(B_G, \mathcal{A}) \longrightarrow 0 \end{aligned}$$

since the sheafification of the presheaf $T \mapsto H^i(T_{fl}, \mathcal{A}_T)$ is trivial for $i \geqslant 1$ (as it follows from [**10**, V, Proposition 5.1] applied to the identity map $\mathrm{Id} : S_{fl} \to S_{fl}$). □

3.5. *The Hochschild–Serre spectral sequence*

We consider a sequence of S-group schemes

$$1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1,$$

which is exact with respect to the fppf-topology.

PROPOSITION 3.12. *For any abelian object $\mathcal{A}$ of B_G, there is a natural G/N-action on $H^j_S(B_N, \mathcal{A})$ and there is a spectral sequence*

$$H^i(G/N, H^j_S(B_N, \mathcal{A})) \Longrightarrow H^{i+j}(G, \mathcal{A}).$$

Proof. The quotient map $G \to G/N$ induces a morphism of classifying topoi

$$f : B_G \longrightarrow B_{G/N},$$

and hence a spectral sequence

$$H^i(B_{G/N}, R^j(f_*)\mathcal{A}) \Longrightarrow H^{i+j}(B_G, \mathcal{A}),$$

which is functorial in the abelian object $\mathcal{A}$ of B_G. In view of the canonical equivalences

$$B_{G/N}/E_{G/N} \simeq S_{fl} \quad \text{and} \quad B_G/(G/N) \simeq B_N,$$

we have a pull-back square

$$\begin{array}{ccc} B_N & \xrightarrow{\pi} & S_{fl} \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B_G & \xrightarrow{f} & B_{G/N} \end{array}$$

where the vertical maps are localization maps. As previously this 'localization pull-back' yields isomorphisms for all $n \geqslant 0$,

$$p^* R^n(f_*) \simeq R^n(\pi_*) p'^*.$$

Moreover, the functor $p'^* : B_G \to B_N$ maps a G-object F to F on which N acts via $N \to G$, so that, for $\mathcal{A}$ an abelian object of B_G, $R^n(\pi_*)p'^*\mathcal{A}$ is really what we (slightly abusively) denote by $H^n_S(B_N, \mathcal{A})$. □

Recall that there is a canonical map $\tau : \mathcal{A} \to \underline{\mathrm{Crois}}_S(G, \mathcal{A})$, and that this map is the zero map if G acts trivially on $\mathcal{A}$.

COROLLARY 3.13. *There is an exact sequence*

$$\begin{aligned} 0 \longrightarrow H^1(B_{G/N}, \mathcal{A}^N) \longrightarrow H^1(B_G, \mathcal{A}) \longrightarrow H^0(B_{G/N}, \underline{\mathrm{Crois}}_S(N, \mathcal{A})/\mathrm{Im}(\tau)) \\ \longrightarrow H^2(B_{G/N}, \mathcal{A}^N) \longrightarrow H^2(B_G, \mathcal{A}). \end{aligned}$$

If N acts trivially on $\mathcal{A}$, then we obtain an exact sequence

$$0 \longrightarrow H^1(B_{G/N}, \mathcal{A}) \longrightarrow H^1(B_G, \mathcal{A}) \longrightarrow H^0(B_{G/N}, \underline{\mathrm{Hom}}(N, \mathcal{A})) \longrightarrow H^2(B_{G/N}, \mathcal{A}) \longrightarrow H^2(B_G, \mathcal{A}).$$

Proof. This is the five-term exact sequence given by the Hochschild–Serre spectral sequence of Proposition 3.12. □

4. *Invariants of symmetric bundles*

In the following sections, we fix a scheme $Y \to \mathrm{Spec}(\mathbb{Z}[1/2])$ in which 2 is invertible. The principal goal of this section is to associate to any symmetric bundle over Y cohomological invariants that generalize the classical Hasse–Witt invariants associated to quadratic forms on vector spaces over fields.

4.1. *Symmetric bundles*

A bilinear form on Y consists of a locally free $\mathcal{O}_Y$-module V (which one may see as a vector bundle on Y) and a morphism of $\mathcal{O}_Y$-modules

$$B : V \otimes_{\mathcal{O}_Y} V \longrightarrow \mathcal{O}_Y$$

such that, for any affine open subscheme Z of Y, the induced map

$$B_Z : V(Z) \times V(Z) \longrightarrow \mathcal{O}_Y(Z)$$

is a symmetric bilinear form on the $\mathcal{O}_Y(Z)$-module $V(Z)$. Let $V^\vee$ be the dual of V. The form B induces a morphism of bundles

$$\varphi_B : V \longrightarrow V^\vee,$$

which is self-adjoint. We call B *non-degenerate* or *unimodular* if φ_B is an isomorphism. A *symmetric bundle* is a pair (V, B) consisting of a Y-symmetric bundle V endowed with a unimodular form B. In general, we will denote such a bundle by (V, q), where q is the quadratic form associated to B. Since 2 is invertible in Y, we will refer to (V, q) either as a symmetric bundle or as a quadratic form over Y. In the case where $Y = \mathrm{Spec}(R)$ is affine, a symmetric bundle (V, q) is given by a pair (M, B), where M is a locally free R-module and B is a unimodular, symmetric, bilinear form on M.

Let (V, q) be a symmetric bundle over Y and let $f : T \to Y$ be a morphism of schemes. We define the pull-back of (V, q) by f as the symmetric bundle $(f^*(V), f^*(q))$ on T where $f^*(V)$ is the pull-back of V endowed with the form $f^*(q)$ defined on any affine open subsets U' and U of T and Y, such that $f(U') \subset U$, by scalar extension from q. We denote by (V_T, q_T) the resulting symmetric bundle on T.

An isometry of symmetric bundles $u : (V, q) \to (E, r)$ on Y is an isomorphism of locally free $\mathcal{O}_Y$-modules $u : V \to E$ such that $r(u(x)) = q(x)$ for any open affine subscheme U of Y and any x in $V(U)$. We denote this set by $\mathrm{Isom}(q, r)$. It follows from [**3**, III, Section 5, no. 2] that $T \to \mathrm{Isom}(q_T, r_T)$ is a sheaf of sets on $\mathbf{Sch}/Y$ endowed with the fppf-topology. We denote this sheaf by $\mathbf{Isom}(q, r)$, or by $\mathbf{Isom}_Y(q, r)$. We define the *orthogonal group* $\mathbf{O}(q)$ as the group $\mathbf{Isom}(q, q)$ of Y_{fl}. This sheaf is representable by a smooth algebraic group scheme over Y, which we also denote by $\mathbf{O}(q)$. We denote by $(\mathcal{O}_Y^n, t_n = x_1^2 + \cdots + x_n^2)$ the standard form over Y of rank n and by $\mathbf{O}(n)$ (or by $\mathbf{O}(n)_Y$ if we wish to stress that the base scheme is Y) the orthogonal group of t_n.

4.2. *Twisted forms*

Let (V, q) be a symmetric bundle over Y. A symmetric bundle (F, r) is called a *twisted form of* (V, q) if there exists an fppf-covering $\{U_i \to Y,\ i \in I\}$ such that there exists an isometry

$$(V \otimes_Y U_i, q \otimes_Y U_i) \simeq (F \otimes_Y U_i, r \otimes_Y U_i) \quad \forall i \in I.$$

Recall that a *groupoid* is a small category whose morphisms are all isomorphisms. We denote by $\mathbf{Twist}(q)$ the groupoid whose objects are twisted forms of (V, q) and morphisms are isometries. Let $\mathbf{Twist}(q)/_\sim$ be the set of isometry classes of twists of (V, q), which we consider as a set pointed by the class of (V, q). If (F, r) is a twist of (V, q), then $\mathbf{Isom}(q, r)$ is an $\mathbf{O}(q)$-torsor of Y_{fl}. We denote by $\mathbf{Tors}(Y_{fl}, \mathbf{O}(q))/_\sim$ the pointed set of isometry classes of $\mathbf{Tors}(Y_{fl}, \mathbf{O}(q))$, pointed by the class of the trivial torsor.

PROPOSITION 4.1. *The canonical functor*

$$\begin{array}{ccc} \mathbf{Twist}(q) & \longrightarrow & \mathbf{Tors}(Y_{fl}, \mathbf{O}(q)), \\ (F, r) & \longmapsto & \mathbf{Isom}(q, r) \end{array}$$

is an equivalence.

Proof. By [3, III, § 5, no. 2.1], the functor above induces an isomorphism of pointed sets:

$$\begin{array}{ccc} \mathbf{Twist}(q)/_\sim & \longrightarrow & \mathbf{Tors}(Y_{fl}, \mathbf{O}(q))/_\sim, \\ (F, r) & \longmapsto & [\mathbf{Isom}(q, r)], \end{array}$$

where $[\mathbf{Isom}(q, r)]$ is the isometry class of the torsor $\mathbf{Isom}(q, r)$. Since $\mathbf{Twist}(q)$ and $\mathbf{Tors}(Y_{fl}, \mathbf{O}(q))$ are both groupoids, we are reduced to showing that the automorphism group of $\mathbf{Isom}(q, r)$ in $\mathbf{Tors}(Y_{fl}, \mathbf{O}(q))$ is in bijection with the automorphism group of (F, r) in $\mathbf{Twist}(q)$. In other words, one has to show that the map

$$\mathbf{O}(r)(Y) \longrightarrow \mathrm{Hom}_{\mathbf{Tors}(Y_{fl}, \mathbf{O}(q))}(\mathbf{Isom}(q, r), \mathbf{Isom}(q, r))$$

is bijective. But this follows from the fact that $\mathbf{Isom}(q, r)$ is a left $\mathbf{O}(r)$-torsor. □

Since $\mathbf{O}(q)$ is smooth, the functor $\mathbf{Tors}(Y_{\mathrm{et}}, \mathbf{O}(q)) \to \mathbf{Tors}(Y_{fl}, \mathbf{O}(q))$ is an equivalence. Hence, any twist of q is already split by an étale covering family. This can also be seen as follows: let $(\mathcal{O}_Y^n, t_n = x_1^2 + \cdots + x_n^2)$ be the standard form over Y of rank n and let $\mathbf{O}(n)$ be the orthogonal group of t_n. Since on any strictly henselian local ring (in which 2 is invertible) a quadratic form of rank n is isometric to the standard form, any symmetric bundle (V, q) on Y is locally isometric to the standard form t_n for the étale topology. We denote by $\mathbf{Quad}_n(Y)$ the groupoid whose objects are symmetric bundles of rank n over Y and whose morphisms are isometries. There are canonical equivalences of categories

$$\begin{aligned} \mathbf{Quad}_n(Y) &\simeq \mathbf{Twist}(t_n) && (17) \\ &\simeq \mathbf{Tors}(Y_{\mathrm{et}}, \mathbf{O}(n)) && (18) \\ &\simeq \mathbf{Tors}(Y_{fl}, \mathbf{O}(n)) && (19) \\ &\simeq \mathbf{Homtop}_{Y_{fl}}(Y_{fl}, B_{\mathbf{O}(n)})^{\mathrm{op}}. && (20) \end{aligned}$$

Given a symmetric bundle (V, q) on Y, we denote by $\{q\} : Y_{fl} \to B_{\mathbf{O}(n)}$ the morphism of topoi associated to the quadratic form q by this equivalence.

PROPOSITION 4.2. *There is an equivalence of categories*

$$\begin{array}{ccc} \mathbf{Quad}_n(Y) & \longrightarrow & \mathbf{Homtop}_{Y_{fl}}(Y_{fl}, B_{\mathbf{O}(n)})^{\mathrm{op}}, \\ (V, q) & \longmapsto & \{q\}. \end{array}$$

4.3. *Invariants in low degree:* $\det[q]$ *and* $[C_q]$

For a symmetric bundle (V,q) over Y, there are canonical cohomology classes in $H^1(B_{\mathbf{O}(q)},\mathbf{Z}/2\mathbf{Z})$ and $H^2(B_{\mathbf{O}(q)},\mathbf{Z}/2\mathbf{Z})$.

4.3.1. *Degree* 1 The determinant map

$$\det_{\mathbf{O}(q)}:\mathbf{O}(q)\longrightarrow\mu_2\xrightarrow{\sim}\mathbf{Z}/2\mathbf{Z}$$

is a morphism of Y-group schemes. By Proposition 3.8, there is a canonical map

$$\mathrm{Hom}_Y(\mathbf{O}(q),\mathbf{Z}/2\mathbf{Z})\longrightarrow H^1(B_{\mathbf{O}(q)},\mathbf{Z}/2\mathbf{Z}).$$

This yields a class $\det[q]\in H^1(B_{\mathbf{O}(q)},\mathbf{Z}/2\mathbf{Z})$.

Definition 4.3. The class $\det[q]\in H^1(B_{\mathbf{O}(q)},\mathbf{Z}/2\mathbf{Z})$ is the class defined by the morphism $\det_{\mathbf{O}(q)}:\mathbf{O}(q)\to\mathbf{Z}/2\mathbf{Z}$.

The cohomology class $\det[q]$ is represented by the morphism

$$B_{\det_{\mathbf{O}(q)}}:B_{\mathbf{O}(q)}\longrightarrow B_{\mathbf{Z}/2\mathbf{Z}}.$$

The $\mathbf{Z}/2\mathbf{Z}$-torsor of $B_{\mathbf{O}(q)}$ corresponding to this morphism is given by $\mathbf{O}(q)/\mathbf{SO}(q)$ with its natural left $\mathbf{O}(q)$-action and its right $\mathbf{Z}/2\mathbf{Z}$-action via $\mathbf{O}(q)/\mathbf{SO}(q)\simeq\mathbf{Z}/2\mathbf{Z}$. In other words, we may write

$$\det[q]=[\mathbf{O}(q)/\mathbf{SO}(q)]\in H^1(B_{\mathbf{O}(q)},\mathbf{Z}/2\mathbf{Z}).$$

4.3.2. *Degree* 2 One can define the Clifford algebra of the symmetric bundle (V,q); this is a sheaf of algebras over Y. This leads us to consider the group $\tilde{\mathbf{O}}(q)$ which, in this context, is the generalization of the group $\mathrm{Pin}(q)$ (see [**5**, Section 1.9, **13**, Appendix 4]). The group $\tilde{\mathbf{O}}(q)$ is a smooth group scheme over Y which is an extension of $\mathbf{O}(q)$ by $\mathbf{Z}/2\mathbf{Z}$, that is, there is an exact sequence of groups in Y_{fl}

$$1\longrightarrow\mathbf{Z}/2\mathbf{Z}\longrightarrow\tilde{\mathbf{O}}(q)\longrightarrow\mathbf{O}(q)\longrightarrow 1. \tag{21}$$

Such a sequence defines a class $C_q\in\mathrm{Ext}_Y(\mathbf{O}(q),\mathbf{Z}/2\mathbf{Z})$ and therefore (see Proposition 3.8) a cohomology class in $H^2(B_{\mathbf{O}(q)},\mathbf{Z}/2\mathbf{Z})$ that we denote by $[C_q]$.

Definition 4.4. The class $[C_q]\in H^2(B_{\mathbf{O}(q)},\mathbf{Z}/2\mathbf{Z})$ is the class defined by the extension $C_q\in\mathrm{Ext}_Y(\mathbf{O}(q),\mathbf{Z}/2\mathbf{Z})$.

4.4. *The fundamental morphisms* T_q *and* Θ_q

Let $(\mathcal{O}_Y^n,t_n=x_1^2+\cdots+x_n^2)$ denote the standard form over Y of rank n and let $\mathbf{O}(n):=\mathbf{Isom}(t_n,t_n)$ be the orthogonal group of t_n. Let $\pi:B_{\mathbf{O}(q)}\to Y_{fl}$ be the morphism of topoi associated to the group homomorphism $\mathbf{O}(q)\to\{e\}$. The sheaf $\mathbf{Isom}(t_n,q)$ has a natural left action of $\mathbf{O}(q)$ and a natural right action of $\mathbf{O}(n)$:

$$\begin{array}{ccc}\mathbf{O}(q)\times\mathbf{Isom}(t_n,q)\times\mathbf{O}(n) & \longrightarrow & \mathbf{Isom}(t_n,q),\\ (\tau,f,\sigma) & \longmapsto & \tau\circ f\circ\sigma.\end{array}$$

These actions are compatible; more precisely, $\mathbf{Isom}(t_n,q)$ is naturally an object of $B_{\mathbf{O}(q)}$ that carries a right action of $\pi^*\mathbf{O}(n)$.

Lemma 4.5. *The sheaf* $\mathbf{Isom}(t_n,q)$ *is a* $\pi^*\mathbf{O}(n)$*-torsor of* $B_{\mathbf{O}(q)}$.

