ZETA FUNCTIONS OF REGULAR ARITHMETIC SCHEMES AT $s = 0$

BAPTISTE MORIN

Abstract

Lichtenbaum conjectured the existence of a Weil-étale cohomology in order to describe the vanishing order and the special value of the zeta function of an arithmetic scheme $\mathcal{X}$ at $s = 0$ in terms of Euler–Poincaré characteristics. Assuming the (conjectured) finite generation of some étale motivic cohomology groups we construct such a cohomology theory for regular schemes proper over $\mathrm{Spec}(\mathbb{Z})$. *In particular, we obtain (unconditionally) the right Weil-étale cohomology for geometrically cellular schemes over number rings. We state a conjecture expressing the vanishing order and the special value up to sign of the zeta function $\zeta(\mathcal{X}, s)$ at $s = 0$ in terms of a perfect complex of abelian groups $R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z})$. Then we relate this conjecture to Soulé's conjecture and to the Tamagawa number conjecture of Bloch–Kato, and deduce its validity in simple cases.*

Contents

1. Introduction

Lichtenbaum [35] conjectured the existence of a Weil-étale cohomology in order to describe the vanishing order and the special value of the zeta function of an arithmetic scheme at $s = 0$ in terms of Euler–Poincaré characteristics. More precisely, we have the following.

DUKE MATHEMATICAL JOURNAL
Vol. 163, No. 7, © 2014 DOI 10.1215/00127094-2681387
Received 3 May 2012. Revision received 25 July 2013.
2010 *Mathematics Subject Classification*. Primary 14F20, 11G40.
Author's work supported by Marie Curie Fellowship 253346 and by Agence Nationale de la Recherche grants ANR-12-JS01-0007 and ANR-12-BS01-0002.

CONJECTURE 1.1 (Lichtenbaum [35, Introduction])
On the category of separated schemes of finite type $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$*, there exists a cohomology theory given by abelian groups* $H^i_{W,c}(\mathcal{X},\mathbb{Z})$ *and real vector spaces* $H^i_W(\mathcal{X},\tilde{\mathbb{R}})$ *and* $H^i_{W,c}(\mathcal{X},\tilde{\mathbb{R}})$ *such that the following holds.*

(a) *The groups* $H^i_{W,c}(\mathcal{X},\mathbb{Z})$ *are finitely generated and zero for i large.*

(b) *The natural map from* $\mathbb{Z}$*- to* $\tilde{\mathbb{R}}$*-coefficients induces isomorphisms*

$$H^i_{W,c}(\mathcal{X},\mathbb{Z}) \otimes \mathbb{R} \simeq H^i_{W,c}(\mathcal{X},\tilde{\mathbb{R}}).$$

(c) *There exists a canonical class* $\theta \in H^1_W(\mathcal{X},\tilde{\mathbb{R}})$ *such that cup product with* θ *turns the sequence*

$$\cdots \xrightarrow{\cup\theta} H^i_{W,c}(\mathcal{X},\tilde{\mathbb{R}}) \xrightarrow{\cup\theta} H^{i+1}_{W,c}(\mathcal{X},\tilde{\mathbb{R}}) \xrightarrow{\cup\theta} \cdots$$

into a bounded acyclic complex of finite-dimensional vector spaces.

(d) *The vanishing order of the zeta function* $\zeta(\mathcal{X},s)$ *at* $s=0$ *is given by the formula*

$$\mathrm{ord}_{s=0}\,\zeta(\mathcal{X},s) = \sum_{i\geq 0}(-1)^i \cdot i \cdot \mathrm{rank}_{\mathbb{Z}}\, H^i_{W,c}(\mathcal{X},\mathbb{Z}).$$

(e) *The leading coefficient* $\zeta^*(\mathcal{X},0)$ *in the Taylor expansion of* $\zeta(\mathcal{X},s)$ *at* $s=0$ *is given up to sign by*

$$\mathbb{Z}\cdot\lambda\big(\zeta^*(\mathcal{X},0)^{-1}\big) = \bigotimes_{i\in\mathbb{Z}} \det\nolimits_{\mathbb{Z}} H^i_{W,c}(\mathcal{X},\mathbb{Z})^{(-1)^i},$$

where $\lambda : \mathbb{R} \xrightarrow{\sim} (\bigotimes_{i\in\mathbb{Z}} \det_{\mathbb{Z}} H^i_{W,c}(\mathcal{X},\mathbb{Z})^{(-1)^i}) \otimes \mathbb{R}$ *is induced by (b) and (c).*

Such a cohomology theory for smooth varieties over finite fields was defined in [34], but similar attempts to construct such cohomology groups for flat arithmetic schemes failed. Lichtenbaum [35] gave the first construction for number rings. He defined a Weil-étale topology which bears the same relation to the usual étale topology as the Weil group does to the Galois group. Under a vanishing statement, he was able to show that his cohomology miraculously yields the value of Dedekind zeta functions at $s=0$. But this cohomology with coefficients in $\mathbb{Z}$ was then shown by Flach to be infinitely generated and hence nonvanishing in even degrees $i \geq 4$ (see [12]). Consequently, Lichtenbaum's complex computing the cohomology with $\mathbb{Z}$-coefficients needs to be artificially truncated in the case of number rings, and is not helpful for flat schemes of dimension greater than 1. However, the Weil-étale topology yields the expected cohomology with $\tilde{\mathbb{R}}$-coefficients, and this fact extends to higher-dimensional arithmetic schemes (see [13]).

The first goal of this paper is to define the right Weil-étale cohomology with $\mathbb{Z}$-coefficients for arithmetic schemes satisfying Conjecture 1.2 (see Theorem 1.3) in order to state a precise version of Conjecture 1.1. We consider a regular scheme $\mathcal{X}$ proper over $\operatorname{Spec}(\mathbb{Z})$ of pure Krull dimension d, and we denote by $\mathbb{Z}(d)$ Bloch's cycle complex. The following conjecture is suggested by [33, Sections 7, 8].

CONJECTURE 1.2 (Lichtenbaum)
The étale motivic cohomology groups $H^i(\mathcal{X}_{\text{ét}}, \mathbb{Z}(d))$ are finitely generated for $0 \leq i \leq 2d$.

Using some purity for the cycle complex $\mathbb{Z}(d)$ on the étale site (which is the key result of [19]) we construct in Section 5 a class $\mathcal{L}(\mathbb{Z})$ (see Definition 5.9) of separated schemes of finite type over $\operatorname{Spec}(\mathbb{Z})$ satisfying Conjecture 1.2. This class $\mathcal{L}(\mathbb{Z})$ contains any geometrically cellular scheme over a number ring (see Definition 5.13), and includes the class $A(\mathbb{F}_q)$ of smooth projective varieties over $\mathbb{F}_q$, which can be constructed out of products of smooth projective curves by unions, base extensions, blowups, and quasidirect summands in the category of Chow motives. (The class $A(\mathbb{F}_q)$ was introduced by Soulé [43].)

In order to state our first main result, we need to fix some notation. For a scheme $\mathcal{X}$ separated and of finite type over $\operatorname{Spec}(\mathbb{Z})$, we consider the quotient topological space $\mathcal{X}_\infty := \mathcal{X}(\mathbb{C})/G_{\mathbb{R}}$ where $\mathcal{X}(\mathbb{C})$ is endowed with the complex topology and we set $\overline{\mathcal{X}} := (\mathcal{X}, \mathcal{X}_\infty)$. We denote by $\overline{\mathcal{X}}_{\text{ét}}$ the Artin–Verdier étale topos of $\overline{\mathcal{X}}$, which comes with a closed embedding $u_\infty : \operatorname{Sh}(\mathcal{X}_\infty) \to \overline{\mathcal{X}}_{\text{ét}}$ in the sense of topos theory, where $\operatorname{Sh}(\mathcal{X}_\infty)$ is the category of sheaves on the topological space $\mathcal{X}_\infty$ (see [13, Section 4]). The Weil-étale topos over the archimedean place $\mathcal{X}_{\infty,W}$ is associated to the trivial action of the topological group $\mathbb{R}$ on the topological space $\mathcal{X}_\infty$ (see [13, Section 6]). Our first main result is the following.

THEOREM 1.3
Let $\mathcal{X}$ be a regular scheme proper over $\operatorname{Spec}(\mathbb{Z})$. *Assume that $\mathcal{X} \in \mathcal{L}(\mathbb{Z})$, or more generally, assume that the connected components of $\mathcal{X}$ satisfy Conjecture 1.2. Then there exist complexes $R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})$ and $R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z})$ such that the following properties hold.*

- *If $\mathcal{X}$ has pure dimension d, there is an exact triangle*

$$R\operatorname{Hom}\big(\tau_{\geq 0} R\Gamma\big(\mathcal{X}, \mathbb{Q}(d)\big), \mathbb{Q}[-2d-2]\big) \to R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}) \to R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}). \quad (1)$$

- *The complex $R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})$ is contravariantly functorial.*
- *There exists a unique morphism $i_\infty^* : R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}) \to R\Gamma(\mathcal{X}_{\infty,W}, \mathbb{Z})$ which renders the following square commutative:*

$$\begin{array}{ccc} R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}) & \longrightarrow & R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}) \\ \downarrow u_\infty^* & & \downarrow i_\infty^* \\ R\Gamma(\mathcal{X}_\infty,\mathbb{Z}) & \longrightarrow & R\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z}) \end{array}$$

We define $R\Gamma_{W,c}(\mathcal{X},\mathbb{Z})$ so that there is an exact triangle

$$R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}) \to R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}) \stackrel{i_\infty^*}{\to} R\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z}).$$

- *The cohomology groups $H^i_W(\overline{\mathcal{X}},\mathbb{Z})$ and $H^i_{W,c}(\mathcal{X},\mathbb{Z})$ are finitely generated for all i and zero for i large.*
- *The cohomology groups $H^i_W(\overline{\mathcal{X}},\mathbb{Z})$ form an integral model for l-adic cohomology: for any prime number l and any $i \in \mathbb{Z}$ there is a canonical isomorphism*

$$H^i_W(\overline{\mathcal{X}},\mathbb{Z}) \otimes \mathbb{Z}_l \simeq H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}_l).$$

- *If $\mathcal{X}$ has characteristic p, then there is a canonical isomorphism in the derived category*

$$R\Gamma(\mathcal{X}_W,\mathbb{Z}) \xrightarrow{\sim} R\Gamma_W(\mathcal{X},\mathbb{Z}),$$

where $R\Gamma(\mathcal{X}_W,\mathbb{Z})$ is the cohomology of the Weil-étale topos (see [34]) and $R\Gamma_W(\mathcal{X},\mathbb{Z})$ is the complex defined in this paper. Moreover, the exact triangle (1) is isomorphic to Geisser's triangle (see [17, Corollary 5.2] for $\mathcal{G}^\cdot = \mathbb{Z}$).
- *If $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$ is the spectrum of a totally imaginary number ring, then there is a canonical isomorphism in the derived category*

$$R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}) \xrightarrow{\sim} \tau_{\leq 3} R\Gamma(\overline{\mathcal{X}}_W,\mathbb{Z}),$$

where $\tau_{\leq 3} R\Gamma(\overline{\mathcal{X}}_W,\mathbb{Z})$ is the truncation of Lichtenbaum's complex (see [35]) and $R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})$ is the complex defined in this paper.

Theorem 1.3 is proven in Definition 2.9, Proposition 2.10, Corollary 2.12, Corollary 2.11, Theorem 2.13, Theorem 2.15, and Proposition 3.1. Notice that the same formalism is used to treat flat arithmetic schemes and schemes over finite fields (see [35, Introduction, Question 1]). The basic idea behind the proof of Theorem 1.3 can be explained as follows. The Weil group is defined as an extension of the Galois group by the idèle class group corresponding to the fundamental class of class field theory (more precisely, as the limit of these group extensions). In this paper we use étale duality for arithmetic schemes rather than class field theory, in order to obtain a canonical "extension" of the étale $\mathbb{Z}$-cohomology by the dual of motivic $\mathbb{Q}(d)$-cohomology. More precisely, our Weil-étale complex is defined as the cone of a map

$$\alpha_{\mathcal{X}} : R\,\mathrm{Hom}\big(\tau_{\geq 0} R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big),\mathbb{Q}[-2d-2]\big) \longrightarrow R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}),$$

where $\alpha_{\mathcal{X}}$ is constructed out of étale duality. Here the scheme $\mathcal{X}$ is of pure dimension d. This idea was suggested by works of Burns [7], Geisser [17], and the author [38]. The techniques involved in this paper rely on results due to Geisser and Levine on Bloch's cycle complex (see [16], [19], [20], [21], [30]).

From now on, Conjecture 1.1 is understood as a list of expected properties for the complex $R\Gamma_{W,c}(\mathcal{X},\mathbb{Z})$ of Theorem 1.3. The second goal of this paper is to relate Conjecture 1.1 to Soulé's conjecture [44, Conjecture 2.2] and to the Tamagawa number conjecture of Bloch–Kato (in the formulation of Fontaine and Perrin-Riou, see [14, Section 1] and [15, Chapitre III, Conjecture 4.5.2]). To this end, we need to assume the following conjecture, which is a special case of a natural refinement for arithmetic schemes of the classical conjecture of Beilinson relating motivic cohomology to Deligne cohomology (see [40, Conjectures 6.1(1), 6.3(1)]). Let $\mathcal{X}$ be a regular, proper, and flat arithmetic scheme of pure dimension d.

CONJECTURE 1.4 (Beilinson)
The Beilinson regulator

$$H^{2d-1-i}\big(\mathcal{X},\mathbb{Q}(d)\big)_{\mathbb{R}} \to H^{2d-1-i}_{\mathcal{D}}\big(\mathcal{X}_{/\mathbb{R}},\mathbb{R}(d)\big)$$

is an isomorphism for $i \geq 1$ and there is an exact sequence

$$0 \to H^{2d-1}\big(\mathcal{X},\mathbb{Q}(d)\big)_{\mathbb{R}} \to H^{2d-1}_{\mathcal{D}}\big(\mathcal{X}_{/\mathbb{R}},\mathbb{R}(d)\big) \to CH^0(\mathcal{X}_{\mathbb{Q}})^*_{\mathbb{R}} \to 0.$$

Now we can state our second main result. If $\mathcal{X}$ is defined over a number ring $\mathcal{O}_F$, we set $\mathcal{X}_F = \mathcal{X} \otimes_{\mathcal{O}_F} F$ and $\mathcal{X}_{\overline{F}} = \mathcal{X} \otimes_{\mathcal{O}_F} \overline{F}$.

THEOREM 1.5
Assume that $\mathcal{X}$ satisfies Conjectures 1.2 and 1.4.

- *Conjectures 1.1(a), 1.1(b), and 1.1(c) hold for $\mathcal{X}$.*
- *Assume that $\mathcal{X}$ is projective over $\mathbb{Z}$. Then Conjecture 1.1(d), that is, the identity*

  $$\mathrm{ord}_{s=0}\,\zeta(\mathcal{X},s) = \sum_{i\geq 0}(-1)^i \cdot i \cdot \mathrm{rank}_{\mathbb{Z}}\, H^i_{W,c}(\mathcal{X},\mathbb{Z}),$$

  is equivalent to Soulé's conjecture [44, Conjecture 2.2] for the vanishing order of $\zeta(\mathcal{X},s)$ at $s=0$.

- *Assume that $\mathcal{X}$ is smooth projective over a number ring $\mathcal{O}_F$, and assume that the representations $H^i(\mathcal{X}_{\overline{F},\text{ét}}, \mathbb{Q}_l)$ of G_F satisfy $H^1_f(F, H^i(\mathcal{X}_{\overline{F},\text{ét}}, \mathbb{Q}_l)) = 0$. Then Conjecture 1.1(e), that is, the identity*

$$\mathbb{Z} \cdot \lambda\big(\zeta^*(\mathcal{X}, 0)^{-1}\big) = \bigotimes_{i\in\mathbb{Z}} \det_{\mathbb{Z}} H^i_{W,c}(\mathcal{X}, \mathbb{Z})^{(-1)^i},$$

is equivalent to the Tamagawa number conjecture [15, Chapitre III, Conjecture 4.5.2] for the motive $\bigoplus_{i=0}^{2d-2} h^i(\mathcal{X}_F)[-i]$.

The proof of the last statement of Theorem 1.5 was already given in [13], assuming expected properties of Weil-étale cohomology. This proof is based on the fact that $\bigotimes_{i\in\mathbb{Z}} \det_{\mathbb{Z}} H^i_{W,c}(\mathcal{X}, \mathbb{Z})^{(-1)^i}$ provides the fundamental line (in the sense of [14]) with a canonical $\mathbb{Z}$-structure. This result shows that the Weil-étale point of view is compatible with the Tamagawa number conjecture of Bloch–Kato, answering a question of Lichtenbaum (see [35, Introduction, Question 2]).

We obtain examples of flat arithmetic schemes satisfying Conjecture 1.1.

THEOREM 1.6
For any number field F, Conjecture 1.1 holds for $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$.

Let $\mathcal{X}$ be a smooth projective scheme over the number ring $\mathcal{O}_{\mathcal{X}}(\mathcal{X}) = \mathcal{O}_F$, where F is an abelian number field. Assume that $\mathcal{X}_F$ admits a smooth cellular decomposition (see Definition 5.13), and assume that $\mathcal{X} \in \mathcal{L}(\mathbb{Z})$ (see Definition 5.9). Then Conjecture 4.2 holds for $\mathcal{X}$.

We refer to [10] for a proof of a dynamical system analogue of Conjecture 1.1.

Notations
We denote by $\mathbb{Z}$ the ring of integers, and by $\mathbb{Q}$, $\mathbb{Q}_p$, $\mathbb{R}$, and $\mathbb{C}$ the fields of rational, p-adic, real, and complex numbers, respectively. An arithmetic scheme is a scheme which is separated and of finite type over $\mathrm{Spec}(\mathbb{Z})$. An arithmetic scheme $\mathcal{X}$ is said to be proper (respectively, flat) if the map $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ is proper (respectively, flat).

For a field k, we choose a separable closure $\bar{k}/k$ and we denote by $G_k = \mathrm{Gal}(\bar{k}/k)$ the absolute Galois group of k. For a proper scheme $\mathcal{X}$ over $\mathrm{Spec}(\mathbb{Z})$ we denote by $\mathcal{X}_\infty := \mathcal{X}(\mathbb{C})/G_{\mathbb{R}}$ the quotient topological space of $\mathcal{X}(\mathbb{C})$ by $G_{\mathbb{R}}$ where $\mathcal{X}(\mathbb{C})$ is given with the complex topology. We consider the Artin–Verdier étale topos $\overline{\mathcal{X}}_{\text{ét}}$ given with the open-closed decomposition of topoi

$$\varphi : \mathcal{X}_{\text{ét}} \to \overline{\mathcal{X}}_{\text{ét}} \leftarrow \mathrm{Sh}(\mathcal{X}_\infty) : u_\infty,$$

where $\mathcal{X}_{\text{ét}}$ is the usual étale topos of the scheme $\mathcal{X}$ (i.e., the category of sheaves of sets on the small étale site of $\mathcal{X}$) and $\mathrm{Sh}(\mathcal{X}_\infty)$ is the category of sheaves of sets on the topological space $\mathcal{X}_\infty$ (see [13, Section 4]). For any abelian sheaf A on $\mathcal{X}_{\text{ét}}$ and any $n > 0$, the sheaf $R^n\varphi_* A$ is a 2-torsion sheaf concentrated on $\mathcal{X}(\mathbb{R})$. In particular, if $\mathcal{X}(\mathbb{R}) = \emptyset$, then $\varphi_* \simeq R\varphi_*$. For $\mathcal{X}(\mathbb{C}) = \emptyset$ one has $\mathcal{X}_{\text{ét}} = \overline{\mathcal{X}}_{\text{ét}}$. For $T = \mathcal{X}_{\text{ét}}$, $\overline{\mathcal{X}}_{\text{ét}}$, or $\mathrm{Sh}(\mathcal{X}_\infty)$ or more generally for any Grothendieck topos T, we denote by $\Gamma(T, -)$ the global section functor, by $R\Gamma(T, -)$ its total right derived functor, and by $H^i(T, -) := H^i(R\Gamma(T, -))$ its cohomology.

Let $\mathbb{Z}(n) := z^n(-, 2n - *)$ be Bloch's cycle complex (see [2], [16], [18], [31], [30]), which we consider as a complex of abelian sheaves on the small étale (or Zariski) site of $\mathcal{X}$. For an abelian group A we define $A(n)$ to be $\mathbb{Z}(n) \otimes A$. Note that $\mathbb{Z}(n)$ is a complex of flat sheaves; hence the tensor product and derived tensor product with $\mathbb{Z}(n)$ agree. We denote by $H^i(\mathcal{X}_{\text{ét}}, A(n))$ the étale hypercohomology of $A(n)$, and by $H^i(\mathcal{X}, A(n)) := H^i(\mathcal{X}_{\text{Zar}}, A(n))$ its Zariski hypercohomology. For $\mathcal{X}(\mathbb{R}) = \emptyset$, we still denote by $A(n) = \varphi_* A(n)$ the pushforward of the cycle complex $A(n)$ on $\overline{\mathcal{X}}_{\text{ét}}$, and by $H^i(\overline{\mathcal{X}}_{\text{ét}}, A(n))$ the étale hypercohomology of $A(n)$. (In this paper $A = \mathbb{Z}$, $\mathbb{Z}/m\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{Q}/\mathbb{Z}$.) Notice that the complex of étale sheaves $\mathbb{Z}/m\mathbb{Z}(n)$ is not in general quasiisomorphic to $\mu_m^{\otimes n}[0]$ (however, see [16, Theorem 1.2(4)]).

We denote by $\mathcal{D}$ the derived category of the category of abelian groups. More generally, we write $\mathcal{D}(R)$ for the derived category of the category of R-modules, where R is a commutative ring. An exact (i.e., distinguished) triangle in $\mathcal{D}(R)$ is (somewhat abusively) depicted as follows:

$$C' \to C \to C'',$$

where C, C', and C'' are objects of $\mathcal{D}(R)$. For an object C of $\mathcal{D}(R)$ we write $C_{\leq n}$ (respectively, $C_{\geq n}$) for the truncated complex so that $H^i(C_{\leq n}) = H^i(C)$ for $i \leq n$ and $H^i(C_{\leq n}) = 0$ otherwise (respectively, so that $H^i(C_{\geq n}) = H^i(C)$ for $i \geq n$ and $H^i(C_{\geq n}) = 0$ otherwise).

Let R be a principal ideal domain. For a finitely generated free R-module L of rank r, we set $\det_R(L) = \bigwedge_R^r L$ and $\det_R(L)^{-1} = \mathrm{Hom}_R(\det_R(L), R)$. For a finitely generated R-module M, one may choose a resolution given by an exact sequence $0 \to L_{-1} \to L_0 \to M \to 0$, where L_{-1} and L_0 are finitely generated free R-modules. Then

$$\det_R(M) := \det_R(L_0) \otimes_R \det_R(L_{-1})^{-1}$$

does not depend on the resolution. If $C \in \mathcal{D}(R)$ is a complex such that $H^i(C)$ is finitely generated for all $i \in \mathbb{Z}$ and $H^i(C) = 0$ for almost all i, we denote $\det_R(C) := \bigotimes_{i \in \mathbb{Z}} \det_R H^i(C)^{(-1)^i}$.

For an abelian group A we write A_{tor} and A_{div} for the maximal torsion and the maximal divisible subgroups of A, respectively, and we set $A_{\text{codiv}} = A/A_{\text{div}}$ and

$A_{\mathrm{cotor}} = A/A_{\mathrm{tor}}$. We denote by ${}_nA$ and A_n the kernel and the cokernel of the map $n : A \to A$ (multiplication by n). We write $A^D = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ for the Pontryagin dual of a finite abelian group A.

2. Weil-étale cohomology

2.1. The conjectures $\mathbf{L}(\mathcal{X}_{\mathrm{\acute{e}t}}, d)$ *and* $\mathbf{L}(\mathcal{X}_{\mathrm{\acute{e}t}}, d)_{\geq 0}$

Let $\mathcal{X}$ be a proper, regular, and connected arithmetic scheme of dimension d. The following conjecture is an étale version of (a special case of) the motivic Bass conjecture (see [26, Conjecture 37]).

CONJECTURE 2.1 ($\mathbf{L}(\mathcal{X}_{\mathrm{\acute{e}t}}, d)$)
For $i \leq 2d$, the abelian group $H^i(\mathcal{X}_{\mathrm{\acute{e}t}}, \mathbb{Z}(d))$ is finitely generated.

In order to define the Weil-étale cohomology we shall consider schemes satisfying a weak version of the previous conjecture.

CONJECTURE 2.2 ($\mathbf{L}(\mathcal{X}_{\mathrm{\acute{e}t}}, d)_{\geq 0}$)
For $0 \leq i \leq 2d$, the abelian group $H^i(\mathcal{X}_{\mathrm{\acute{e}t}}, \mathbb{Z}(d))$ is finitely generated.

2.2. The morphism $\alpha_{\mathcal{X}}$

Let $\mathcal{X}$ be a proper, regular, and connected arithmetic scheme. The goal of this section is to construct (conditionally) a certain morphism $\alpha_{\mathcal{X}}$ in the derived category of abelian groups $\mathcal{D}$, in order to define in Section 2.3 the Weil-étale complex $R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})$ as the cone of $\alpha_{\mathcal{X}}$.

2.2.1. The case $\mathcal{X}(\mathbb{R}) = \emptyset$

In this section, $\mathcal{X}$ denotes a proper, regular, and connected arithmetic scheme of dimension d such that $\mathcal{X}(\mathbb{R})$ is empty. In particular, φ_* is exact (since $\mathcal{X}(\mathbb{R}) = \emptyset$); hence $H^i(\overline{\mathcal{X}}_{\mathrm{\acute{e}t}}, \mathbb{Z}(d)) := H^i(\overline{\mathcal{X}}_{\mathrm{\acute{e}t}}, \varphi_*\mathbb{Z}(d)) \simeq H^i(\mathcal{X}_{\mathrm{\acute{e}t}}, \mathbb{Z}(d))$.