Proof. On the one hand, $\mathbf{Isom}(t_n, q)$ is an $\mathbf{O}(n)$-torsor of Y_{fl}, since there exists an fppf-covering (or equivalently, an étale covering) $U \to Y$ and an isometry

$$(V \otimes_Y U, q \otimes_Y U) \simeq (\mathcal{O}_Y^n \otimes_Y U, t_n \otimes_Y U).$$

We obtain a covering $U \to Y$ in Y_{fl} and an $\mathbf{O}(n)$-equivariant isomorphism in $U_{fl} \simeq Y_{fl}/U$:

$$U \times \mathbf{Isom}(t_n, q) \xrightarrow{\sim} U \times \mathbf{O}(n).$$

On the other hand, there is a canonical equivalence $Y_{fl} \simeq B_{\mathbf{O}(q)}/E_{\mathbf{O}(q)}$ such that the composite morphism

$$f : Y_{fl} \xrightarrow{\sim} B_{\mathbf{O}(q)}/E_{\mathbf{O}(q)} \longrightarrow B_{\mathbf{O}(q)}$$

is the map induced by the morphism of groups $1 \to \mathbf{O}(q)$; in other words, the inverse image functor f^* forgets the $\mathbf{O}(q)$-action. We may therefore view, respectively, the right $(E_{\mathbf{O}(q)} \times \pi^*\mathbf{O}(n))$-objects $E_{\mathbf{O}(q)} \times \mathbf{Isom}(t_n, q)$ and $E_{\mathbf{O}(q)} \times \pi^*\mathbf{O}(n)$ of the topos $B_{\mathbf{O}(q)}/E_{\mathbf{O}(q)}$ as the right $\mathbf{O}(n)$-objects $\mathbf{Isom}(t_n, q)$ (with no $\mathbf{O}(q)$-action) and $\mathbf{O}(n)$ of the topos Y_{fl}. We obtain an $\mathbf{O}(n)$-equivariant isomorphism

$$U \times E_{\mathbf{O}(q)} \times \mathbf{Isom}(t_n, q) \xrightarrow{\sim} U \times E_{\mathbf{O}(q)} \times \pi^*\mathbf{O}(n)$$

in the topos

$$B_{\mathbf{O}(q)}/(U \times E_{\mathbf{O}(q)}) \simeq (B_{\mathbf{O}(q)}/E_{\mathbf{O}(q)})/(U \times E_{\mathbf{O}(q)}) \simeq Y_{fl}/f^*U \simeq U_{fl}.$$

The result follows since $U \times E_{\mathbf{O}(q)} \to *$ covers the final object of $B_{\mathbf{O}(q)}$. □

Definition 4.6. We denote by T_q the sheaf $\mathbf{Isom}(t_n, q)$ endowed with its structure of $\pi^*(\mathbf{O}(n))$-torsor of $B_{\mathbf{O}(q)}$, and we define

$$T_q : B_{\mathbf{O}(q)} \longrightarrow B_{\mathbf{O}(n)}$$

to be the morphism associated to the torsor T_q.

Proposition 4.7. *The map $T_q : B_{\mathbf{O}(q)} \to B_{\mathbf{O}(n)}$ is an equivalence.*

Proof. Let $X_q : B_{\mathbf{O}(n)} \to B_{\mathbf{O}(q)}$ be the map associated to the $\mathbf{O}(q)$-torsor of $B_{\mathbf{O}(n)}$ given by $\mathbf{Isom}(q, t_n)$, and consider

$$X_q \circ T_q : B_{\mathbf{O}(q)} \longrightarrow B_{\mathbf{O}(n)} \longrightarrow B_{\mathbf{O}(q)}.$$

We have

$$(X_q \circ T_q)^* E_{\mathbf{O}(q)} = T_q^* X_q^*(E_{\mathbf{O}(q)}) = T_q^*(\mathbf{Isom}(q, t_n)) = \mathbf{Isom}(t_n, q) \wedge^{\mathbf{O}(n)} \mathbf{Isom}(q, t_n).$$

But the map

$$\begin{array}{ccc} \mathbf{Isom}(t_n, q) \times \mathbf{Isom}(q, t_n) & \longrightarrow & \mathbf{Isom}(q, q), \\ (f, g) & \longmapsto & f \circ g \end{array}$$

induces a $\pi^*\mathbf{O}(q)$-equivariant isomorphism

$$\mathbf{Isom}(t_n, q) \wedge^{\mathbf{O}(n)} \mathbf{Isom}(q, t_n) \simeq \mathbf{Isom}(q, q) = E_{\mathbf{O}(q)}.$$

Hence, we have a canonical isomorphism

$$(X_q \circ T_q)^* E_{\mathbf{O}(q)} = \mathbf{Isom}(t_n, q) \wedge^{\mathbf{O}(n)} \mathbf{Isom}(q, t_n) \simeq E_{\mathbf{O}(q)}$$

of $\pi^*\mathbf{O}(q)$-torsors in $B_{\mathbf{O}(q)}$. By Theorem 2.3, we obtain an isomorphism $X_q \circ T_q \simeq \mathrm{Id}_{B_{\mathbf{O}(q)}}$ of morphisms of topoi. Similarly, we have

$$(T_q \circ X_q)^* E_{\mathbf{O}(n)} = \mathbf{Isom}(q, t_n) \wedge^{\mathbf{O}(q)} \mathbf{Isom}(t_n, q) \simeq E_{\mathbf{O}(n)},$$

hence $T_q \circ X_q \simeq \mathrm{Id}_{B_{\mathbf{O}(n)}}$. □

Let $\eta : Y_{fl} \to B_{\mathbf{O}(q)}$ be the morphism of topoi induced by the morphism of groups $1 \to \mathbf{O}(q)$, while $\pi : B_{\mathbf{O}(q)} \to Y_{fl}$ is induced by the morphism of groups $\mathbf{O}(q) \to 1$. Note that

$$\mathrm{Id}_{Y_{fl}} \cong \pi \circ \eta : Y_{fl} \longrightarrow B_{\mathbf{O}(q)} \longrightarrow Y_{fl}.$$

However, $\eta \circ \pi \ncong \mathrm{Id}_{B_{\mathbf{O}(q)}}$, since $(\eta \circ \pi)^*$ sends an object of $B_{\mathbf{O}(q)}$ given by a sheaf $\mathcal{F}$ of Y_{fl} endowed with a (possibly non-trivial) $\mathbf{O}(q)$-action to the object of $B_{\mathbf{O}(q)}$ given by $\mathcal{F}$ with trivial $\mathbf{O}(q)$-action. The pull-back $\eta^*\mathbf{Isom}(t_n, q)$ is an $\mathbf{O}(n)$-torsor of Y_{fl}, and $\pi^*\eta^*(\mathbf{Isom}(t_n, q))$ is a $\pi^*(\mathbf{O}(n))$-torsor of $B_{\mathbf{O}(q)}$. We denote this torsor by Θ_q . This is the sheaf $\mathbf{Isom}(t_n, q)$ endowed with the trivial left action of $\mathbf{O}(q)$ and its natural right action of $\mathbf{O}(n)$.

Definition 4.8. We consider the $\pi^*(\mathbf{O}(n))$-torsor of $B_{\mathbf{O}(q)}$ given by

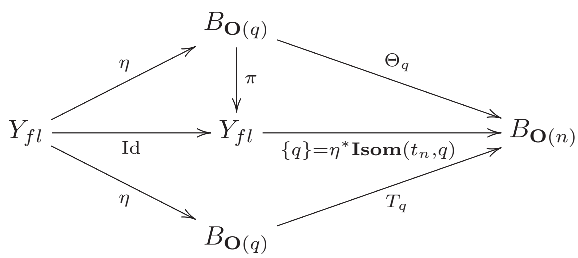
$$\Theta_q := \pi^*\eta^*(\mathbf{Isom}(t_n, q)),$$

and we define

$$\Theta_q : B_{\mathbf{O}(q)} \longrightarrow B_{\mathbf{O}(n)}$$

to be the morphism associated to the torsor Θ_q.

We have a commutative diagram:



In other words, we have canonical isomorphisms:

$$\pi \circ \eta \simeq id, \quad \{q\} \circ \pi \simeq \Theta_q, \quad T_q \circ \eta \simeq \Theta_q \circ \eta \simeq \{q\}.$$

4.5. *Hasse–Witt invariants*

In [**14**, Theorem 2.8], Jardine proved the following result, which is a basic source for the definitions of our invariants.

Theorem 4.9. *Let Y be a scheme in which 2 is invertible and let A denote the algebra $H^*_{\mathrm{et}}(Y, \mathbf{Z}/2\mathbf{Z})$. Assume that Y is the disjoint union of its connected components. Then there is a canonical isomorphism of graded A-algebras of the form*

$$H^*(\mathrm{Top}(B(\mathbf{O}(n), Y)_{\mathrm{Et}}), \mathbf{Z}/2\mathbf{Z}) \simeq A[HW_1, \ldots, HW_n],$$

where the polynomial generator HW_i has degree i.

By Theorem 3.7, there is a canonical isomorphism:

$$H^*(B_{\mathbf{O}(n)}, \mathbf{Z}/2\mathbf{Z}) \simeq H^*(\mathrm{Top}(B(\mathbf{O}(n), Y)_{\mathrm{Et}}), \mathbf{Z}/2\mathbf{Z}).$$

We use this isomorphism to identify these two groups and from now on we view HW_i as an element of $H^i(B_{\mathbf{O}(n)}, \mathbf{Z}/2\mathbf{Z})$.

Definition 4.10. The universal Hasse–Witt ith-invariant of the quadratic form q is

$$HW_i(q) = T_q^*(HW_i) \in H^i(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z}).$$

When q is the standard form t_n, the invariants $HW_i(q)$ coincide in degree 1 and 2 with the invariants we introduced in subsection 4.3, that is, we have

$$HW_1 = HW_1(t_n) = \det[t_n] \quad \text{and} \quad HW_2 = HW_2(t_n) = [C_n].$$

Corollary 4.11. *There is a canonical isomorphism of graded A-algebras of the form*

$$H^*(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z}) \simeq A[HW_1(q), \ldots, HW_n(q)],$$

where the polynomial generator $HW_i(q)$ has degree i.

Proof. This follows from Theorem 4.9 and Proposition 4.7. □

We note that the 'usual' ith Hasse–Witt invariant of q is the class of $H^i(Y_{fl}, \mathbf{Z}/2\mathbf{Z})$ obtained by pulling back $HW_i(q)$ by η. To be more precise, the ith Hasse–Witt invariant of q is defined by

$$w_i(q) = \eta^*(HW_i(q)) = \{q\}^*(HW_i).$$

Since $\eta^* \circ \pi^* \simeq \mathrm{Id}$, the group homomorphism

$$\eta^* : H^i(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z}) \longrightarrow H^i(Y_{fl}, \mathbf{Z}/2\mathbf{Z})$$

is split for any $i \geqslant 0$. Therefore, we may identify via π^* the group $H^i(Y_{fl}, \mathbf{Z}/2\mathbf{Z})$ as a direct factor of $H^i(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z})$. Since $\Theta_q \simeq \{q\} \circ \pi$, we note that under this identification ${\Theta_q}^*(HW_i)$ identifies with $w_i(q)$. This leads to the following definition.

Definition 4.12. The ith Hasse–Witt invariant of q is defined by

$$w_i(q) = \Theta_q^*(HW_i) \in H^i(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z}).$$

Let G_Y be a Y-group scheme and let (V, q, ρ) be a G_Y-equivariant symmetric bundle on Y. The group scheme homomorphism $\rho : G_Y \to \mathbf{O}(q)$ induces a morphism of topoi $\rho : B_{G_Y} \to B_{\mathbf{O}(q)}$. We obtain

$$T_q \circ \rho : B_{G_Y} \longrightarrow B_{\mathbf{O}(q)} \longrightarrow B_{\mathbf{O}(n)}.$$

This morphism corresponds to the $\mathbf{O}(n)$-torsor $\rho^*(T_q)$ of B_{G_Y}, which is given by the sheaf $\mathbf{Isom}(t_n, q)$ endowed with a left action of G_Y via ρ and a right action of $\mathbf{O}(n)$.

Definition 4.13. The ith equivariant Hasse–Witt invariant of (V, q, ρ) is defined by

$$w_i(q, \rho) = \rho^*(HW_i(q)) = \rho^* T_q^*(HW_i) \in H^i(B_{G_Y}, \mathbf{Z}/2\mathbf{Z}).$$

5. *Universal comparison formulas*

Let (V, q) be a symmetric bundle on the scheme Y. We assume that 2 is invertible in Y and that $Y = \coprod_{\alpha \in A} Y_\alpha$ is the disjoint union of its connected components. This second condition is rather weak but not automatic.

By Corollary 4.11, we have a canonical isomorphism

$$H^*(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z}) \simeq A[HW_1(q), \ldots, HW_n(q)],$$

where $A = H^*_{\mathrm{et}}(Y_{fl}, \mathbf{Z}/2\mathbf{Z})$. Under this identification, we have classes

$$w_1(q), w_2(q) \in A \quad \text{and} \quad \det[q],\ [C_q] \in A[HW_1(q), \ldots, HW_n(q)]$$

defined in subsection 4.5. Theorems 5.2 and 5.3 provide an explicit expression of $\det[q]$ and $[C_q]$ as polynomials in $HW_1(q)$ and $HW_2(q)$ with coefficients in A written in terms of $w_1(q), w_2(q) \in A$. More precisely, we will prove the following theorem.

THEOREM 5.1. *Let (V, q) be a symmetric bundle of rank n on the scheme Y in which 2 is invertible. Assume that Y is the disjoint union of its connected components. Then we have*

$$\det[q] = w_1(q) + HW_1(q)$$

and

$$[C_q] = (w_1(q) \cdot w_1(q) + w_2(q)) + w_1(q) \cdot HW_1(q) + HW_2(q)$$

in the polynomial ring

$$H^*(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z}) \simeq A[HW_1(q), \ldots, HW_n(q)].$$

These formulas are the source for many other comparison formulas that either have been proved in previous papers [**2**, **4**] or that we shall establish, using the following principle. For any topos $\mathcal{E}$ given with an $\mathbf{O}(q)$-torsor, we have a canonical map $f : \mathcal{E} \to B_{\mathbf{O}(q)}$, and we obtain comparison formulas in $H^*(\mathcal{E}, \mathbf{Z}/2\mathbf{Z})$ by applying the functor f^* to the universal comparison formulas of Theorem 5.1.

We split Theorem 5.1 in two theorems according to the degree.

THEOREM 5.2. *Let (V, q) be a symmetric bundle of rank n on the scheme Y in which 2 is invertible. Then*

$$HW_1(q) = w_1(q) + \det[q]$$

in $H^1(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z})$.

Proof. The group $H^1(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z})$ can be understood as the group of isomorphism classes of $\mathbf{Z}/2\mathbf{Z}$-torsors of the topos $B_{\mathbf{O}(q)}$. Hence, the theorem may be proved by describing an isomorphism between the torsors representing both sides of the required equality. The cohomology class $HW_1(q) = T_q^*(HW_1) = T_q^*(\det[t_n])$ is represented by the morphism

$$B_{\det_{\mathbf{O}(n)}} \circ T_q : B_{\mathbf{O}(q)} \longrightarrow B_{\mathbf{O}(n)} \longrightarrow B_{\mathbf{Z}/2\mathbf{Z}},$$

where $B_{\det_{\mathbf{O}(n)}}$ is the map of classifying topoi induced by the morphism of groups $\det_{\mathbf{O}(n)} : \mathbf{O}(n) \to \mathbf{Z}/2\mathbf{Z}$. Therefore, $HW_1(q)$ is represented by the $\mathbf{Z}/2\mathbf{Z}$-torsor:

$$\begin{aligned}(B_{\det_{\mathbf{O}(n)}} \circ T_q)^* E_{\mathbf{Z}/2\mathbf{Z}} &= T_q^* B^*_{\det_{\mathbf{O}(n)}} E_{\mathbf{Z}/2\mathbf{Z}} = T_q^*(\mathbf{O}(n)/\mathbf{SO}(n)) \\ &= T_q \wedge^{\mathbf{O}(n)} (\mathbf{O}(n)/\mathbf{SO}(n)).\end{aligned}$$

Similarly, $w_1(q)$ is represented by the $\mathbf{Z}/2\mathbf{Z}$-torsor $\Theta_q \wedge^{\mathbf{O}(n)} (\mathbf{O}(n)/\mathbf{SO}(n))$. Note that $\mathbf{O}(q)$ acts on $T_q \wedge^{\mathbf{O}(n)} \mathbf{O}(n)/\mathbf{SO}(n)$ via its left action on T_q, while it acts trivially on $\Theta_q \wedge^{\mathbf{O}(n)} \mathbf{O}(n)/\mathbf{SO}(n)$. In both cases $\mathbf{Z}/2\mathbf{Z}$ acts by right multiplication on $\mathbf{O}(n)/\mathbf{SO}(n) \simeq \mathbf{Z}/2\mathbf{Z}$. The group $\mathbf{O}(q)$ acts on $T_q \wedge^{\mathbf{O}(n)} (\mathbf{O}(n)/\mathbf{SO}(n))$ as follows:

$$\begin{array}{ccc} \mathbf{O}(q) \times (T_q \wedge^{\mathbf{O}(n)} (\mathbf{O}(n)/\mathbf{SO}(n))) & \longrightarrow & T_q \wedge^{\mathbf{O}(n)} (\mathbf{O}(n)/\mathbf{SO}(n)), \\ (f, [\sigma, \bar{g}]) & \longmapsto & [f \circ \sigma, \bar{g}]. \end{array}$$

We now consider $\mathbf{Z}/2\mathbf{Z}$ as an object of Y_{fl} endowed with a right action of $\mathbf{O}(q)$ via $\det_{\mathbf{O}(q)}$ and right multiplication on the one hand, and with a left action of $\mathbf{O}(n)$ via $\det_{\mathbf{O}(n)}$ and left multiplication on the other hand. Then $\mathbf{O}(q)$ acts on the left on $\Theta_q \wedge^{\mathbf{O}(n)} \mathbf{Z}/2\mathbf{Z}$ as follows:

$$\begin{array}{ccc} \mathbf{O}(q) \times (\Theta_q \wedge^{\mathbf{O}(n)} \mathbf{Z}/2\mathbf{Z}) & \longrightarrow & \Theta_q \wedge^{\mathbf{O}(n)} \mathbf{Z}/2\mathbf{Z}, \\ (f, [\sigma, \epsilon]) & \longmapsto & [\sigma, \epsilon \cdot \det_{\mathbf{O}(q)}(f)^{-1}]. \end{array}$$

Let us show that the map

$$\begin{array}{cccc} \iota: & T_q \wedge^{\mathbf{O}(n)} \mathbf{O}(n)/\mathbf{SO}(n) & \longrightarrow & \Theta_q \wedge^{\mathbf{O}(n)} \mathbf{Z}/2\mathbf{Z}, \\ & [\sigma, \bar{g}] & \longmapsto & [\sigma, \det_{\mathbf{O}(n)}(g)] \end{array}$$

is an isomorphism of $\mathbf{Z}/2\mathbf{Z}$-torsors of $B_{\mathbf{O}(q)}$. The map ι is an isomorphism of $\mathbf{Z}/2\mathbf{Z}$-torsors of Y_{fl} since $\det_{\mathbf{O}(n)}$ induces $\mathbf{O}(n)/\mathbf{SO}(n) \simeq \mathbf{Z}/2\mathbf{Z}$. It remains to check that ι respects the left action of $\mathbf{O}(q)$. On points, we have

$$\begin{aligned} \iota(f * [\sigma, \bar{g}]) &= \iota[f \circ \sigma, \bar{g}] = \iota[\sigma \circ (\sigma^{-1} \circ f^{-1} \circ \sigma)^{-1}, \overline{\sigma^{-1} f^{-1} \sigma} \cdot \bar{g}] \\ &= [\sigma, \det_{t_n}(g) \det_{\mathbf{O}(q)}(f)^{-1}] = f * \iota[\sigma, \bar{g}] \end{aligned}$$

since $\sigma^{-1} \circ f \circ \sigma$ is a section of $\mathbf{O}(n)$. We remark that

$$\mathbf{O}(n)/\mathbf{SO}(n) \wedge^{\mathbf{Z}/2\mathbf{Z}} \mathbf{O}(q)/\mathbf{SO}(q)$$

is canonically isomorphic to $\mathbf{Z}/2\mathbf{Z}$ and is naturally given with a right action of $\mathbf{O}(q)$ and a left action of $\mathbf{O}(n)$. Hence, ι yields a canonical isomorphism of $\mathbf{Z}/2\mathbf{Z}$-torsors in the topos $B_{\mathbf{O}(q)}$:

$$T_q \wedge^{\mathbf{O}(n)} \mathbf{O}(n)/\mathbf{SO}(n) \simeq \Theta_q \wedge^{\mathbf{O}(n)} (\mathbf{O}(n)/\mathbf{SO}(n) \wedge^{\mathbf{Z}/2\mathbf{Z}} \mathbf{O}(q)/\mathbf{SO}(q)). \tag{22}$$

Recall that the class $\det[q] \in H^1(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z})$ is represented by the $\mathbf{Z}/2\mathbf{Z}$-torsor $\mathbf{O}(q)/\mathbf{SO}(q)$. We have

$$\begin{aligned} w_1(q) + \det[q] &= [(\Theta_q \wedge^{\mathbf{O}(n)} \mathbf{O}(n)/\mathbf{SO}(n)) \wedge^{\mathbf{Z}/2\mathbf{Z}} \mathbf{O}(q)/\mathbf{SO}(q)] && (23) \\ &= [\Theta_q \wedge^{\mathbf{O}(n)} (\mathbf{O}(n)/\mathbf{SO}(n) \wedge^{\mathbf{Z}/2\mathbf{Z}} \mathbf{O}(q)/\mathbf{SO}(q))] && (24) \\ &= [T_q \wedge^{\mathbf{O}(n)} \mathbf{O}(n)/\mathbf{SO}(n)] && (25) \\ &= HW_1(q). && (26) \end{aligned}$$

Here, (24) is given by the associativity of the contracted product (see [**7**, 1.3.5]) and isomorphism (25) is just (22). □

THEOREM 5.3. *Let (V, q) be a symmetric bundle of rank n on the scheme Y in which 2 is invertible. Assume that Y is the disjoint union of its connected components. Then*

$$HW_2(q) = w_2(q) + w_1(q) \cup \det[q] + [C_q]$$

in $H^2(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z})$.

Proof. For the convenience of the reader we split the proof into several steps.
Step 0: Cech cocycles.

Let $\mathcal{E}$ be a topos, let A be an abelian object in $\mathcal{E}$ and let U be a covering of the final object of $\mathcal{E}$, that is, $U \to *$ is an epimorphism. We consider the covering $\mathcal{U} := \{U \to *\}$. The Cech complex $\check{C}^*(\mathcal{U}, A)$ with value in A with respect to the covering $\mathcal{U}$ is

$$0 \longrightarrow A(U) \xrightarrow{d_0} A(U \times U) \xrightarrow{d_1} A(U \times U \times U) \xrightarrow{d_2} \cdots \xrightarrow{d_n} A(U^{n+2}) \xrightarrow{d_{n+1}} \cdots,$$

where, for $f \in A(U^{n+1}) = \mathrm{Hom}_{\mathcal{E}}(U^{n+1}, A)$, we have

$$d_n(f) = \sum_{1 \leqslant i \leqslant n+2} (-1)^{i-1} f \circ p_{1,2,\ldots,\hat{i},\ldots,n+2} \in A(U^{n+2}) = \mathrm{Hom}_{\mathcal{E}}(U^{n+2}, A).$$

Here, $p_{1,2,\ldots,\hat{i},\ldots,n+2} : U^{n+2} \to U^{n+1}$ is the projection obtained by omitting the ith coordinate. We denote by $\mathcal{Z}^n(\mathcal{U}, A) := \mathrm{Ker}(d_n)$ the group of n-cocycles, and the Cech cohomology with respect to the covering U with coefficients in A is defined as follows:

$$\check{H}^n(\mathcal{U}, A) := \mathrm{Ker}(d_n)/\mathrm{Im}(d_{n-1}).$$

For a refinement $V \to U$, that is, V covers the final object and is given with a map to U, we have an induced map of complexes $\check{C}^*(\mathcal{U}, A) \to \check{C}^*(\mathcal{V}, A)$, hence a map $\check{H}^n(\mathcal{U}, A) \to \check{H}^n(\mathcal{V}, A)$ for any n, which can be shown to be independent of the map $V \to U$, using the fact that the cohomology $\check{H}^*(\mathcal{U}, A) = \check{H}^*(\mathcal{R}_U, A)$ (respectively, $\check{H}^*(\mathcal{V}, A) = \check{H}^*(\mathcal{R}_V, A)$) only depends on the sieve $\mathcal{R}_U$ (respectively, $\mathcal{R}_V$) generated by U (respectively, by V). We then define

$$\check{H}^n(\mathcal{E}, A) := \varinjlim \check{H}^n(\mathcal{U}, A).$$

There are always canonical maps $\check{H}^n(\mathcal{E}, A) \to H^n(\mathcal{E}, A)$ (given by the Cartan–Leray spectral sequence) but this map is not an isomorphism in general. However, the map

$$\check{H}^1(\mathcal{E}, A) \longrightarrow H^1(\mathcal{E}, A)$$

is an isomorphism for any topos $\mathcal{E}$. Fix a ring R in the topos $\mathcal{E}$, and let A and B be R-modules. We have cup-products

$$\check{H}^n(\mathcal{E}, A) \times \check{H}^m(\mathcal{E}, B) \xrightarrow{\cup} \check{H}^{n+m}(\mathcal{E}, A \otimes_R B)$$

induced by the maps

$$\begin{array}{ccc} \check{C}^n(\mathcal{U}, A) \times \check{C}^m(\mathcal{U}, B) & \longrightarrow & \check{C}^{n+m}(\mathcal{U}, A \otimes_R B), \\ (f : U^{n+1} \to A, g : U^{m+1} \to B) & \longmapsto & f \circ p_{1,\ldots,n+1} \otimes g \circ p_{n+1,\ldots,n+m+1}, \end{array}$$

where $p_{1,\ldots,n+1}$ is the projection on the $(1, \ldots, n+1)$-components. We obtain a cup-product

$$H^1(\mathcal{E}, A) \times H^1(\mathcal{E}, B) \xrightarrow{\sim} \check{H}^1(\mathcal{E}, A) \times \check{H}^1(\mathcal{E}, B) \longrightarrow \check{H}^2(\mathcal{E}, A \otimes_R B) \longrightarrow H^2(\mathcal{E}, A \otimes_R B).$$

For $A = B = R$, composing with the multiplication map $R \times R \to R$, we obtain the following lemma.

Lemma 5.4. *For any ringed topos $(\mathcal{E}, R)$ there is a cup-product*

$$H^1(\mathcal{E}, R) \times H^1(\mathcal{E}, R) \longrightarrow H^2(\mathcal{E}, R)$$

compatible with cup-product of Cech cocycles.

For a (not necessarily commutative) group G in $\mathcal{E}$, we denote by $H^1(\mathcal{E}, G)$ the pointed set of isomorphism classes of G-torsors in $\mathcal{E}$. Let $\mathcal{U} = \{U \to *\}$ be a covering. The definitions of $\mathcal{C}^1(\mathcal{U}, G)$, $\mathcal{Z}^1(\mathcal{U}, G)$ and $\check{H}^1(\mathcal{U}, G)$ extend to the non-abelian case. Indeed, we set $\mathcal{C}^1(\mathcal{U}, G) := G(\mathcal{U} \times \mathcal{U})$ and

$$\mathcal{Z}^1(\mathcal{U}, G) := \{s \in G(U \times U),\ (s \circ p_{23})(s \circ p_{13})^{-1}(s \circ p_{12}) = 1\},$$

where $p_{ij} : U \times U \times U \to U \times U$ is the projection on the (i,j)-components. There is a natural action of $\mathcal{C}^0(\mathcal{U}, G) := G(U)$ on $\mathcal{Z}^1(\mathcal{U}, G)$ that is defined as follows: if $\sigma \in G(U)$ and $s \in \mathcal{Z}^1(\mathcal{U}, G)$, then

$$\sigma \star s = (\sigma \circ p_1) \cdot s \cdot (\sigma \circ p_2)^{-1} \in \mathcal{Z}^1(\mathcal{U}, G),$$

where $p_1, p_2 : U \times U \to U$ are the projections. Then one defines

$$\check{H}^1(\mathcal{U}, G) := \mathcal{Z}^1(\mathcal{U}, G)/G(U)$$

to be the quotient of $\mathcal{Z}^1(\mathcal{U}, G)$ by this group action. Note that $\mathcal{Z}^1(\mathcal{U}, G)$ and hence $\check{H}^1(\mathcal{U}, G)$, both have the structure of a pointed set. Then there is an isomorphism

$$\check{H}^1(\mathcal{E}, A) := \varinjlim \check{H}^1(\mathcal{U}, G) \xrightarrow{\sim} H^1(\mathcal{E}, A)$$

of pointed sets.

The 1*-cocycle associated to a torsor.*

Let G be a group in the topos $\mathcal{E}$ and let T be a G-torsor in $\mathcal{E}$. In order to obtain a 1-cocycle that represents T, we proceed as follows. By definition $T \to *$ is a covering of the final object in $\mathcal{E}$ which trivializes T. More precisely, the canonical map

$$\begin{array}{rccc} \mu : & T \times G & \longrightarrow & T \times T, \\ & (t, g) & \longmapsto & (t, t \cdot g) \end{array}$$

is an isomorphism in $\mathcal{E}/T$. Here, the assignment $(t, g) \mapsto (t, t \cdot g)$ makes sense on sections. Indeed, for any object X in $\mathcal{E}$, the set $T(X)$ carries a right action of the group $G(X)$ such that the map

$$\begin{array}{rccc} \mu(X) : & T(X) \times G(X) & \longrightarrow & T(X) \times T(X), \\ & (t, g) & \longmapsto & (t, t \cdot g) \end{array}$$

is a bijection, that is, $G(X)$ acts simply and transitively on $T(X)$. Let $\mu^{-1} : T \times T \to T \times G$ be the inverse map. For any X in $\mathcal{E}$, we have

$$\begin{array}{rccc} \mu^{-1}(X) : & T(X) \times T(X) & \longrightarrow & T(X) \times G(X), \\ & (t, u) & \longmapsto & (t, t^{-1}u), \end{array}$$

where, by the notation $g = t^{-1}u$, we mean the unique element of the group $G(X)$ such that $t \cdot g = u$. If $f : U \to T$ is a morphism of $\mathcal{E}$ such that $U \to *$ is a covering, then we obtain a 1-cocycle representing T, by considering

$$c_T \in \mathcal{Z}^1(\{U \longrightarrow *\}, G) \subset G(U \times U)$$

defined (on sections) by

$$\begin{array}{rccc} c_T : & U \times U & \longrightarrow & G, \\ & (t, u) & \longmapsto & f(t)^{-1} f(u), \end{array}$$

where $\mathcal{Z}^1(\{U \to *\}, G)$ is the pointed set of 1-cocycles with respect to the cover $\{U \to *\}$ with values in G. Recall that

$$\mathcal{Z}^1(\{U \longrightarrow *\}, G) := \{s \in G(U \times U),\ (s \circ p_{23})(s \circ p_{13})^{-1}(s \circ p_{12}) = 1\},$$

where $p_{ij} : U \times U \times U \to U \times U$ is the projection on the (i,j)-components. The cocycle c_T represents the G-torsor T in the sense that

$$\mathcal{Z}^1(\{U \longrightarrow *\}, G) \twoheadrightarrow \check{H}^1(\{U \longrightarrow *\}, G) \longrightarrow \check{H}^1(\mathcal{E}, G) \xrightarrow{\sim} H^1(\mathcal{E}, G)$$

maps c_T to $[T]$, where $[T]$ is the class of the G-torsor T.

The 2*-cocycle associated to a central extension with local sections.*

We consider an exact sequence of groups in $\mathcal{E}$

$$1 \longrightarrow A \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1 \tag{27}$$

such that A is *central* in $\tilde{G}$. We have a boundary map $H^1(\mathcal{E}, G) \to H^2(\mathcal{E}, A)$ defined as follows:

$$\begin{array}{cccc} \delta : & H^1(\mathcal{E}, G) & \longrightarrow & H^2(\mathcal{E}, A), \\ & [T] & \longmapsto & T^*([C]), \end{array}$$

where $[C] \in H^2(B_G, A)$ is the class defined (see Proposition 3.8) by the extension (27). The class $\delta(T)$ is trivial if and only if the G-torsor can be lifted into a $\tilde{G}$-torsor $\tilde{T}$ (that is, if and only if there exists a $\tilde{G}$-torsor $\tilde{T}$ and an isomorphism of G-torsors $\tilde{T} \wedge^{\tilde{G}} G \simeq T$). Let T be a G-torsor, let $\mathcal{U} = \{U \to *\}$ be a covering and let $c_T \in \mathcal{Z}^1(\mathcal{U}, G)$ be a 1-cocycle representing T. Assume that there exists a lifting $\tilde{c}_T : U \times U \to \tilde{G}$ of c_T. Then

$$\check{\delta}(c_T) := \tilde{c}_T \circ p_{23} - \tilde{c}_T \circ p_{13} + \tilde{c}_T \circ p_{12}$$

is a 2-cocycle with values in A, that is, one has $\check{\delta}(c_T) \in \mathcal{Z}^2(\mathcal{U}, A)$. The class of $\check{\delta}(c_T)$ in $H^2(\mathcal{U}, A)$ does not depend on the choice of $\tilde{c}_T$. Moreover, the image of $\check{\delta}(c_T)$ in $H^2(\mathcal{E}, A)$ is $\delta([T]) = T^*([C])$ (see Giraud IV.3.5.4).

Step 1: First reduction.

The aim of this step is to prove the following result. Denote by

$$B_{r_q} : B_{\tilde{\mathbf{O}}(q)} \longrightarrow B_{\mathbf{O}(q)}$$

the map induced by the morphism $r_q : \tilde{\mathbf{O}}(q) \to \mathbf{O}(q)$ (see Step 2 for a precise definition of r_q).

PROPOSITION 5.5. *The identity*

$$HW_2(q) = w_2(q) + w_1(q) \cup \det[q] + [C_q] \tag{28}$$

in $H^2(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z})$ is equivalent to the identity

$$B^*_{r_q}(HW_2(q)) = B^*_{r_q}(w_2(q) + w_1(q) \cup \det[q] + [C_q]) \tag{29}$$

in $H^2(B_{\tilde{\mathbf{O}}(q)}, \mathbf{Z}/2\mathbf{Z})$.