LEMMA 2.3
We have that

$$\begin{aligned} H^i\big(\overline{\mathcal{X}}_{\mathrm{\acute{e}t}}, \mathbb{Z}(d)\big) &= 0 \quad \textit{for } i > 2d+2 \\ &\simeq \mathbb{Q}/\mathbb{Z} \quad \textit{for } i = 2d+2 \\ &= 0 \quad \textit{for } i = 2d+1 \\ &\simeq H^i\big(\mathcal{X}, \mathbb{Q}(d)\big) \quad \textit{for } i < 0. \end{aligned}$$

Proof
For any positive integer n and any $i \in \mathbb{Z}$, one has

$$H^{i-1}\big(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}(d)\big) = \operatorname{Ext}^{i-1}_{\mathcal{X},\mathbb{Z}/n\mathbb{Z}}\big(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}(d)\big) \tag{2}$$

$$\simeq \operatorname{Ext}^{i}_{\mathcal{X}}\big(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}(d)\big) \tag{3}$$

$$\simeq H^{2d+2-i}(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})^D, \tag{4}$$

where (3) and (4) are given by [19, Lemma 2.4 and Theorem 7.8], respectively, and $H^{2d+2-i}(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})^D$ denotes the Pontryagin dual of the finite group $H^{2d+2-i}(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})$.

The complex $\mathbb{Z}(d)$ consists of flat sheaves; hence the tensor product and derived tensor product with $\mathbb{Z}(d)$ agree. Therefore, applying $\mathbb{Z}(d) \otimes^L (-)$ to the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ yields an exact triangle:

$$\mathbb{Z}(d) \to \mathbb{Q}(d) \to \mathbb{Q}/\mathbb{Z}(d). \tag{5}$$

Moreover, we have $H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Q}(d)) \simeq H^i(\mathcal{X}, \mathbb{Q}(d))$ for any i (see [16, Proposition 3.6]) and $H^i(\mathcal{X}, \mathbb{Q}(d)) = 0$ for any $i > 2d$ (see [31, Lemma 11.1]). Hence (5) gives

$$H^i\big(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}(d)\big) \simeq H^{i-1}\big(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Q}/\mathbb{Z}(d)\big) \simeq \varinjlim H^{i-1}\big(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}(d)\big)$$

for $i \geq 2d+2$. The result for $i \geq 2d+2$ then follows from (4).

By [23, Exposé X, Théorème 6.2], the scheme $\mathcal{X}$ has l-cohomological dimension $2d+1$ for any prime number l (note that $\mathcal{X}(\mathbb{R}) = \emptyset$); hence (4) shows that $H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Q}/\mathbb{Z}(d)) = 0$ for $i < 0$. The result for $i < 0$ now follows from $H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Q}(d)) \simeq H^i(\mathcal{X}, \mathbb{Q}(d))$ and from the exact triangle (5).

It remains to treat the case $i = 2d+1$. Assume that $\mathcal{X}$ is flat over $\mathbb{Z}$. Then we have (see [27, Theorem 6.1(2)])

$$H^{2d}\big(\mathcal{X}, \mathbb{Q}(d)\big) \simeq CH^d(\mathcal{X}) \otimes \mathbb{Q} = 0; \tag{6}$$

hence $H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Q}(d)) = 0$ for $i \geq 2d$. Here $CH^d(\mathcal{X})$ denotes the Chow group of cycles of codimension d. We obtain

$$\begin{aligned}
H^{2d+1}\big(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}(d)\big) &\simeq H^{2d}\big(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Q}/\mathbb{Z}(d)\big) \\
&\simeq \varinjlim\big(H^1(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})^D\big) \\
&\simeq \big(\varprojlim \operatorname{Hom}\big(\pi_1(\overline{\mathcal{X}}_{\text{ét}}), \mathbb{Z}/n\mathbb{Z}\big)\big)^D \\
&\simeq \operatorname{Hom}\big(\pi_1(\overline{\mathcal{X}}_{\text{ét}})^{\text{ab}}, \widehat{\mathbb{Z}}\big)^D \\
&= 0,
\end{aligned}$$

where the first isomorphism (resp., the second) is given by (5) (resp., by (4)), and the last isomorphism follows from the fact that the abelian fundamental group $\pi_1(\overline{\mathcal{X}}_{\text{ét}})^{\text{ab}}$ is finite (see [27, Theorem 9.10] and [49, Corollary 3]).

Assume now that $\mathcal{X}$ is a smooth proper scheme over a finite field (hence $\overline{\mathcal{X}} = \mathcal{X}$). One has

$$\begin{aligned} H^{2d}\big(\mathcal{X}_{\text{ét}}, \mathbb{Q}/\mathbb{Z}(d)\big) &\simeq \varinjlim \big(H^1(\mathcal{X}_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})^D\big) & (7)\\ &\simeq \varinjlim \big(\operatorname{Hom}\big(\pi_1(\mathcal{X}_{\text{ét}}), \mathbb{Z}/n\mathbb{Z}\big)^D\big) & (8)\\ &\simeq \varinjlim \big(CH^d(\mathcal{X}) \otimes \mathbb{Z}/n\mathbb{Z}\big) & (9)\\ &\simeq CH^d(\mathcal{X}) \otimes \mathbb{Q}/\mathbb{Z}, & (10)\end{aligned}$$

where (7) is given by (4) while (9) follows from class field theory (see [49, Corollary 3]). The exact triangle (5) yields an exact sequence

$$H^{2d}\big(\mathcal{X}, \mathbb{Q}(d)\big) \xrightarrow{p} H^{2d}\big(\mathcal{X}_{\text{ét}}, \mathbb{Q}/\mathbb{Z}(d)\big) \to H^{2d+1}\big(\mathcal{X}_{\text{ét}}, \mathbb{Z}(d)\big) \to 0. \qquad (11)$$

Moreover, the isomorphism (10) is the direct limit of the maps

$$CH^d(\mathcal{X})_n = H^{2d}\big(\mathcal{X}, \mathbb{Z}(d)\big)_n \to H^{2d}\big(\mathcal{X}, \mathbb{Z}/n\mathbb{Z}(d)\big) \to H^{2d}\big(\mathcal{X}_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}(d)\big).$$

It follows that the map p in the sequence (11) can be identified with the morphism

$$CH^d(\mathcal{X}) \otimes \mathbb{Q} \to CH^d(\mathcal{X}) \otimes \mathbb{Q}/\mathbb{Z}$$

induced by the surjection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. Hence p is surjective, since $CH^d(\mathcal{X})$ is finitely generated of rank one (see [27, Theorem 6.1]), so that $H^{2d+1}(\mathcal{X}_{\text{ét}}, \mathbb{Z}(d)) = 0$. □

Notation 2.4
We set $H^j(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}(d))_{\geq 0} := H^j(R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}(d))_{\geq 0})$.

LEMMA 2.5
Assume that $\mathcal{X}$ satisfies $\mathbf{L}(\mathcal{X}_{\text{ét}}, d)_{\geq 0}$. Then the natural map

$$H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}) \to \operatorname{Hom}\big(H^{2d+2-i}\big(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}(d)\big)_{\geq 0}, \mathbb{Q}/\mathbb{Z}\big)$$

is an isomorphism of abelian groups for $i \geq 1$.

Proof
The map of the lemma is given by the pairing

$$H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}) \times \operatorname{Ext}^{2d+2-i}_{\overline{\mathcal{X}}}\big(\mathbb{Z}, \mathbb{Z}(d)\big) \to H^{2d+2}\big(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}(d)\big) \simeq \mathbb{Q}/\mathbb{Z}.$$

For $i=1$ the result follows from Lemma 2.3, since one has $H^1(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})=\operatorname{Hom}_{\text{cont}}(\pi_1(\overline{\mathcal{X}}_{\text{ét}},p),\mathbb{Z})=0$ because the fundamental group $\pi_1(\overline{\mathcal{X}}_{\text{ét}},p)$ is profinite (hence compact).

The scheme $\mathcal{X}$ is connected and normal; hence $H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Q})=H^i(\mathcal{X}_{\text{ét}},\mathbb{Q})=\mathbb{Q},0$ for $i=0$ and $i\geq 1$, respectively. We obtain

$$H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})=H^{i-1}(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Q}/\mathbb{Z})=\varinjlim H^{i-1}(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}/n\mathbb{Z})$$

for $i\geq 2$, since $\mathcal{X}$ is quasicompact and quasiseparated. For any positive integer n the canonical map

$$\operatorname{Ext}^{2d+2-i}_{\overline{\mathcal{X}}}\big(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}(d)\big)\times H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}/n\mathbb{Z})\to\mathbb{Q}/\mathbb{Z}$$

is a perfect pairing of finite groups (see [19, Theorem 7.8]). The short exact sequence

$$0\to\mathbb{Z}\to\mathbb{Z}\to\mathbb{Z}/n\mathbb{Z}\to 0$$

yields an exact sequence

$$\begin{aligned}H^{j-1}\big(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d)\big)\to H^{j-1}\big(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d)\big)&\to\operatorname{Ext}^{j}_{\overline{\mathcal{X}}}\big(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}(d)\big)\\&\to H^{j}\big(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d)\big)\to H^{j}\big(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d)\big).\end{aligned}$$

We obtain a short exact sequence

$$0\to H^{j-1}\big(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d)\big)_n\to\operatorname{Ext}^{j}_{\overline{\mathcal{X}}}\big(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}(d)\big)\to{}_nH^{j}\big(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d)\big)\to 0$$

for any j. By the left exactness of projective limits, the sequence

$$0\to\varprojlim H^{j-1}\big(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d)\big)_n\to\varprojlim\operatorname{Ext}^{j}_{\overline{\mathcal{X}}}\big(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}(d)\big)\to\varprojlim{}_nH^{j}\big(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d)\big)$$

is exact. The module $\varprojlim{}_nH^j(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d))$ vanishes for $0\leq j\leq 2d+1$ since $H^j(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d))$ is assumed to be finitely generated for such j. We have $\varprojlim{}_nH^j(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d))=0$ for $j<0$ since $H^j(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d))$ is uniquely divisible for $j<0$ (see Lemma 2.3). This yields an isomorphism of profinite groups

$$\begin{aligned}\varprojlim H^{j-1}\big(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d)\big)_n&\xrightarrow{\sim}\varprojlim\operatorname{Ext}^{j}_{\overline{\mathcal{X}}}\big(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}(d)\big)\\&\xrightarrow{\sim}\big(\varinjlim H^{2d+2-j}(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}/n\mathbb{Z})\big)^D,\end{aligned}$$

where the last isomorphism follows from the duality above (and from the fact that $\operatorname{Ext}^{j}_{\overline{\mathcal{X}}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}(d))$ is finite for any n). This gives isomorphisms of torsion groups

$$\begin{aligned} H^{2d+2-(j-1)}(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}) &\xrightarrow{\sim} \big(\varinjlim H^{2d+2-j}(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}/n\mathbb{Z})\big)^{DD} \\ &\xrightarrow{\sim} \big(\varprojlim H^{j-1}\big(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d)\big)_n\big)^{D} \\ &\xrightarrow{\sim} \operatorname{Hom}\big(H^{j-1}\big(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d)\big)_{\geq 0},\mathbb{Q}/\mathbb{Z}\big) \end{aligned}$$

for any $j \leq 2d+1$. (Note that $2d+2-(j-1) \geq 2 \Leftrightarrow j \leq 2d+1$.) The last isomorphism above follows from the fact that $H^j(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d))$ is finitely generated for $0 \leq j \leq 2d$ and uniquely divisible for $j < 0$. Hence for any $i \geq 2$ the natural map

$$H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}) \to \operatorname{Hom}\big(H^{2d+2-i}\big(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d)\big)_{\geq 0},\mathbb{Q}/\mathbb{Z}\big)$$

is an isomorphism. □

Recall that an abelian group A is of *cofinite type* if $A \simeq \operatorname{Hom}_{\mathbb{Z}}(B,\mathbb{Q}/\mathbb{Z})$ where B is a finitely generated abelian group. If A is of cofinite type, then there exists an isomorphism $A \simeq (\mathbb{Q}/\mathbb{Z})^r \oplus T$, where $r \in \mathbb{N}$ and T is a finite abelian group. If $\mathcal{X}$ satisfies $\mathbf{L}(\mathcal{X}_{\text{ét}},d)_{\geq 0}$, then $H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})$ is of cofinite type for $i \geq 1$ by Lemma 2.5.

THEOREM 2.6
Assume that $\mathcal{X}$ satisfies $\mathbf{L}(\mathcal{X}_{\text{ét}},d)_{\geq 0}$. There exists a unique morphism in $\mathcal{D}$

$$\alpha_{\mathcal{X}} : R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{Q}[-2d-2]\big) \to R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})$$

such that $H^i(\alpha_{\mathcal{X}})$ is the following composite map:

$$\begin{aligned} \operatorname{Hom}\big(H^{2d+2-i}\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{Q}\big) &\xrightarrow{\sim} \operatorname{Hom}\big(H^{2d+2-i}\big(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d)\big)_{\geq 0},\mathbb{Q}\big) \\ &\to \operatorname{Hom}\big(H^{2d+2-i}\big(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d)\big)_{\geq 0},\mathbb{Q}/\mathbb{Z}\big) \\ &\xleftarrow{\sim} H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}) \end{aligned}$$

for any $i \geq 2$.

Proof
In order to ease the notation, we set $D_{\mathcal{X}} := R\operatorname{Hom}(R\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{Q}[-2d-2])$. We consider the spectral sequence (see [46, Chapitre III, Section 4.6.10])

$$E_2^{p,q} = \prod_{i\in\mathbb{Z}} \operatorname{Ext}^p\big(H^{i-q}(D_{\mathcal{X}}),H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})\big) \Rightarrow H^{p+q}\big(R\operatorname{Hom}\big(D_{\mathcal{X}},R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})\big)\big). \tag{12}$$

An edge morphism from (12) gives a morphism

$$H^0\big(R\,\mathrm{Hom}\big(D_{\mathcal{X}}, R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})\big)\big) \to \prod_{i\in\mathbb{Z}} \mathrm{Ext}^0\big(H^i(D_{\mathcal{X}}), H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})\big) \tag{13}$$

such that the composite map

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}\big(D_{\mathcal{X}}, R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})\big) &\xrightarrow{\sim} H^0\big(R\,\mathrm{Hom}\big(D_{\mathcal{X}}, R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})\big)\big)\\ &\to \prod_{i\in\mathbb{Z}} \mathrm{Ext}^0\big(H^i(D_{\mathcal{X}}), H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})\big)\end{aligned}$$

is the obvious one.

By Lemma 2.5, $H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})$ is of cofinite type for $i \neq 0$. It follows that if both $i \neq 0$ and $p \neq 0$, then $\mathrm{Ext}^p(H^{i-q}(D_{\mathcal{X}}), H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})) = 0$, since $H^{i-q}(D_{\mathcal{X}})$ is uniquely divisible. Indeed, $\mathrm{Ext}^p(H^{i-q}(D_{\mathcal{X}}), (\mathbb{Q}/\mathbb{Z})^r) = 0$ for $p \geq 1$, since $(\mathbb{Q}/\mathbb{Z})^r$ is divisible. Moreover, if T is a finite abelian group of order n, then $\mathrm{Ext}^p(H^{i-q}(D_{\mathcal{X}}), T) = 0$ since $\mathrm{Ext}^p(H^{i-q}(D_{\mathcal{X}}), T)$ is both uniquely divisible (since $H^{i-q}(D_{\mathcal{X}})$ is) and killed by n (since T is). For $i = 0$, one has $\mathrm{Ext}^p(H^{0-q}(D_{\mathcal{X}}), H^0(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})) = 0$ as long as $p \geq 2$, because $H^0(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}) = \mathbb{Z}$ has an injective resolution of length 1. In particular, $E_2^{p,q} = 0$ for $p \neq 0, 1$; hence the spectral sequence (12) degenerates at E_2. Moreover, one has

$$E_2^{1,-1} = \prod_{i\in\mathbb{Z}} \mathrm{Ext}^1\big(H^{i+1}(D_{\mathcal{X}}), H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})\big) = \mathrm{Ext}^1\big(H^1(D_{\mathcal{X}}), H^0(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})\big) = 0$$

since $H^1(D_{\mathcal{X}}) = 0$. It follows that the edge morphism (13) is an isomorphism.

For $i \leq 1$, any map $H^i(\alpha_{\mathcal{X}}) : H^i(D_{\mathcal{X}}) \to H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})$ must be trivial since $H^i(D_{\mathcal{X}}) = 0$. For any $i \geq 2$, we consider the morphism

$$\begin{aligned}\alpha^i_{\mathcal{X}} : H^i(D_{\mathcal{X}}) &= \mathrm{Hom}\big(H^{2d+2-i}\big(\mathcal{X}, \mathbb{Q}(d)\big)_{\geq 0}, \mathbb{Q}\big)\\ &\xrightarrow{\sim} \mathrm{Hom}\big(H^{2d+2-i}\big(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}(d)\big)_{\geq 0}, \mathbb{Q}\big)\\ &\to \mathrm{Hom}\big(H^{2d+2-i}\big(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}(d)\big)_{\geq 0}, \mathbb{Q}/\mathbb{Z}\big) \xleftarrow{\sim} H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}),\end{aligned}$$

where the last isomorphism is given by Lemma 2.5. But (13) is an isomorphism; hence there exists a unique map in $\mathcal{D}$

$$\alpha_{\mathcal{X}} : R\,\mathrm{Hom}\big(R\Gamma\big(\mathcal{X}, \mathbb{Q}(d)\big)_{\geq 0}, \mathbb{Q}[-2d-2]\big) \to R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})$$

such that $H^i(\alpha_{\mathcal{X}}) = \alpha^i_{\mathcal{X}}$. □

2.2.2. The general case

In this section, we allow $\mathcal{X}(\mathbb{R}) \neq \emptyset$. So $\mathcal{X}$ denotes a proper, regular, and connected arithmetic scheme of dimension d. It is possible to define a dualizing complex $\mathbb{Z}(d)^{\overline{\mathcal{X}}}$

on $\overline{\mathcal{X}}_{\text{ét}}$, to prove Artin–Verdier duality in this setting, and to define $\alpha_{\mathcal{X}}$ in the very same way as in Section 2.2.1. For the sake of conciseness, we shall use a somewhat trickier construction, which is based on Milne's cohomology with compact support (see [37]).

We denote by

$$R\hat{\Gamma}_c(\mathcal{X}_{\text{ét}},\mathcal{F}) := R\hat{\Gamma}_c\big(\mathrm{Spec}(\mathbb{Z})_{\text{ét}}, Rf_*\mathcal{F}\big) \tag{14}$$

Milne's cohomology with compact support, where $R\hat{\Gamma}_c(\mathrm{Spec}(\mathbb{Z})_{\text{ét}},-)$ is defined as in [37, Section II.2] and $f:\mathcal{X}\to\mathrm{Spec}(\mathbb{Z})$ is the structure map. Milne's definition yields a canonical map $R\hat{\Gamma}_c(\mathrm{Spec}(\mathbb{Z})_{\text{ét}},-)\to R\Gamma(\mathrm{Spec}(\mathbb{Z})_{\text{ét}},-)$ inducing

$$R\hat{\Gamma}_c(\mathcal{X}_{\text{ét}},\mathbb{Z})\to R\Gamma(\mathcal{X}_{\text{ét}},\mathbb{Z}). \tag{15}$$

We need the following lemma.

LEMMA 2.7
There is a morphism $R\hat{\Gamma}_c(\mathcal{X}_{\text{ét}},\mathbb{Z})\xrightarrow{c} R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})$ *satisfying the following properties. The composite map*

$$R\hat{\Gamma}_c(\mathcal{X}_{\text{ét}},\mathbb{Z})\xrightarrow{c} R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})\xrightarrow{\varphi^*} R\Gamma(\mathcal{X}_{\text{ét}},\mathbb{Z})$$

coincides with (15), $H^i(c)$ *is an isomorphism for* i *large, and* $H^i(c)$ *has finite* 2*-torsion kernel and cokernel for any* $i\in\mathbb{Z}$.

Proof
Recall from [13, Section 4] that there is an isomorphism of functors $u_\infty^*\varphi_*\simeq\pi_*\alpha^*$ where $\alpha:\mathrm{Sh}(G_{\mathbb{R}},\mathcal{X}(\mathbb{C}))\to\mathcal{X}_{\text{ét}}$ and $\pi:\mathrm{Sh}(G_{\mathbb{R}},\mathcal{X}(\mathbb{C}))\to\mathrm{Sh}(\mathcal{X}_\infty)$ are the canonical maps while $u_\infty:\mathrm{Sh}(\mathcal{X}_\infty)\to\overline{\mathcal{X}}_{\text{ét}}$ and $\varphi:\mathcal{X}_{\text{ét}}\to\overline{\mathcal{X}}_{\text{ét}}$ are complementary closed and open embeddings (see [13, Section 4]). Here $\mathrm{Sh}(G_{\mathbb{R}},\mathcal{X}(\mathbb{C}))$ (resp., $\mathrm{Sh}(\mathcal{X}_\infty)$) denotes the topos of $G_{\mathbb{R}}$-equivariant sheaves of sets on $\mathcal{X}(\mathbb{C})$ (resp., the topos of sheaves on $\mathcal{X}_\infty$). It is easy to see that α^* sends injective abelian sheaves to π_*-acyclic ones. We obtain

$$u_\infty^*R\varphi_*\simeq R(u_\infty^*\varphi_*)\simeq R(\pi_*\alpha^*)\simeq(R\pi_*)\alpha^*.$$

Consider the map

$$R\varphi_*\mathbb{Z}\to u_{\infty,*}u_\infty^*R\varphi_*\mathbb{Z}\simeq u_{\infty,*}R\pi_*\mathbb{Z}\to u_{\infty,*}\tau^{>0}R\pi_*\mathbb{Z}.$$

Since $\tau^{\leq 0}R\pi_*\mathbb{Z}$ is the constant sheaf $\mathbb{Z}=u_\infty^*\mathbb{Z}$ put in degree 0, we obtain an exact triangle

$$\mathbb{Z}^{\overline{\mathcal{X}}} \to R\varphi_*\mathbb{Z} \to u_{\infty,*}\tau^{>0}R\pi_*\mathbb{Z}, \tag{16}$$

where $\mathbb{Z}^{\overline{\mathcal{X}}}$ denotes the constant sheaf $\mathbb{Z}$ on $\overline{\mathcal{X}}_{\text{ét}}$. Moreover, we have $R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, u_{\infty,*}\tau^{>0}R\pi_*\mathbb{Z}) \simeq R\Gamma(\mathcal{X}_\infty, \tau^{>0}R\pi_*\mathbb{Z}) \simeq R\Gamma(\mathcal{X}(\mathbb{R}), \tau^{>0}R\pi_*\mathbb{Z})$ because $u_{\infty,*}$ is exact and $\tau^{>0}R\pi_*\mathbb{Z}$ is concentrated on $\mathcal{X}(\mathbb{R}) \subset \mathcal{X}_\infty$. Therefore, applying $R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, -)$ to (16), we get an exact triangle

$$R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}) \to R\Gamma(\mathcal{X}_{\text{ét}}, \mathbb{Z}) \to R\Gamma\big(\mathcal{X}(\mathbb{R}), \tau^{>0}R\pi_*\mathbb{Z}\big). \tag{17}$$

On the other hand, there is an exact triangle

$$R\hat{\Gamma}_c(\mathcal{X}_{\text{ét}}, \mathbb{Z}) \to R\Gamma(\mathcal{X}_{\text{ét}}, \mathbb{Z}) \to R\hat{\Gamma}\big(G_\mathbb{R}, \mathcal{X}(\mathbb{R}), \mathbb{Z}\big), \tag{18}$$

where $R\hat{\Gamma}(G_\mathbb{R}, \mathcal{X}(\mathbb{R}), -) := R\hat{\Gamma}(G_\mathbb{R}, R\Gamma(\mathcal{X}(\mathbb{R}), -))$ denotes equivariant Tate cohomology (which was introduced in [45]). Notice that the canonical map $R\hat{\Gamma}(G_\mathbb{R}, \mathcal{X}(\mathbb{C}), \mathbb{Z}) \to R\hat{\Gamma}(G_\mathbb{R}, \mathcal{X}(\mathbb{R}), \mathbb{Z})$ is an isomorphism (see [45]).

Below, we shall define a canonical map $c_\infty : R\hat{\Gamma}(G_\mathbb{R}, \mathcal{X}(\mathbb{R}), \mathbb{Z}) \to R\Gamma(\mathcal{X}(\mathbb{R}), \tau^{>0}R\pi_*\mathbb{Z})$ such that $H^i(c_\infty)$ is an isomorphism for i large and $H^i(c_\infty)$ has finite 2-torsion kernel and cokernel for any $i \in \mathbb{Z}$. The map c_∞ is compatible (in the obvious sense) with the identity map of $R\Gamma(\mathcal{X}_{\text{ét}}, \mathbb{Z})$; hence there exists a morphism $c : R\hat{\Gamma}_c(\mathcal{X}_{\text{ét}}, \mathbb{Z}) \to R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})$ such that (c, Id, c_∞) is a morphism of exact triangles from (18) to (17). Hence the result will follow.

It remains to define c_∞. We denote by $R\Gamma(G_\mathbb{R}, \mathcal{X}(\mathbb{R}), -) := R\Gamma(G_\mathbb{R}, R\Gamma(\mathcal{X}(\mathbb{R}), -))$ the usual equivariant cohomology. We have isomorphisms

$$\begin{aligned} R\hat{\Gamma}\big(G_\mathbb{R}, \mathcal{X}(\mathbb{R}), \mathbb{Z}\big)_{>d} &\simeq R\Gamma\big(G_\mathbb{R}, \mathcal{X}(\mathbb{R}), \mathbb{Z}\big)_{>d} && (19)\\ &\simeq R\Gamma\big(\mathcal{X}(\mathbb{R}), R\pi_*\mathbb{Z}\big)_{>d} && (20)\\ &\simeq R\Gamma\big(\mathcal{X}(\mathbb{R}), (R\pi_*\mathbb{Z})_{>0}\big)_{>d} && (21)\\ &\simeq R\Gamma\big(\mathcal{X}(\mathbb{R}), R\Gamma(G_\mathbb{R}, \mathbb{Z})_{>0}\big)_{>d} && (22)\\ &\simeq R\Gamma\big(\mathcal{X}(\mathbb{R}), R\hat{\Gamma}(G_\mathbb{R}, \mathbb{Z})_{>0}\big)_{>d} && (23)\\ &\simeq R\Gamma\big(\mathcal{X}(\mathbb{R}), R\hat{\Gamma}(G_\mathbb{R}, \mathbb{Z})\big)_{>d}, && (24) \end{aligned}$$

where we consider $R\hat{\Gamma}(G_\mathbb{R}, \mathbb{Z})$ and $R\Gamma(G_\mathbb{R}, \mathbb{Z})$ as complexes of constant sheaves on $\mathcal{X}(\mathbb{R})$. The canonical map $R\Gamma(G_\mathbb{R}, \mathcal{X}(\mathbb{R}), \mathbb{Z}) \to R\hat{\Gamma}(G_\mathbb{R}, \mathcal{X}(\mathbb{R}), \mathbb{Z})$ and the corresponding morphism of hypercohomology spectral sequences give the isomorphism (19), since $\mathcal{X}(\mathbb{R})$ is of topological dimension d. Similarly, (21) and (24) can be deduced from the corresponding morphisms of spectral sequences. Finally, (22) is given by $(R\pi_*\mathbb{Z})_{|\mathcal{X}(\mathbb{R})} \simeq R\Gamma(G_\mathbb{R}, \mathbb{Z})$. Since $G_\mathbb{R}$ is cyclic, we have an isomorphism $R\hat{\Gamma}(G_\mathbb{R}, \mathbb{Z}) \xrightarrow{\sim} R\hat{\Gamma}(G_\mathbb{R}, \mathbb{Z})[2]$ in $\mathcal{D}$. Applying $R\Gamma(\mathcal{X}(\mathbb{R}), -)$ we obtain

$$R\Gamma\big(\mathcal{X}(\mathbb{R}), R\hat{\Gamma}(G_{\mathbb{R}},\mathbb{Z})\big) \xrightarrow{\sim} R\Gamma\big(\mathcal{X}(\mathbb{R}), R\hat{\Gamma}(G_{\mathbb{R}},\mathbb{Z})\big)[2].$$

The same is true for equivariant Tate cohomology: we have an isomorphism

$$R\hat{\Gamma}\big(G_{\mathbb{R}}, \mathcal{X}(\mathbb{R}),\mathbb{Z}\big) \xrightarrow{\sim} R\hat{\Gamma}\big(G_{\mathbb{R}}, \mathcal{X}(\mathbb{R}),\mathbb{Z}\big)[2].$$

Hence for any integer $k \geq 1$, (24) induces an isomorphism

$$R\hat{\Gamma}\big(G_{\mathbb{R}}, \mathcal{X}(\mathbb{R}),\mathbb{Z}\big)_{>d-2k} \simeq R\Gamma\big(\mathcal{X}(\mathbb{R}), R\hat{\Gamma}(G_{\mathbb{R}},\mathbb{Z})\big)_{>d-2k}.$$

Taking k large enough, we obtain a canonical isomorphism

$$R\hat{\Gamma}\big(G_{\mathbb{R}}, \mathcal{X}(\mathbb{R}),\mathbb{Z}\big)_{>0} \simeq R\Gamma\big(\mathcal{X}(\mathbb{R}), R\hat{\Gamma}(G_{\mathbb{R}},\mathbb{Z})\big)_{>0}. \tag{25}$$

The map $R\hat{\Gamma}(G_{\mathbb{R}},\mathbb{Z}) \to R\hat{\Gamma}(G_{\mathbb{R}},\mathbb{Z})_{>0}$ induces

$$R\Gamma\big(\mathcal{X}(\mathbb{R}), R\hat{\Gamma}(G_{\mathbb{R}},\mathbb{Z})\big) \longrightarrow R\Gamma\big(\mathcal{X}(\mathbb{R}), R\hat{\Gamma}(G_{\mathbb{R}},\mathbb{Z})_{>0}\big). \tag{26}$$

Since $R\Gamma(\mathcal{X}(\mathbb{R}), R\hat{\Gamma}(G_{\mathbb{R}},\mathbb{Z})_{>0})$ is acyclic in degrees less than or equal to 0, (26) is induced by a unique map

$$R\Gamma\big(\mathcal{X}(\mathbb{R}), R\hat{\Gamma}(G_{\mathbb{R}},\mathbb{Z})\big)_{>0} \longrightarrow R\Gamma\big(\mathcal{X}(\mathbb{R}), R\hat{\Gamma}(G_{\mathbb{R}},\mathbb{Z})_{>0}\big) \tag{27}$$

and c_∞ is defined as the composition

$$\begin{aligned} R\hat{\Gamma}\big(G_{\mathbb{R}}, \mathcal{X}(\mathbb{R}),\mathbb{Z}\big) \longrightarrow R\hat{\Gamma}\big(G_{\mathbb{R}}, \mathcal{X}(\mathbb{R}),\mathbb{Z}\big)_{>0} &\xrightarrow{(25)} R\Gamma\big(\mathcal{X}(\mathbb{R}), R\hat{\Gamma}(G_{\mathbb{R}},\mathbb{Z})\big)_{>0} \\ &\xrightarrow{(27)} R\Gamma\big(\mathcal{X}(\mathbb{R}), R\hat{\Gamma}(G_{\mathbb{R}},\mathbb{Z})_{>0}\big) \simeq R\Gamma\big(\mathcal{X}(\mathbb{R}), \tau^{>0}R\pi_*\mathbb{Z}\big). \quad \square \end{aligned}$$

If $\mathcal{X}(\mathbb{R}) = \emptyset$, then the maps c and φ^* of Lemma 2.7 are isomorphisms in $\mathcal{D}$. (This follows from (18) and from the fact that φ_* is exact in this case.) Therefore, the following result generalizes Theorem 2.6.

PROPOSITION 2.8

Assume that $\mathcal{X}$ satisfies $\mathbf{L}(\mathcal{X}_{\text{ét}}, d)_{\geq 0}$. Then there exists a unique morphism in $\mathcal{D}$

$$\alpha_{\mathcal{X}} : R\,\mathrm{Hom}\big(R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0}, \mathbb{Q}[-2d-2]\big) \to R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})$$

such that $H^i(\alpha_{\mathcal{X}})$ is the following composite map:

$$\begin{aligned} \mathrm{Hom}\big(H^{2d+2-i}\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{Q}\big) &\xrightarrow{\sim} \mathrm{Hom}\big(H^{2d+2-i}\big(\mathcal{X}_{\text{ét}},\mathbb{Z}(d)\big)_{\geq 0},\mathbb{Q}\big) \\ &\longrightarrow \mathrm{Hom}\big(H^{2d+2-i}\big(\mathcal{X}_{\text{ét}},\mathbb{Z}(d)\big)_{\geq 0},\mathbb{Q}/\mathbb{Z}\big) \\ &\xleftarrow{\sim} \hat{H}^i_c(\mathcal{X}_{\text{ét}},\mathbb{Z}) \xrightarrow{H^i(c)} H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}), \end{aligned}$$

for any $i \geq 2$.

Proof
We consider the composite map

$$\begin{aligned} R\operatorname{Hom}_{\mathcal{X}}\big(\mathbb{Z},\mathbb{Z}(d)\big) &\to R\operatorname{Hom}_{\mathrm{Spec}(\mathbb{Z})}\big(Rf_*\mathbb{Z}, Rf_*\mathbb{Z}(d)\big)\\ &\xrightarrow{p} R\operatorname{Hom}_{\mathrm{Spec}(\mathbb{Z})}\big(Rf_*\mathbb{Z},\mathbb{Z}(1)[-2d+2]\big)\\ &\xrightarrow{q} R\operatorname{Hom}_{\mathcal{D}}\big(R\hat{\Gamma}_c\big(\mathrm{Spec}(\mathbb{Z})_{\text{ét}}, Rf_*\mathbb{Z}\big),\mathbb{Q}/\mathbb{Z}[-2d-2]\big)\\ &= R\operatorname{Hom}_{\mathcal{D}}\big(R\hat{\Gamma}_c(\mathcal{X}_{\text{ét}},\mathbb{Z}),\mathbb{Q}/\mathbb{Z}[-2d-2]\big),\end{aligned}$$

where p is induced by the pushforward map of [19, Corollary 7.2(b)] and q is given by 1-dimensional Artin–Verdier duality (see [37, Chapter II, Theorem 3.1]) since $\mathbb{Z}(1)\simeq\mathbb{G}_m[-1]$ (see [19, Lemma 7.4]). This composite map induces

$$R\Gamma\big(\mathcal{X}_{\text{ét}},\mathbb{Z}(d)\big)_{\geq 0}\to R\operatorname{Hom}\big(R\hat{\Gamma}_c(\mathcal{X}_{\text{ét}},\mathbb{Z}),\mathbb{Q}/\mathbb{Z}[-2d-2]\big)$$

since $R\hat{\Gamma}_c(\mathcal{X}_{\text{ét}},\mathbb{Z})$ is acyclic in degrees greater than $2d+2$. Indeed, $\hat{H}^i_c(\mathrm{Spec}(\mathbb{Z})_{\text{ét}}, F)=0$ for any $i>3$ and any torsion abelian sheaf F (see [37, Theorem II.3.1(b)]); hence by proper base change the hypercohomology spectral sequence associated to (14) gives $\hat{H}^i_c(\mathcal{X}_{\text{ét}},\mathbb{Q}/\mathbb{Z})=0$ for $i>2d+1$. By adjunction, we obtain the product map

$$R\Gamma\big(\mathcal{X}_{\text{ét}},\mathbb{Z}(d)\big)_{\geq 0}\otimes^L_{\mathbb{Z}} R\hat{\Gamma}_c(\mathcal{X}_{\text{ét}},\mathbb{Z})\longrightarrow\mathbb{Q}/\mathbb{Z}[-2d-2]. \tag{28}$$

For any positive integer n and any $i\in\mathbb{Z}$, the canonical map

$$\operatorname{Ext}^{2d+2-i}_{\mathcal{X}}\big(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}(d)\big)\times\hat{H}^i_c(\mathcal{X}_{\text{ét}},\mathbb{Z}/n\mathbb{Z})\to\mathbb{Q}/\mathbb{Z} \tag{29}$$

is a perfect pairing of finite groups (see [19, Theorem 7.8]). Note that (29) is compatible with (28). The arguments of the proof of Lemma 2.5 show that the morphism

$$\hat{H}^i_c(\mathcal{X}_{\text{ét}},\mathbb{Z})\xrightarrow{\sim}\operatorname{Hom}\big(H^{2d+2-i}\big(\mathcal{X}_{\text{ét}},\mathbb{Z}(d)\big)_{\geq 0},\mathbb{Q}/\mathbb{Z}\big), \tag{30}$$

induced by (28), is an isomorphism of abelian groups for $i\geq 2$. The argument of the proof of Theorem 2.6 provides us with a unique map

$$\tilde{\alpha}_{\mathcal{X}}: R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{Q}[-2d-2]\big)\to R\hat{\Gamma}_c(\mathcal{X}_{\text{ét}},\mathbb{Z})$$

such that $H^i(\tilde{\alpha}_{\mathcal{X}})$ is the following composite map:

$$\begin{aligned}&\operatorname{Hom}\big(H^{2d+2-i}\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{Q}\big)\\ &\xrightarrow{\sim}\operatorname{Hom}\big(H^{2d+2-i}\big(\mathcal{X}_{\text{ét}},\mathbb{Z}(d)\big)_{\geq 0},\mathbb{Q}\big)\\ &\to\operatorname{Hom}\big(H^{2d+2-i}\big(\mathcal{X}_{\text{ét}},\mathbb{Z}(d)\big)_{\geq 0},\mathbb{Q}/\mathbb{Z}\big)\xleftarrow{\sim}\hat{H}^i_c(\mathcal{X}_{\text{ét}},\mathbb{Z})\end{aligned}$$

for any $i\geq 2$. Then we define $\alpha_{\mathcal{X}}$ as the composite map

$$R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{Q}[-\delta]\big)\xrightarrow{\tilde{\alpha}_{\mathcal{X}}} R\hat{\Gamma}_c(\mathcal{X}_{\text{ét}},\mathbb{Z})\xrightarrow{c} R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}),$$

where c is the map of Lemma 2.7. It remains to show that $\alpha_{\mathcal{X}}$ does not depend on the choice of c. (Note that c is not uniquely defined.) The map $\varphi^*\circ c$ is well defined (since it coincides with (15) by Lemma 2.7); hence $\varphi^*\circ\alpha_{\mathcal{X}}: D_{\mathcal{X}}\to R\Gamma(\mathcal{X}_{\text{ét}},\mathbb{Z})$ is as well, where $D_{\mathcal{X}}:=R\operatorname{Hom}(R\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{Q}[-2d-2])$. By (17) and using the fact that $D_{\mathcal{X}}$ is a complex of $\mathbb{Q}$-vector spaces while $R\Gamma(\mathcal{X}(\mathbb{R}),\tau^{>0}R\pi_*\mathbb{Z})$ is a complex of $\mathbb{Z}/2\mathbb{Z}$-vector spaces, we see that composition with φ^* induces an isomorphism

$$\varphi^*\circ:\operatorname{Hom}_{\mathcal{D}}\big(D_{\mathcal{X}},R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})\big)\xrightarrow{\sim}\operatorname{Hom}_{\mathcal{D}}\big(D_{\mathcal{X}},R\Gamma(\mathcal{X}_{\text{ét}},\mathbb{Z})\big)$$

so that $\alpha_{\mathcal{X}}$ does not depend on the choice of c. The result follows. □

2.3. *The complex* $R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})$

Throughout this section $\mathcal{X}$ denotes a proper, regular, and connected arithmetic scheme of dimension d satisfying $\mathbf{L}(\mathcal{X}_{\text{ét}},d)_{\geq 0}$.

Definition 2.9

There exists an exact triangle

$$R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{Q}[-2d-2]\big)\xrightarrow{\alpha_{\mathcal{X}}} R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})\longrightarrow R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})$$

well defined up to a unique isomorphism in $\mathcal{D}$. We define

$$H^i_W(\overline{\mathcal{X}},\mathbb{Z}):=H^i\big(R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})\big).$$

The existence of such an exact triangle follows from axiom TR1 of triangulated categories. Its uniqueness is given by Theorem 2.12 and can be stated as follows. If we have two objects $R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})$ and $R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})'$ given with exact triangles as in Definition 2.9, then there exists a *unique* map $f: R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})\to R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})'$ in $\mathcal{D}$, which yields a morphism of exact triangles

$$\begin{array}{ccccc}
R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{Q}[-2d-2]\big) & \xrightarrow{\alpha_{\mathcal{X}}} & R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}) & \longrightarrow & R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})\\
\downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle f}\\
R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{Q}[-2d-2]\big) & \xrightarrow{\alpha_{\mathcal{X}}} & R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}) & \longrightarrow & R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})'
\end{array}$$

PROPOSITION 2.10

The following assertions are true.

(1) *The group* $H^i_W(\overline{\mathcal{X}},\mathbb{Z})$ *is finitely generated for any* $i\in\mathbb{Z}$. *One has* $H^i_W(\overline{\mathcal{X}},\mathbb{Z})=0$ *for* $i<0$, $H^0_W(\overline{\mathcal{X}},\mathbb{Z})=\mathbb{Z}$, *and* $H^i_W(\overline{\mathcal{X}},\mathbb{Z})=0$ *for* i *large.*

(2) *If $\mathcal{X}(\mathbb{R}) = \emptyset$, then there is an exact sequence*

$$0 \to H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})_{\text{codiv}} \to H^i_W(\overline{\mathcal{X}}, \mathbb{Z}) \\ \to \text{Hom}_{\mathbb{Z}}\big(H^{2d+2-(i+1)}\big(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}(d)\big)_{\geq 0}, \mathbb{Z}\big) \to 0$$

for any $i \in \mathbb{Z}$.

(3) *For any $\mathcal{X}$ and any $i \in \mathbb{Z}$, there is an exact sequence*

$$0 \to H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})_{\text{codiv}} \to H^i_W(\overline{\mathcal{X}}, \mathbb{Z}) \to \text{Ker}\big(H^{i+1}(\alpha_{\mathcal{X}})\big) \to 0,$$

where $\text{Ker}(H^{i+1}(\alpha_{\mathcal{X}})) \subseteq \text{Hom}_{\mathbb{Z}}(H^{2d+2-(i+1)}(\mathcal{X}_{\text{ét}}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q})$ *is a $\mathbb{Z}$-lattice.*

Proof

In order to ease the notation we set $\delta := 2d + 2$. The exact triangle of Definition 2.9 yields a long exact sequence

$$\cdots \to \text{Hom}\big(H^{\delta-i}\big(\mathcal{X}, \mathbb{Q}(d)\big)_{\geq 0}, \mathbb{Q}\big) \to H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}) \to H^i_W(\overline{\mathcal{X}}, \mathbb{Z}) \\ \to \text{Hom}\big(H^{\delta-(i+1)}\big(\mathcal{X}, \mathbb{Q}(d)\big)_{\geq 0}, \mathbb{Q}\big) \to H^{i+1}(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}) \to \cdots$$

and a short exact sequence

$$0 \to \text{Coker}\, H^i(\alpha_{\mathcal{X}}) \to H^i_W(\overline{\mathcal{X}}, \mathbb{Z}) \to \text{Ker}\, H^{i+1}(\alpha_{\mathcal{X}}) \to 0. \tag{31}$$

Assume for the moment that $\mathcal{X}(\mathbb{R}) = \emptyset$. For $i \geq 1$, the morphism

$$H^i(\alpha_{\mathcal{X}}) : \text{Hom}\big(H^{\delta-i}\big(\mathcal{X}, \mathbb{Q}(d)\big)_{\geq 0}, \mathbb{Q}\big) \to H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})$$

is the following composite map:

$$\text{Hom}\big(H^{\delta-i}\big(\mathcal{X}, \mathbb{Q}(d)\big)_{\geq 0}, \mathbb{Q}\big) \xrightarrow{\sim} \text{Hom}\big(H^{\delta-i}\big(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}(d)\big)_{\geq 0}, \mathbb{Q}\big) \\ \to \text{Hom}\big(H^{\delta-i}\big(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}(d)\big)_{\geq 0}, \mathbb{Q}/\mathbb{Z}\big) \simeq H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}).$$

Since $H^{\delta-i}(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}(d))_{\geq 0}$ is assumed to be finitely generated, the image of the morphism $H^i(\alpha_{\mathcal{X}})$ is the maximal divisible subgroup of $H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})$, which we denote by $H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})_{\text{div}}$. For $i = 0$ one has

$$H^0(\alpha_{\mathcal{X}}) : 0 = \text{Hom}\big(H^{\delta}\big(\mathcal{X}, \mathbb{Q}(d)\big), \mathbb{Q}\big) \to H^0(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}) = \mathbb{Z}$$

and $H^{\delta-i}(\mathcal{X}, \mathbb{Q}(d)) = H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}) = 0$ for $i < 0$. We obtain

$$\text{Coker}\, H^i(\alpha_{\mathcal{X}}) = H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})/H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})_{\text{div}} =: H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})_{\text{codiv}}$$

for any $i\in\mathbb{Z}$. By the definition of $\alpha_{\mathcal{X}}$, the kernel of $H^i(\alpha_{\mathcal{X}})$ can be identified with the kernel of the map $\mathrm{Hom}(H^{\delta-i}(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d))_{\geq 0},\mathbb{Q})\to\mathrm{Hom}(H^{\delta-i}(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d))_{\geq 0},\mathbb{Q}/\mathbb{Z})$. In other words, one has

$$\operatorname{Ker} H^i(\alpha_{\mathcal{X}})\simeq \mathrm{Hom}\big(H^{\delta-i}\big(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d)\big)_{\geq 0},\mathbb{Z}\big).$$

The exact sequence (31) therefore takes the form

$$0\to H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})_{\text{codiv}}\to H^i_W(\overline{\mathcal{X}},\mathbb{Z})\to \mathrm{Hom}_{\mathbb{Z}}\big(H^{2d+2-(i+1)}\big(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d)\big)_{\geq 0},\mathbb{Z}\big)\to 0.$$

We obtain the second claim of the proposition.

Assume now that $\mathcal{X}(\mathbb{R})\neq\emptyset$. Then for $i\geq 2$, the morphism $H^i(\tilde{\alpha}_{\mathcal{X}})$ is the map

$$\begin{aligned}\mathrm{Hom}\big(H^{\delta-i}\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{Q}\big)&\xrightarrow{\sim}\mathrm{Hom}\big(H^{\delta-i}\big(\mathcal{X}_{\text{ét}},\mathbb{Z}(d)\big)_{\geq 0},\mathbb{Q}\big)\\ &\to \mathrm{Hom}\big(H^{\delta-i}\big(\mathcal{X}_{\text{ét}},\mathbb{Z}(d)\big)_{\geq 0},\mathbb{Q}/\mathbb{Z}\big)\simeq \hat{H}^i_c(\mathcal{X}_{\text{ét}},\mathbb{Z})\end{aligned}$$

and $H^i(\alpha_{\mathcal{X}})$ is defined as the composite map

$$H^i(\alpha_{\mathcal{X}}):\mathrm{Hom}\big(H^{\delta-i}\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{Q}\big)\xrightarrow{H^i(\tilde{\alpha}_{\mathcal{X}})}\hat{H}^i_c(\mathcal{X}_{\text{ét}},\mathbb{Z})\longrightarrow H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}).$$

The same argument as in the case $\mathcal{X}(\mathbb{R})=\emptyset$ shows that $\mathrm{Im}(H^i(\tilde{\alpha}_{\mathcal{X}}))=\hat{H}^i_c(\mathcal{X}_{\text{ét}},\mathbb{Z})_{\text{div}}$. Let us show that $h:\hat{H}^i_c(\mathcal{X}_{\text{ét}},\mathbb{Z})_{\text{div}}\to H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})_{\text{div}}$ is surjective for any $i\in\mathbb{Z}$. This is obvious for $i\leq 1$, since $H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})_{\text{div}}=0$ for $i\leq 1$. For $i\geq 2$, the maps $\hat{H}^i_c(\mathcal{X}_{\text{ét}},\mathbb{Z})\to H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})$ and $\hat{H}^i_c(\mathcal{X}_{\text{ét}},\mathbb{Z})_{\text{div}}\to\hat{H}^i_c(\mathcal{X}_{\text{ét}},\mathbb{Z})$ both have finite cokernels. (Note that $\hat{H}^i_c(\mathcal{X}_{\text{ét}},\mathbb{Z})$ is of cofinite type for $i\geq 2$ by (30).) It follows immediately that the cokernel of $h:\hat{H}^i_c(\mathcal{X}_{\text{ét}},\mathbb{Z})_{\text{div}}\to H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})_{\text{div}}$ is of finite exponent and divisible and hence vanishes. So h is indeed surjective. We obtain $\mathrm{Im}(H^i(\alpha_{\mathcal{X}}))=H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})_{\text{div}}$; hence

$$\operatorname{Coker} H^i(\alpha_{\mathcal{X}})=H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})_{\text{codiv}}$$

for any $i\in\mathbb{Z}$. We now compute $\operatorname{Ker} H^i(\alpha_{\mathcal{X}})$. By the definition of $\tilde{\alpha}_{\mathcal{X}}$, the kernel of $H^i(\tilde{\alpha}_{\mathcal{X}})$ can be identified with the kernel of the map $\mathrm{Hom}(H^{\delta-i}(\mathcal{X}_{\text{ét}},\mathbb{Z}(d))_{\geq 0},\mathbb{Q})\to\mathrm{Hom}(H^{\delta-i}(\mathcal{X}_{\text{ét}},\mathbb{Z}(d))_{\geq 0},\mathbb{Q}/\mathbb{Z})$. Hence one has

$$\operatorname{Ker} H^i(\tilde{\alpha}_{\mathcal{X}})\simeq\mathrm{Hom}\big(H^{\delta-i}\big(\mathcal{X}_{\text{ét}},\mathbb{Z}(d)\big)_{\geq 0},\mathbb{Z}\big).\tag{32}$$

We denote by T_i the kernel of the surjective map $h:\hat{H}^i_c(\mathcal{X}_{\text{ét}},\mathbb{Z})_{\text{div}}\to H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})_{\text{div}}$ so that there is an exact sequence

$$0\longrightarrow T_i\longrightarrow\hat{H}^i_c(\mathcal{X}_{\text{ét}},\mathbb{Z})_{\text{div}}\xrightarrow{h}H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})_{\text{div}}\longrightarrow 0.\tag{33}$$

Consider the obvious injective map $f : \operatorname{Ker} H^i(\tilde{\alpha}_{\mathcal{X}}) \to \operatorname{Ker} H^i(\alpha_{\mathcal{X}})$, and consider the following morphism of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker} H^i(\tilde{\alpha}_{\mathcal{X}}) & \longrightarrow & \operatorname{Hom}\big(H^{\delta-i}(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}\big) & \xrightarrow{H^i(\tilde{\alpha}_{\mathcal{X}})} & \hat{H}^i_c(\mathcal{X}_{\text{ét}}, \mathbb{Z})_{\text{div}} & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow = & & \downarrow h & & \\
0 & \longrightarrow & \operatorname{Ker} H^i(\alpha_{\mathcal{X}}) & \longrightarrow & \operatorname{Hom}\big(H^{\delta-i}(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}\big) & \xrightarrow{H^i(\alpha_{\mathcal{X}})} & H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})_{\text{div}} & \longrightarrow & 0
\end{array}$$

The snake lemma then yields an isomorphism $T_i = \operatorname{Ker}(h) \simeq \operatorname{Coker}(f)$. Using (32), we obtain an exact sequence

$$0 \to \operatorname{Hom}\big(H^{\delta-i}(\mathcal{X}_{\text{ét}}, \mathbb{Z}(d))_{\geq 0}, \mathbb{Z}\big) \to \operatorname{Ker} H^i(\alpha_{\mathcal{X}}) \to T_i \to 0.$$

Moreover, T_i is finite and 2-torsion. We obtain the third claim of the proposition.