Proof. By functoriality (28) implies (29). Let us show the converse. Assume that (29) holds, so that (see Lemma 5.6)

$$B^*_{r_q}(HW_2(q) + w_2(q) + w_1(q) \cup \det[q]) = 0.$$

Recall that $Y = \coprod_\alpha Y_\alpha$ is the disjoint union of its connected components. Since cohomology sends disjoint sums to direct products, we may assume Y to be connected. By Lemma 5.7, we have either

$$HW_2(q) + w_2(q) + w_1(q) \cup \det[q] = 0$$

or

$$HW_2(q) + w_2(q) + w_1(q) \cup \det[q] = [C_q].$$

Assume that

$$HW_2(q) + w_2(q) + w_1(q) \cup \det[q] = 0.$$

By Theorem 5.2, we would obtain

$$HW_2(q) + w_1(q) \cdot HW_1(q) + (w_1(q) \cdot w_1(q) + w_2(q)) = 0$$

in the polynomial ring $A[HW_1(q), \dots, HW_n(q)]$. This is a contradiction since $w_1(q), w_2(q) \in A$. Identity (28) follows. □

LEMMA 5.6. *Consider an extension*

$$1 \longrightarrow A \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

of a group G by an abelian group A in some topos $\mathcal{E}$. Let $C \in Ext(G,A)$ be the class of this extension and let $[C]$ be its cohomology class in $H^2(B_G, A)$. Then C vanishes in $Ext(\tilde{G}, A)$. A fortiori, $[C]$ vanishes in $H^2(B_{\tilde{G}}, A)$.

Proof. The natural map

$$\mathrm{Ext}(G,A) \longrightarrow \mathrm{Ext}(\tilde{G},A)$$

sends the class C of the extension $1 \to A \to \tilde{G} \to G \to 1$ to the class $\tilde{C}$ of

$$1 \longrightarrow A \longrightarrow \tilde{G} \times_G \tilde{G} \longrightarrow \tilde{G} \longrightarrow 1,$$

which is split by the diagonal $\tilde{G} \to \tilde{G} \times_G \tilde{G}$, so that $\tilde{C} = 0$.

Giraud's exact sequence (Proposition 3.8) is functorial in G, so that one has a commutative square:

$$\begin{array}{ccc} \mathrm{Ext}(G,A) & \longrightarrow & H^2(B_G, A) \\ \downarrow & & \downarrow \\ \mathrm{Ext}(\tilde{G},A) & \longrightarrow & H^2(B_{\tilde{G}}, A) \end{array}$$

hence $[C] \in H^2(B_G, A)$ maps to $[\tilde{C}] = 0 \in H^2(B_{\tilde{G}}, A)$. □

Recall that we denote by C_q the class of the canonical extension

$$1 \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow \tilde{\mathbf{O}}(q) \longrightarrow \mathbf{O}(q) \longrightarrow 1.$$

LEMMA 5.7. *The Hochschild–Serre spectral sequence associated to the above extension of group schemes induces an exact sequence*

$$0 \longrightarrow H^0(Y, \mathbf{Z}/2\mathbf{Z}) \longrightarrow H^2(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z}) \longrightarrow H^2(B_{\tilde{\mathbf{O}}(q)}, \mathbf{Z}/2\mathbf{Z}).$$

Moreover, if Y is connected, then $\mathbf{Z}/2\mathbf{Z} \to H^2(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z})$ maps 1 to $[C_q]$.

Proof. The scheme Y is the disjoint union of its connected components. Since cohomology sends disjoint unions to direct products, we may suppose Y to be connected. By Corollary 3.13, we have an exact sequence

$$\begin{aligned} 0 &\longrightarrow H^1(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z}) \longrightarrow H^1(B_{\tilde{\mathbf{O}}(q)}, \mathbf{Z}/2\mathbf{Z}) \longrightarrow H^0(B_{\mathbf{O}(q)}, \underline{\mathrm{Hom}}(\mathbf{Z}/2\mathbf{Z}, \mathbf{Z}/2\mathbf{Z})) \\ &\longrightarrow H^2(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z}) \longrightarrow H^2(B_{\tilde{\mathbf{O}}(q)}, \mathbf{Z}/2\mathbf{Z}). \end{aligned}$$

But $\underline{\mathrm{Hom}}(\mathbf{Z}/2\mathbf{Z}, \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$ and $H^0(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$, hence we obtain an exact sequence

$$\mathbf{Z}/2\mathbf{Z} \longrightarrow H^2(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z}) \xrightarrow{r_q^*} H^2(B_{\tilde{\mathbf{O}}(q)}, \mathbf{Z}/2\mathbf{Z}).$$

By Lemma 5.6, the class $[C_q]$ lies in $\mathrm{Ker}(r_q^*)$. It remains to show that $[C_q] \neq 0$ in $H^2(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z})$. Let $U \to Y$ be an étale map (or any map) such that q_U is isomorphic to the standard form $t_{n,U}$ on U. Then there is an equivalence (using Proposition 2.8)

$$B_{\mathbf{O}(q_U)} \simeq B_{\mathbf{O}(q)}/U \simeq B_{\mathbf{O}(n)}/U \simeq B_{\mathbf{O}(t_{n,U})}$$

such that $[C_{q_U}]$ maps to $[C_{t_{n,U}}] = HW_2 \in H^2(B_{\mathbf{O}(t_{n,U})}, \mathbf{Z}/2\mathbf{Z})$, which is non-zero by Theorem 4.9. Hence, $[C_q]$ maps to $[C_{q_U}] \neq 0$, hence $[C_q] \neq 0$. □

Step 2: The maps θ and $\tilde{\theta}$.

For a brief summary of the Clifford algebras and Clifford groups associated to a symmetric bundle, see [**4**, Section 1.9]. Let (V, q) be a symmetric bundle on Y. We denote by $C(q)$ its Clifford algebra and by $C^*(q)$ its Clifford group. We then consider the sheaf of algebras and the sheaf of groups for the flat topology, defined by the functors $C(q) : T \to C(q_T)$ and $C^*(q) : T \to C^*(q_T)$, respectively. The norm map $N : C(q) \to C(q)$ restricts to a morphism of groups

$$N : C^*(q) \longrightarrow \mathbf{G}_m,$$

where $\mathbf{G}_m$ denotes the multiplicative group. We let $\tilde{\mathbf{O}}(q)$ denote the kernel of this homomorphism. This is a sheaf of groups for the flat topology that is representable by a smooth group scheme over Y. The group scheme $\tilde{\mathbf{O}}(q)$ splits as $\tilde{\mathbf{O}}_+(q) \coprod \tilde{\mathbf{O}}_-(q)$. Let x be in $\tilde{\mathbf{O}}_\varepsilon(q)$ with $\varepsilon = \pm 1$, then we define $r_q(x)$ as the element of $\mathbf{O}(q)$

$$\begin{aligned} r_q(x) : V &\longrightarrow V, \\ v &\longmapsto \varepsilon x v x^{-1}. \end{aligned}$$

This defines a group homomorphism $r_q : \tilde{\mathbf{O}}(q) \to \mathbf{O}(q)$. One can show that, for each $x \in \tilde{\mathbf{O}}_\varepsilon(q)$, the element $r_q(x)$ belongs to $\mathbf{O}_+(q) = \mathbf{SO}(q)$ or $\mathbf{O}_-(q) = \mathbf{O}(q) \setminus \mathbf{SO}(q)$ depending on whether $\varepsilon = 1$ or -1, respectively.

We start by considering the affine case $Y = \operatorname{Spec}(R)$, where R is a commutative ring in which 2 is invertible. In this situation, $\mathbf{O}(q)$ and $\tilde{\mathbf{O}}(q)$ can be, respectively, considered as the orthogonal group and the Pinor group of the form (V, q), where V is a finitely generated projective R-module and q is a non-degenerate form on V.

LEMMA 5.8. *Let (V, q) and (W, f) be symmetric bundles over R and let $t \in \mathbf{Isom}(q, f)(R)$.*

(i) *The map t extends in a unique way to a graded isomorphism of Clifford algebras*

$$\tilde{\psi}_t : C(q) \longrightarrow C(f).$$

(ii) *The isomorphism $\tilde{\psi}_t$ induces, by restriction, an isomorphism of groups, again denoted by $\tilde{\psi}_t$, such that the following diagram is commutative*:

$$\begin{array}{ccccccc} \{1\} & \longrightarrow & (\mathbf{Z}/2\mathbf{Z})(R) & \longrightarrow & \tilde{\mathbf{O}}(q)(R) & \xrightarrow{r_q} & \mathbf{O}(q)(R) \\ & & \Big\downarrow{\scriptstyle Id} & & \Big\downarrow{\scriptstyle \tilde{\psi}_t} & & \Big\downarrow{\scriptstyle \psi_t} \\ \{1\} & \longrightarrow & (\mathbf{Z}/2\mathbf{Z})(R) & \longrightarrow & \tilde{\mathbf{O}}(f)(R) & \xrightarrow{r_f} & \mathbf{O}(f)(R) \end{array}$$

where ψ_t is the group isomorphism $u \to tut^{-1}$.

(iii) *For any $a \in \mathbf{O}(f)(R)$ and $t \in \mathbf{Isom}(q, f)(R)$, we have the equalities*

$$\tilde{\psi}_{at} = \tilde{\psi}_a \circ \tilde{\psi}_t \quad \text{and} \quad \psi_{at} = a \psi_t a^{-1}.$$

(iv) *Suppose that $(V, q) = (W, f)$. We assume that t has a lift $s(t)$ in $\tilde{\mathbf{O}}(q)(R)$. Then $\tilde{\psi}_t = i_{s(t)}$ (respectively, $\varepsilon i_{s(t)}$ on $\tilde{\mathbf{O}}_\varepsilon(q)(R)$) if $t \in \mathbf{O}_+(q)(R)$ (respectively, $\mathbf{O}_-(q)(R)$).*

Proof. It follows from the very definition of the Clifford algebra that t extends to a graded isomorphism $\tilde{\psi}_t : C(q) \to C(f)$, which itself induces by restriction an isomorphism $\tilde{\psi}_t : \tilde{\mathbf{O}}(q) \to \tilde{\mathbf{O}}(f)$. For $a \in \tilde{\mathbf{O}}_\varepsilon(q)$, one has by definition

$$r_f \circ \tilde{\psi}_t(a)(x) = \varepsilon \tilde{\psi}_t(a) x \tilde{\psi}_t(a)^{-1} \quad \forall x \in W.$$

Since $\tilde{\psi}_t$ is induced by a morphism of R-algebras, the right-hand side of this equality can be written as

$$\varepsilon\tilde{\psi}_t(a\tilde{\psi}_t^{-1}(x)a^{-1}).$$

Since $\tilde{\psi}_t$ coincides with t on V and since $at^{-1}(x)a^{-1} \in V$ (recall that $a \in \tilde{\mathbf{O}}(q)$) we conclude that

$$r_f \circ \tilde{\psi}_t(a)(x) = \psi_t \circ r_q(a)(x) \quad \forall x \in W.$$

Therefore, (i) and (ii) are proved. The second equality of (iii) is immediate. In order to prove the first equality, it suffices to observe that both sides of this equality coincide when restricted to V.

If we now assume that t has a lift in $\tilde{\mathbf{O}}(q)(R)$, then we obtain two automorphisms of $C(q)$, namely $\tilde{\psi}_t$ and $\imath_{s(t)}$. Moreover, since $s(t) \in \tilde{\mathbf{O}}_\varepsilon(q)$, it follows from the definition of r_q that $t(x) = \varepsilon s(t)xs(t)^{-1}$ for any x in V. If $t \in \mathbf{O}_+(q)$, then $\varepsilon = 1$ and $\tilde{\psi}_t$ and $i_{s(t)}$ coincide on V, and therefore coincide on $C(q)$. If $t \in \mathbf{O}_-(q)$, then $\varepsilon = -1$ and therefore $\tilde{\psi}_t$ and $i_{s(t)}$ will coincide on $C_+(q)$ and will differ by a minus sign on $C_-(q)$, and the result follows. □

We now return to the forms q and t_n on Y and the sheaf $\mathbf{Isom}(t_n, q)$ of Y_{fl}. Let us define morphisms in Y_{fl}:

$$\underline{\theta} : \mathbf{Isom}(t_n, q) \times \mathbf{O}(q) \longrightarrow \mathbf{Isom}(t_n, q) \times \mathbf{O}(n),$$
$$\underline{\tilde{\theta}} : \mathbf{Isom}(t_n, q) \times \tilde{\mathbf{O}}(q) \longrightarrow \mathbf{Isom}(t_n, q) \times \tilde{\mathbf{O}}(n)$$

as follows. The morphisms $\underline{\theta}$ and $\underline{\tilde{\theta}}$ can be defined on sections. Furthermore, the class of affine schemes yields a generating subcategory of Y_{fl}, hence one can define the morphisms $\underline{\theta}$ and $\underline{\tilde{\theta}}$ on sections over affine schemes of the form $\mathrm{Spec}(R) \to Y$. For any $\mathrm{Spec}(R) \to Y$ and $t \in \mathbf{Isom}(t_n, q)(R)$, we set $\underline{\theta}_t = \psi_{t^{-1}}$ and $\underline{\tilde{\theta}}_t = \tilde{\psi}_{t^{-1}}$. Then the maps

$$\begin{array}{rccc} \underline{\theta}(R): & \mathbf{Isom}(t_n, q)(R) \times \mathbf{O}(q)(R) & \longrightarrow & \mathbf{Isom}(t_n, q)(R) \times \mathbf{O}(n)(R), \\ & (t, x) & \longmapsto & (t, \underline{\theta}_t(x)) \end{array}$$

and

$$\begin{array}{rccc} \underline{\tilde{\theta}}(R): & \mathbf{Isom}(t_n, q)(R) \times \tilde{\mathbf{O}}(q)(R) & \longrightarrow & \mathbf{Isom}(t_n, q)(R) \times \tilde{\mathbf{O}}(n)(R), \\ & (t, x) & \longmapsto & (t, \underline{\tilde{\theta}}_t(x)) \end{array}$$

are both functorial in $\mathrm{Spec}(R) \to Y$, and yield the morphisms $\underline{\theta}$ and $\underline{\tilde{\theta}}$. We denote by

$$\tilde{\pi} : B_{\tilde{\mathbf{O}}(q)} \xrightarrow{B_{r_q}} B_{\mathbf{O}(q)} \xrightarrow{\pi} Y_{fl}$$

the canonical map. Recall that

$$\Theta_q := \pi^* \mathbf{Isom}(t_n, q).$$

Similarly, we denote by $\tilde{\Theta}_q$ the object of $B_{\tilde{\mathbf{O}}(q)}$ given by

$$\tilde{\Theta}_q := B_{r_q}^* \Theta_q = \tilde{\pi}^* \mathbf{Isom}(t_n, q),$$

which is the sheaf $\mathbf{Isom}(t_n, q)$ with trivial $\tilde{\mathbf{O}}(q)$-action. Pulling back $\underline{\theta}$ and $\underline{\tilde{\theta}}$ along the morphism $\tilde{\pi} : B_{\tilde{\mathbf{O}}(q)} \to Y_{fl}$, we obtain morphisms in $B_{\tilde{\mathbf{O}}(q)}$:

$$\tilde{\pi}^*\underline{\theta} : \tilde{\Theta}_q \times \tilde{\pi}^*\mathbf{O}(q) \longrightarrow \tilde{\Theta}_q \times \tilde{\pi}^*\mathbf{O}(n) \quad \text{and} \quad \tilde{\pi}^*\underline{\tilde{\theta}} : \tilde{\Theta}_q \times \tilde{\pi}^*\tilde{\mathbf{O}}(q) \longrightarrow \tilde{\Theta}_q \times \tilde{\pi}^*\tilde{\mathbf{O}}(n),$$

where the above maps are just $\underline{\theta}$ and $\underline{\tilde{\theta}}$ seen as equivariant maps between objects of Y_{fl} with trivial $\tilde{\mathbf{O}}(q)$-action. Moreover, $\tilde{\pi}^*\underline{\theta}$ and $\tilde{\pi}^*\underline{\tilde{\theta}}$ are defined over $\tilde{\Theta}_q$, that is, $\tilde{\pi}^*\underline{\theta}$ and $\tilde{\pi}^*\underline{\tilde{\theta}}$ commute with the projection to $\tilde{\Theta}_q$. In other words, $\tilde{\pi}^*\underline{\theta}$ and $\tilde{\pi}^*\underline{\tilde{\theta}}$ are maps in $B_{\tilde{\mathbf{O}}(q)}/\tilde{\Theta}_q$, which we simply denote by $\theta := \tilde{\pi}^*\underline{\theta}$ and $\tilde{\theta} := \tilde{\pi}^*\underline{\tilde{\theta}}$. We summarize what we have constructed so far.