It follows from the third claim that $H^i_W(\overline{\mathcal{X}}, \mathbb{Z})$ is finitely generated for any $i \in \mathbb{Z}$. Indeed, $\operatorname{Ker} H^{i+1}(\alpha_{\mathcal{X}})$ is finitely generated since it is a $\mathbb{Z}$-lattice in a finite-dimensional $\mathbb{Q}$-vector space. Moreover, $\hat{H}^i_c(\mathcal{X}_{\text{ét}}, \mathbb{Z})$ is of cofinite type for $i \geq 2$ and finitely generated for $i \leq 1$. Hence $\hat{H}^i_c(\mathcal{X}_{\text{ét}}, \mathbb{Z})_{\text{codiv}}$ is finite for any $i \in \mathbb{Z}$ and so is $H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})_{\text{codiv}}$ by (33).

Moreover, one has $H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}) = 0$ for $i < 0$ and $H^0(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}) = \mathbb{Z}$. Moreover, one has $H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}) \simeq \hat{H}^i_c(\mathcal{X}_{\text{ét}}, \mathbb{Z})$ for i large (by Lemma 2.7) and $\hat{H}^i_c(\mathcal{X}_{\text{ét}}, \mathbb{Z}) = 0$ for $i > 2d + 2$ by (30); hence $H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}) = 0$ for i large. Finally, $\operatorname{Ker} H^{i+1}(\alpha_{\mathcal{X}}) \subset \operatorname{Hom}(H^{2d+2-(i+1)}(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}) = 0$ for $i \leq 0$ as well as for $i \geq 2d + 2$. The first claim of the proposition follows. □

We denote by $H^i_{\text{cont}}(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}_l) := H^i(R\varprojlim R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}/l^\nu\mathbb{Z}))$ the continuous l-adic cohomology.

COROLLARY 2.11

Let $\mathcal{X}$ be a proper, regular, and connected arithmetic scheme of dimension d satisfying $\mathbf{L}(\mathcal{X}_{\text{ét}}, d)_{\geq 0}$. For any prime number l and any $i \in \mathbb{Z}$, there is an isomorphism

$$H^i_W(\overline{\mathcal{X}}, \mathbb{Z}) \otimes \mathbb{Z}_l \simeq H^i_{\text{cont}}(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}_l).$$

Proof

Consider the exact triangle

$$R\operatorname{Hom}\big(R\Gamma(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}[-2d-2]\big) \to R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}) \to R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}).$$

Let n be any positive integer. Applying the functor $-\otimes^L_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ we obtain an isomorphism

$$R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}) \simeq R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}) \otimes^L_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\sim} R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}) \otimes^L_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$$

since $R\operatorname{Hom}(R\Gamma(\mathcal{X}, \mathbb{Q}(d_{\mathcal{X}}))_{\geq 0}, \mathbb{Q}[-2d-2]) \otimes^L_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \simeq 0$. Let l be a prime number, and let ν be a positive integer. We obtain a short exact sequence

$$0 \to H^i_W(\overline{\mathcal{X}}, \mathbb{Z})_{l^\nu} \to H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}/l^\nu\mathbb{Z}) \to {}_{l^\nu}H^{i+1}_W(\overline{\mathcal{X}}, \mathbb{Z}) \to 0 \tag{34}$$

for any $i \in \mathbb{Z}$. By the left exactness of projective limits we get

$$0 \to \varprojlim H^i_W(\overline{\mathcal{X}}, \mathbb{Z})_{l^\nu} \to \varprojlim H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}/l^\nu\mathbb{Z}) \to \varprojlim {}_{l^\nu}H^{i+1}_W(\overline{\mathcal{X}}, \mathbb{Z}).$$

But $\varprojlim_{l^\nu} H^{i+1}_W(\overline{\mathcal{X}}, \mathbb{Z}) = 0$ since $H^{i+1}_W(\overline{\mathcal{X}}, \mathbb{Z})$ is finitely generated. Moreover, we have

$$H^i_{\text{cont}}(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}_l) \simeq \varprojlim H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}/l^\nu\mathbb{Z})$$

since $H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}/l^\nu\mathbb{Z})$ is finite by (34). □

THEOREM 2.12
Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of relative dimension c between proper, regular, and connected arithmetic schemes such that $\mathbf{L}(\mathcal{X}_{\text{ét}}, d_{\mathcal{X}})_{\geq 0}$ and $\mathbf{L}(\mathcal{Y}_{\text{ét}}, d_{\mathcal{Y}})_{\geq 0}$ hold, where $d_{\mathcal{X}}$ (respectively, $d_{\mathcal{Y}}$) denotes the dimension of $\mathcal{X}$ (respectively, of $\mathcal{Y}$). We choose complexes $R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})$ and $R\Gamma_W(\overline{\mathcal{Y}}, \mathbb{Z})$ as in Definition 2.9. Assume either that $c \geq 0$ or that $\mathbf{L}(\mathcal{X}_{\text{ét}}, d_{\mathcal{X}})$ holds.

Then there exists a unique *map in $\mathcal{D}$*

$$f^*_W : R\Gamma_W(\overline{\mathcal{Y}}, \mathbb{Z}) \to R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}),$$

which sits in a morphism of exact triangles

$$\begin{array}{ccccc} R\operatorname{Hom}\big(R\Gamma(\mathcal{X}, \mathbb{Q}(d_{\mathcal{X}}))_{\geq 0}, \mathbb{Q}[-2d_{\mathcal{X}}-2]\big) & \longrightarrow & R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}) & \longrightarrow & R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}) \\ \uparrow & & \uparrow f^*_{\text{ét}} & & \uparrow f^*_W \\ R\operatorname{Hom}\big(R\Gamma(\mathcal{Y}, \mathbb{Q}(d_{\mathcal{Y}}))_{\geq 0}, \mathbb{Q}[-2d_{\mathcal{Y}}-2]\big) & \longrightarrow & R\Gamma(\overline{\mathcal{Y}}_{\text{ét}}, \mathbb{Z}) & \longrightarrow & R\Gamma_W(\overline{\mathcal{Y}}, \mathbb{Z}) \end{array}$$

In particular, if $\mathcal{X}$ satisfies $\mathbf{L}(\mathcal{X}_{\text{ét}}, d_{\mathcal{X}})_{\geq 0}$, then $R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})$ is well defined up to a unique *isomorphism in $\mathcal{D}$.*

Proof
Let $\mathcal{X}$ and $\mathcal{Y}$ be connected, proper, and regular arithmetic schemes of dimension $d_{\mathcal{X}}$ and $d_{\mathcal{Y}}$, respectively. We set $\delta_{\mathcal{X}} := 2d_{\mathcal{X}} + 2$ and $\delta_{\mathcal{Y}} := 2d_{\mathcal{Y}} + 2$. We choose

complexes $R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})$ and $R\Gamma_W(\overline{\mathcal{Y}}, \mathbb{Z})$ as in Definition 2.9. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of relative dimension $c = d_{\mathcal{X}} - d_{\mathcal{Y}}$. The morphism f is proper, and the map

$$z^n(\mathcal{X}, *) \to z^{n-c}(\mathcal{Y}, *)$$

induces a morphism

$$f_*\mathbb{Q}(d_{\mathcal{X}}) \to \mathbb{Q}(d_{\mathcal{Y}}))[-2c]$$

of complexes of abelian Zariski sheaves on $\mathcal{Y}$. We need to see that

$$f_*\mathbb{Q}(d_{\mathcal{X}}) \simeq Rf_*\mathbb{Q}(d_{\mathcal{X}}). \tag{35}$$

By localizing over the base, it is enough to show this fact for f a proper map over a discrete valuation ring. Over a discrete valuation ring, Zariski hypercohomology of the cycle complex coincides with its cohomology as a complex of abelian group (see [30] and [16, Theorem 3.2]). This yields (35) and a morphism of complexes

$$R\Gamma\big(\mathcal{X}, \mathbb{Q}(d_{\mathcal{X}})\big) \simeq R\Gamma\big(\mathcal{Y}, f_*\mathbb{Q}(d_{\mathcal{X}})\big) \to R\Gamma\big(\mathcal{Y}, \mathbb{Q}(d_{\mathcal{Y}})\big)[-2c].$$

If $c \geq 0$, this induces a morphism

$$R\Gamma\big(\mathcal{X}, \mathbb{Q}(d_{\mathcal{X}})\big)_{\geq 0} \to R\Gamma\big(\mathcal{Y}, \mathbb{Q}(d_{\mathcal{Y}})\big)_{\geq 0}[-2c].$$

If $c < 0$, then $\mathbf{L}(\mathcal{X}_{\text{ét}}, d_{\mathcal{X}})$ holds by assumption. It follows that $H^i(\mathcal{X}, \mathbb{Q}(d_{\mathcal{X}})) = 0$ for $i < 0$. Indeed, we have $H^i(\mathcal{X}_{\text{ét}}, \mathbb{Z}(d)) \simeq H^i(\mathcal{X}, \mathbb{Q}(d_{\mathcal{X}}))$ for $i < 0$ (by Lemma 2.3 for $\mathcal{X}(\mathbb{R}) = \emptyset$ and by (29) for the general case). Hence for $i < 0$, $H^i(\mathcal{X}, \mathbb{Q}(d_{\mathcal{X}})) \simeq H^i(\mathcal{X}_{\text{ét}}, \mathbb{Z}(d))$ is both uniquely divisible and finitely generated by $\mathbf{L}(\mathcal{X}_{\text{ét}}, d_{\mathcal{X}})$; hence it must be trivial. So we may consider the map

$$\begin{aligned} R\Gamma\big(\mathcal{X}, \mathbb{Q}(d_{\mathcal{X}})\big)_{\geq 0} \xleftarrow{\sim} R\Gamma\big(\mathcal{X}, \mathbb{Q}(d_{\mathcal{X}})\big) &\to R\Gamma\big(\mathcal{Y}, \mathbb{Q}(d_{\mathcal{Y}})\big)[-2c] \\ &\to R\Gamma\big(\mathcal{Y}, \mathbb{Q}(d_{\mathcal{Y}})\big)_{\geq 0}[-2c]. \end{aligned}$$

In both cases we get a morphism

$$R\operatorname{Hom}\big(R\Gamma\big(\mathcal{Y}, \mathbb{Q}(d_{\mathcal{Y}})\big)_{\geq 0}, \mathbb{Q}[-\delta_{\mathcal{Y}}]\big) \to R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X}, \mathbb{Q}(d_{\mathcal{X}})\big)_{\geq 0}, \mathbb{Q}[-\delta_{\mathcal{X}}]\big)$$

such that the following square is commutative:

$$\begin{array}{ccc} R\operatorname{Hom}\big(R\Gamma\big(\mathcal{Y}, \mathbb{Q}(d_{\mathcal{Y}})\big)_{\geq 0}, \mathbb{Q}[-\delta_{\mathcal{Y}}]\big) & \xrightarrow{\alpha_{\mathcal{Y}}} & R\Gamma(\overline{\mathcal{Y}}_{\text{ét}}, \mathbb{Z}) \\ \downarrow & & \downarrow \\ R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X}, \mathbb{Q}(d_{\mathcal{X}})\big)_{\geq 0}, \mathbb{Q}[-\delta_{\mathcal{X}}]\big) & \xrightarrow{\alpha_{\mathcal{X}}} & R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}) \end{array}$$

Showing that the diagram above is indeed commutative is tedious but straightforward using the pushforward map defined in [19, Corollary 7.2(b)]. Hence there exists a morphism

$$f_W^* : R\Gamma_W(\overline{\mathcal{Y}}, \mathbb{Z}) \to R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})$$

inducing a morphism of exact triangles. We claim that such a morphism f_W^* is unique. In order to ease the notations, we set

$$D_{\mathcal{X}} := R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X}, \mathbb{Q}(d_{\mathcal{X}})\big)_{\geq 0}, \mathbb{Q}[-\delta_{\mathcal{X}}]\big) \qquad \text{and}$$
$$D_{\mathcal{Y}} := R\operatorname{Hom}\big(R\Gamma\big(\mathcal{Y}, \mathbb{Q}(d_{\mathcal{Y}})\big)_{\geq 0}, \mathbb{Q}[-\delta_{\mathcal{Y}}]\big).$$

The complexes $R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})$ and $R\Gamma_W(\overline{\mathcal{Y}}, \mathbb{Z})$ are both perfect complexes of abelian groups, since they are bounded complexes with finitely generated cohomology groups. Applying the functor $\operatorname{Hom}_{\mathcal{D}}(-, R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}))$ to the exact triangle

$$D_{\mathcal{Y}} \to R\Gamma(\overline{\mathcal{Y}}_{\text{ét}}, \mathbb{Z}) \to R\Gamma_W(\overline{\mathcal{Y}}, \mathbb{Z}) \to D_{\mathcal{Y}}[1],$$

we obtain an exact sequence of abelian groups

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}}\big(D_{\mathcal{Y}}[1], R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})\big) &\to \operatorname{Hom}_{\mathcal{D}}\big(R\Gamma_W(\overline{\mathcal{Y}}, \mathbb{Z}), R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})\big)\\ &\to \operatorname{Hom}_{\mathcal{D}}\big(R\Gamma(\overline{\mathcal{Y}}_{\text{ét}}, \mathbb{Z}), R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})\big).\end{aligned}$$

On one hand, $\operatorname{Hom}_{\mathcal{D}}(D_{\mathcal{Y}}[1], R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}))$ is uniquely divisible since $D_{\mathcal{Y}}[1]$ is a complex of $\mathbb{Q}$-vector spaces. On the other hand, the abelian group $\operatorname{Hom}_{\mathcal{D}}(R\Gamma_W(\overline{\mathcal{Y}}, \mathbb{Z}), R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}))$ is finitely generated as it follows from the spectral sequence

$$\prod_{i\in\mathbb{Z}} \operatorname{Ext}^p\big(H_W^i(\overline{\mathcal{Y}}, \mathbb{Z}), H_W^{q+i}(\overline{\mathcal{X}}, \mathbb{Z})\big) \Rightarrow H^{p+q}\big(R\operatorname{Hom}\big(R\Gamma_W(\overline{\mathcal{Y}}, \mathbb{Z}), R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})\big)\big)$$

since $R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})$ and $R\Gamma_W(\overline{\mathcal{Y}}, \mathbb{Z})$ are both perfect. Hence the morphism

$$\operatorname{Hom}_{\mathcal{D}}\big(R\Gamma_W(\overline{\mathcal{Y}}, \mathbb{Z}), R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})\big) \to \operatorname{Hom}_{\mathcal{D}}\big(R\Gamma(\overline{\mathcal{Y}}_{\text{ét}}, \mathbb{Z}), R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})\big)$$

is injective, which implies the uniqueness of the morphism f_W^* sitting in the morphism of exact triangles of the theorem.

Assume now that $\mathcal{X}$ satisfies $\mathbf{L}(\mathcal{X}_{\text{ét}}, d_{\mathcal{X}})_{\geq 0}$, and let $R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})$ and $R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})'$ be two complexes given with exact triangles as in Definition 2.9. Then the identity map $\operatorname{Id} : \mathcal{X} \to \mathcal{X}$ induces a unique isomorphism $R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}) \simeq R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})'$ in $\mathcal{D}$. □

2.4. Relationship with Lichtenbaum's definition over finite fields

Let Y be a scheme of finite type over a finite field k. We denote by G_k and W_k the Galois group and the Weil group of k, respectively. The (small) Weil-étale topos Y_W^{sm} is the category of W_k-equivariant sheaves of sets on the étale site of $Y \otimes_k \overline{k}$. The big Weil-étale topos is defined (see [13]) as the fiber product

$$Y_W := Y_{\text{ét}} \times_{\overline{\text{Spec}(\mathbb{Z})}_{\text{ét}}} \overline{\text{Spec}(\mathbb{Z})}_W \simeq Y_{\text{ét}} \times_{B^{\text{sm}}_{G_k}} B_{W_k},$$

where $B^{\text{sm}}_{G_k}$ (resp., B_{W_k}) denotes the small classifying topos of G_k (resp., the big classifying topos of W_k). The topoi Y_W and Y_W^{sm} are cohomologically equivalent (see [13, Corollary 2]). Therefore, by [17] one has an exact triangle in the derived category of abelian sheaves on $Y_{\text{ét}}$

$$\mathbb{Z} \to R\gamma_*\mathbb{Z} \to \mathbb{Q}[-1] \to \mathbb{Z}[1],$$

where $\gamma : Y_W \to Y_{\text{ét}}$ is the first projection. Applying $R\Gamma(Y_{\text{ét}}, -)$ and rotating, we get

$$R\Gamma\big(Y_{\text{ét}}, \mathbb{Q}[-2]\big) \xrightarrow{a_Y} R\Gamma(Y_{\text{ét}}, \mathbb{Z}) \to R\Gamma(Y_W, \mathbb{Z}) \to R\Gamma\big(Y_{\text{ét}}, \mathbb{Q}[-2]\big)[1].$$

THEOREM 2.13

Let Y be a d-dimensional connected projective smooth scheme over k satisfying $\mathbf{L}(Y_{\text{ét}}, d)_{\geq 0}$. Then there is an isomorphism in $\mathcal{D}$

$$R\Gamma(Y_W, \mathbb{Z}) \xrightarrow{\sim} R\Gamma_W(Y, \mathbb{Z}), \tag{36}$$

where $R\Gamma(Y_W, \mathbb{Z})$ is the cohomology of the Weil-étale topos and $R\Gamma_W(Y, \mathbb{Z})$ is the complex defined in this paper. Moreover, the exact triangle of Definition 2.9 is isomorphic to Geisser's triangle (see [17, Corollary 5.2]).

Proof

We shall define a commutative square in $\mathcal{D}$

$$\begin{array}{ccc} R\Gamma\big(Y_{\text{ét}}, \mathbb{Q}[-2]\big) & \xrightarrow{\quad a_Y \quad} & R\Gamma(Y_{\text{ét}}, \mathbb{Z}) \\ \simeq \downarrow & & \downarrow \text{Id} \\ R\,\text{Hom}\big(R\Gamma\big(Y, \mathbb{Q}(d)\big)_{\geq 0}, \mathbb{Q}[-2d-2]\big) & \xrightarrow{\alpha_Y} & R\Gamma(Y_{\text{ét}}, \mathbb{Z}) \end{array}$$

where the vertical maps are isomorphisms. The existence of the isomorphism (36) and the compatibility with Geisser's triangle (see [17, Corollary 5.2] for $\mathcal{G}^{\cdot} = \mathbb{Z}$) will immediately follow. Moreover, the uniqueness of (36) will follow from the argument of the proof of Theorem 2.12.

One is therefore reduced to defining the commutative square above. Replacing k with a finite extension if necessary, one may suppose that Y is geometrically connected over k. One has $H^{2d}(Y,\mathbb{Q}(d)) = CH^d(Y)_{\mathbb{Q}} \simeq \mathbb{Q}$ (see [27, Theorem 6.1]) and $H^i(Y,\mathbb{Q}(d)) = 0$ for $i > 2d$. This yields a map $R\Gamma(Y,\mathbb{Q}(d)) \to \mathbb{Q}[-2d]$. The morphism

$$\begin{aligned} R\,\mathrm{Hom}_Y\big(\mathbb{Q},\mathbb{Q}(d)\big) &\to R\,\mathrm{Hom}\big(R\Gamma(Y,\mathbb{Q}), R\Gamma\big(Y,\mathbb{Q}(d)\big)\big) \\ &\to R\,\mathrm{Hom}\big(R\Gamma(Y,\mathbb{Q}),\mathbb{Q}[-2d]\big) \end{aligned}$$

induces a morphism

$$R\Gamma(Y_{\text{ét}},\mathbb{Q}) \simeq R\Gamma(Y,\mathbb{Q}) \to R\,\mathrm{Hom}\big(R\Gamma\big(Y,\mathbb{Q}(d)\big)_{\geq 0},\mathbb{Q}[-2d]\big). \tag{37}$$

It follows from conjecture $\mathbf{L}(Y_{\text{ét}},d)_{\geq 0}$, Lemma 2.5, and the fact that $H^i(Y_{\text{ét}},\mathbb{Z})$ is finite for $i \neq 0,2$ that the group $H^i(Y_{\text{ét}},\mathbb{Z}(d))_{\geq 0}$ is finite for $i \neq 2d, 2d+2$ and torsion for $i \neq 2d+2$. It follows easily that (37) is a quasi-isomorphism.

It remains to check the commutativity of the square above. The complex

$$D_Y := R\,\mathrm{Hom}\big(R\Gamma\big(Y,\mathbb{Q}(d)\big)_{\geq 0},\mathbb{Q}[-2d-2]\big) \simeq R\Gamma\big(Y_{\text{ét}},\mathbb{Q}[-2]\big)$$

is concentrated in degree 2. It follows that both a_Y and α_Y uniquely factor through the truncated complex $R\Gamma(Y_{\text{ét}},\mathbb{Z})_{\leq 2}$. It is therefore enough to show that the square

$$\begin{array}{ccc} R\Gamma\big(Y_{\text{ét}},\mathbb{Q}[-2]\big) & \longrightarrow & R\Gamma(Y_{\text{ét}},\mathbb{Z})_{\leq 2} \\ \downarrow & & \downarrow \mathrm{Id} \\ R\,\mathrm{Hom}\big(\Gamma\big(Y,\mathbb{Q}(d)\big)_{\geq 0},\mathbb{Q}[-2d-2]\big) & \longrightarrow & R\Gamma(Y_{\text{ét}},\mathbb{Z})_{\leq 2} \end{array}$$

commutes. The exact triangle

$$\mathbb{Z}[0] \to R\Gamma(Y_{\text{ét}},\mathbb{Z})_{\leq 2} \to \pi_1(Y_{\text{ét}})^D[-2] \to \mathbb{Z}[1]$$

induces an exact sequence of abelian groups

$$\mathrm{Hom}_{\mathfrak{D}}\big(D_Y,\mathbb{Z}[0]\big) \to \mathrm{Hom}_{\mathfrak{D}}\big(D_Y, R\Gamma(Y_{\text{ét}},\mathbb{Z})_{\leq 2}\big) \to \mathrm{Hom}_{\mathfrak{D}}\big(D_Y,\pi_1(Y_{\text{ét}})^D[-2]\big)$$

which shows that the map

$$\mathrm{Hom}_{\mathfrak{D}}\big(D_Y, R\Gamma(Y_{\text{ét}},\mathbb{Z})_{\leq 2}\big) \to \mathrm{Hom}_{\mathfrak{D}}\big(D_Y,\pi_1(Y_{\text{ét}})^D[-2]\big)$$

is injective. Indeed, $\mathbb{Z}[0] \simeq [\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}]$ has an injective resolution of length 1 and D_Y is concentrated in degree 2; hence $\mathrm{Hom}_{\mathfrak{D}}(D_Y,\mathbb{Z}[0]) = 0$. In view of the quasi-isomorphism

$$D_Y \simeq \mathrm{Hom}\big(H^{2d}\big(Y, \mathbb{Q}(d)\big), \mathbb{Q}\big)[-2] \simeq H^0(Y, \mathbb{Q})[-2],$$

one is reduced to showing the commutativity of the following square (of abelian groups):

$$\begin{array}{ccc} H^0(Y_{\acute{e}t}, \mathbb{Q}) & \xrightarrow{d_2^{0,1}} & H^2(Y_{\acute{e}t}, \mathbb{Z}) \\ \downarrow & & \downarrow \mathrm{Id} \\ \mathrm{Hom}\big(H^{2d}\big(Y, \mathbb{Q}(d)\big), \mathbb{Q}\big) & \xrightarrow{H^2(\alpha_Y)} & H^2(Y_{\acute{e}t}, \mathbb{Z}) \end{array}$$

By construction the map $H^2(\alpha_Y)$ is the following composition:

$$\begin{aligned} \mathrm{Hom}\big(H^{2d}\big(Y, \mathbb{Q}(d)\big), \mathbb{Q}\big) &\simeq \mathrm{Hom}\big(CH^d(Y), \mathbb{Q}\big) \to \mathrm{Hom}\big(CH^d(Y), \mathbb{Q}/\mathbb{Z}\big) \\ &\xleftarrow{\sim} \pi_1(Y_{\acute{e}t})^D \simeq H^2(Y_{\acute{e}t}, \mathbb{Z}), \end{aligned}$$

where the isomorphism $CH^d(Y)^D \xleftarrow{\sim} \pi_1(Y_{\acute{e}t})^D$ is the dual of the map $CH^d(Y) \to \pi_1(Y_{\acute{e}t})^{\mathrm{ab}}$ given by class field theory, which is injective with dense image (see [49, Corollary 3]). The top horizontal map in the last commutative square is the differential $d_2^{0,1}$ of the spectral sequence

$$H^i\big(Y_{\acute{e}t}, R^j(\gamma_*)\mathbb{Z}\big) \Rightarrow H^{i+j}(Y_W, \mathbb{Z}).$$

There is a canonical isomorphism

$$H^0\big(Y_{\acute{e}t}, R^1(\gamma_*)\mathbb{Z}\big) = \varinjlim_{k'/k} \mathrm{Hom}(W_{k'}, \mathbb{Z}) = \mathrm{Hom}(W_k, \mathbb{Q}),$$

and k'/k runs over the finite extensions of k, as follows from the isomorphism of pro-discrete groups $\pi_1(Y'_W, p) \simeq \pi_1(Y'_{\acute{e}t}, p) \times_{G_k} W_k$, which is valid for any Y' connected étale over Y. Then the left-hand side vertical map in the last square above is the map

$$\deg^* : \mathrm{Hom}(W_k, \mathbb{Q}) \to \mathrm{Hom}\big(CH^d(Y), \mathbb{Q}\big)$$

induced by the degree map

$$\deg : CH^d(Y) \to \mathbb{Z} \simeq W_k,$$

and the differential $d_2^{0,1}$ is the following map:

$$\mathrm{Hom}(W_k, \mathbb{Q}) = \mathrm{Hom}\big(\pi_1(Y_W, p), \mathbb{Q}\big) \to \mathrm{Hom}_c\big(\pi_1(Y_W, p), \mathbb{Q}/\mathbb{Z}\big) \simeq \pi_1(Y_{\acute{e}t}, p)^D,$$

where $\mathrm{Hom}_c(-,-)$ denotes the group of continuous morphisms. One is therefore reduced to observing that the square

$$\begin{array}{ccc} \mathrm{Hom}(W_k,\mathbb{Q}) & \xrightarrow{d_2^{0,1}} & \pi_1(Y_{\acute{e}t})^D \\ \downarrow{\scriptstyle \deg^*} & & \downarrow{\scriptstyle \mathrm{Id}} \\ \mathrm{Hom}\big(CH^d(Y),\mathbb{Q}\big) & \xrightarrow{H^2(\alpha_Y)} & \pi_1(Y_{\acute{e}t})^D \end{array}$$

commutes. □

We shall need the following result in Section 4.3.

PROPOSITION 2.14
Let $f: Y \to \mathcal{X}$ be a morphism of proper regular arithmetic schemes such that $\mathcal{X}$ is flat over Spec($\mathbb{Z}$) *and Y has characteristic p. Assume that $\mathcal{X}$ has pure dimension d, and assume that $\mathbf{L}(\mathcal{X}_{\acute{e}t}, d)_{\geq 0}$ holds. Then there exists a* unique *map in $\mathcal{D}$*

$$\tilde{f}_W^*: R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}) \to R\Gamma(Y_W,\mathbb{Z}),$$

which renders the following square commutative:

$$\begin{array}{ccc} R\Gamma(Y_{\acute{e}t},\mathbb{Z}) & \longrightarrow & R\Gamma(Y_W,\mathbb{Z}) \\ \uparrow{\scriptstyle f_{\acute{e}t}^*} & & \uparrow{\scriptstyle \tilde{f}_W^*} \\ R\Gamma(\overline{\mathcal{X}}_{\acute{e}t},\mathbb{Z}) & \longrightarrow & R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}) \end{array}$$

where $f_{\acute{e}t}^*$ is induced by the map $Y_{\acute{e}t} \to \overline{\mathcal{X}}_{\acute{e}t}$ and $R\Gamma(Y_W,\mathbb{Z})$ is the cohomology of the Weil-étale topos.

Proof
We shall define a morphism of exact triangles

$$\begin{array}{ccccc} R\Gamma(Y,\mathbb{Q})[-2] & \xrightarrow{a_Y} & R\Gamma(Y_{\acute{e}t},\mathbb{Z}) & \longrightarrow & R\Gamma(Y_W,\mathbb{Z}) \\ \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle f_{\acute{e}t}^*} & & \uparrow \\ R\,\mathrm{Hom}\big(R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{Q}[-\delta]\big) & \xrightarrow{\alpha_{\mathcal{X}}} & R\Gamma(\overline{\mathcal{X}}_{\acute{e}t},\mathbb{Z}) & \longrightarrow & R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}) \end{array}$$

where the left-hand side vertical map is the zero map and $\delta := 2d + 2$. The existence of $\tilde{f}_W^*$ will follow from the commutativity of the left-hand side square and the uniqueness of $\tilde{f}_W^*$ will follow from the facts that $R\Gamma(Y_W,\mathbb{Z})$ and $R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})$ are both perfect (by [34, Theorem 7.4] and Proposition 2.10, respectively) and that

$R\operatorname{Hom}(R\Gamma(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}[-\delta])$ is a complex of $\mathbb{Q}$-vector spaces, as in the proof of Theorem 2.12.

In order to show that such a morphism of exact triangles does exist, we only need to check that the composite map

$$R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X}, \mathbb{Q}(d)\big)_{\geq 0}, \mathbb{Q}[-\delta]\big) \xrightarrow{\alpha_{\mathcal{X}}} R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}) \xrightarrow{f_{\text{ét}}^*} R\Gamma(Y_{\text{ét}}, \mathbb{Z}) \tag{38}$$

is the zero map. The complex of $\mathbb{Q}$-vector spaces $D_{\mathcal{X}} := R\operatorname{Hom}(R\Gamma(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}[-\delta])$ is concentrated in degrees greater than or equal to 3, because $\mathcal{X}$ is flat (see (6)). Moreover, we have that $H^0(Y_{\text{ét}}, \mathbb{Z}) = \mathbb{Z}^{\pi_0(Y)}$, $H^1(Y_{\text{ét}}, \mathbb{Z}) = 0$, $H^2(Y_{\text{ét}}, \mathbb{Z}) \simeq (\mathbb{Q}/\mathbb{Z})^{\pi_0(Y)} \oplus A$ is the direct sum of a divisible group and a finite group A, and $H^i(Y_{\text{ét}}, \mathbb{Z})$ is finite for $i > 2$. The spectral sequence

$$\prod_{i \in \mathbb{Z}} \operatorname{Ext}^p\big(H^i(D_{\mathcal{X}}), H^{q+i}(Y_{\text{ét}}, \mathbb{Z})\big) \Rightarrow H^{p+q}\big(R\operatorname{Hom}\big(D_{\mathcal{X}}, R\Gamma(Y_{\text{ét}}, \mathbb{Z})\big)\big)$$

then shows that

$$\operatorname{Hom}_{\mathcal{D}}\big(D_{\mathcal{X}}, R\Gamma(Y_{\text{ét}}, \mathbb{Z})\big) = H^0\big(R\operatorname{Hom}\big(D_{\mathcal{X}}, R\Gamma(Y_{\text{ét}}, \mathbb{Z})\big)\big) = 0.$$

Hence (38) must be the zero map, and the result follows. □

In cases where Theorem 2.12 and Proposition 2.14 both apply, we have a commutative diagram

$$\begin{array}{ccc} R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}) & \xrightarrow{\tilde{f}_W^*} & R\Gamma(Y_W, \mathbb{Z}) \\ & \searrow^{f_W^*} & \downarrow \simeq \\ & & R\Gamma_W(Y, \mathbb{Z}) \end{array}$$

where f_W^* is the map defined in Theorem 2.12, $\tilde{f}_W^*$ is the map defined in Proposition 2.14, and the vertical isomorphism is defined in Theorem 2.13. Indeed, up to the identification given by Theorem 2.13, the map f_W^* sits in the commutative square of Proposition 2.14 and hence must coincide with $\tilde{f}_W^*$.

2.5. Relationship with Lichtenbaum's definition for number rings

In this section we consider a totally imaginary number field F and we set $\mathcal{X} = \operatorname{Spec}(\mathcal{O}_F)$. The complex $R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})$ is well defined since $\mathbf{L}(\mathcal{X}_{\text{ét}}, \mathbb{Z}(1))$ holds (see Theorem 5.1).

THEOREM 2.15
There is a canonical isomorphism in $\mathcal{D}$

$$R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}) \xrightarrow{\sim} R\Gamma(\overline{\mathcal{X}}_W,\mathbb{Z})_{\leq 3},$$

where $R\Gamma(\overline{\mathcal{X}}_W,\mathbb{Z})_{\leq 3}$ *is the truncation of Lichtenbaum's complex (see [35]) and* $R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})$ *is the complex defined in this paper.*

Proof
By [38, Theorem 9.5], we have a quasi-isomorphism

$$R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, R\mathbb{Z}) \xrightarrow{\sim} R\Gamma(\overline{\mathcal{X}}_W,\mathbb{Z})$$

inducing

$$R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, R_W\mathbb{Z}) \xrightarrow{\sim} R\Gamma(\overline{\mathcal{X}}_W,\mathbb{Z})_{\leq 3}, \tag{39}$$

where $R\mathbb{Z}$ is the complex defined in [38, Theorem 8.5] and $R_W\mathbb{Z} := R\mathbb{Z}_{\leq 2}$. The complex $R\Gamma(\overline{\mathcal{X}}_W,\mathbb{Z})$, defined in [35], is the cohomology of the Weil-étale topos $\overline{\mathcal{X}}_W$, which is defined in [39]. We have an exact triangle

$$\mathbb{Z}[0] \to R_W\mathbb{Z} \to R^2_W\mathbb{Z}[-2]$$

in the derived category of étale sheaves on $\overline{\mathcal{X}}$. Rotating and applying $R\Gamma(\overline{\mathcal{X}}_{\text{ét}},-)$ we get an exact triangle

$$R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, R^2_W\mathbb{Z})[-3] \to R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}) \to R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, R_W\mathbb{Z}).$$

We have canonical quasi-isomorphisms (see [38, Theorem 9.4 and isomorphism (35)])

$$R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, R^2_W\mathbb{Z})[-3] \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F^{\times},\mathbb{Q})[-3] \simeq R\,\mathrm{Hom}\big(R\Gamma\big(\mathcal{X},\mathbb{Q}(1)\big),\mathbb{Q}[-4]\big).$$

It follows that the morphism $R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, R^2_W\mathbb{Z})[-3] \to R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})$ in the triangle above is determined by the induced map

$$\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F^{\times},\mathbb{Q}) = H^0(\overline{\mathcal{X}}_{\text{ét}}, R^2_W\mathbb{Z}) \to H^3(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F^{\times},\mathbb{Q}/\mathbb{Z})$$

and so is the morphism $\alpha_{\mathcal{X}}$. In both cases this map is the obvious one. Hence the square

$$\begin{array}{ccc}
R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, R^2_W\mathbb{Z})[-3] & \longrightarrow & R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}) \\
\uparrow \simeq & & \uparrow \mathrm{Id} \\
R\,\mathrm{Hom}\big(R\Gamma\big(\mathcal{X},\mathbb{Q}(1)\big),\mathbb{Q}[-4]\big) & \longrightarrow & R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})
\end{array}$$

is commutative. Hence there exists an isomorphism $R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})\simeq R\Gamma(\overline{\mathcal{X}}_{et},R_W\mathbb{Z})$. The uniqueness of this isomorphism can be shown as in the proof of Theorem 2.12. Composing this isomorphism with (39), we obtain the result. □

3. Weil-étale cohomology with compact support

We recall below the definition given in [13] of the Weil-étale topos and some results concerning its cohomology with $\tilde{\mathbb{R}}$-coefficients. Then we define Weil-étale cohomology with compact support and $\mathbb{Z}$-coefficients and we study the expected map from $\mathbb{Z}$- to $\mathbb{R}$-coefficients.

3.1. Cohomology with $\tilde{\mathbb{R}}$-coefficients

Let $\mathcal{X}$ be any proper regular connected arithmetic scheme. The Weil-étale topos is defined as a 2-fiber product of topoi

$$\overline{\mathcal{X}}_W := \overline{\mathcal{X}}_{et} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{et}} \overline{\mathrm{Spec}(\mathbb{Z})}_W.$$

There is a canonical morphism

$$\mathfrak{f}:\overline{\mathcal{X}}_W\to\overline{\mathrm{Spec}(\mathbb{Z})}_W\to B_{\mathbb{R}},$$

where $B_{\mathbb{R}}$ is Grothendieck's classifying topos of the topological group $\mathbb{R}$ (see [23] or [13]). Consider the sheaf $y\mathbb{R}$ on $B_{\mathbb{R}}$ represented by $\mathbb{R}$ with the standard topology and trivial $\mathbb{R}$-action. Then one defines the sheaf

$$\tilde{\mathbb{R}} := \mathfrak{f}^*(y\mathbb{R})$$

on $\overline{\mathcal{X}}_W$. By [13], the following diagram consists of two pullback squares of topoi, and the rows give open-closed decompositions:

$$\begin{array}{ccccc}
\mathcal{X}_W & \xrightarrow{\phi} & \overline{\mathcal{X}}_W & \xleftarrow{i_\infty} & \mathcal{X}_{\infty,W}\\
\downarrow & & \downarrow & & \downarrow\\
\mathcal{X}_{et} & \xrightarrow{\varphi} & \overline{\mathcal{X}}_{et} & \xleftarrow{u_\infty} & \mathrm{Sh}(\mathcal{X}_\infty)
\end{array}$$

Here the map

$$\overline{\mathcal{X}}_W := \overline{\mathcal{X}}_{et} \times_{\overline{\mathrm{Spec}(\mathbb{Z})}_{et}} \overline{\mathrm{Spec}(\mathbb{Z})}_W \longrightarrow \overline{\mathcal{X}}_{et}$$

is the first projection, $\mathrm{Sh}(\mathcal{X}_\infty)$ is the category of sheaves on the space $\mathcal{X}_\infty$, and

$$\mathcal{X}_{\infty,W} = B_{\mathbb{R}}\times\mathrm{Sh}(\mathcal{X}_\infty),$$

where the product is taken over the final topos. As shown in [13], the topos $\overline{\mathcal{X}}_W$ has the right $\tilde{\mathbb{R}}$-cohomology with and without compact supports. We have

$$R\Gamma_W(\overline{\mathcal{X}},\tilde{\mathbb{R}}) := R\Gamma(\overline{\mathcal{X}}_W,\tilde{\mathbb{R}}) \simeq R\Gamma(B_{\mathbb{R}},\tilde{\mathbb{R}}) \simeq \mathbb{R}[-1]\oplus\mathbb{R}.$$

Concerning the cohomology with compact support, one has

$$R\Gamma_{W,c}(\mathcal{X},\tilde{\mathbb{R}}) := R\Gamma(\overline{\mathcal{X}}_W,\phi_!\tilde{\mathbb{R}}) \simeq R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\varphi_!\mathbb{R})[-1]\oplus R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\varphi_!\mathbb{R}), \tag{40}$$

where the complex $R\Gamma_c(\mathcal{X}_{\text{ét}},\mathbb{R}) := R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\varphi_!\mathbb{R})$ is quasi-isomorphic to

$$\mathrm{Cone}\big(\mathbb{R}[0]\to R\Gamma(\mathcal{X}_\infty,\mathbb{R})\big)[-1].$$

Cup product with the fundamental class $\theta\in H^1(\mathcal{X}_W,\tilde{\mathbb{R}})$ yields a morphism

$$\cup\,\theta : R\Gamma_{W,c}(\mathcal{X},\tilde{\mathbb{R}}) \to R\Gamma_{W,c}(\mathcal{X},\tilde{\mathbb{R}})[1] \tag{41}$$

such that the induced sequence

$$\cdots\to H^{i-1}_{W,c}(\mathcal{X},\tilde{\mathbb{R}})\to H^{i}_{W,c}(\mathcal{X},\tilde{\mathbb{R}})\to H^{i+1}_{W,c}(\mathcal{X},\tilde{\mathbb{R}})\to\cdots$$

is a bounded acyclic complex of finite-dimensional $\mathbb{R}$-vector spaces. In view of $R\Gamma_{W,c}(\mathcal{X},\tilde{\mathbb{R}})[1] = R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\varphi_!\mathbb{R})\oplus R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\varphi_!\mathbb{R})[1]$, the map (41) is simply given by projection and inclusion

$$R\Gamma_{W,c}(\mathcal{X},\tilde{\mathbb{R}}) \twoheadrightarrow R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\varphi_!\mathbb{R}) \hookrightarrow R\Gamma_{W,c}(\mathcal{X},\tilde{\mathbb{R}})[1].$$

3.2. Cohomology with $\mathbb{Z}$-coefficients

In the remaining part of this section, $\mathcal{X}$ denotes a proper, regular, and connected arithmetic scheme of dimension d satisfying $\mathbf{L}(\mathcal{X}_{\text{ét}},d)_{\geq 0}$. If $\mathcal{X}$ has characteristic p, then we set $R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}) := R\Gamma_W(\mathcal{X},\mathbb{Z})$ and the cohomology with compact support is defined as $H^i_{W,c}(\mathcal{X},\mathbb{Z}) := H^i_W(\mathcal{X},\mathbb{Z})$. The case when $\mathcal{X}$ is flat over $\mathbb{Z}$ is the case of interest. The closed embedding $u_\infty : \mathrm{Sh}(\mathcal{X}_\infty)\to\mathcal{X}_{\text{ét}}$ induces a morphism $u_\infty^* : R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})\to R\Gamma(\mathcal{X}_\infty,\mathbb{Z})$.

PROPOSITION 3.1

There exists a unique morphism $i_\infty^* : R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})\to R\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z})$ which makes the following diagram commutative:

$$\begin{array}{ccccc}
R\,\mathrm{Hom}\big(R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{Q}[-2d-2]\big) & \longrightarrow & R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}) & \longrightarrow & R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}) \\
\downarrow & & \downarrow{\scriptstyle u_\infty^*} & & {\scriptstyle \exists!}\downarrow{\scriptstyle i_\infty^*} \\
0 & \longrightarrow & R\Gamma(\mathcal{X}_\infty,\mathbb{Z}) & \longrightarrow & R\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z})
\end{array}$$

Proof
Recall from [13] that the second projection

$$\mathcal{X}_{\infty,W} := B_{\mathbb{R}} \times \mathrm{Sh}(\mathcal{X}_\infty) \to \mathrm{Sh}(\mathcal{X}_\infty) = \mathcal{X}_{\infty,\text{ét}}$$

induces a quasi-isomorphism $R\Gamma(\mathcal{X}_\infty, \mathbb{Z}) \xrightarrow{\sim} R\Gamma(\mathcal{X}_{\infty,W}, \mathbb{Z})$. Hence the existence of the map i_∞^* will follow (axiom TR3 of triangulated categories) from the fact that the map

$$\beta : R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X}, \mathbb{Q}(d)\big)_{\geq 0}, \mathbb{Q}[-2d-2]\big) \xrightarrow{\alpha_{\mathcal{X}}} R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}) \xrightarrow{u_\infty^*} R\Gamma(\mathcal{X}_\infty, \mathbb{Z}) \quad (42)$$

is the zero map. Again, we set $D_{\mathcal{X}} = R\operatorname{Hom}(R\Gamma(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}[-2d-2])$ for brevity. Then the uniqueness of i_∞^* follows from the exact sequence

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}}\big(D_{\mathcal{X}}[1], R\Gamma(\mathcal{X}_{\infty,W}, \mathbb{Z})\big) &\xrightarrow{0} \operatorname{Hom}_{\mathcal{D}}\big(R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}), R\Gamma(\mathcal{X}_{\infty,W}, \mathbb{Z})\big) \\ &\to \operatorname{Hom}_{\mathcal{D}}\big(R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}), R\Gamma(\mathcal{X}_{\infty,W}, \mathbb{Z})\big),\end{aligned}$$

whose exactness follows from the fact that $\operatorname{Hom}_{\mathcal{D}}(D_{\mathcal{X}}[1], R\Gamma(\mathcal{X}_{\infty,W}, \mathbb{Z}))$ is divisible while $\operatorname{Hom}_{\mathcal{D}}(R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}), R\Gamma(\mathcal{X}_{\infty,W}, \mathbb{Z}))$ is finitely generated.

It remains to show that the morphism (42) is indeed trivial. Since $D_{\mathcal{X}}$ is a bounded complex of $\mathbb{Q}$-vector spaces acyclic in degrees less than 2, one can choose a (noncanonical) isomorphism $D_{\mathcal{X}} \simeq \bigoplus_{k\geq 2} H^k(D_{\mathcal{X}})[-k]$. Then β is identified with the collection of maps $(\beta_k)_{k\geq 2}$ in $\mathcal{D}$, with

$$\beta_k : H^k(D_{\mathcal{X}})[-k] \to \bigoplus_{k\geq 2} H^k(D_{\mathcal{X}})[-k] \simeq D_{\mathcal{X}} \to R\Gamma(\mathcal{X}_\infty, \mathbb{Z}).$$

It is enough to show that $\beta_k = 0$ for $k \geq 2$. We fix such a k, and we consider the spectral sequence

$$\begin{aligned}E_2^{p,q} &= \operatorname{Ext}^p\big(H^k D_{\mathcal{X}}, H^{q+k}(\mathcal{X}_\infty, \mathbb{Z})\big) \\ &\Rightarrow H^{p+q}\big(R\operatorname{Hom}\big(H^k(D_{\mathcal{X}})[-k], R\Gamma(\mathcal{X}_\infty, \mathbb{Z})\big)\big).\end{aligned}$$

The group $H^k D_{\mathcal{X}}$ is uniquely divisible and $H^{q+k}(\mathcal{X}_\infty, \mathbb{Z})$ is finitely generated (hence has an injective resolution of length 1), so that $\operatorname{Ext}^p(H^k(D_{\mathcal{X}}), H^{q+k}(\mathcal{X}_\infty, \mathbb{Z})) = 0$ for $p \neq 1$. Hence the spectral sequence above degenerates and gives a canonical isomorphism

$$\operatorname{Hom}_{\mathcal{D}}\big(H^k D_{\mathcal{X}}[-k], R\Gamma(\mathcal{X}_\infty, \mathbb{Z})\big) \simeq \operatorname{Ext}^1\big(H^k D_{\mathcal{X}}, H^{k-1}(\mathcal{X}_\infty, \mathbb{Z})\big).$$

Moreover, the long exact sequence for $\operatorname{Ext}^*(H^k D_{\mathcal{X}}, -)$ yields

$$\operatorname{Ext}^1\big(H^k D_{\mathcal{X}}, H^{k-1}(\mathcal{X}_\infty, \mathbb{Z})\big) \simeq \operatorname{Ext}^1\big(H^k D_{\mathcal{X}}, H^{k-1}(\mathcal{X}_\infty, \mathbb{Z})_{\text{cotor}}\big)$$

since the maximal torsion subgroup of $H^{k-1}(\mathcal{X}_\infty, \mathbb{Z})$ is finite and $H^k D_{\mathcal{X}}$ is uniquely divisible. The short exact sequence

$$0 \to H^{k-1}(\mathcal{X}_\infty, \mathbb{Z})_{\mathrm{cotor}} \to H^{k-1}(\mathcal{X}_\infty, \mathbb{Q}) \to H^{k-1}(\mathcal{X}_\infty, \mathbb{Q}/\mathbb{Z})_{\mathrm{div}} \to 0$$

is an injective resolution of the $\mathbb{Z}$-module $H^{k-1}(\mathcal{X}_\infty, \mathbb{Z})_{\mathrm{cotor}}$. This yields an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}\big(H^k D_{\mathcal{X}}, H^{k-1}(\mathcal{X}_\infty, \mathbb{Q})\big) \to \mathrm{Hom}\big(H^k D_{\mathcal{X}}, H^{k-1}(\mathcal{X}_\infty, \mathbb{Q}/\mathbb{Z})_{\mathrm{div}}\big) \\ &\to \mathrm{Ext}^1\big(H^k D_{\mathcal{X}}, H^{k-1}(\mathcal{X}_\infty, \mathbb{Z})_{\mathrm{cotor}}\big) \to 0. \end{aligned}$$

Let us define a natural lifting

$$\tilde{\beta}_k \in \mathrm{Hom}\big(H^k D_{\mathcal{X}}, H^{k-1}(\mathcal{X}_\infty, \mathbb{Q}/\mathbb{Z})_{\mathrm{div}}\big)$$

of $\beta_k \in \mathrm{Ext}^1(H^k D_{\mathcal{X}}, H^{k-1}(\mathcal{X}_\infty, \mathbb{Z})_{\mathrm{cotor}})$ and show that this lifting $\tilde{\beta}_k$ is already zero. One can assume that $k \geq 2$. Recall that $H^k D_{\mathcal{X}} = \mathrm{Hom}(H^{\delta-k}(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q})$. We have the following commutative diagram:

$$\begin{array}{ccccc}
 & & \mathrm{Hom}\big(H^{\delta-k}(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}\big) & \overset{H^k(\alpha_{\mathcal{X}})}{\searrow} & \\
 & & \downarrow & & \\
 & & H^{k-1}(\overline{\mathcal{X}}_{\mathrm{\acute{e}t}}, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\simeq} & H^k(\overline{\mathcal{X}}_{\mathrm{\acute{e}t}}, \mathbb{Z}) \\
 & & \downarrow & & \downarrow \\
 & & H^{k-1}(\mathcal{X}_{\mathbb{Q},\mathrm{\acute{e}t}}, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\simeq} & H^k(\mathcal{X}_{\mathbb{Q},\mathrm{\acute{e}t}}, \mathbb{Z}) \\
 & & \downarrow & & \downarrow \\
 & & H^{k-1}(\mathcal{X}_{\overline{\mathbb{Q}},\mathrm{\acute{e}t}}, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\simeq} & H^k(\mathcal{X}_{\overline{\mathbb{Q}},\mathrm{\acute{e}t}}, \mathbb{Z}) \\
 & & \downarrow \simeq & & \downarrow \\
H^{k-1}\big(\mathcal{X}(\mathbb{C}), \mathbb{Q}\big) & \longrightarrow & H^{k-1}\big(\mathcal{X}(\mathbb{C}), \mathbb{Q}/\mathbb{Z}\big) & \longrightarrow & H^k\big(\mathcal{X}(\mathbb{C}), \mathbb{Z}\big)
\end{array}$$

The morphism given by the central column of the diagram above factors through

$$H^{k-1}(\mathcal{X}_\infty, \mathbb{Q}/\mathbb{Z})_{\mathrm{div}} \subseteq H^{k-1}(\mathcal{X}_\infty, \mathbb{Q}/\mathbb{Z}) \to H^{k-1}\big(\mathcal{X}(\mathbb{C}), \mathbb{Q}/\mathbb{Z}\big)$$

and yields the desired lifting

$$\tilde{\beta}_k : \mathrm{Hom}\big(H^{\delta-k}(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{Q}\big) \to H^{k-1}(\mathcal{X}_\infty, \mathbb{Q}/\mathbb{Z})_{\mathrm{div}}.$$