Lemma 5.9. *The following diagram is a commutative diagram in $B_{\tilde{\mathbf{O}}(q)}/\tilde{\Theta}_q$.*

$$\begin{array}{ccccccccc} \tilde{\Theta}_q & \longrightarrow & \tilde{\Theta}_q \times \mathbf{Z}/2\mathbf{Z} & \longrightarrow & \tilde{\Theta}_q \times \tilde{\pi}^*\tilde{\mathbf{O}}(q) & \xrightarrow{r_q} & \tilde{\Theta}_q \times \tilde{\pi}^*\mathbf{O}(q) & \longrightarrow & \tilde{\Theta}_q \\ & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \tilde{\theta}} & & \downarrow{\scriptstyle \theta} & & \\ \tilde{\Theta}_q & \longrightarrow & \tilde{\Theta}_q \times \mathbf{Z}/2\mathbf{Z} & \longrightarrow & \tilde{\Theta}_q \times \tilde{\pi}^*\tilde{\mathbf{O}}(n) & \xrightarrow{r_n} & \tilde{\Theta}_q \times \tilde{\pi}^*\mathbf{O}(n) & \longrightarrow & \tilde{\Theta}_q \end{array}$$

Moreover, the horizontal rows are exact sequences of sheaves of groups on $B_{\tilde{\mathbf{O}}(q)}/\tilde{\Theta}_q$ and the vertical arrows are isomorphisms. Note that $\tilde{\Theta}_q$ is the trivial group in $B_{\tilde{\mathbf{O}}(q)}/\tilde{\Theta}_q$.

We now use the morphisms θ and $\tilde{\theta}$ to associate to any object $p : W \to \tilde{\Theta}_q$ of $B_{\tilde{\mathbf{O}}(q)}/\tilde{\Theta}_q$ isomorphisms of groups

$$\theta_p : \tilde{\pi}^*\mathbf{O}(q)(W) \xrightarrow{\sim} \tilde{\pi}^*\mathbf{O}(n)(W) \quad \text{and} \quad \tilde{\theta}_p : \tilde{\pi}^*\tilde{\mathbf{O}}(q)(W) \xrightarrow{\sim} \tilde{\pi}^*\tilde{\mathbf{O}}(n)(W),$$

where W is the object of $B_{\tilde{\mathbf{O}}(q)}$ underlying p. For example, we have $\tilde{\pi}^*\mathbf{O}(n)(W) := \mathrm{Hom}_{B_{\tilde{\mathbf{O}}(q)}}(W, \tilde{\pi}^*\mathbf{O}(n))$. These isomorphisms can be described as follows:

$$\theta_p : (f : W \to \tilde{\pi}^*\mathbf{O}(q)) \longrightarrow (\theta_p(f) : W \xrightarrow{p\times f} \tilde{\Theta}_q \times \tilde{\pi}^*\mathbf{O}(q) \xrightarrow{\theta} \tilde{\Theta}_q \times \tilde{\pi}^*\mathbf{O}(n) \xrightarrow{\mathrm{pr}} \tilde{\pi}^*\mathbf{O}(n)),$$

$$\tilde{\theta}_p : (f : W \to \tilde{\pi}^*\mathbf{O}(q)) \longrightarrow (\tilde{\theta}_p(f) : W \xrightarrow{p\times f} \tilde{\Theta}_q \times \tilde{\pi}^*\tilde{\mathbf{O}}(q) \xrightarrow{\tilde{\theta}} \tilde{\Theta}_q \times \tilde{\pi}^*\tilde{\mathbf{O}}(n) \xrightarrow{\mathrm{pr}} \tilde{\pi}^*\tilde{\mathbf{O}}(n)),$$

where pr denotes the projection on the second component. We then have a commutative diagram of groups

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{Z}/2\mathbf{Z}(W) & \longrightarrow & \tilde{\pi}^*\tilde{\mathbf{O}}(q)(W) & \xrightarrow{r_{q,W}} & \tilde{\pi}^*\mathbf{O}(q)(W) \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \tilde{\theta}_p} & & \downarrow{\scriptstyle \theta_p} \\ 1 & \longrightarrow & \mathbf{Z}/2\mathbf{Z}(W) & \longrightarrow & \tilde{\pi}^*\tilde{\mathbf{O}}(n)(W) & \xrightarrow{r_{n,W}} & \tilde{\pi}^*\mathbf{O}(n)(W) \end{array}$$

We (also) denote by $\eta : Y_{fl} \to B_{\tilde{\mathbf{O}}(q)}$ the morphism of topoi induced by the morphism of groups $1 \to \tilde{\mathbf{O}}(q)$. The functor η^* forgets the $\tilde{\mathbf{O}}(q)$-action. It has a left adjoint $\eta_!$, which is defined as follows: $\eta_!(Z)$ is the object of $B_{\tilde{\mathbf{O}}(q)}$ given by $E_{\tilde{\mathbf{O}}(q)} \times Z$ on which $\tilde{\mathbf{O}}(q)$ acts via the first factor. Furthermore, we may consider the maps $\eta^*\theta$ and $\eta^*\tilde{\theta}$ as maps in the topos $Y_{fl}/\mathbf{Isom}(t_n, q)$. For a map $t : Z \to \mathbf{Isom}(t_n, q)$ in Y_{fl} we define

$$(\eta^*\theta)_t : \mathbf{O}(q)(Z) \longrightarrow \mathbf{O}(n)(Z) \quad \text{and} \quad (\eta^*\tilde{\theta})_t : \tilde{\mathbf{O}}(q)(Z) \longrightarrow \tilde{\mathbf{O}}(n)(Z)$$

in a similar manner. In particular, for $t : Z = \mathrm{Spec}(R) \to \mathbf{Isom}(t_n, q)$, one has

$$(\eta^*\theta)_t = \psi_{t^{-1}} \quad \text{and} \quad (\eta^*\tilde{\theta})_t = \tilde{\psi}_{t^{-1}} \tag{30}$$

as defined in Lemma 5.8.

Lemma 5.10. *Let $Z = Spec(R)$ be an object of $B_{\tilde{\mathbf{O}}(q)}$ with trivial $\tilde{\mathbf{O}}(q)$-action and let $Z \to \tilde{\Theta}_q$ be a map in $B_{\tilde{\mathbf{O}}(q)}$. By adjunction one has canonical identifications*

$$(-)_{|R} : \tilde{\pi}^*\tilde{\mathbf{O}}(n)(E_{\tilde{\mathbf{O}}(q)} \times Z) \xrightarrow{\sim} \tilde{\mathbf{O}}(n)(\eta^*Z) = \tilde{\mathbf{O}}(n)(R), \tag{31}$$

$$(-)_{|R} : \tilde{\pi}^*\tilde{\mathbf{O}}(q)(E_{\tilde{\mathbf{O}}(q)} \times Z) \xrightarrow{\sim} \tilde{\mathbf{O}}(q)(\eta^*Z) = \tilde{\mathbf{O}}(q)(R), \tag{32}$$

$$(-)_{|R} : \tilde{\Theta}_q(E_{\tilde{\mathbf{O}}(q)} \times Z) \xrightarrow{\sim} \mathbf{Isom}(t_n, q)(\eta^*Z) = \mathbf{Isom}(t_n, q)(R) \tag{33}$$

such that

$$\theta_p(\sigma)_{|R} = (\eta^*\theta)_{p_{|R}}(\sigma_{|R}) := \psi_{p_{|R}^{-1}}(\sigma_{|R}) \quad \text{and} \quad \tilde{\theta}_p(\sigma)_{|R} = (\eta^*\tilde{\theta})_{p_{|R}}(\sigma_{|R}) := \tilde{\psi}_{p_{|R}^{-1}}(\sigma_{|R}) \tag{34}$$

for any

$$(p, \sigma) \in \tilde{\Theta}_q(E_{\tilde{\mathbf{O}}(q)} \times Z) \times \tilde{\pi}^*\tilde{\mathbf{O}}(q)(E_{\tilde{\mathbf{O}}(q)} \times Z).$$

Proof. The map (31) is defined as follows. Given

$$\sigma : E_{\tilde{\mathbf{O}}(q)} \times Z \longrightarrow \tilde{\pi}^*\tilde{\mathbf{O}}(q) \in \tilde{\pi}^*\tilde{\mathbf{O}}(q)(E_{\tilde{\mathbf{O}}(q)} \times Z),$$

we define

$$\sigma_{|R} : \eta^* Z \xrightarrow{e} \eta^* E_{\tilde{\mathbf{O}}(q)} \times \eta^* Z \xrightarrow{\eta^*(\sigma)} \eta^*\tilde{\pi}^*\tilde{\mathbf{O}}(q) = \tilde{\mathbf{O}}(q),$$

where $e : \eta^* Z \longrightarrow \eta^* E_{\tilde{\mathbf{O}}(q)} \times \eta^* Z$ is the map given by the unit section of $\tilde{\mathbf{O}}(q) = \eta^* E_{\tilde{\mathbf{O}}(q)}$. This morphism $\sigma \mapsto \sigma_{|R}$ is an inverse of the adjunction map:

$$\begin{aligned}\tilde{\mathbf{O}}(n)(\eta^* Z) &= \mathrm{Hom}_{Y_{fl}}(\eta^* Z, \tilde{\mathbf{O}}(n)) = \mathrm{Hom}_{Y_{fl}}(\eta^* Z, \eta^*\tilde{\pi}^*\tilde{\mathbf{O}}(n)) \\ &= \mathrm{Hom}_{B_{\tilde{\mathbf{O}}(q)}}(\eta_!\eta^* Z, \tilde{\pi}^*\tilde{\mathbf{O}}(n)) \\ &= \mathrm{Hom}_{B_{\tilde{\mathbf{O}}(q)}}(E_{\tilde{\mathbf{O}}(q)} \times Z, \tilde{\pi}^*\tilde{\mathbf{O}}(n)) = \tilde{\pi}^*\tilde{\mathbf{O}}(n)(E_{\tilde{\mathbf{O}}(q)} \times Z).\end{aligned}$$

Replacing successively $\tilde{\mathbf{O}}(n)$ with $\tilde{\mathbf{O}}(q)$ and $\mathbf{Isom}(t_n, q)$, we obtain, respectively, (32) and (33). Then (34) follows immediately from the definitions: for some $(p, \sigma) \in \tilde{\Theta}_q(E_{\tilde{\mathbf{O}}(q)} \times Z) \times \tilde{\pi}^*\tilde{\mathbf{O}}(q)(E_{\tilde{\mathbf{O}}(q)} \times Z)$ we have

$$\tilde{\theta}_p(\sigma) : E_{\tilde{\mathbf{O}}(q)} \times Z \longrightarrow \tilde{\Theta}_q \times \tilde{\pi}^*\tilde{\mathbf{O}}(q) \xrightarrow{\tilde{\theta}} \tilde{\Theta}_q \times \tilde{\pi}^*\tilde{\mathbf{O}}(n) \longrightarrow \tilde{\pi}^*\tilde{\mathbf{O}}(n)$$

and

$$\tilde{\theta}_p(\sigma)_{|R} : \eta^* Z \longrightarrow \eta^* E_{\tilde{\mathbf{O}}(q)} \times \eta^* Z \longrightarrow \eta^*\tilde{\Theta}_q \times \eta^*\tilde{\pi}^*\tilde{\mathbf{O}}(q) \xrightarrow{\eta^*\tilde{\theta}} \eta^*\tilde{\Theta}_q \times \eta^*\tilde{\pi}^*\tilde{\mathbf{O}}(n) \longrightarrow \eta^*\tilde{\pi}^*\tilde{\mathbf{O}}(n),$$

which is just

$$(\eta^*\tilde{\theta})_{p_{|R}}(\sigma_{|R}) : \mathrm{Spec}(R) \xrightarrow{(p_{|R}, \sigma_{|R})} \mathbf{Isom}(t_n, q) \times \tilde{\mathbf{O}}(q) \xrightarrow{\eta^*\tilde{\theta}} \mathbf{Isom}(t_n, q) \times \tilde{\mathbf{O}}(n) \longrightarrow \tilde{\mathbf{O}}(n).$$

□

We shall also need the following result.

LEMMA 5.11. *The class of objects of the form* $E_{\tilde{\mathbf{O}}(q)} \times \mathrm{Spec}(R)$, *where* $\mathrm{Spec}(R)$ *is an affine scheme endowed with its trivial* $\tilde{\mathbf{O}}(q)$*-action, is a generating family of the topos* $B_{\tilde{\mathbf{O}}(q)}$.

To be more precise, here we denote by $\mathrm{Spec}(R)$ an object of $B_{\tilde{\mathbf{O}}(q)}$ given by a sheaf on Y_{fl}, represented by a Y-scheme of the form $\mathrm{Spec}(R) \to Y$ (the map itself is not necessarily affine) endowed with its trivial $\tilde{\mathbf{O}}(q)$-action.

Proof. We need to show that, for any object $\mathcal{F}$ in $B_{\tilde{\mathbf{O}}(q)}$, there exists an epimorphic family

$$\{E_{\tilde{\mathbf{O}}(q)} \times \mathrm{Spec}(R_i) \longrightarrow \mathcal{F},\ i \in I\}$$

of morphisms in $B_{\tilde{\mathbf{O}}(q)}$. Recall that we denote by $\eta : Y_{fl} \to B_{\tilde{\mathbf{O}}(q)}$ the morphism of topoi induced by the morphism of groups $1 \to \tilde{\mathbf{O}}(q)$. The functor η^* forgets the $\tilde{\mathbf{O}}(q)$-action. It has a left adjoint $\eta_!$, which is defined as follows: $\eta_!(Z)$ is the object of $B_{\tilde{\mathbf{O}}(q)}$ given by $E_{\tilde{\mathbf{O}}(q)} \times Z$ on which $\tilde{\mathbf{O}}(q)$ acts via the first factor.

Using the Yoneda lemma, one sees that the class of Y-schemes $Y' \to Y$ forms a generating family of the topos Y_{fl}. Since any Y-scheme Y' can be covered by open affine subschemes, the class of objects of the form $\mathrm{Spec}(R) \to Y$ is a generating family of Y_{fl}. Now let $\mathcal{F}$ be an object of $B_{\tilde{\mathbf{O}}(q)}$, and let

$$\{g_i : \mathrm{Spec}(R_i) \longrightarrow \eta^*\mathcal{F},\ i \in I\}$$

be an epimorphic family in Y_{fl}. By adjunction we obtain a family

$$\{f_i : \eta_!(\mathrm{Spec}(R_i)) \longrightarrow \mathcal{F},\ i \in I\}$$

of morphisms in $B_{\tilde{\mathbf{O}}(q)}$ such that

$$g_i = \eta^*(f_i) \circ \tau_i : \mathrm{Spec}(R_i) \longrightarrow \eta^*\eta_!(\mathrm{Spec}(R_i)) \longrightarrow \eta^*\mathcal{F},$$

where $\tau_i : \mathrm{Spec}(R_i) \to \eta^*\eta_!(\mathrm{Spec}(R_i))$ is the adjunction map.

Let $u, v : \mathcal{F} \rightrightarrows \mathcal{G}$ be a pair of maps in $B_{\tilde{\mathbf{O}}(q)}$ such that $u \circ f_i = v \circ f_i$ for each $i \in I$. In particular, we have

$$\eta^*(u) \circ \eta^*(f_i) \circ \tau_i = \eta^*(v) \circ \eta^*(f_i) \circ \tau_i$$

for each $i \in I$, hence $\eta^*(u) \circ g_i = \eta^*(v) \circ g_i$ for each $i \in I$. It follows that $\eta^*(u) = \eta^*(v)$ since the family $\{g_i, i \in I\}$ is epimorphic. We obtain $u = v$ since η^* is faithful. Hence, the family $\{f_i, i \in I\}$ is epimorphic as well, and the result follows. □

Step 3: The cocycles α, β and γ.