Here the morphism

$$H^{k-1}(\mathcal{X}_\infty, \mathbb{Q}/\mathbb{Z}) \to H^{k-1}\big(\mathcal{X}(\mathbb{C}), \mathbb{Q}/\mathbb{Z}\big) \tag{43}$$

is induced by the projection $\mathcal{X}(\mathbb{C}) \to \mathcal{X}_\infty$. Using standard spectral sequences for equivariant cohomology, it is easy to see that the kernel of the morphism (43) is of finite exponent. (More precisely, this kernel is finite and killed by a power of 2.) It follows that $\tilde{\beta}_k = 0$ if and only if the map

$$\mathrm{Hom}\big(H^{\delta-k}\big(\mathcal{X}, \mathbb{Q}(d)\big)_{\geq 0}, \mathbb{Q}\big) \to H^{k-1}\big(\mathcal{X}(\mathbb{C}), \mathbb{Q}/\mathbb{Z}\big), \tag{44}$$

given by the central column of the previous diagram, is the zero map. Moreover, the map

$$H^{k-1}(\mathcal{X}_{\mathbb{Q},\text{ét}}, \mathbb{Q}/\mathbb{Z}) \to H^{k-1}(\mathcal{X}_{\overline{\mathbb{Q}},\text{ét}}, \mathbb{Q}/\mathbb{Z})$$

factors through $H^{k-1}(\mathcal{X}_{\overline{\mathbb{Q}},\text{ét}}, \mathbb{Q}/\mathbb{Z})^{G_\mathbb{Q}}$ and hence so does the map (44). In order to show that $\tilde{\beta}_k = 0$ it is therefore enough to show that

$$\big(H^{k-1}(\mathcal{X}_{\overline{\mathbb{Q}},\text{ét}}, \mathbb{Q}/\mathbb{Z})^{G_\mathbb{Q}}\big)_{\text{div}} = \bigoplus_l \big(H^{k-1}(\mathcal{X}_{\overline{\mathbb{Q}},\text{ét}}, \mathbb{Q}_l/\mathbb{Z}_l)^{G_\mathbb{Q}}\big)_{\text{div}} = 0.$$

Let l be a fixed prime number. Let $U \subseteq \mathrm{Spec}(\mathbb{Z})$ on which l is invertible and such that $\mathcal{X}_U \to U$ is smooth. Let $p \in U$. By smooth and proper base change we have that

$$H^{k-1}(\mathcal{X}_{\overline{\mathbb{Q}},\text{ét}}, \mathbb{Q}_l/\mathbb{Z}_l)^{I_p} \simeq H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,\text{ét}}, \mathbb{Q}_l/\mathbb{Z}_l).$$

Recall that $H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,\text{ét}}, \mathbb{Z}_l)$ is a finitely generated $\mathbb{Z}_l$-module. We have an exact sequence

$$0 \to H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,\text{ét}}, \mathbb{Z}_l)_{\text{cotor}} \to H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,\text{ét}}, \mathbb{Q}_l) \to H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,\text{ét}}, \mathbb{Q}_l/\mathbb{Z}_l)_{\text{div}} \to 0.$$

We get

$$\begin{aligned} 0 &\to \big(H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,\text{ét}}, \mathbb{Z}_l)_{\text{cotor}}\big)^{G_{\mathbb{F}_p}} \to H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,\text{ét}}, \mathbb{Q}_l)^{G_{\mathbb{F}_p}} \\ &\to \big(H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,\text{ét}}, \mathbb{Q}_l/\mathbb{Z}_l)_{\text{div}}\big)^{G_{\mathbb{F}_p}} \to H^1\big(G_{\mathbb{F}_p}, H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,\text{ét}}, \mathbb{Z}_l)_{\text{cotor}}\big). \end{aligned}$$

Again, $H^1(G_{\mathbb{F}_p}, H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,\text{ét}}, \mathbb{Z}_l)_{\text{cotor}})$ is a finitely generated $\mathbb{Z}_l$-module; hence we get a surjective map

$$H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,\text{ét}}, \mathbb{Q}_l)^{G_{\mathbb{F}_p}} \to \Big(\big(H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,\text{ét}}, \mathbb{Q}_l/\mathbb{Z}_l)_{\text{div}}\big)^{G_{\mathbb{F}_p}}\Big)_{\text{div}} \to 0.$$

Note that

$$\big((H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,\text{ét}},\mathbb{Q}_l/\mathbb{Z}_l)_{\text{div}}\big)^{G_{\mathbb{F}_p}}\big)_{\text{div}}=\big(H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,\text{ét}},\mathbb{Q}_l/\mathbb{Z}_l)^{G_{\mathbb{F}_p}}\big)_{\text{div}}.$$

But $H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,\text{ét}},\mathbb{Q}_l)$ is pure of weight $k-1>0$ by [9]; hence there is no nontrivial element in $H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,\text{ét}},\mathbb{Q}_l)$ fixed by the Frobenius. This shows that

$$\begin{aligned}\big(H^{k-1}(\mathcal{X}_{\overline{\mathbb{Q}},\text{ét}},\mathbb{Q}_l/\mathbb{Z}_l)^{G_{\mathbb{Q}_p}}\big)_{\text{div}}&=\big(H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,\text{ét}},\mathbb{Q}_l/\mathbb{Z}_l)^{G_{\mathbb{F}_p}}\big)_{\text{div}}\\&=H^{k-1}(\mathcal{X}_{\overline{\mathbb{F}}_p,\text{ét}},\mathbb{Q}_l)^{G_{\mathbb{F}_p}}=0.\end{aligned}$$

A fortiori, one has $(H^{k-1}(\mathcal{X}_{\overline{\mathbb{Q}},\text{ét}},\mathbb{Q}_l/\mathbb{Z}_l)^{G_{\mathbb{Q}}})_{\text{div}}=0$ and the result follows. □

Definition 3.2
There exists an object $R\Gamma_{W,c}(\mathcal{X},\mathbb{Z})$, well defined up to isomorphism in $\mathcal{D}$, *endowed with* an exact triangle

$$R\Gamma_{W,c}(\mathcal{X},\mathbb{Z})\to R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})\xrightarrow{i_\infty^*}R\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z}). \tag{45}$$

The determinant $\det_{\mathbb{Z}}R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}):=\bigotimes_{i\in\mathbb{Z}}\det_{\mathbb{Z}}H^i_{W,c}(\mathcal{X},\mathbb{Z})^{(-1)^i}$ is well defined up to a *canonical* isomorphism.

The cohomology with compact support is defined (up to isomorphism only) as follows:

$$H^i_{W,c}(\mathcal{X},\mathbb{Z}):=H^i\big(R\Gamma_{W,c}(\mathcal{X},\mathbb{Z})\big).$$

To see that $\det_{\mathbb{Z}}R\Gamma_{W,c}(\mathcal{X},\mathbb{Z})$ is indeed well defined, consider another object $R\Gamma_{W,c}(\mathcal{X},\mathbb{Z})'$ of $\mathcal{D}$ endowed with an exact triangle (45). There exists a (nonunique) morphism $u:R\Gamma_{W,c}(\mathcal{X},\mathbb{Z})\to R\Gamma_{W,c}(\mathcal{X},\mathbb{Z})'$ lying in a morphism of exact triangles

$$\begin{array}{ccccc}R\Gamma_{W,c}(\mathcal{X},\mathbb{Z})&\longrightarrow&R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})&\longrightarrow&R\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z})\\ {\scriptstyle\exists u}\big\downarrow{\scriptstyle\simeq}&&{\scriptstyle\mathrm{Id}}\big\downarrow&&{\scriptstyle\mathrm{Id}}\big\downarrow\\ R\Gamma_{W,c}(\mathcal{X},\mathbb{Z})'&\longrightarrow&R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})&\longrightarrow&R\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z})\end{array}$$

The map u induces

$$\det_{\mathbb{Z}}(u):\det_{\mathbb{Z}}R\Gamma_{W,c}(\mathcal{X},\mathbb{Z})\xrightarrow{\sim}\det_{\mathbb{Z}}R\Gamma_{W,c}(\mathcal{X},\mathbb{Z})'.$$

By [28, p. 43, Corollary 2], $\det_{\mathbb{Z}}(u)$ does not depend on the choice of u, since it coincides with the following canonical isomorphism:

$$\begin{aligned}\det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}) &\simeq \det_{\mathbb{Z}} R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}) \otimes \det_{\mathbb{Z}}^{-1} R\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z})\\ &\simeq \det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X},\mathbb{Z})'.\end{aligned}$$

Given a complex of abelian groups C, we write $C_{\mathbb{R}}$ for $C \otimes \mathbb{R}$.

PROPOSITION 3.3

We set $\delta := 2d + 2$. *There is a* canonical and functorial *direct sum decomposition in* $\mathcal{D}$

$$R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})_{\mathbb{R}} \simeq R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{R}) \oplus R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{R}[-\delta]\big)[1]$$

such that the following square commutes:

$$\begin{array}{ccc} R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})_{\mathbb{R}} & \xrightarrow{i_\infty^*\otimes\mathbb{R}} & R\Gamma(\mathcal{X}_{\infty,W},\mathbb{Z})_{\mathbb{R}} \\ \simeq\downarrow & & \simeq\downarrow \\ R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{R}) \oplus R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{R}[-\delta]\big)[1] & \xrightarrow{(u_\infty^*\otimes\mathbb{R},0)} & R\Gamma(\mathcal{X}_\infty,\mathbb{R}) \end{array}$$

Proof

Applying $(-) \otimes \mathbb{R}$ to the exact triangle of Definition 2.9, we obtain an exact triangle

$$R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{R}[-\delta]\big) \to R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{R}) \to R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})_{\mathbb{R}}.$$

But the map

$$R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{R}[-\delta]\big) \to R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{R})$$

is trivial since $R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{R}) \simeq \mathbb{R}[0]$ is injective and $R\operatorname{Hom}(R\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{R}[-\delta])$ is acyclic in degrees less than or equal to 1. This shows the existence of the direct sum decomposition. We write $D_{\mathbb{R}} := R\operatorname{Hom}(R\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{R}[-\delta])$ and $R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{R}) \simeq \mathbb{R}[0]$ for brevity. The exact sequence

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}}\big(D_{\mathbb{R}}[1],\mathbb{R}[0]\big) &\to \operatorname{Hom}_{\mathcal{D}}\big(R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}),\mathbb{R}[0]\big)\\ &\to \operatorname{Hom}_{\mathcal{D}}\big(\mathbb{R}[0],\mathbb{R}[0]\big) \to \operatorname{Hom}_{\mathcal{D}}\big(D_{\mathbb{R}},\mathbb{R}[0]\big)\end{aligned}$$

yields an isomorphism $\operatorname{Hom}_{\mathcal{D}}(R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}),\mathbb{R}[0]) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}}(\mathbb{R}[0],\mathbb{R}[0])$. Hence there exists a unique map $s_{\overline{\mathcal{X}}} : R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z})_{\mathbb{R}} \to \mathbb{R}[0]$ such that $s_{\overline{\mathcal{X}}} \circ \gamma_{\overline{\mathcal{X}}}^* = \mathrm{Id}_{\mathbb{R}[0]}$

where $\gamma^*_{\overline{\mathcal{X}}} : \mathbb{R}[0] \to R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})$ is the given map. The functorial behavior of $s_{\overline{\mathcal{X}}}$ follows from the fact that it is the unique map such that $s_{\overline{\mathcal{X}}} \circ \gamma^*_{\overline{\mathcal{X}}} = \mathrm{Id}_{\mathbb{R}[0]}$. The direct sum decomposition is therefore canonical and functorial. The commutativity of the square follows from Proposition 3.1. □

Recall that we denote $R\Gamma_c(\mathcal{X}_{\text{ét}}, \mathbb{R}) := R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \varphi_!\mathbb{R})$.

PROPOSITION 3.4

We set $\delta := 2d + 2$. *There is a* noncanonical *direct sum decomposition*

$$R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z})_{\mathbb{R}} \simeq R\Gamma_c(\mathcal{X}_{\text{ét}}, \mathbb{R}) \oplus R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X}, \mathbb{Q}(d)\big)_{\geq 0}, \mathbb{R}[-\delta]\big)[1] \tag{46}$$

inducing a canonical *isomorphism*

$$\det_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z})_{\mathbb{R}} \simeq \det_{\mathbb{R}} R\Gamma_c(\mathcal{X}_{\text{ét}}, \mathbb{R}) \otimes \det^{-1}_{\mathbb{R}} R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X}, \mathbb{Q}(d)\big)_{\geq 0}, \mathbb{R}[-\delta]\big).$$

Proof

Again we write $D_{\mathbb{R}} := R\operatorname{Hom}(R\Gamma(\mathcal{X}, \mathbb{Q}(d))_{\geq 0}, \mathbb{R}[-\delta])$ and $R\Gamma(\mathcal{X}_{\text{ét}}, \mathbb{R}) \simeq \mathbb{R}[0]$ for brevity. (Recall that $\mathcal{X}$ is connected.) Consider the following morphism of exact triangles:

$$\begin{array}{ccccc}
R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z})_{\mathbb{R}} & \longrightarrow & R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})_{\mathbb{R}} & \xrightarrow{i^*_\infty \otimes \mathbb{R}} & R\Gamma(\mathcal{X}_{\infty,W}, \mathbb{Z})_{\mathbb{R}} \\
\downarrow{\scriptstyle \exists u} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\
R\Gamma_c(\mathcal{X}_{\text{ét}}, \mathbb{R}) \oplus D_{\mathbb{R}}[1] & \longrightarrow & R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{R}) \oplus D_{\mathbb{R}}[1] & \longrightarrow & R\Gamma(\mathcal{X}_\infty, \mathbb{R})
\end{array}$$

Here all the maps but the isomorphism u are canonical. The noncanonical direct sum decomposition (46) follows. A choice of such an isomorphism u induces

$$\det_{\mathbb{R}}(u) : \det_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z})_{\mathbb{R}} \xrightarrow{\sim} \det_{\mathbb{R}}\big(R\Gamma_c(\mathcal{X}_{\text{ét}}, \mathbb{R}) \oplus D_{\mathbb{R}}[1]\big).$$

But $\det_{\mathbb{R}}(u)$ coincides (see [28, p. 43, Corollary 2]) with the following (canonical) isomorphism:

$$\begin{aligned}
\det_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z})_{\mathbb{R}} &\simeq \det_{\mathbb{R}} R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z})_{\mathbb{R}} \otimes \det^{-1}_{\mathbb{R}} R\Gamma(\mathcal{X}_{\infty,W}, \mathbb{Z})_{\mathbb{R}} \\
&\simeq \det_{\mathbb{R}}\big(R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{R}) \oplus D_{\mathbb{R}}[1]\big) \otimes \det^{-1}_{\mathbb{R}} R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{R}) \\
&\simeq \det_{\mathbb{R}}\big(R\Gamma_c(\mathcal{X}_{\text{ét}}, \mathbb{R}) \oplus D_{\mathbb{R}}[1]\big),
\end{aligned}$$

and hence does not depend on the choice of u. □

3.3. The conjecture $\mathbf{B}(\mathcal{X},d)$ *and the regulator map*

Let $\mathcal{X}$ be a proper flat regular connected arithmetic scheme of dimension d with generic fiber $X=\mathcal{X}_{\mathbb{Q}}$. The "integral part in the motivic cohomology" $H_M^{2d-1-i}(X_{/\mathbb{Z}},\mathbb{Q}(d))$ is defined as the image of the morphism

$$H^{2d-1-i}\big(\mathcal{X},\mathbb{Q}(d)\big)\to H^{2d-1-i}\big(X,\mathbb{Q}(d)\big).$$

Let $H_{\mathcal{D}}^{p}(X_{/\mathbb{R}},\mathbb{R}(q))$ denote the real Deligne cohomology, and let

$$\rho_\infty^i: H_M^{2d-1-i}\big(X_{/\mathbb{Z}},\mathbb{Q}(d)\big)_{\mathbb{R}}\to H_{\mathcal{D}}^{2d-1-i}\big(X_{/\mathbb{R}},\mathbb{R}(d)\big)$$

be the Beilinson regulator. According to a classical conjecture of Beilinson, the map ρ_∞^i should be an isomorphism for $i\geq 1$ and there should be an exact sequence

$$0\to H_M^{2d-1}\big(X_{/\mathbb{Z}},\mathbb{Q}(d)\big)_{\mathbb{R}}\xrightarrow{\rho_\infty^0} H_{\mathcal{D}}^{2d-1}\big(X_{/\mathbb{R}},\mathbb{R}(d)\big)\to CH^0(X)_{\mathbb{R}}^*\to 0$$

for $i=0$. Moreover, the natural map

$$H^{2d-1-i}\big(\mathcal{X},\mathbb{Q}(d)\big)_{\mathbb{R}}\to H^{2d-1-i}\big(X,\mathbb{Q}(d)\big)_{\mathbb{R}}$$

is expected to be injective for $i\geq 0$. This suggests the following conjecture, where we consider the composite map

$$H^{2d-1-i}\big(\mathcal{X},\mathbb{Q}(d)\big)_{\mathbb{R}}\to H^{2d-1-i}\big(X,\mathbb{Q}(d)\big)_{\mathbb{R}}\to H_{\mathcal{D}}^{2d-1-i}\big(X_{/\mathbb{R}},\mathbb{R}(d)\big).$$

CONJECTURE 3.5 ($\mathbf{B}(\mathcal{X},d)$)

The map

$$H^{2d-1-i}\big(\mathcal{X},\mathbb{Q}(d)\big)_{\mathbb{R}}\to H_{\mathcal{D}}^{2d-1-i}\big(X_{/\mathbb{R}},\mathbb{R}(d)\big)$$

is an isomorphism for $1\leq i\leq 2d-1$ *and there is an exact sequence*

$$0\to H^{2d-1}\big(\mathcal{X},\mathbb{Q}(d)\big)_{\mathbb{R}}\to H_{\mathcal{D}}^{2d-1}\big(X_{/\mathbb{R}},\mathbb{R}(d)\big)\to CH^0(X)_{\mathbb{R}}^*\to 0$$

for $i=0$.

By [22], there is a canonical morphism of complexes

$$\Gamma\big(X,\mathbb{Q}(d)\big)\to C_{\mathcal{D}}\big(X_{/\mathbb{R}},\mathbb{R}(d)\big)$$

inducing Beilinson's regulator on cohomology (see [6]), where the complex $C_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(d))$ computes the real Deligne cohomology. Let $j: X_{\mathrm{Zar}}\to\mathcal{X}_{\mathrm{Zar}}$ be the natural embedding. Consider the map

$$\begin{aligned} R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big) &\to R\Gamma\big(\mathcal{X}, j_*\mathbb{Q}(d)\big) \to R\Gamma\big(\mathcal{X}, Rj_*\mathbb{Q}(d)\big) \\ &\simeq R\Gamma\big(X,\mathbb{Q}(d)\big) \simeq \Gamma\big(X,\mathbb{Q}(d)\big), \end{aligned}$$

where the last isomorphism follows from the fact that, over a field, the Zariski hypercohomology of the cycle complex coincides with its cohomology. Then we consider the composite map

$$\rho_\infty : R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big) \to \Gamma\big(X,\mathbb{Q}(d)\big) \to C_{\mathcal{D}}\big(X_{/\mathbb{R}},\mathbb{R}(d)\big),$$

and we denote by $\mathcal{D}(\mathbb{R})$ the derived category of $\mathbb{R}$-vector spaces.

THEOREM 3.6

Let $\mathcal{X}$ be a proper flat regular connected scheme of dimension d satisfying $\mathbf{L}(\mathcal{X}_{ét}, d)_{\geq 0}$ and $\mathbf{B}(\mathcal{X}, d)$. A choice of a direct sum decomposition (46) induces, in a canonical way, an isomorphism in $\mathcal{D}(\mathbb{R})$:

$$R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}) \otimes \mathbb{R} \xrightarrow{\sim} R\Gamma_{W,c}(\mathcal{X},\tilde{\mathbb{R}}).$$

Proof

Duality for Deligne cohomology (see [5, Corollary 2.28])

$$H^i_{\mathcal{D}}\big(X_{/\mathbb{R}},\mathbb{R}(p)\big) \times H^{2d-1-i}_{\mathcal{D}}\big(X_{/\mathbb{R}},\mathbb{R}(d-p)\big) \to \mathbb{R} \tag{47}$$

yields an isomorphism in $\mathcal{D}(\mathbb{R})$

$$C_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}) \to R\operatorname{Hom}\big(C_{\mathcal{D}}\big(X_{/\mathbb{R}},\mathbb{R}(d)\big),\mathbb{R}[-2d+1]\big).$$

Composing with $R\Gamma(\mathcal{X}_\infty,\mathbb{R}) \xrightarrow{\sim} C_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R})$ we obtain

$$R\Gamma(\mathcal{X}_\infty,\mathbb{R}) \xrightarrow{\sim} R\operatorname{Hom}\big(C_{\mathcal{D}}\big(X_{/\mathbb{R}},\mathbb{R}(d)\big),\mathbb{R}[-2d+1]\big).$$

One has a morphism of complexes $C_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(d)) \to CH^0(X)^*_{\mathbb{R}}[-2d+1]$ and we define $\tilde{C}_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(d))$ to be its mapping fiber. Applying the functor $R\operatorname{Hom}(-, \mathbb{R}[-2d+1])$, we obtain an exact triangle (note that X is irreducible)

$$\begin{aligned} \mathbb{R}[0] &\to R\operatorname{Hom}\big(C_{\mathcal{D}}\big(X_{/\mathbb{R}},\mathbb{R}(d)\big),\mathbb{R}[-2d+1]\big) \\ &\to R\operatorname{Hom}\big(\tilde{C}_{\mathcal{D}}\big(X_{/\mathbb{R}},\mathbb{R}(d)\big),\mathbb{R}[-2d+1]\big). \end{aligned}$$

We have a morphism of exact triangles

$$\begin{array}{ccccc} \mathbb{R}[0] & \longrightarrow & R\Gamma(\mathcal{X}_\infty,\mathbb{R}) & \longrightarrow & R\Gamma_c(\mathcal{X}_{ét},\mathbb{R})[1] \\ \downarrow & & \downarrow & & \downarrow \exists! \\ \mathbb{R}[0] & \to & R\operatorname{Hom}\big(C_{\mathcal{D}}\big(X_{/\mathbb{R}},\mathbb{R}(d)\big),\mathbb{R}[-2d+1]\big) & \to & R\operatorname{Hom}\big(\tilde{C}_{\mathcal{D}}\big(X_{/\mathbb{R}},\mathbb{R}(d)\big),\mathbb{R}[-2d+1]\big) \end{array}$$

where the vertical map on the right-hand side is uniquely determined. This yields a canonical quasi-isomorphism

$$R\Gamma_c(\mathcal{X}_{\text{ét}},\mathbb{R})[-1] \xrightarrow{\sim} R\operatorname{Hom}\big(\tilde{C}_{\mathcal{D}}\big(X_{/\mathbb{R}},\mathbb{R}(d)\big),\mathbb{R}[-2d-1]\big). \tag{48}$$

It follows from Conjecture $\mathbf{B}(\mathcal{X},d)$ that the morphism of complexes $\rho_\infty : R\Gamma(\mathcal{X}, \mathbb{Q}(d))_{\geq 0} \to C_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(d))$ induces a quasi-isomorphism

$$\tilde{\rho}_{\infty,\mathbb{R}} : R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0,\mathbb{R}} \xrightarrow{\sim} \tilde{C}_{\mathcal{D}}\big(X_{/\mathbb{R}},\mathbb{R}(d)\big).$$

Applying the functor $R\operatorname{Hom}(-,\mathbb{R}[-2d-1])$ and composing with the map (48), we obtain an isomorphism

$$\begin{aligned} R\Gamma_c(\mathcal{X}_{\text{ét}},\mathbb{R})[-1] &\xrightarrow{\sim} R\operatorname{Hom}\big(\tilde{C}_{\mathcal{D}}\big(X_{/\mathbb{R}},\mathbb{R}(d)\big),\mathbb{R}[-2d-1]\big)\\ &\xrightarrow{\sim} R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{R}[-2d-1]\big).\end{aligned}$$

The inverse isomorphism $R\operatorname{Hom}(R\Gamma(\mathcal{X},\mathbb{Q}(d))_{\geq 0},\mathbb{R}[-2d-1]) \xrightarrow{\sim} R\Gamma_c(\mathcal{X}_{\text{ét}},\mathbb{R})[-1]$ and the direct sum decomposition from (46) provide us with the desired map:

$$\begin{aligned} R\Gamma_{W,c}(\mathcal{X},\mathbb{Z})_{\mathbb{R}} &\xrightarrow{\sim} R\Gamma_c(\mathcal{X}_{\text{ét}},\mathbb{Z})_{\mathbb{R}} \oplus R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{R}[-2d-1]\big)\\ &\xrightarrow{\sim} R\Gamma_c(\mathcal{X}_{\text{ét}},\mathbb{R}) \oplus R\Gamma_c(\mathcal{X}_{\text{ét}},\mathbb{R})[-1] \xrightarrow{\sim} R\Gamma_{W,c}(\mathcal{X},\tilde{\mathbb{R}}). \qquad \square\end{aligned}$$

COROLLARY 3.7

Let $\mathcal{X}$ be a proper regular connected scheme of dimension d satisfying $\mathbf{L}(\mathcal{X}_{\text{ét}},d)_{\geq 0}$ and $\mathbf{B}(\mathcal{X},d)$. Then there is a canonical *isomorphism*

$$\det\nolimits_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X},\mathbb{Z})_{\mathbb{R}} \xrightarrow{\sim} \det\nolimits_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X},\tilde{\mathbb{R}}).$$

Proof

If $\mathcal{X}$ is flat over $\mathbb{Z}$, the result follows from Proposition 3.4 and Theorem 3.6. So we may assume that $\mathcal{X}$ is smooth and proper over a finite field. Then we have $R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}) = R\Gamma_W(\mathcal{X},\mathbb{Z})$ and a canonical isomorphism

$$R\Gamma_W(\mathcal{X},\mathbb{Z})_{\mathbb{R}} \simeq R\Gamma(\mathcal{X}_{\text{ét}},\mathbb{R}) \oplus R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{R}[-2d-1]\big).$$

It follows from $\mathbf{L}(\mathcal{X}_{\text{ét}},d)_{\geq 0}$ that the natural map

$$R\Gamma(\mathcal{X}_{\text{ét}},\mathbb{R})[-1] \to R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{R}[-2d-1]\big)$$

is a quasi-isomorphism (see the proof of Theorem 2.13). This yields a canonical map

$$R\Gamma_W(\mathcal{X},\mathbb{Z})_{\mathbb{R}} \xrightarrow{\sim} R\Gamma(\mathcal{X}_{\text{ét}},\mathbb{R}) \oplus R\operatorname{Hom}\big(R\Gamma\big(\mathcal{X},\mathbb{Q}(d)\big)_{\geq 0},\mathbb{R}[-2d-1]\big) \tag{49}$$

$$\xrightarrow{\sim} R\Gamma(\mathcal{X}_{\text{ét}},\mathbb{R}) \oplus R\Gamma(\mathcal{X}_{\text{ét}},\mathbb{R})[-1] \xrightarrow{\sim} R\Gamma_W(\mathcal{X},\tilde{\mathbb{R}}). \tag{50}$$

$\square$

4. Zeta functions at $s = 0$

Recall that the zeta function of a scheme $\mathcal{X}$ of finite type over $\mathrm{Spec}(\mathbb{Z})$ is defined as an infinite product

$$\zeta(\mathcal{X}, s) := \prod_{x \in \mathcal{X}_0} \frac{1}{1 - N(x)^{-s}}, \tag{51}$$

where $\mathcal{X}_0$ denotes the set of closed points of $\mathcal{X}$ and $N(x)$ is the cardinality of the residue field of $x \in \mathcal{X}_0$. The product (51) converges for $\Re(s) > \dim(\mathcal{X})$. It is expected to have a meromorphic continuation to the whole complex plane. We refer to [28] for generalities on the determinant functor.