Recall that we denote by

$$\tilde{\pi} : B_{\tilde{\mathbf{O}}(q)} \xrightarrow{B_{r_q}} B_{\mathbf{O}(q)} \xrightarrow{\pi} Y_{fl}$$

the canonical map. We set

$$\tilde{\Theta}_q := B_{r_q}^*\Theta_q, \quad \tilde{T}_q := B_{r_q}^* T_q, \quad \tilde{E}_{\mathbf{O}(q)} := B_{r_q}^* E_{\mathbf{O}(q)} = \tilde{\mathbf{O}}(q)/(\mathbf{Z}/2\mathbf{Z}).$$

We now look for a covering of $B_{\tilde{\mathbf{O}}(q)}$ that splits both the $\tilde{\pi}^*\mathbf{O}(n)$-torsors $\tilde{T}_q$ and $\tilde{\Theta}_q$ and also the $\tilde{\pi}^*\mathbf{O}(q)$-torsor $\tilde{E}_{\mathbf{O}(q)}$. The map

$$\begin{array}{ccc} \tilde{E}_{\mathbf{O}(q)} \times \tilde{\Theta}_q & \longrightarrow & \tilde{E}_{\mathbf{O}(q)} \times \tilde{T}_q, \\ (x, t) & \longmapsto & (x, xt) \end{array}$$

is an isomorphism of $B_{\tilde{\mathbf{O}}(q)}$. It follows that $\{U = \tilde{E}_{\mathbf{O}(q)} \times \tilde{\Theta}_q \to *\}$ is a covering of the final object in $B_{\tilde{\mathbf{O}}(q)}$ trivializing $\tilde{T}_q$, $\tilde{\Theta}_q$ and $\tilde{E}_{\mathbf{O}(q)}$. We now can use the construction recalled in Step 1 to obtain 1-cocycles of U that represent each of these torsors. We then consider the map $f : U \to \tilde{T}_q$, defined by

$$f : U = \tilde{E}_{\mathbf{O}(q)} \times \tilde{\Theta}_q \longrightarrow \tilde{E}_{\mathbf{O}(q)} \times \tilde{T}_q \longrightarrow \tilde{T}_q, \quad (x, t) \longrightarrow (x, xt) \longrightarrow xt,$$

in order to obtain the 1-cocycle $\gamma \in \mathcal{Z}^1(\{U \to *\}, \tilde{\pi}^*\mathbf{O}(n))$ which represents $\tilde{T}_q$:

$$\begin{array}{cccc} \gamma : & U \times U & \longrightarrow & \tilde{\pi}^*\mathbf{O}(n), \\ & (x, t, y, u) & \longmapsto & (xt)^{-1}(yu). \end{array}$$

We apply again this construction where the role of f is now played successively by the projections $U = \tilde{E}_{\mathbf{O}(q)} \times \tilde{\Theta}_q \to \tilde{\Theta}_q$ and $U = \tilde{E}_{\mathbf{O}(q)} \times \tilde{\Theta}_q \to \tilde{E}_{\mathbf{O}(q)}$. We obtain firstly

$$\begin{array}{cccc} \beta : & U \times U & \longrightarrow & \tilde{\pi}^*\mathbf{O}(n), \\ & (x, t, y, u) & \longmapsto & t^{-1}u \end{array}$$

for a representative of $\tilde{\Theta}_q$ and secondly

$$\begin{array}{cccc} \alpha : & U \times U & \longrightarrow & \tilde{\pi}^*\mathbf{O}(q), \\ & (x, t, y, u) & \longmapsto & x^{-1}y \end{array}$$

for a representative of $\tilde{E}_{\mathbf{O}(q)}$. Of course, we have

$$\alpha \in \mathcal{Z}^1(\{U \longrightarrow *\}, \tilde{\pi}^*\mathbf{O}(q)) \quad \text{and} \quad \beta, \gamma \in \mathcal{Z}^1(\{U \longrightarrow *\}, \tilde{\pi}^*\mathbf{O}(n)).$$

Finally, applying the construction described in Step 2, we consider the group isomorphism

$$\theta_p : \tilde{\pi}^*\mathbf{O}(q)(U \times U) \xrightarrow{\sim} \tilde{\pi}^*\mathbf{O}(n)(U \times U)$$

associated to

$$\begin{array}{cccc} p: & U \times U & \longrightarrow & \widetilde{\Theta}_q, \\ & (x,t,y,u) & \longrightarrow & t. \end{array}$$

Considering α as an element of the group $\tilde{\pi}^*\mathbf{O}(q)(U \times U)$ and β, γ as elements of the group $\tilde{\pi}^*\mathbf{O}(n)(U \times U)$, we have

$$\gamma = \theta_p(\alpha) \cdot \beta \in \tilde{\pi}^*\mathbf{O}(n)(U \times U), \tag{35}$$

since we may write

$$\gamma(x,t,y,u) = (t^{-1}(x^{-1}y)t)(t^{-1}u) \in \tilde{\pi}^*\mathbf{O}(n)(U \times U).$$

Note that (35) only makes sense in the group $\tilde{\pi}^*\mathbf{O}(n)(U \times U)$, since $\mathcal{Z}^1(\{U \to *\}, \tilde{\pi}^*\mathbf{O}(n))$ only carries the structure of a pointed set.

We continue to view α and β as elements of the groups $\tilde{\pi}^*\mathbf{O}(q)(U \times U)$ and $\tilde{\pi}^*\mathbf{O}(n)(U \times U)$, respectively, and we consider the maps

$$r_q : \tilde{\pi}^*\tilde{\mathbf{O}}(q) \twoheadrightarrow \tilde{\pi}^*\mathbf{O}(q) \quad \text{and} \quad r_n : \tilde{\pi}^*\tilde{\mathbf{O}}(n) \twoheadrightarrow \tilde{\pi}^*\mathbf{O}(n).$$

LEMMA 5.12. *There exist an epimorphism $U' \to U$ together with elements*

$$\widetilde{\alpha_{|U' \times U'}} \in \tilde{\pi}^*\tilde{\mathbf{O}}(q)(U' \times U') \quad \textit{and} \quad \widetilde{\beta_{|U' \times U'}} \in \tilde{\pi}^*\tilde{\mathbf{O}}(n)(U' \times U')$$

such that

$$\alpha_{|U' \times U'} = r_q(\widetilde{\alpha_{|U' \times U'}}) \quad \textit{and} \quad \beta_{|U' \times U'} = r_n(\widetilde{\beta_{|U' \times U'}}).$$

Proof. First, we show that β has a lift. The map β can be factored in the following manner:

$$\beta : U \times U = \tilde{E}_{\mathbf{O}(q)} \times \tilde{\Theta}_q \times \tilde{E}_{\mathbf{O}(q)} \times \tilde{\Theta}_q \longrightarrow \tilde{\Theta}_q \times \tilde{\Theta}_q \xrightarrow{b} \tilde{\pi}^*\mathbf{O}(n),$$

where

$$\begin{array}{cccc} b: & \tilde{\Theta}_q \times \tilde{\Theta}_q & \longrightarrow & \tilde{\pi}^*\mathbf{O}(n), \\ & (t,u) & \longmapsto & t^{-1}u. \end{array}$$

By base change, it is enough to show that there exists an epimorphism $V \to \tilde{\Theta}_q$ and a commutative diagram in $B_{\tilde{\mathbf{O}}(q)}$:

$$\begin{array}{ccc} V \times V & \longrightarrow & \tilde{\pi}^*\tilde{\mathbf{O}}(n) \\ \downarrow & & \downarrow \\ \tilde{\Theta}_q \times \tilde{\Theta}_q & \longrightarrow & \tilde{\pi}^*\mathbf{O}(n) \end{array}$$

The objects $\tilde{\Theta}_q \times \tilde{\Theta}_q$, $\tilde{\pi}^*\mathbf{O}(n)$ and $\tilde{\pi}^*\tilde{\mathbf{O}}(n)$ of $B_{\tilde{\mathbf{O}}(q)}$ occurring in this square are all given with the trivial action of $\tilde{\mathbf{O}}(q)$, hence it is enough to show that there exist an epimorphism

$V \to \mathbf{Isom}(t_n, q)$ in Y_{fl} and a commutative diagram in Y_{fl}:

$$\begin{array}{ccc} V \times V & \longrightarrow & \tilde{\mathbf{O}}(n) \\ \downarrow & & \downarrow \\ \mathbf{Isom}(t_n, q) \times \mathbf{Isom}(t_n, q) & \longrightarrow & \mathbf{O}(n) \end{array}$$

Take an étale covering $Y' \to Y$ trivializing q, that is, such that there is an isometry $f : q_{Y'} \xrightarrow{\sim} t_{n,Y'}$, that is, such that there is a section

$$f : Y' \longrightarrow Y' \times \mathbf{Isom}(q, t_n) \quad \text{in } Y_{fl}/Y'.$$

Composition with f

$$Y' \times \mathbf{Isom}(t_n, q) \overset{(f,1)}{\longrightarrow} Y' \times \mathbf{Isom}(q, t_n) \times \mathbf{Isom}(t_n, q) \overset{(1,-\circ-)}{\longrightarrow} Y' \times \mathbf{O}(n)$$

yields an isomorphism of $\mathbf{O}(n)$-torsors over Y':

$$Y' \times \mathbf{Isom}(t_n, q) \simeq Y' \times \mathbf{O}(n).$$

Indeed, this map is clearly $\mathbf{O}(n)$-equivariant; it is an isomorphism whose inverse is induced by composition with $f^{-1} : t_{n,Y'} \xrightarrow{\sim} q_{Y'}$ in a similar way. We consider the maps

$$V = Y' \times \tilde{\mathbf{O}}(n) \longrightarrow Y' \times \mathbf{O}(n) \simeq Y' \times \mathbf{Isom}(t_n, q) \longrightarrow \mathbf{Isom}(t_n, q)$$

and

$$\begin{array}{rcl} V \times V = Y' \times \tilde{\mathbf{O}}(n) \times Y' \times \tilde{\mathbf{O}}(n) & \longrightarrow & \tilde{\mathbf{O}}(n), \\ (y', \sigma, z', \tau) & \longmapsto & \sigma^{-1}\tau. \end{array}$$

It is then straightforward to check that the above square is commutative.

It remains to show that α has a lift. We consider the epimorphism in $B_{\tilde{\mathbf{O}}(q)}$

$$U' = E_{\tilde{\mathbf{O}}(q)} \times \tilde{\Theta}_q \overset{(r,\mathrm{Id})}{\longrightarrow} \tilde{E}_{\mathbf{O}(q)} \times \tilde{\Theta}_q = U.$$

Here, $r : E_{\tilde{\mathbf{O}}(q)} \to \tilde{E}_{\mathbf{O}(q)}$ is the map $\tilde{\mathbf{O}}(q) \to \mathbf{O}(q)$ seen as an $\tilde{\mathbf{O}}(q)$-equivariant map, where $\tilde{\mathbf{O}}(q)$ acts by left multiplication on both $\tilde{\mathbf{O}}(q)$ and $\mathbf{O}(q)$. Then

$$\begin{array}{ccc} U' \times U' & \longrightarrow & \tilde{\pi}^*\tilde{\mathbf{O}}(q) \\ \downarrow & & \downarrow \\ U \times U & \longrightarrow & \tilde{\pi}^*\mathbf{O}(q) \end{array}$$

is a commutative diagram in $B_{\tilde{\mathbf{O}}(q)}$ where the top horizontal map is defined as follows:

$$\begin{array}{rcl} U' \times U' = E_{\tilde{\mathbf{O}}(q)} \times \tilde{\Theta}_q \times E_{\tilde{\mathbf{O}}(q)} \times \tilde{\Theta}_q & \longrightarrow & \tilde{\pi}^*\tilde{\mathbf{O}}(q), \\ (\sigma, t, \tau, u) & \longmapsto & \sigma^{-1}\tau. \end{array}$$

We have shown that there exists epimorphisms $U_\alpha \to U$ and $U_\beta \to U$ such that $\alpha_{|U_\alpha \times U_\alpha}$ and $\beta_{|U_\beta \times U_\beta}$ have lifts $\widetilde{\alpha_{|U_\alpha \times U_\alpha}}$ and $\widetilde{\beta_{|U_\beta \times U_\beta}}$, respectively. The conclusion of the lemma with $U' = U_\alpha \times_U U_\beta \twoheadrightarrow U$ follows. $\square$

It follows from (35) that

$$\gamma_{|U' \times U'} = (\theta_p(\alpha) \cdot \tilde{\beta})_{|U' \times U'} = \theta_{p_{|U' \times U'}}(\alpha_{|U' \times U'}) \cdot \beta_{|U' \times U'}. \tag{36}$$

Using Lemma 5.11, we obtain

$$\widetilde{\gamma_{|U' \times U'}} = \tilde{\theta}_{p_{|U' \times U'}}(\widetilde{\alpha_{|U' \times U'}}) \cdot \widetilde{\beta_{|U' \times U'}} \in \tilde{\pi}^*\tilde{\mathbf{O}}(q)(U' \times U') \tag{37}$$

is a lift of $\gamma_{|U'\times U'} \in \tilde{\pi}^*\tilde{\mathbf{O}}(q)(U'\times U')$. From now on, we write U for U', p for $p_{|U'\times U'}: U'\times U' \to W \to \tilde{\Theta}_q$, α for $\alpha_{|U'\times U'}$, β for $\beta_{|U'\times U'}$ and γ for $\gamma_{|U'\times U'}$. We have lifts $\tilde{\alpha} \in \tilde{\pi}^*\tilde{\mathbf{O}}(q)(U\times U)$, $\tilde{\beta} \in \tilde{\pi}^*\tilde{\mathbf{O}}(n)(U\times U)$ and $\tilde{\gamma} \in \tilde{\pi}^*\tilde{\mathbf{O}}(n)(U\times U)$ of $\alpha \in \tilde{\pi}^*\mathbf{O}(q)(U\times U)$, $\beta \in \tilde{\pi}^*\mathbf{O}(n)(U\times U)$ and $\gamma \in \tilde{\pi}^*\mathbf{O}(n)(U\times U)$, respectively. Moreover, we have

$$\tilde{\gamma} = \tilde{\theta}_p(\tilde{\alpha})\cdot\tilde{\beta} \in \tilde{\pi}^*\tilde{\mathbf{O}}(q)(U\times U). \tag{38}$$

Step 4: Reduction to an identity of cocycles

The extension of group objects in Y_{fl}

$$1 \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow \tilde{\mathbf{O}}(n) \longrightarrow \mathbf{O}(n) \longrightarrow 1$$

gives a morphism

$$\delta_n^2 : H^1(B_{\tilde{\mathbf{O}}(q)}, \tilde{\pi}^*\mathbf{O}(n)) \longrightarrow H^2(B_{\tilde{\mathbf{O}}(q)}, \mathbf{Z}/2\mathbf{Z}).$$

Note that one has

$$\delta_n^2(\tilde{T}_q) = \tilde{T}_q^*[C_n] = B_{r_q}^* T_q^*[C_n] = B_{r_q}^* HW_2(q) \tag{39}$$

and

$$\delta_n^2(\tilde{\Theta}_q) = \tilde{\Theta}_q^*[C_n] = B_{r_q}^*\Theta_q^*[C_n] = B_{r_q}^* w_2(q). \tag{40}$$

Similarly, the group extension

$$1 \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow \tilde{\mathbf{O}}(q) \longrightarrow \mathbf{O}(q) \longrightarrow 1$$

gives a morphism

$$\delta_q^2 : H^1(B_{\tilde{\mathbf{O}}(q)}, \tilde{\pi}^*\mathbf{O}(q)) \longrightarrow H^2(B_{\tilde{\mathbf{O}}(q)}, \mathbf{Z}/2\mathbf{Z})$$

such that

$$\delta_q^2(\tilde{E}_{\mathbf{O}(q)}) = \tilde{E}_{\mathbf{O}(q)}^*[C_q] = B_{r_q}^*[C_q] = 0. \tag{41}$$

PROPOSITION 5.13. *One is reduced to show*

$$\delta_n^2(\tilde{T}_q) = \delta_n^2(\tilde{\Theta}_q) + B_{r_q}^* w_1(q) \cup B_{r_q}^*\det[q] + \delta_q^2(\tilde{E}_{\mathbf{O}(q)}) \tag{42}$$

in $H^2(B_{\tilde{\mathbf{O}}(q)}, \mathbf{Z}/2\mathbf{Z})$, *which in turn will follow from an identity of cocycles*

$$\delta_n^2(\gamma) = \delta_n^2(\beta) + (\det_{\mathbf{O}(n)}\circ\beta) \cup (\det_{\mathbf{O}(q)}\circ\alpha) + \delta_q^2(\alpha) \tag{43}$$

in $\check{H}^2(\{U \to *\}, \mathbf{Z}/2\mathbf{Z})$.

Proof. By Proposition 5.5, it is enough to show that (43) implies (42) and that (42) implies (29). The fact that (42) implies (29) follows immediately from (39)–(41) and the fact that $B_{r_q}^*$ respects sums and cup-products.