4.1. The main conjecture

The following theorem summarizes some results obtained previously (see [13] for Theorems 4.1(a) and 4.1(b), Proposition 2.10 and Definition 3.2 for Theorem 4.1(c), and Theorem 3.6 and (50) for Theorem 4.1(d)).

THEOREM 4.1

Let $\mathcal{X}$ be a proper regular arithmetic scheme of pure dimension d.

(a) *The compact support cohomology groups $H^i_{W,c}(\mathcal{X}, \tilde{\mathbb{R}})$ are finite-dimensional vector spaces over $\mathbb{R}$, vanish for almost all i, and satisfy*

$$\sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{R}} H^i_{W,c}(\mathcal{X}, \tilde{\mathbb{R}}) = 0.$$

(b) *Cup product with the fundamental class $\theta \in H^1_W(\mathcal{X}, \tilde{\mathbb{R}})$ yields an acyclic complex*

$$\cdots \xrightarrow{\cup\theta} H^i_{W,c}(\mathcal{X}, \tilde{\mathbb{R}}) \xrightarrow{\cup\theta} H^{i+1}_{W,c}(\mathcal{X}, \tilde{\mathbb{R}}) \xrightarrow{\cup\theta} \cdots .$$

(c) *If $\mathbf{L}(\mathcal{X}_{\text{ét}}, d)_{\geq 0}$ holds, then the compact support cohomology groups $H^i_{W,c}(\mathcal{X}, \mathbb{Z})$ are finitely generated over $\mathbb{Z}$ and they vanish for almost all i.*

(d) *If $\mathbf{B}(\mathcal{X}, d)$ also holds, then there are isomorphisms*

$$H^i_{W,c}(\mathcal{X}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} H^i_{W,c}(\mathcal{X}, \tilde{\mathbb{R}}).$$

Consider now a proper regular connected arithmetic scheme $\mathcal{X}$ of dimension d satisfying $\mathbf{L}(\mathcal{X}_{\text{ét}}, d)_{\geq 0}$ and $\mathbf{B}(\mathcal{X}, d)$. By Corollary 3.7 we have canonical isomorphisms

$$\begin{aligned}\big(\det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z})\big)_{\mathbb{R}} &\simeq \det_{\mathbb{R}}\big(R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z})_{\mathbb{R}}\big) \\ &\simeq \det_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X}, \tilde{\mathbb{R}}).\end{aligned}$$

Moreover, the exact sequence in Theorem 4.1(b) above provides us with

$$\lambda : \mathbb{R} \xrightarrow{\sim} \bigotimes_{i\in\mathbb{Z}} \det_{\mathbb{R}} H^i_{W,c}(\mathcal{X}, \tilde{\mathbb{R}})^{(-1)^i} \xrightarrow{\sim} \det_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X}, \tilde{\mathbb{R}}) \xrightarrow{\sim} \big(\det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z})\big)_{\mathbb{R}}.$$

The following conjecture presupposes that $\zeta(\mathcal{X}, s)$ has a meromorphic continuation to a neighborhood of $s = 0$.

CONJECTURE 4.2

(a) *The vanishing order of $\zeta(\mathcal{X}, s)$ at $s = 0$ is given by*

$$\operatorname{ord}_{s=0} \zeta(\mathcal{X}, s) = \sum_{i\in\mathbb{Z}} (-1)^i \cdot i \cdot \operatorname{rank}_{\mathbb{Z}} H^i_{W,c}(\mathcal{X}, \mathbb{Z}).$$

(b) *The leading coefficient $\zeta^*(\mathcal{X}, 0)$ in the Taylor expansion of $\zeta(\mathcal{X}, s)$ at $s = 0$ is given up to sign by*

$$\mathbb{Z} \cdot \lambda\big(\zeta^*(\mathcal{X}, 0)^{-1}\big) = \det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}).$$

4.2. Relation to Soulé's conjecture

Let $\mathcal{X}$ be a regular connected arithmetic scheme of dimension d. We assume in this section that $\mathcal{X}$ is moreover flat and projective over $\operatorname{Spec}(\mathbb{Z})$. The following conjecture is Bloch's reformulation (see [2, Section 7]) of Soulé's conjecture [44, Conjecture 2.2] in terms of motivic cohomology (thanks to [31, Theorem 14.7(5)]). It presupposes that $\zeta(\mathcal{X}, s)$ has a meromorphic continuation near $s = 0$ and that the $\mathbb{Q}$-vector space $H^i(\mathcal{X}, \mathbb{Q}(d))$ is finite-dimensional for all i and zero for almost all i.

CONJECTURE 4.3 (Soulé)

One has

$$\operatorname{ord}_{s=0} \zeta(\mathcal{X}, s) = \sum_i (-1)^{i+1} \dim_{\mathbb{Q}} H^{2d-i}\big(\mathcal{X}, \mathbb{Q}(d)\big).$$

If $\mathcal{X}$ satisfies $\mathbf{L}(\mathcal{X}_{\text{ét}}, d)$, then $H^i(\mathcal{X}, \mathbb{Q}(d)) = 0$ for $i < 0$ (see the proof of Theorem 2.12). Hence the conjunction of $\mathbf{L}(\mathcal{X}_{\text{ét}}, d)$ and $\mathbf{B}(\mathcal{X}, d)$ is equivalent to the conjunction of Conjectures 1.2 and 1.4 for $\mathcal{X}$.

PROPOSITION 4.4

Assume that $\mathcal{X}$ satisfies $\mathbf{L}(\mathcal{X}_{\text{ét}}, d)$ and $\mathbf{B}(\mathcal{X}, d)$. Then Conjecture 4.2(a) is equivalent to Conjecture 4.3.

Proof
Assuming $\mathbf{L}(\mathcal{X}_{\text{ét}}, d)$ and $\mathbf{B}(\mathcal{X}, d)$ one has

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}}(-1)^i \cdot i \cdot \mathrm{rank}_{\mathbb{Z}} H^i_{W,c}(\mathcal{X},\mathbb{Z}) \\
&\quad = \sum_{i\in\mathbb{Z}}(-1)^i \cdot i \cdot \dim_{\mathbb{Q}} H^i_{W,c}(\mathcal{X},\mathbb{Z})_{\mathbb{Q}} \\
&\quad = \sum_{i\in\mathbb{Z}}(-1)^i \cdot i \cdot \big(\dim_{\mathbb{Q}} H^i_c(\mathcal{X}_{\text{ét}},\mathbb{Q}) + \dim_{\mathbb{Q}} H^{2d+2-i-1}\big(\mathcal{X},\mathbb{Q}(d)\big)^*\big) \\
&\quad = \sum_{i\in\mathbb{Z}}(-1)^i \cdot i \cdot \big(\dim_{\mathbb{Q}} H^{2d-i}\big(\mathcal{X},\mathbb{Q}(d)\big)^* + \dim_{\mathbb{Q}} H^{2d+1-i}\big(\mathcal{X},\mathbb{Q}(d)\big)^*\big) \\
&\quad = \sum_{i\in\mathbb{Z}}(-1)^i \dim_{\mathbb{Q}} H^{2d+1-i}\big(\mathcal{X},\mathbb{Q}(d)\big).
\end{aligned}$$

The second equality follows from Proposition 3.4. Since $H^i(\mathcal{X},\mathbb{Q}(d)) = 0$ for $i < 0$ by $\mathbf{L}(\mathcal{X}_{\text{ét}}, d)$, the third equality follows from Conjecture $\mathbf{B}(\mathcal{X}, d)$ and from duality for Deligne cohomology (see the proof of Theorem 3.6). The fourth equality follows from $\mathbf{L}(\mathcal{X}_{\text{ét}}, d)$ since it implies that $H^{2d-i}(\mathcal{X},\mathbb{Q}(d))$ is finite-dimensional and zero for almost all i. The result follows. ☐

4.3. Relation to the Tamagawa number conjecture

In this section we consider the Tamagawa number conjecture of Bloch and Kato in the formulation of Fontaine and Perrin-Riou (see [14] and [15]).

Throughout this section we let $\mathcal{X}$ be a smooth projective scheme over a number ring $\mathcal{O}_F$, and we assume that $\mathcal{X}$ is connected of dimension d. We set $\mathcal{X}_F := \mathcal{X} \otimes_{\mathcal{O}_F} F$ and $\mathcal{X}_{\mathfrak{p}} := \mathcal{X} \otimes_{\mathcal{O}_F} \mathbb{F}_{\mathfrak{p}}$ where $\mathbb{F}_{\mathfrak{p}} := \mathcal{O}_F/\mathfrak{p}$ for any maximal ideal $\mathfrak{p} \subset \mathcal{O}_F$. We assume further that $\mathcal{X}$ satisfies $\mathbf{L}(\mathcal{X}_{\text{ét}}, d)_{\geq 0}$ and $\mathbf{B}(\mathcal{X}, d)$. In order to ease the notation, we also assume $\mathcal{O}_F = \mathcal{O}_{\mathcal{X}}(\mathcal{X})$. In particular, $\mathcal{X}_F$ and $\mathcal{X}_{\mathfrak{p}}$ (for any finite prime $\mathfrak{p}$) are geometrically irreducible (see [36, Corollary 5.3.17]).

Note that there is no loss of generality caused by the assumption $\mathcal{O}_F = \mathcal{O}_{\mathcal{X}}(\mathcal{X})$. Indeed, if $\mathcal{X}$ is connected, smooth, and projective over $\mathcal{O}_F$, then $\mathcal{O}_{\mathcal{X}}(\mathcal{X})$ is a number ring (finite over $\mathcal{O}_F$) and $\mathcal{X}$ is also smooth and projective over $\mathcal{O}_{\mathcal{X}}(\mathcal{X})$.

4.3.1. Motivic L-functions

Recall that for any connected, smooth, and projective scheme X/F and $0 \leq i \leq 2 \cdot \dim(X)$ one defines the L-function

$$L\big(h^i(X), s\big) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}\big(h^i(X), s\big) = \prod_{\mathfrak{p}} P_{\mathfrak{p}}\big(h^i(X), N(\mathfrak{p})^{-s}\big)^{-1} \tag{52}$$

as an Euler product over all finite primes $\mathfrak{p}$ of F where $N(\mathfrak{p})$ is the cardinal of $\mathbb{F}_{\mathfrak{p}}$ and

$$P_{\mathfrak{p}}\big(h^i(X), T\big) = \det_{\mathbb{Q}_l}\big(1 - \mathrm{Fr}_{\mathfrak{p}}^{-1} \cdot T \mid H^i(X_{\overline{F},\text{ét}}, \mathbb{Q}_l)^{I_{\mathfrak{p}}}\big)$$

is a polynomial (conjecturally) with rational coefficients and independent of the chosen prime l as long as $\mathfrak{p}$ does not divide l. Here $\mathrm{Fr}_{\mathfrak{p}}$ denotes a Frobenius element. The product (52) is known to converge for $\Re(s) > i/2 + 1$ by [9].

Assume now that X/F is the generic fiber $\mathcal{X} \otimes_{\mathcal{O}_F} F$ of the smooth proper scheme $\mathcal{X}$ over $\mathcal{O}_F$. By smooth and proper base change, one has

$$H^i(\mathcal{X}_{\overline{\mathbb{F}}_{\mathfrak{p}},\text{ét}}, \mathbb{Q}_l) \simeq H^i(\mathcal{X}_{\overline{F},\text{ét}}, \mathbb{Q}_l) \simeq H^i(\mathcal{X}_{\overline{F},\text{ét}}, \mathbb{Q}_l)^{I_p},$$

and by Grothendieck's formula, one has

$$\zeta(\mathcal{X}, s) := \prod_{x \in \mathcal{X}_0} \frac{1}{1 - N(x)^{-s}} = \prod_{\mathfrak{p}} \zeta(\mathcal{X}_{\mathfrak{p}}, s) = \prod_{i=0}^{2\dim(\mathcal{X}_F)} L\big(h^i(\mathcal{X}_F), s\big)^{(-1)^i},$$

where $\mathcal{X}_0$ is the set of closed points in $\mathcal{X}$.

4.3.2. Statement of the Tamagawa number conjecture

Let $\mathcal{X}/\mathcal{O}_F$ be a scheme satisfying the assumptions of the introduction of Section 4.3. We set $X := \mathcal{X}_F$. Recall that the "integral part in the motivic cohomology" $H^j_{\mathcal{M}}(X_{/\mathbb{Z}}, \mathbb{Q}(d))$ is defined as the image of the map $H^j(\mathcal{X}, \mathbb{Q}(d)) \to H^j(X, \mathbb{Q}(d))$. But this map is injective for $j \geq 0$ by $\mathbf{B}(\mathcal{X}, d)$; hence we may identify $H^j_{\mathcal{M}}(X_{/\mathbb{Z}}, \mathbb{Q}(d)) = H^j(\mathcal{X}, \mathbb{Q}(d))$ for $j \geq 0$. Define the fundamental line

$$\Delta_f\big(h^i(X)\big) = \det_{\mathbb{Q}}^{-1}\big(H^i\big(X(\mathbb{C}), \mathbb{Q}\big)^+\big) \otimes_{\mathbb{Q}} \det_{\mathbb{Q}} H^{2d-1-i}_{\mathcal{M}}\big(X_{/\mathbb{Z}}, \mathbb{Q}(d)\big)^*$$

for $0 < i \leq 2d - 2$ and

$$\begin{aligned}\Delta_f\big(h^0(X)\big) = \det_{\mathbb{Q}} CH^0(X)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \det_{\mathbb{Q}}^{-1}\big(H^0\big(X(\mathbb{C}), \mathbb{Q}\big)^+\big) \\ \otimes_{\mathbb{Q}} \det_{\mathbb{Q}} H^{2d-1}_{\mathcal{M}}\big(X_{/\mathbb{Z}}, \mathbb{Q}(d)\big)^*\end{aligned}$$

for $i = 0$. There is an isomorphism

$$\vartheta^i_\infty : \mathbb{R} \simeq \Delta_f\big(h^i(X)\big)_{\mathbb{R}}$$

induced by $\mathbf{B}(\mathcal{X}, d)$ and duality for Deligne cohomology (47). Assuming that $L(h^i(X), 0)$ has a meromorphic continuation near $s = 0$, we denote by $L^*(h^i(X), 0)$ the leading coefficient in the Taylor expansion of $L(h^i(X), 0)$ at $s = 0$.

CONJECTURE 4.5 (Beilinson–Deligne)

There is an identity of $\mathbb{Q}$-subspaces of $\Delta_f(h^i(X))_{\mathbb{R}}$:

$$\mathbb{Q} \cdot \vartheta^i_\infty\big(L^*\big(h^i(X), 0\big)^{-1}\big) = \Delta_f\big(h^i(X)\big).$$

Now fix a prime number l, and let $U \subseteq \mathrm{Spec}(\mathcal{O}_F)$ be an open subscheme in which l is invertible. For any locally constant l-adic sheaf V on U (i.e., any $\mathbb{Q}_l$-representation of the fundamental group $\pi_1(U_{\text{ét}}, \bar{u})$ with base point $\bar{u} : \mathrm{Spec}(\overline{F}) \to U$) and any finite prime $\mathfrak{p}$ of F not dividing l, one defines a complex concentrated in degrees 0 and 1

$$R\Gamma_f(F_{\mathfrak{p}}, V) = R\Gamma(\mathbb{F}_{\mathfrak{p}}, V^{I_{\mathfrak{p}}}) = V^{I_{\mathfrak{p}}} \stackrel{1-\mathrm{Fr}_{\mathfrak{p}}^{-1}}{\longrightarrow} V^{I_{\mathfrak{p}}},$$

where $I_{\mathfrak{p}}$ is the inertia subgroup at $\mathfrak{p}$. For $\mathfrak{p}$ dividing l define

$$R\Gamma_f(F_{\mathfrak{p}}, V) = D_{\mathrm{cris},\mathfrak{p}}(V) \stackrel{(1-\phi,\iota)}{\longrightarrow} D_{\mathrm{cris},\mathfrak{p}}(V) \oplus D_{dR,\mathfrak{p}}(V)/F^0 D_{dR,\mathfrak{p}}(V),$$

where $D_{\mathrm{cris},\mathfrak{p}}(V) = (B_{\mathrm{cris},\mathfrak{p}} \otimes_{\mathbb{Q}_p} V)^{G_{F_{\mathfrak{p}}}}$ and $D_{dR,\mathfrak{p}}(V) = (B_{dR,\mathfrak{p}} \otimes_{\mathbb{Q}_p} V)^{G_{F_{\mathfrak{p}}}}$ (see [14] and [15]). In both cases there is a map of complexes $R\Gamma_f(F_{\mathfrak{p}}, V) \to R\Gamma(F_{\mathfrak{p}}, V)$ and one defines $R\Gamma_{/f}(F_{\mathfrak{p}}, V)$ as the mapping cone. Then one defines a global complex $R\Gamma_f(F, V)$ as the mapping fiber of the composite map

$$R\Gamma(U_{\text{ét}}, V) \to \bigoplus_{\mathfrak{p}\notin U} R\Gamma(F_{\mathfrak{p}}, V) \to \bigoplus_{\mathfrak{p}\notin U} R\Gamma_{/f}(F_{\mathfrak{p}}, V).$$

There is an exact triangle in the derived category of $\mathbb{Q}_l$-vector spaces

$$R\Gamma_c(U_{\text{ét}}, V) \to R\Gamma_f(F, V) \to \bigoplus_{\mathfrak{p}\notin U} R\Gamma_f(F_{\mathfrak{p}}, V), \tag{53}$$

where the primes $\mathfrak{p} \notin U$ include archimedean $\mathfrak{p}$ with the convention $R\Gamma_f(\mathbb{R}, V) = R\Gamma(\mathbb{R}, V)$ and $R\Gamma_f(\mathbb{C}, V) = R\Gamma(\mathbb{C}, V)$.

Notation 4.6
For an archimedean prime $\mathfrak{p}$ of F, we choose a complex embedding $\sigma_{\mathfrak{p}} : F \to \mathbb{C}$ representing $\mathfrak{p}$, and we set $\mathcal{X}_{\mathfrak{p}} := \mathcal{X}_{F,\sigma_{\mathfrak{p}}}(\mathbb{C})/G_{F_{\mathfrak{p}}}$, where $\mathcal{X}_{F,\sigma_{\mathfrak{p}}}(\mathbb{C}) := \mathrm{Hom}_{\mathrm{Spec}(F)}(\mathrm{Spec}(\mathbb{C}), \mathcal{X}_F)$ is the space of complex points of $\mathcal{X}_F$ lying over $\sigma_{\mathfrak{p}}$. One has $\mathcal{X}_\infty = \coprod_{\mathfrak{p}|\infty} \mathcal{X}_{\mathfrak{p}}$. Finally, we set $\mathcal{X}_{\mathfrak{p},\text{ét}} := \mathrm{Sh}(\mathcal{X}_{\mathfrak{p}})$ and $\mathcal{X}_{\mathfrak{p},W} := \mathrm{Sh}(\mathcal{X}_{\mathfrak{p}}) \times B_{\mathbb{R}}$.

The following result is proven in [13, Proposition 9.1]. Note that [13, Conjecture 7] is known in the smooth case.

PROPOSITION 4.7
Let $\pi : \mathcal{X} \to \mathrm{Spec}(\mathcal{O}_F)$ be as above, and let $\overline{\mathcal{X}}_{\text{ét}}$ be its Artin–Verdier étale topos. Let $U \subseteq \mathrm{Spec}(\mathcal{O}_F)$ be an open subscheme in which the prime number l is invertible. Then there is an isomorphism of exact triangles in the derived category of $\mathbb{Q}_l$-vector spaces:

$$\begin{array}{ccccc}
R\Gamma_c(\mathcal{X}_{U,\text{ét}}, \mathbb{Q}_l) & \longrightarrow & R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Q}_l) & \longrightarrow & \bigoplus_{\mathfrak{p} \notin U} R\Gamma(\mathcal{X}_{\mathfrak{p},\text{ét}}, \mathbb{Q}_l) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{i=0}^{2d-2} R\Gamma_c(U_{\text{ét}}, V_l^i)[-i] & \rightarrow & \bigoplus_{i=0}^{2d-2} R\Gamma_f(F, V_l^i)[-i] & \rightarrow & \bigoplus_{\mathfrak{p} \notin U} \bigoplus_{i=0}^{2d-2} R\Gamma_f(F_{\mathfrak{p}}, V_l^i)[-i]
\end{array}$$

Here $V_l^i := H^i(X_{\overline{F},\text{ét}}, \mathbb{Q}_l)$ is viewed as a $\mathbb{Q}_l$-representation of G_F, the bottom exact triangle is a sum over triangles (53), and $R\Gamma_c(\mathcal{X}_{U,\text{ét}}, \mathbb{Q}_l) := R\Gamma(\overline{\mathcal{X}}_{\text{ét}}, j_!\mathbb{Q}_l)$ where $j : \mathcal{X}_{U,\text{ét}} \to \overline{\mathcal{X}}_{\text{ét}}$ is the canonical open embedding.

The statement of the Tamagawa number conjecture requires the following.

CONJECTURE 4.8 (Bloch–Kato)
We have that $H^1_f(F, V_l^i) = 0$ for any i.

One can show that $H^0_f(F, V_l^0) \cong CH^0(\mathcal{X}_F)_{\mathbb{Q}_l}$, $H^0_f(F, V_l^i) = 0$ for $i > 0$, and $H^3_f(F, V_l^i) = 0$. Therefore, by Corollary 2.11 and Proposition 4.7, Conjecture 4.8 induces an isomorphism

$$\begin{aligned}
\rho_l^i : H^2_f(F, V_l^i) &\xrightarrow{\sim} H^{i+2}(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Q}_l) \\
&\xrightarrow{\sim} H^{i+2}_W(\overline{\mathcal{X}}, \mathbb{Z})_{\mathbb{Q}_l} \xrightarrow{\sim} H^{2d-1-i}\big(\mathcal{X}, \mathbb{Q}(d)\big)^*_{\mathbb{Q}_l}. \qquad (54)
\end{aligned}$$

Moreover, Artin's comparison isomorphism yields

$$H^i\big(\mathcal{X}(\mathbb{C}), \mathbb{Q}\big)^+_{\mathbb{Q}_l} \simeq \Big(\bigoplus_{\sigma : F \hookrightarrow \mathbb{C}} H^i\big(\mathcal{X}_{F,\sigma}(\mathbb{C}), \mathbb{Q}\big)\Big)^+_{\mathbb{Q}_l} \qquad (55)$$

$$\simeq \bigoplus_{\mathfrak{p}|\infty} H^i(\mathcal{X}_{\overline{F},\text{ét}}, \mathbb{Q}_l)^{G_{\mathfrak{p}}} = \bigoplus_{\mathfrak{p}|\infty} (V_l^i)^{G_{\mathfrak{p}}}, \qquad (56)$$

where σ runs over the complex embeddings of F and $\mathcal{X}_{F,\sigma}(\mathbb{C}) = \mathrm{Hom}_{\mathrm{Spec}(F)}(\mathrm{Spec}(\mathbb{C}), \mathcal{X}_F)$ with respect to the map $\sigma : \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(F)$. We obtain an isomorphism (for $0 \le i \le 2d - 2$)

$$\vartheta_l^i : \Delta_f\big(h^i(X)\big)_{\mathbb{Q}_l} \xrightarrow{\sim} \det_{\mathbb{Q}_l} R\Gamma_f(F, V_l^i) \otimes \bigotimes_{\mathfrak{p}|\infty} \det^{-1}_{\mathbb{Q}_l} R\Gamma(F_{\mathfrak{p}}, V_l^i) \qquad (57)$$

$$\xrightarrow{\sim} \det_{\mathbb{Q}_l} R\Gamma_c(U_{\text{ét}}, V_l^i) \otimes \bigotimes_{\mathfrak{p} \in Z} \det_{\mathbb{Q}_l} R\Gamma_f(F_{\mathfrak{p}}, V_l^i) \qquad (58)$$

$$\xrightarrow{\sim} \det_{\mathbb{Q}_l} R\Gamma_c(U_{\text{ét}}, V_l^i), \qquad (59)$$

where $Z = \mathrm{Spec}(\mathcal{O}_F) - U$ is the closed complement of U. Indeed, (57) is induced by (54) and (56), (58) is induced by (53), and (59) is induced by the isomorphism (for $\mathfrak{p} \in Z$)

$$\iota_{\mathfrak{p}} : \det_{\mathbb{Q}_l} R\Gamma_f(F_{\mathfrak{p}}, V_l^i) \cong \mathbb{Q}_l,$$

which is in turn induced by the identity map on $(V_l^i)^{I_{\mathfrak{p}}}$ and $D_{\mathrm{cris},\mathfrak{p}}(V_l^i)$. Below is the Tamagawa number conjecture in the formulation of Fontaine and Perrin-Riou (see [14] and [15]). It presupposes Conjecture 4.5.