Let us show that (43) implies (42). The 2-cocycles $\delta_n^2(\gamma)$, $\delta_n^2(\beta)$ and $\delta_q^2(\alpha)$, all elements of $\mathcal{Z}^2(\{U \to *\}, \mathbf{Z}/2\mathbf{Z})$, represent the cohomology classes $\delta_n^2(\tilde{T}_q)$, $\delta_n^2(\tilde{\Theta}_q)$ and $\delta_q^2(\tilde{E}_{\mathbf{O}(q)})$, respectively. Then we observe that $\det_{\mathbf{O}(n)}\circ\beta$ and $\det_{\mathbf{O}(q)}\circ\alpha$ are 1-cocycles representing the maps

$$B_{\tilde{\mathbf{O}}(q)} \xrightarrow{B_{r_q}} B_{\mathbf{O}(q)} \xrightarrow{\Theta_q} B_{\mathbf{O}(n)} \xrightarrow{\det_{\mathbf{O}(n)}} B_{\mathbf{Z}/2\mathbf{Z}}$$

and

$$B_{\tilde{\mathbf{O}}(q)} \xrightarrow{B_{r_q}} B_{\mathbf{O}(q)} \xrightarrow{E_{\mathbf{O}(q)}=\mathrm{Id}} B_{\mathbf{O}(q)} \xrightarrow{\det_{\mathbf{O}(q)}} B_{\mathbf{Z}/2\mathbf{Z}},$$

respectively. By definition, these two maps correspond to the cohomology classes $B_{r_q}^* w_1(q)$ and $B_{r_q}^*\det[q]$, respectively. The result follows since the map $\check{H}^2(\{U \to *\}, \mathbf{Z}/2\mathbf{Z}) \to H^2(B_{\tilde{\mathbf{O}}(q)}, \mathbf{Z}/2\mathbf{Z})$ is compatible with cup-products. □

Step 5: Proof of (43)
We still denote by

$$p_{ij} : U \times U \times U \longrightarrow U \times U$$

the projection on the (i, j)-components. Then we have

$$\delta_n^2(\gamma) = (\tilde{\gamma}p_{23})(\tilde{\gamma}p_{13})^{-1}(\tilde{\gamma}p_{12}) \in \mathcal{Z}^2(\{U \longrightarrow *\}, \mathbf{Z}/2\mathbf{Z}) \subset \tilde{\pi}^*\tilde{\mathbf{O}}(n)(U \times U \times U),$$

a 2-cocycle representative of $\delta_n^2(\tilde{T}_q) \in H^2(B_{\tilde{\mathbf{O}}(q)}, \mathbf{Z}/2\mathbf{Z})$. Of course, $\delta_n^2(\gamma)$ is only well defined in $\check{H}^2(\{U \to *\}, \mathbf{Z}/2\mathbf{Z})$; that is, a different choice for the lift $\tilde{\gamma}$ gives a cohomologous 2-cocycle. By (38), we have

$$\delta_n^2(\gamma) = ((\tilde{\theta}_p(\tilde{\alpha})p_{23})(\tilde{\beta}p_{23}))((\tilde{\theta}_p(\tilde{\alpha})p_{13})(\tilde{\beta}p_{13}))^{-1}((\tilde{\theta}_p(\tilde{\alpha})p_{12})(\tilde{\beta}p_{12})).$$

Our first goal is to understand the terms $\tilde{\theta}_p(\tilde{\alpha})p_{ij}$. To this end, we introduce the natural projections $\mathfrak{p}_i$ for $1 \leqslant i \leqslant 3$:

$$\mathfrak{p}_i : U \times U \times U \xrightarrow{\mathrm{pr}_i} U \longrightarrow \tilde{E}_{\mathbf{O}(q)} \times \tilde{\Theta}_q \xrightarrow{\mathrm{pr}_2} \tilde{\Theta}_q.$$

Recall from Step 2 that

$$\tilde{\theta}_p(\tilde{\alpha})p_{23} = \mathrm{pr} \circ \tilde{\theta} \circ (p \times \tilde{\alpha}) \circ p_{23},$$

where

$$p : U \times U \xrightarrow{\mathrm{pr}_1} U \longrightarrow \tilde{E}_{\mathbf{O}(q)} \times \tilde{\Theta}_q \xrightarrow{\mathrm{pr}_2} \tilde{\Theta}_q.$$

We now observe that

$$(p \times \tilde{\alpha}) \circ p_{23} = (p \circ p_{23}) \times (\tilde{\alpha} \circ p_{23}) = \mathfrak{p}_2 \times \tilde{\alpha} \circ p_{23}.$$

Hence, we deduce that

$$\tilde{\theta}_p(\tilde{\alpha})p_{23} = \tilde{\theta}_{\mathfrak{p}_2}(\tilde{\alpha}p_{23}) \in \tilde{\pi}^*\tilde{\mathbf{O}}(n)(U \times U \times U).$$

Similarly, we have

$$\tilde{\theta}_p(\tilde{\alpha})p_{13} = \tilde{\theta}_{\mathfrak{p}_1}(\tilde{\alpha}p_{13}) \quad \text{and} \quad \tilde{\theta}_p(\tilde{\alpha})p_{12} = \tilde{\theta}_{\mathfrak{p}_1}(\tilde{\alpha}p_{12})$$

in the group $\tilde{\pi}^*\tilde{\mathbf{O}}(n)(U \times U \times U)$. This yields

$$\delta_n^2(\gamma) = \tilde{\theta}_{\mathfrak{p}_2}(\tilde{\alpha}p_{23})(\tilde{\beta}p_{23})(\tilde{\beta}p_{13})^{-1}\tilde{\theta}_{\mathfrak{p}_1}(\tilde{\alpha}p_{13}^{-1})\tilde{\theta}_{\mathfrak{p}_1}(\tilde{\alpha}p_{12})(\tilde{\beta}p_{12}). \tag{44}$$

Moreover, we have

$$\tilde{\theta}_{\mathfrak{p}_1}(\tilde{\alpha}p_{13}^{-1})\tilde{\theta}_{\mathfrak{p}_1}(\tilde{\alpha}p_{12}) = \tilde{\theta}_{\mathfrak{p}_1}(\tilde{\alpha}p_{13}^{-1}\tilde{\alpha}p_{12}) = \tilde{\theta}_{\mathfrak{p}_1}(\tilde{\alpha}p_{23}^{-1})\tilde{\theta}_{\mathfrak{p}_1}(\tilde{\alpha}p_{23}\tilde{\alpha}p_{13}^{-1}\tilde{\alpha}p_{12}).$$

Since $\tilde{\alpha}p_{23}\tilde{\alpha}p_{13}^{-1}\tilde{\alpha}p_{12}$ is in the kernel of $r_{q,Z}$ and since $\tilde{\theta}_{\mathfrak{p}_1}$ coincides with the identity on this kernel, we can write

$$\tilde{\theta}_{\mathfrak{p}_1}(\tilde{\alpha}p_{23}^{-1})\tilde{\theta}_{\mathfrak{p}_1}(\tilde{\alpha}p_{23}\tilde{\alpha}p_{13}^{-1}\tilde{\alpha}p_{12}) = \tilde{\theta}_{\mathfrak{p}_1}(\tilde{\alpha}p_{23}^{-1})(\tilde{\alpha}p_{23}\tilde{\alpha}p_{13}^{-1}\tilde{\alpha}p_{12}). \tag{45}$$

Since $(\tilde{\beta}p_{23})(\tilde{\beta}p_{13})^{-1}(\tilde{\beta}p_{12})$ is in the kernel of $r_{n,Z}$, it belongs to the centre of $\tilde{\pi}^*\tilde{\mathbf{O}}(n)(U \times U \times U)$, and it follows from (44) and (45) that we have

$$\delta_n^2(\gamma) = \delta_n^2(\beta) \cdot \xi \cdot \delta_q^2(\alpha) \in \tilde{\pi}^*\tilde{\mathbf{O}}(n)(U \times U \times U),$$

where ξ is defined as follows:

$$\xi = \tilde{\theta}_{\mathfrak{p}_2}(\tilde{\alpha}p_{23})(\tilde{\beta}p_{12})^{-1}\tilde{\theta}_{\mathfrak{p}_1}(\tilde{\alpha}p_{23}^{-1})(\tilde{\beta}p_{12}) \in \mathcal{Z}^2(\{U \longrightarrow *\}, \mathbf{Z}/2\mathbf{Z}).$$

Clearly, the result (43) would follow from an identity

$$\xi = \det_{\mathbf{O}(n)}(\beta) \cup \det_{\mathbf{O}(q)}(\alpha) \in \mathcal{Z}^2(\{U \longrightarrow *\}, \mathbf{Z}/2\mathbf{Z})$$

in the group of 2-cocycles $\mathcal{Z}^2(\{U \to *\}, \mathbf{Z}/2\mathbf{Z})$. Since $\mathcal{Z}^2(\{U \to *\}, \mathbf{Z}/2\mathbf{Z}) \subset \mathbf{Z}/2\mathbf{Z}(U \times U \times U)$, it is of course equivalent to showing that

$$\xi = \det_{\mathbf{O}(n)}(\beta) \cup \det_{\mathbf{O}(q)}(\alpha) \in \mathbf{Z}/2\mathbf{Z}(U \times U \times U). \tag{46}$$

Let us first make the cup-product $\det_{\mathbf{O}(n)}(\beta) \cup \det_{\mathbf{O}(q)}(\alpha)$ more explicit: It is given by

$$m \circ (\det_{\mathbf{O}(n)}(\beta)p_{12}, \det_{\mathbf{O}(q)}(\alpha)p_{23}) : U \times U \times U \longrightarrow \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \longrightarrow \mathbf{Z}/2\mathbf{Z},$$

where $m : \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$ is the standard multiplication,

$$\det_{\mathbf{O}(n)}(\beta) := \det_{\mathbf{O}(n)} \circ \beta : \begin{array}{ccc} U \times U & \longrightarrow & \mu_2 = \mathbf{Z}/2\mathbf{Z}, \\ (x_1, t_1, x_2, t_2) & \longmapsto & \det_{\mathbf{O}(n)}(t_1^{-1}t_2) \end{array}$$

and

$$\det_{\mathbf{O}(q)}(\alpha) := \det_{\mathbf{O}(q)} \circ \alpha : \begin{array}{ccc} U \times U & \longrightarrow & \mu_2 = \mathbf{Z}/2\mathbf{Z}, \\ (x_1, t_1, x_2, t_2) & \longmapsto & \det_{\mathbf{O}(q)}(x_1^{-1}x_2). \end{array}$$

By Lemma 5.11, the class of objects of the form $E_{\tilde{\mathbf{O}}(q)} \times \mathrm{Spec}(R) \to Y$, where $\mathrm{Spec}(R)$ is an affine Y-scheme endowed with its trivial $\tilde{\mathbf{O}}(q)$-action, is a generating family of the topos $B_{\tilde{\mathbf{O}}(q)}$. Therefore, in order to prove (46), it is enough to show

$$\xi \circ u = (\det_{\mathbf{O}(n)}(\beta) \cup \det_{\mathbf{O}(q)}(\alpha)) \circ u \in \mathbf{Z}/2\mathbf{Z}(E_{\tilde{\mathbf{O}}(q)} \times \mathrm{Spec}(R)) \tag{47}$$

for any map u in $B_{\tilde{\mathbf{O}}(q)}$ of the form

$$u : E_{\tilde{\mathbf{O}}(q)} \times \mathrm{Spec}(R) \longrightarrow U \times U \times U,$$

where $\mathrm{Spec}(R)$ is an affine scheme. Moreover, by adjunction we have an isomorphism (see the proof of Lemma 5.11)

$$\begin{array}{ccc} \mathrm{Hom}_{B_{\tilde{\mathbf{O}}(q)}}(E_{\tilde{\mathbf{O}}(q)} \times \mathrm{Spec}(R), \mathbf{Z}/2\mathbf{Z}) & \longrightarrow & \mathrm{Hom}_{Y_{fl}}(\mathrm{Spec}(R), \mathbf{Z}/2\mathbf{Z}), \\ f & \longmapsto & f_{|R} \end{array}$$

sending $f : E_{\tilde{\mathbf{O}}(q)} \times \mathrm{Spec}(R) \to \mathbf{Z}/2\mathbf{Z}$ to

$$f_{|R} : \mathrm{Spec}(R) \longrightarrow \eta^*(E_{\tilde{\mathbf{O}}(q)} \times \mathrm{Spec}(R)) \xrightarrow{\eta^* f} \mathbf{Z}/2\mathbf{Z}.$$

Using the bijection $f \mapsto f_{|R}$ above, we are reduced to showing the identity

$$(\xi \circ u)_{|R} = m \circ (\det_{\mathbf{O}(n)}(\beta p_{12} u)_{|R}, \det_{\mathbf{O}(q)}(\alpha p_{23} u)_{|R}) \tag{48}$$

in $\mathbf{Z}/2\mathbf{Z}(\mathrm{Spec}(R)) = \mathbf{Z}/2\mathbf{Z}^{\pi_0(\mathrm{Spec}(R))}$. Note by the way that one may suppose $\mathrm{Spec}(R)$ to be connected and reduced. Indeed, (48) can be shown after restriction to $\mathrm{Spec}(R_j)^{\mathrm{red}} \to \mathrm{Spec}(R)$ for any connected component $\mathrm{Spec}(R_j)^{\mathrm{red}}$ of $\mathrm{Spec}(R)$, given its unique structure of reduced closed affine subscheme (note that a connected component is always closed but not necessarily open). By Lemma 5.10, we have

$$(\xi \circ u)_{|R} = (\tilde{\theta}_{\mathfrak{p}_2 u}(\tilde{\alpha} p_{23} u) \cdot (\tilde{\beta} p_{12} u)^{-1} \cdot \tilde{\theta}_{\mathfrak{p}_1 u}(\tilde{\alpha} p_{23} u)^{-1} \cdot (\tilde{\beta} p_{12} u))_{|R} \tag{49}$$

$$= (\tilde{\theta}_{\mathfrak{p}_2 u}(\tilde{\alpha} p_{23} u))_{|R} \cdot (\tilde{\beta} p_{12} u)_{|R}^{-1} \cdot (\tilde{\theta}_{\mathfrak{p}_1 u}(\tilde{\alpha} p_{23} u))_{|R}^{-1} \cdot (\tilde{\beta} p_{12} u)_{|R} \tag{50}$$

$$= \eta^* \tilde{\theta}_{\mathfrak{p}_2 u_{|R}}(\tilde{\alpha} p_{23} u_{|R}) \cdot \tilde{\beta} p_{12} u_{|R}^{-1} \cdot \eta^* \tilde{\theta}_{\mathfrak{p}_1 u_{|R}}(\tilde{\alpha} p_{23} u_{|R})^{-1} \cdot \tilde{\beta} p_{12} u_{|R}, \tag{51}$$

where $\tilde{\alpha} p_{ij} u_{|R} \in \tilde{\mathbf{O}}(q)(R)$, $\tilde{\beta} p_{ij} u_{|R} \in \tilde{\mathbf{O}}(n)(R)$ and $\eta^* \tilde{\theta}_{\mathfrak{p}_i u_{|R}} : \tilde{\mathbf{O}}(q)(R) \to \tilde{\mathbf{O}}(n)(R)$ is the map induced by $\mathfrak{p}_i u_{|R} \in \mathbf{Isom}(t_n, q)(R)$; see Step 2. Moreover, one has

$$\mathfrak{p}_2 = \mathfrak{p}_1 \star \beta p_{12},$$

where $\star : \tilde{\Theta}_q \times \tilde{\pi}^* \mathbf{O}(n) \to \tilde{\Theta}_q$ is the $\tilde{\pi}^* \mathbf{O}(n)$-torsor structure map of $\tilde{\Theta}_q$. Applying successively $(-) \circ u$ and $(-)_{|R}$, we obtain

$$\mathfrak{p}_2 u_{|R} = (\mathfrak{p}_1 u_{|R}) \circ (\beta p_{12} u_{|R}),$$

where the right-hand side $(\mathfrak{p}_1 u_{|R}) \circ (\beta p_{12} u_{|R}) \in \mathbf{Isom}(t_n, q)(R)$ is the composition of $\mathfrak{p}_1 u_{|R} \in \mathbf{Isom}(t_n, q)(R)$ and $\beta p_{12} u_{|R} \in \mathbf{O}(n)(R)$. It then follows from (30) and Lemma 5.8 that

$$\begin{aligned}\eta^* \tilde{\theta}_{\mathfrak{p}_2 u_{|R}} &= \tilde{\psi}_{\mathfrak{p}_2 u_{|R}^{-1}} = \tilde{\psi}_{(\beta p_{12} u_{|R})^{-1} (\mathfrak{p}_1 u_{|R})^{-1}} = \tilde{\psi}_{(\beta p_{12} u_{|R})^{-1}} \circ \tilde{\psi}_{(\mathfrak{p}_1 u_{|R})^{-1}} \\ &= \tilde{\psi}_{\beta p_{12} u_{|R}^{-1}} \circ \eta^* \tilde{\theta}_{\mathfrak{p}_1 u_{|R}}. \end{aligned} \quad (52)$$

By Lemma 5.8(iii) and (iv), if $\beta p_{12} u_{|R} \in \mathbf{O}_+(n)(R)$, then $\tilde{\psi}_{(\beta p_{12} u_{|R})^{-1}} = i_{(\tilde{\beta} p_{12} u_{|R})^{-1}}$, since $\tilde{\beta} p_{12} u_{|R}$ is a lift of $\beta p_{12} u_{|R}$. For $\beta p_{12} u_{|R} \in \mathbf{O}_+(n)(R)$, we obtain

$$\eta^* \tilde{\theta}_{\mathfrak{p}_2 u_{|R}}(-) = (\tilde{\beta} p_{12} u_{|R})^{-1} \cdot \eta^* \tilde{\theta}_{\mathfrak{p}_1 u_{|R}}(-) \cdot (\tilde{\beta} p_{12} u_{|R}),$$

hence, by (51), $(\xi \circ u)_{|R} = 0$. We now suppose that $\beta p_{12} u_{|R} \in \mathbf{O}_-(n)(R)$ and $\alpha p_{23} u_{|R} \in \mathbf{O}_+(q)(R)$. It follows that $\tilde{\alpha} p_{23} u_{|R} \in \tilde{\mathbf{O}}_+(q)(R)$ and that $\eta^* \tilde{\theta}_{\mathfrak{p}_1 u_{|R}}(\tilde{\alpha} p_{23} u_{|R}) \in \tilde{\mathbf{O}}_+(n)(R)$. Since $\tilde{\psi}_{(\beta p_{12} u_{|R})^{-1}}$ coincides with $i_{(\tilde{\beta} p_{12} u_{|R})^{-1}}$ on $\tilde{\mathbf{O}}_+(n)(R)$, we deduce from (51) and (52) that $(\xi \circ u)_{|R} = 0$. We now assume that $\alpha p_{23} u_{|R} \in \mathbf{O}_-(q)(R)$ and $\tilde{\alpha} p_{23} u_{|R} \in \tilde{\mathbf{O}}_-(q)(R)$. Using $\tilde{\psi}_{(\beta p_{12} u_{|R})^{-1}} = -i_{(\tilde{\beta} p_{12} u_{|R})^{-1}}$ on $\tilde{\mathbf{O}}_-(n)(R)$, we conclude that $(\xi \circ u)_{|R} = 1$ in this last case. A comparison in each case of the values of $(\xi \circ u)_{|R}$ and

$$((\det_{\mathbf{O}(n)}(\beta) \cup \det_{\mathbf{O}(q)}(\alpha)) \circ u)_{|R} = \det_{\mathbf{O}(n)}(\beta p_{12} u_{|R}) \cdot \det_{\mathbf{O}(q)}(\alpha p_{23} u_{|R})$$

yields

$$(\xi \circ u)_{|R} = ((\det_{\mathbf{O}(n)}(\beta) \cup \det_{\mathbf{O}(q)}(\alpha)) \circ u)_{|R} \in \mathbf{Z}/2\mathbf{Z}(\mathrm{Spec}(R)) = \mathbf{Z}/2\mathbf{Z}$$

for any $\mathrm{Spec}(R)$ connected. The result follows. □

REMARK 5.14. Let us define

$$H(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z})^* = \left\{ 1 + a_1 + a_2 \in \bigoplus_{0 \leqslant i \leqslant 2} H^i(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z}); a_i \in H^i(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z}) \right\}.$$

We define an abelian group structure on $H(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z})^*$ by

$$(1 + a_1 + a_2)(1 + b_1 + b_2) = 1 + (a_1 + b_1) + a_2 + b_2 + a_1 \cup b_1.$$

Moreover, T_q and Θ_q induce morphisms of abelian group from $H(B_{\mathbf{O}(n)}, \mathbf{Z}/2\mathbf{Z})^*$ to $H(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z})^*$. We associate to (V, q) the element

$$s_q = 1 + \det[q] + [C_q] \in H(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z})^*$$

and we simply write s_n for s_{t_n}. Then Theorem 5.1 yields the identity

$$s_q = T_q^*(s_n) {\Theta_q}^*(s_n)^{-1}.$$

6. *Consequences of the main theorem*

6.1. *Serre's formula*

Our aim is to deduce comparison formulas from Theorem 5.1 which extend the work of Serre (see [**12**, Chapitre III, Annexe, (2.2.1), (2.2.2)]) to symmetric bundles over an arbitrary base scheme. This formula is also referred to as the 'real Fröhlich–Kahn–Snaith formula' in [**13**]. A direct proof of this result is given in [**2**, Theorem 0.2].

We consider an $\mathbf{O}(q)$-torsor α of Y_{fl}. We also denote by $\alpha : Y_{fl} \to B_{\mathbf{O}(q)}$ the classifying map for this torsor, and by $[\alpha]$ its class in $H^1(Y_{fl}, \mathbf{O}(q))$. We define by

$$\delta_q^1 : H^1(Y_{fl}, \mathbf{O}(q)) \longrightarrow H^1(Y_{fl}, \mathbf{Z}/2\mathbf{Z})$$

the map induced by the determinant map $\det_q : \mathbf{O}(q) \to \mathbf{Z}/2\mathbf{Z}$, and by

$$\delta_q^2 : H^1(Y_{fl}, \mathbf{O}(q)) \longrightarrow H^2(Y_{fl}, \mathbf{Z}/2\mathbf{Z})$$

the boundary map associated to the group extension C_q in Y_{fl}

$$1 \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow \tilde{\mathbf{O}}(q) \longrightarrow \mathbf{O}(q) \longrightarrow 1. \tag{53}$$

In other words, we have

$$\delta_q^1[\alpha] := \alpha^*(\det[q]) \quad \text{and} \quad \delta_q^2[\alpha] := \alpha^*[C_q]. \tag{54}$$

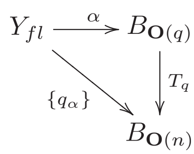
As in Section 4.2, we associate to α a symmetric bundle (V_α, q_α) on Y.

COROLLARY 6.1. *For any $\mathbf{O}(q)$-torsor α of Y_{fl}, we have*

(i) $w_1(q_\alpha) = w_1(q) + \delta_q^1[\alpha]$ *in* $H^1(Y, \mathbf{Z}/2\mathbf{Z})$;
(ii) $w_2(q_\alpha) = w_2(q) + w_1(q) \cdot \delta_q^1[\alpha] + \delta_q^2[\alpha]$ *in* $H^2(Y, \mathbf{Z}/2\mathbf{Z})$.

Proof. We define $\{q_\alpha\} : Y_{fl} \to B_{\mathbf{O}(n)}$ to be the morphism of topoi associated to the $\mathbf{O}(n)$-torsor $\mathbf{Isom}(t_n, q_\alpha)$ of Y_{fl} and we let $T_q : B_{\mathbf{O}(q)} \to B_{\mathbf{O}(n)}$ be the morphism of topoi defined in Definition 4.6.

LEMMA 6.2. *The following triangle is commutative*:



Proof. It will suffice to describe an isomorphism

$$\{q_\alpha\}^* E_{\mathbf{O}(n)} \simeq \alpha^* T_q^* E_{\mathbf{O}(n)}$$

of $\mathbf{O}(n)$-torsors of Y_{fl}. It follows from the definitions that $\{q_\alpha\}^* E_{\mathbf{O}(n)} = \mathbf{Isom}(t_n, q_\alpha)$ and that

$$\alpha^* T_q^* E_{\mathbf{O}(n)} = \alpha^* \mathbf{Isom}(t_n, q) = \mathbf{Isom}(q, q_\alpha) \wedge^{\mathbf{O}(q)} \mathbf{Isom}(t_n, q).$$

The lemma then follows from the fact that the map

$$\mathbf{Isom}(q, q_\alpha) \times \mathbf{Isom}(t_n, q) \longrightarrow \mathbf{Isom}(t_n, q_\alpha),$$

given by composition, induces an $\mathbf{O}(n)$-equivariant isomorphism

$$\mathbf{Isom}(q, q_\alpha) \wedge^{\mathbf{O}(q)} \mathbf{Isom}(t_n, q) \simeq \mathbf{Isom}(t_n, q_\alpha).$$

As a consequence of the lemma we obtain that

$$\alpha^* T_q^*(HW_i) = \{q_\alpha\}^*(HW_i) = w_i(q_\alpha) \text{ in } H^i(Y, \mathbf{Z}/2\mathbf{Z}) \quad \text{for } i = 1, 2. \tag{55}$$

We now observe that, since $\pi \circ \alpha \simeq id$, we have

$$\alpha^*(w_i(q)) = \alpha^* \pi^*(w_i(q)) = w_i(q). \tag{56}$$

Using (54)–(56), the corollary is just the pull-back of Theorem 5.1 via α^*. □

6.2. *Comparison formulas for Hasse–Witt invariants of orthogonal representations*

Let (V, q, ρ) be an orthogonal representation of G. To be more precise, G is a group scheme over Y, (V, q) is a symmetric bundle over Y and $\rho : G \to \mathbf{O}(q)$ is a morphism of Y-group schemes.

We denote by $B_\rho : B_G \to B_{\mathbf{O}(q)}$ the morphism of classifying topoi induced by the group homomorphism ρ. The Hasse–Witt invariants $w_i(q, \rho)$ of (V, q, ρ) lie in $H^i(B_G, \mathbf{Z}/2\mathbf{Z})$. Indeed, there is a morphism

$$B_G \xrightarrow{B_\rho} B_{\mathbf{O}(q)} \xrightarrow{T_q} B_{\mathbf{O}(n)}$$

canonically associated to (V, q, ρ) and $w_i(q, \rho)$ is simply the pull-back of HW_i along this map:

$$w_i(q, \rho) := (T_q \circ B_\rho)^*(HW_i) = B_\rho^*(HW_i(q)). \tag{57}$$

On the other hand, the morphism of groups

$$\det_q \circ \rho : G \longrightarrow \mathbf{O}(q) \longrightarrow \mathbf{Z}/2\mathbf{Z}$$

defines (see Proposition 3.8) a cohomology class $w_1(\rho) \in H^1(B_G, \mathbf{Z}/2\mathbf{Z})$. Note that one has

$$w_1(\rho) = B_\rho^*(\det[q]). \tag{58}$$

Pulling back the group extension C_q along the map $\rho : G \to \mathbf{O}(q)$, we obtain a group extension

$$1 \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1,$$

where $\tilde{G} := \tilde{\mathbf{O}}(q) \times_{\mathbf{O}(q)} G$. We denote by $C_G \in \mathrm{Ext}_Y(G, \mathbf{Z}/2\mathbf{Z})$ the class of this extension and by $[C_G]$ its cohomology class in $H^2(B_G, \mathbf{Z}/2\mathbf{Z})$ (see Proposition 3.8), so that

$$B_\rho^*([C_q]) = [C_G]. \tag{59}$$

Corollary 6.3. *Let G be a group scheme on Y and let (V, q, ρ) be an orthogonal representation of G. Then, in $H^*(B_G, \mathbf{Z}/2\mathbf{Z})$ we have*

(i) $w_1(q, \rho) = w_1(q) + w_1(\rho)$;
(ii) $w_2(q, \rho) = w_2(q) + w_1(q) \cdot w_1(\rho) + [C_G]$.

Proof. Let us denote by μ and π the morphisms of classifying topoi associated to the group morphisms $G_Y \to 1$ and $\mathbf{O}(q) \to 1$, respectively. We have $\mu \simeq \pi \circ B_\rho$ and therefore

$$\mu^*(w_i(q)) = B_\rho^* \pi^*(w_i(q)), \quad i = 1, 2. \tag{60}$$

As in subsection 4.5, we identify $H^i(Y_{fl}, \mathbf{Z}/2\mathbf{Z})$ as a direct summand of $H^i(B_G, \mathbf{Z}/2\mathbf{Z})$ (respectively, $H^i(B_{\mathbf{O}(q)}, \mathbf{Z}/2\mathbf{Z})$) via μ^* (respectively, π^*). In view of (57)–(60), the corollary is just the pull-back of Theorem 5.1 via B_ρ^*. □

6.3. *Fröhlich twists*

In this section, we extend the work of Fröhlich [**6**] and the results of [**2**], Theorem 0.4, to twists of quadratic forms by G-torsors when the group scheme G is not necessarily constant. Let G be a group scheme over Y and let (V, q, ρ) be an orthogonal representation of G.

Definition 6.4. For any G-torsor X on Y, we define the twist (V_X, q_X) to be the symmetric bundle on Y associated to the morphism

$$\{q_X\} : Y_{fl} \xrightarrow{X} B_G \xrightarrow{B_\rho} B_{\mathbf{O}(q)} \xrightarrow{T_q} B_{\mathbf{O}(n)}.$$

Equivalently, (V_X, q_X) is the twist of (V, q) given by the morphism

$$Y_{fl} \xrightarrow{X} B_G \xrightarrow{B_\rho} B_{\mathbf{O}(q)}.$$

The twist (V_X, q_X) can be made explicit in a number of situations (see, for example, Section 6.4). In order to compare $w(q)$ with $w(q_X)$, we denote by

$$\delta^1_{q,\rho} : H^1(Y_{fl}, G) \longrightarrow H^1(Y_{fl}, \mathbf{Z}/2\mathbf{Z})$$

the map induced by the group homomorphism $\det_q \circ \rho$ and by

$$\delta^2_{q,\rho} : H^1(Y_{fl}, G) \longrightarrow H^2(Y_{fl}, \mathbf{Z}/2\mathbf{Z})$$

the boundary map associated to the group extension C_G in Y_{fl}

$$1 \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1.$$

COROLLARY 6.5. *Let (V, q, ρ) be an orthogonal representation of G and let X be a G-torsor over Y. Then we have*

(i) $w_1(q_X) = w_1(q) + \delta^1_{q,\rho}[X]$;
(ii) $w_2(q_X) = w_2(q) + w_1(q) \cdot \delta^1_{q,\rho}[X] + \delta^2_{q,\rho}[X]$.

Proof. By definition $\delta^1_{q,\rho}[X]$ is the cohomology class associated to the morphism

$$B_{\det_q} \circ B_\rho \circ X : Y_{fl} \longrightarrow B_G \xrightarrow{B_\rho} B_{\mathbf{O}(q)} \xrightarrow{B_{\det_q}} B_{\mathbf{Z}/2\mathbf{Z}}.$$

It follows that

$$\delta^1_{q,\rho}[X] = (B_\rho \circ X)^*(\det[q]). \tag{61}$$

Moreover, one has

$$\delta^2_{q,\rho}[X] := X^*([C_G]) = X^* B^*_\rho([C_q]) = (B_\rho \circ X)^*([C_q]) \tag{62}$$

and

$$w_i(q_X) := (T_q \circ B_\rho \circ X)^*(HW_i) = (B_\rho \circ X)^*(HW_i(q)). \tag{63}$$

and finally, we have

$$w_i(q) = (B_\rho \circ X)^*(w_i(q)) \tag{64}$$

since $B_\rho \circ X$ is defined over Y_{fl}. In view of (61)–(64) the corollary now follows by pulling back the equality in Theorem 5.1 along the morphism $B_\rho \circ X$. □

REMARK 6.6. We have associated to any orthogonal representation $\rho : G \to \mathbf{O}(q)$ an exact sequence of Y-group schemes:

$$1 \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1.$$

For any G-torsor X over Y the class $\delta^2_{q,\rho}[X]$ may be seen as the obstruction of the embedding problem associated to the scheme X and the exact sequence. Corollary 6.4 provides us with a formula for this obstruction in terms of Hasse–Witt invariants of symmetric bundles:

$$\delta^2_{q,\rho}[X] = w_2(q_X) + w_2(q) + w_1(q)^2 + w_1(q)w_1(q_X).$$

In the particular case where $\rho : G_Y \to \mathbf{SO}(q)$ we have this remarkably simple formula

$$\delta^2_{q,\rho}[X] = w_2(q_X) - w_2(q),$$

which expresses this obstruction as a difference of two Hasse–Witt invariants of quadratic forms.

6.4. *An explicit description of the twisted form*

We shall use our previous work to obtain an explicit description of the twists which, in this geometric context, generalizes Fröhlich's construction [6]. Recall that (V_X, q_X) is the symmetric bundle on Y associated to the morphism

$$Y_{fl} \xrightarrow{X} B_G \xrightarrow{B_\rho} B_{\mathbf{O}(q)}.$$

In other words, (V_X, q_X) is determined by the fact that we have an isomorphism of $\mathbf{O}(q)$-torsors

$$\mathbf{Isom}(q, q_X) \simeq X \wedge^G \mathbf{O}(q).$$

Our goal is to provide a concrete description of (V_X, q_X) at least when G satisfies some additional hypotheses. We start by recalling the results of [1] in the affine case.

Definition 6.7. Let R be a commutative Noetherian integral domain with fraction field K. A finite and flat R-algebra A is said to satisfy $\mathbf{H}_2$ when A_K is a commutative separable K-algebra and the image under the counit of the set of integral of A is the square of a principal ideal of R.

Let $S = \mathrm{Spec}(R)$. We assume that $G \to S$ is a group scheme associated to a Hopf algebra A which satisfies $\mathbf{H}_2$ (we will say that G *satisfies* $\mathbf{H}_2$). We consider a G-equivariant symmetric bundle (V, q, ρ) given by a projective R-module V endowed with a non-degenerate quadratic form q and a group homomorphism $\rho : G \to \mathbf{O}(q)$. We have proved in [1, Theorem 3.1] that, for any G-torsor $X = \mathrm{Spec}(B) \to S$ where B is a commutative and finite R-algebra, the twist of (V, q, ρ) by X is defined by

$$(V_X, q_X) = (\mathcal{D}^{-1/2}(B) \otimes_R V, Tr \otimes q)^A.$$

The twist $(V_X; q_X)$ can be roughly described as the symmetric bundle obtained by taking the fixed point by A of the tensor product of (V, q) by the square root of the different of B, endowed with the trace form, (see [1], Sections 1 and 2 for the precise definitions).

We now come back to the general situation. We assume that $G \to S$ satisfies $\mathbf{H_2}$. Moreover, for the sake of simplicity, we suppose that Y is integral and flat over S. Let (V, q, ρ) be a G_Y-equivariant symmetric bundle and let X be a G_Y-torsor. For any affine open subscheme $U \to Y$ we set $X_U = X \times_Y U$ and $G_U = G_Y \times_Y U$. We know that by base change $X_U \to U$ is a G_U-torsor. Moreover, by restriction (V, q) defines an equivariant G_U-symmetric bundle $(V \mid U, q \mid U, \rho \mid U)$ over U. Using the functoriality properties of the different maps involved, it is easy to check that

$$(V_X, q_X) \mid U \simeq ((V \mid U)_{X_U}, (q \mid U)_{X_U}).$$

Since the properties of G are preserved by flat base change, we conclude that $G_U \to U$ satisfies $\mathbf{H_2}$ and therefore that, by the above, $((V \mid U)_{X_U}, (q \mid U)_{X_U})$ has now been explicitly described.

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