CONJECTURE 4.9 (l-part of the Tamagawa number conjecture)
There is an identity of free rank one $\mathbb{Z}_l$*-submodules of* $\det_{\mathbb{Q}_l} R\Gamma_c(U_{\text{ét}}, V_l^i)$

$$\mathbb{Z}_l \cdot \vartheta_l^i \circ \vartheta_\infty^i\big(L^*\big(h^i(X), 0\big)^{-1}\big) = \det_{\mathbb{Z}_l} R\Gamma_c(U_{\text{ét}}, T_l^i)$$

for any constructible $\mathbb{Z}_l$*-sheaf* T_l^i *such that* $T_l^i \otimes_{\mathbb{Z}_l} \mathbb{Q}_l = V_l^i$.

This conjecture is independent of the choice of T_l^i, as well as of the choice of U in which l is invertible (see [14]).

4.3.3
One can reformulate the Tamagawa number conjecture in terms of the L-function

$$L_U\big(h^i(X), s\big) = \prod_{\mathfrak{p} \in U} L_{\mathfrak{p}}\big(h^i(X), s\big)$$

associated to the smooth l-adic sheaf V_l^i over U, using a second isomorphism

$$\tilde{\iota}_{\mathfrak{p}} : \det_{\mathbb{Q}_l} R\Gamma_f(F_{\mathfrak{p}}, V_l^i) \simeq \mathbb{Q}_l,$$

which satisfies

$$\iota_{\mathfrak{p}} = P_{\mathfrak{p}}^*\big(h^i(X), 1\big)^{-1} \tilde{\iota}_{\mathfrak{p}} = L_{\mathfrak{p}}^*\big(h^i(X), 0\big) \log\big(N(\mathfrak{p})^{r_{i,\mathfrak{p}}}\big) \tilde{\iota}_{\mathfrak{p}}, \tag{60}$$

where $r_{i,\mathfrak{p}} = \mathrm{ord}_{T=1} P_{\mathfrak{p}}(h^i(X), T) = -\mathrm{ord}_{s=0} L_{\mathfrak{p}}(h^i(X), s)$. The isomorphism $\tilde{\iota}_{\mathfrak{p}}$ is defined as follows. We shall see below that the complex $R\Gamma_f(F_{\mathfrak{p}}, V_l^i)$ is semisimple at 0; in other words, the identity map (on $(V_l^i)^{I_{\mathfrak{p}}}$ for $\mathfrak{p} \nmid l$ and $D_{\mathrm{cris},\mathfrak{p}}(V_l^i)$ for $\mathfrak{p} \mid l$) induces an isomorphism $H_f^0(F_{\mathfrak{p}}, V_l^i) \xrightarrow{\sim} H_f^1(F_{\mathfrak{p}}, V_l^i)$ and hence a trivialization

$$\tilde{\iota}_{\mathfrak{p}} : \det_{\mathbb{Q}_l} R\Gamma_f(F_{\mathfrak{p}}, V_l^i) \simeq \det_{\mathbb{Q}_l} H_f^0(F_{\mathfrak{p}}, V_l^i) \otimes_{\mathbb{Q}_l} \det_{\mathbb{Q}_l}^{-1} H_f^1(F_{\mathfrak{p}}, V_l^i) \simeq \mathbb{Q}_l.$$

Then we define

$$\tilde{\vartheta}_l^i : \Delta_f\big(h^i(X)\big)_{\mathbb{Q}_l} \xrightarrow{\sim} \det_{\mathbb{Q}_l} R\Gamma_c(U_{\text{ét}}, V_l^i) \otimes \bigotimes_{\mathfrak{p}\in Z} \det_{\mathbb{Q}_l} R\Gamma_f(F_{\mathfrak{p}}, V_l^i)$$
$$\xrightarrow{\sim} \det_{\mathbb{Q}_l} R\Gamma_c(U_{\text{ét}}, V_l^i) \tag{61}$$

for $0 \le i \le 2d-2$, where the first isomorphism is (58) and the second is induced by the $\tilde{\iota}_{\mathfrak{p}}$'s. By (60) we have $\tilde{\vartheta}_l^i = \prod_{\mathfrak{p}\in Z} P_{\mathfrak{p}}^*(h^i(X),1)\cdot \vartheta_l^i$ and we need to define

$$\tilde{\vartheta}_\infty^i := \prod_{\mathfrak{p}\in Z} \log\big(N(\mathfrak{p})\big)^{r_{i,\mathfrak{p}}} \vartheta_\infty^i. \tag{62}$$

In our situation, we have $r_{i,\mathfrak{p}} = 0$ for $i>0$ (for weight reasons) and $r_{0,\mathfrak{p}} = 1$ (because $V_l^0 = \mathbb{Q}_l$ with trivial G_F-action; see below) for any $\mathfrak{p}\in Z$. The Tamagawa number conjecture becomes the following.

CONJECTURE 4.10
There is an identity of free rank one $\mathbb{Z}_l$-submodules of $\det_{\mathbb{Q}_l} R\Gamma_c(U_{\text{ét}}, V_l^i)$

$$\mathbb{Z}_l \cdot \tilde{\vartheta}_l^i \circ \tilde{\vartheta}_\infty^i\big(L_U^*\big(h^i(X),0\big)^{-1}\big) = \det_{\mathbb{Z}_l} R\Gamma_c(U_{\text{ét}}, T_l^i). \tag{63}$$

Now we observe that $R\Gamma_f(F_{\mathfrak{p}}, V_l^i)$ is indeed semisimple at 0, and explain the compatibility between $\tilde{\iota}_{\mathfrak{p}}$ and the canonical trivialization of $(\det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{\mathfrak{p},W},\mathbb{Z}))_{\mathbb{Q}_l}$. Let $\mathfrak{p}\in Z$. By Proposition 4.7 we have

$$\bigoplus_{i\ge 0} R\Gamma_f(F_{\mathfrak{p}}, V_l^i)[-i] \simeq R\Gamma(\mathcal{X}_{\mathfrak{p},\text{ét}},\mathbb{Q}_l) \simeq R\Gamma(\mathcal{X}_{\mathfrak{p},W},\mathbb{Z})_{\mathbb{Q}_l}. \tag{64}$$

But $H^i(\mathcal{X}_{\mathfrak{p},W},\mathbb{Z})_{\mathbb{Q}_l} = \mathbb{Q}_l$ for $i=0,1$ (since $\mathcal{X}_{\mathfrak{p}}$ is connected) and $H^i(\mathcal{X}_{\mathfrak{p},W},\mathbb{Z})_{\mathbb{Q}_l} = 0$ otherwise (by [34, Theorem 7.4]). It follows that $R\Gamma_f(F_{\mathfrak{p}}, V_l^i)$ is acyclic for $i>0$ and hence semisimple at 0. For $i=0$, we have $V_l^0 = \mathbb{Q}_l$ with trivial G_F-action (since $\mathcal{X}_F$ is geometrically connected); hence $(V_l^0)^{I_{\mathfrak{p}}} = \mathbb{Q}_l$ for $\mathfrak{p}\nmid l$ and $D_{\text{cris},\mathfrak{p}}(V_l^0) = (F_{\mathfrak{p}})_0$ for $\mathfrak{p}\mid l$. In both cases, $R\Gamma_f(F_{\mathfrak{p}}, V_l^0)$ is semisimple at 0. Moreover, (60) is given by [8, Lemma 1]. Indeed, for $i>0$ and $\mathfrak{p}\mid l$, one has

$$P_{\mathfrak{p}}(V_l^i,1) = P_l\big((V_l^i)',1\big) = \det_{\mathbb{Q}_l}\big(1-\phi \mid D_{\text{cris},\mathfrak{p}}(V_l^i)\big),$$

where $(V_l^i)' := \operatorname{Ind}_{F_{\mathfrak{p}}/\mathbb{Q}_l}(V_l^i)$ is the l-adic representation of $G_{\mathbb{Q}_l}$ induced by V_l^i (seen as a representation of $G_{F_{\mathfrak{p}}}$). The case $\mathfrak{p}\nmid l$ is similar and the case $i=0$ is obvious.

Note also that $H_f^i(F_{\mathfrak{p}}, V_l^0) \simeq H^i(\mathcal{X}_{\mathfrak{p},W},\mathbb{Z})_{\mathbb{Q}_l}$ for $i=0,1$. Under this identification, the isomorphism $H_f^0(F_{\mathfrak{p}}, V_l^0) \xrightarrow{\sim} H_f^1(F_{\mathfrak{p}}, V_l^0)$ given by the semisimplicity of $R\Gamma_f(F_{\mathfrak{p}}, V_l^0)$ corresponds to the map $H^0(\mathcal{X}_{\mathfrak{p},W},\mathbb{Z})_{\mathbb{Q}_l} \xrightarrow{\cup e} H^1(\mathcal{X}_{\mathfrak{p},W},\mathbb{Z})_{\mathbb{Q}_l}$ given by cup product with the fundamental class $e\in H^1(\mathcal{X}_{\mathfrak{p},W},\mathbb{Z})$. Recall that e is the pullback of the homomorphism in $H^1(\operatorname{Spec}(\mathbb{F}_{\mathfrak{p}})_W,\mathbb{Z}) = \operatorname{Hom}(W_{\mathbb{F}_{\mathfrak{p}}},\mathbb{Z})$ which sends $\operatorname{Fr}_{\mathfrak{p}}$ to 1. It follows that we have a commutative square of isomorphisms

$$\begin{array}{ccc} \bigotimes_{i\geq 0} \det_{\mathbb{Q}_l}^{(-1)^i} R\Gamma_f(F_{\mathfrak{p}}, V_l^i) & \xrightarrow{\otimes_i \tilde{\iota}_p^{(-1)^i}} & \mathbb{Q}_l \\ \downarrow & & \| \\ \det_{\mathbb{Q}_l} R\Gamma(\mathcal{X}_{\mathfrak{p},W}, \mathbb{Z})_{\mathbb{Q}_l} & \xrightarrow{\mu^{-1}_{\mathcal{X}_{\mathfrak{p}},\mathbb{Q}_l}} & \mathbb{Q}_l \end{array} \quad (65)$$

where the left-hand side vertical isomorphism is induced by (64), and the lower horizontal isomorphism is induced by $\mu_{\mathcal{X}_{\mathfrak{p}}} : \mathbb{Q} \xrightarrow{\sim} \det_{\mathbb{Q}} R\Gamma(\mathcal{X}_{\mathfrak{p},W}, \mathbb{Z})_{\mathbb{Q}}$, which is in turn induced by the exact sequence (see [34, Theorem 7.4])

$$\cdots \xrightarrow{\cup e} H^i(\mathcal{X}_{\mathfrak{p},W}, \mathbb{Z})_{\mathbb{Q}} \xrightarrow{\cup e} H^{i+1}(\mathcal{X}_{\mathfrak{p},W}, \mathbb{Z})_{\mathbb{Q}} \xrightarrow{\cup e} \cdots . \quad (66)$$

4.3.4

Let $\mathcal{X}/\mathcal{O}_F$ be a smooth projective scheme satisfying the assumptions of the introduction of Section 4.3. Let $U \subseteq \operatorname{Spec}(\mathcal{O}_F)$ be an open subscheme on which the prime number l is invertible. Consider the morphism

$$R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}) \longrightarrow \bigoplus_{\mathfrak{p}\notin U} R\Gamma(\mathcal{X}_{\mathfrak{p},W}, \mathbb{Z})$$

given by Propositions 2.14 and 3.1, where $\mathcal{X}_{\mathfrak{p},W}$ is the Weil-étale topos of $\mathcal{X}_{\mathfrak{p}}$. Here the primes $\mathfrak{p} \notin U$ include archimedean primes (see Notation 4.6). We define $R\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z})$ such that the triangle

$$R\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z}) \to R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}) \to \bigoplus_{\mathfrak{p}\notin U} R\Gamma(\mathcal{X}_{\mathfrak{p},W}, \mathbb{Z}) \quad (67)$$

is exact. We do not show nor use that $R\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z})$ only depends on $\mathcal{X}_U$. In fact $R\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z})$ is only defined up to a noncanonical isomorphism but $\det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z})$ is canonically defined and we have a canonical isomorphism

$$\det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z}) \simeq \det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}) \otimes \det_{\mathbb{Z}}^{-1} R\Gamma(\mathcal{X}_{Z,W}, \mathbb{Z}), \quad (68)$$

where $Z = \operatorname{Spec}(\mathcal{O}_F) - U$. We define

$$R\Gamma_{W,c}(\mathcal{X}_U, \tilde{\mathbb{R}}) := R\Gamma(\overline{\mathcal{X}}_W, j_!\tilde{\mathbb{R}}),$$

where $j : \mathcal{X}_{U,W} \to \overline{\mathcal{X}}_W$ is the obvious open embedding of topoi (i.e., induced by $\mathcal{X}_{U,\text{ét}} \to \overline{\mathcal{X}}_{\text{ét}}$). The triangle

$$R\Gamma_{W,c}(\mathcal{X}_U, \tilde{\mathbb{R}}) \to R\Gamma_W(\overline{\mathcal{X}}, \tilde{\mathbb{R}}) \to \bigoplus_{\mathfrak{p}\notin U} R\Gamma(\mathcal{X}_{\mathfrak{p},W}, \tilde{\mathbb{R}})$$

is exact, where the map on the right-hand side is induced by the closed embedding $\coprod_{\mathfrak{p}\notin U} \mathcal{X}_{\mathfrak{p},W} \to \overline{\mathcal{X}}_W$ (which is the closed complement of j). We obtain a canonical isomorphism

$$\det_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X}_U, \tilde{\mathbb{R}}) \simeq \det_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X}, \tilde{\mathbb{R}}) \otimes \det_{\mathbb{R}}^{-1} R\Gamma_W(\mathcal{X}_Z, \tilde{\mathbb{R}}). \qquad (69)$$

By Corollary 3.7, (68), and (69) we have a canonical isomorphism

$$\det_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z})_{\mathbb{R}} \xrightarrow{\sim} \det_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X}_U, \tilde{\mathbb{R}}) \qquad (70)$$

and we obtain

$$\lambda_{\mathcal{X}_U} : \mathbb{R} \xrightarrow{\sim} \det_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X}_U, \tilde{\mathbb{R}}) \xrightarrow{\sim} \det_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z})_{\mathbb{R}} \qquad (71)$$

such that the following square of isomorphisms commutes:

$$\begin{array}{ccc} \mathbb{R} \otimes \mathbb{R} & \xrightarrow{\lambda_{\mathcal{X}} \otimes \lambda_{\mathcal{X}_Z}^{-1}} & \det_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z})_{\mathbb{R}} \otimes \det_{\mathbb{R}}^{-1} R\Gamma(\mathcal{X}_{Z,W}, \mathbb{Z})_{\mathbb{R}} \\ \downarrow & & \downarrow \\ \mathbb{R} & \xrightarrow{\lambda_{\mathcal{X}_U}} & \det_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z})_{\mathbb{R}} \end{array} \qquad (72)$$

Here we identify $\lambda_{\mathcal{X}_Z}$ with its dual, the left-hand side vertical map is induced by the product map, and the right-hand side vertical map is induced by (68).

THEOREM 4.11
Let $\mathcal{X}/\mathcal{O}_F$ be a smooth projective scheme over $\mathcal{O}_F$. Assume that $\mathcal{X}$ is connected of dimension d, and assume that $\mathcal{X}$ satisfies $\mathbf{L}(\mathcal{X}_{\text{ét}}, d)_{\geq 0}$ and $\mathbf{B}(\mathcal{X}, d)$. Assume moreover that $H^1_f(F, H^i(\mathcal{X}_{\overline{F},\text{ét}}, \mathbb{Q}_l)) = 0$ for all i, and assume that $\zeta(\mathcal{X}, s)$ has a meromorphic continuation to $s = 0$. Then the Tamagawa number conjecture (Conjecture 4.9) for the motive $\bigoplus_{i=0}^{2d-2} h^i(X)[-i]$ and all l is equivalent to Conjecture 4.2(b) for $\mathcal{X}$.

Proof
We consider an open subscheme $U \subseteq \operatorname{Spec}(\mathcal{O}_F)$ on which l is invertible and we let Z be the closed complement of U. By [34] $\mathcal{X}_{\mathfrak{p}}$ satisfies Conjecture 1.1 for any $\mathfrak{p} \in Z$. Hence the factorization $\zeta(\mathcal{X}, s) = \zeta(\mathcal{X}_U, s) \cdot \zeta(\mathcal{X}_Z, s)$ together with (68), (69), and (72) shows that Conjecture 4.2(b) for $\mathcal{X}$ is equivalent to Conjecture 4.2(b) for $\mathcal{X}_U$.

By Proposition 3.4 and by the definition of $\lambda_{\mathcal{X}}$, we have a canonical isomorphism

$$\det_{\mathbb{Q}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z})_{\mathbb{Q}} \simeq \bigotimes_{i=0}^{2d-2} \Delta_f\big(h^i(X)\big)^{(-1)^i}, \qquad (73)$$

which is compatible (in the obvious sense) with $\lambda_{\mathcal{X}}$ and $\bigotimes_i(\vartheta^i_\infty)^{(-1)^i}$. In particular, Conjecture 4.2(b) implies Conjecture 4.5 for $\bigoplus_{i=0}^{2d-2} h^i(X)[-i]$. Moreover, for any prime $\mathfrak{p}\in Z$, cup product with the canonical class $e\in H^1(\mathcal{X}_{\mathfrak{p},W},\mathbb{Z})$ yields a trivialization (induced by (66))

$$\mu_{\mathcal{X}_{\mathfrak{p}}}:\mathbb{Q}\cong \det_{\mathbb{Q}} R\Gamma(\mathcal{X}_{\mathfrak{p},W},\mathbb{Z})_{\mathbb{Q}}. \tag{74}$$

Then (68), (73), and (74) yield an isomorphism

$$\begin{aligned}\vartheta_W:\det_{\mathbb{Q}} R\Gamma_{W,c}(\mathcal{X}_U,\mathbb{Z})_{\mathbb{Q}} &\cong \det_{\mathbb{Q}} R\Gamma_{W,c}(\mathcal{X},\mathbb{Z})_{\mathbb{Q}}\otimes\bigotimes_{\mathfrak{p}\in Z}\det^{-1}_{\mathbb{Q}} R\Gamma(\mathcal{X}_{\mathfrak{p},W},\mathbb{Z})_{\mathbb{Q}}\\ &\cong\bigotimes_{i=0}^{2d-2}\Delta_f\big(h^i(X)\big)^{(-1)^i}.\end{aligned}$$

We claim that the following diagram of isomorphisms is commutative:

$$\begin{array}{ccc}\mathbb{R} & \xrightarrow{\lambda_{\mathcal{X}_U}} & \det_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X}_U,\mathbb{Z})_{\mathbb{R}}\\ \| & & \downarrow{\scriptstyle \vartheta_{W,\mathbb{R}}}\\ \mathbb{R} & \xrightarrow{\otimes_i(\tilde\vartheta^i_\infty)^{(-1)^i}} & \bigotimes_{i=0}^{2d-2}\Delta_f\big(h^i(X)\big)^{(-1)^i}_{\mathbb{R}}\end{array} \tag{75}$$

where $\lambda_{\mathcal{X}_U}$ (resp., $\tilde\vartheta^i_\infty$) is defined in (71) (resp., in (62)). Similarly, for any prime l invertible on U we have a commutative diagram of isomorphisms

$$\begin{array}{ccc}\det_{\mathbb{Q}_l} R\Gamma_{W,c}(\mathcal{X}_U,\mathbb{Z})_{\mathbb{Q}_l} & \longrightarrow & \det_{\mathbb{Q}_l} R\Gamma_c(\mathcal{X}_{U,\text{ét}},\mathbb{Q}_l)\\ \downarrow{\scriptstyle \vartheta_{W,\mathbb{Q}_l}} & & \downarrow\\ \bigotimes_{i=0}^{2d-2}\Delta_f\big(h^i(X)\big)^{(-1)^i}_{\mathbb{Q}_l} & \xrightarrow{\otimes_i(\tilde\vartheta^i_l)^{(-1)^i}} & \bigotimes_{i=0}^{2d-2}\det^{(-1)^i}_{\mathbb{Q}_l} R\Gamma_c(U_{\text{ét}},V^i_l)\end{array} \tag{76}$$

where the top horizontal isomorphism is induced by the isomorphism

$$\det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X}_U,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}_l\cong\det_{\mathbb{Z}_l} R\Gamma_c(\mathcal{X}_{U,\text{ét}},\mathbb{Z}_l) \tag{77}$$

given by (68) and Corollary 2.11, while the right-hand side vertical isomorphism is induced by the isomorphism

$$\begin{aligned}\det_{\mathbb{Z}_l} R\Gamma_c(\mathcal{X}_{U,\text{ét}},\mathbb{Z}_l)&\cong\det_{\mathbb{Z}_l} R\Gamma_c(U_{\text{ét}},R\pi_*\mathbb{Z}_l)\\ &\cong\bigotimes_{i=0}^{2d-2}\det^{(-1)^i}_{\mathbb{Z}_l} R\Gamma_c(U_{\text{ét}},T^i_l),\end{aligned} \tag{78}$$

where $T_l^i := R^i \pi_* \mathbb{Z}_l$. Hence the $\mathbb{Z}_l$-lattice of $\bigotimes_{i=0}^{2d-2} \Delta_f(h^i(X))_{\mathbb{Q}_l}^{(-1)^i}$ given by the images of $\det_{\mathbb{Z}_l} R\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z})_{\mathbb{Z}_l}$ and $\bigotimes_{i=0}^{2d-2} \det_{\mathbb{Z}_l}^{(-1)^i} R\Gamma_c(U_{\text{ét}}, T_l^i)$ coincide. This shows that Conjecture 4.2(b) implies Conjecture 4.10 (hence Conjecture 4.9) for $\bigoplus_{i=0}^{2d-2} h^i(\mathcal{X}_F)[-i]$ and all l. Conversely, Conjecture 4.10 for $\bigoplus_{i=0}^{2d-2} h^i(\mathcal{X}_F)[-i]$ and all l implies the l-primary part of Conjecture 4.2(b) for $\mathcal{X}[1/l]$ and all l, hence the l-primary part of Conjecture 4.2(b) for $\mathcal{X}$ and all l, and hence Conjecture 4.2(b) for $\mathcal{X}$. Here by the l-primary part of Conjecture 4.2(b) for $\mathcal{X}$ we mean an identity

$$\mathbb{Z}_{(l)} \cdot \lambda_{\mathcal{X}}\big(\zeta^*(\mathcal{X}, 0)^{-1}\big) = \det_{\mathbb{Z}_{(l)}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z})_{\mathbb{Z}_{(l)}},$$

where $\mathbb{Z}_{(l)}$ is the localization of $\mathbb{Z}$ at the prime ideal $l\mathbb{Z}$.

It remains to check that the squares (75) and (76) are indeed commutative. Consider the following diagram:

$$\begin{array}{ccccc}
\det_{\mathbb{Q}_l} R\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z})_{\mathbb{Q}_l} \otimes \mathbb{Q}_l & \xrightarrow{1 \otimes \mu_{\mathcal{X}_Z, \mathbb{Q}_l}} & \det_{\mathbb{Q}_l} R\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z})_{\mathbb{Q}_l} \otimes \det_{\mathbb{Q}_l} R\Gamma(\mathcal{X}_{Z,W}, \mathbb{Z})_{\mathbb{Q}_l} & \xrightarrow{b} & \det_{\mathbb{Q}_l} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z})_{\mathbb{Q}_l} \\
\downarrow{\scriptstyle a \otimes 1} & & \downarrow{\scriptstyle a \otimes f} & & \downarrow{\scriptstyle d} \\
\bigotimes_i \det_{\mathbb{Q}_l}^{(-1)^i} R\Gamma_c(U_{\text{ét}}, V_l^i) \otimes \mathbb{Q}_l & \xrightarrow{1 \otimes \bigotimes_{\mathfrak{p},i} \tilde{\iota}_{\mathfrak{p}}^{(-1)^{i+1}}} & \bigotimes_i \det_{\mathbb{Q}_l} R\Gamma_c(U_{\text{ét}}, V_l^i) \otimes \bigotimes_{\mathfrak{p},i} \det_{\mathbb{Q}_l} R\Gamma_f(F_{\mathfrak{p}}, V_l^i) & \xrightarrow{c} & \bigotimes_i \Delta_f(h^i(X))_{\mathbb{Q}_l}^{(-1)^i}
\end{array}$$

Here we identify the Weil-étale cohomology tensor $\mathbb{Q}_l$ with l-adic étale cohomology. Then the left-hand side vertical isomorphism a is induced by (78), the map f in the central vertical isomorphism is given by (64), and b, c, and d are induced by (68), (58), and (73), respectively. The commutativity of the left-hand side square follows from the commutativity of (65). The commutativity of the right-hand side square follows from Proposition 4.7. By the definition of ϑ_W we have that $d \circ b \circ (1 \otimes \mu_{\mathcal{X}_Z, \mathbb{Q}_l}) = \vartheta_{W, \mathbb{Q}_l}$. By the definition of $\tilde{\vartheta}_l^i$ we have that $\bigotimes_i (\tilde{\vartheta}_l^i)^{(-1)^i} = (c \circ (1 \otimes \bigotimes_{\mathfrak{p},i} \tilde{\iota}_{\mathfrak{p}}^{(-1)^{i+1}})^{-1}$. The commutativity of (76) follows immediately.

The square (75) can be decomposed as follows:

$$\begin{array}{ccccc}
\mathbb{R} \otimes \mathbb{R} & \xrightarrow{\lambda_{\mathcal{X}} \otimes \lambda_{\mathcal{X}_Z}^{-1}} & \det_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z})_{\mathbb{R}} \otimes \det_{\mathbb{R}}^{-1} R\Gamma(\mathcal{X}_{Z,W}, \mathbb{Z})_{\mathbb{R}} & \xrightarrow{b} & \det_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X}_U, \mathbb{Z})_{\mathbb{R}} \\
\downarrow{\scriptstyle 1 \otimes 1} & & \downarrow{\scriptstyle \eta \otimes \mu_{\mathcal{X}_Z, \mathbb{R}}} & & \downarrow{\scriptstyle \vartheta_{W,\mathbb{R}}} \\
\mathbb{R} \otimes \mathbb{R} & \xrightarrow{(\otimes_i (\vartheta_\infty^i)^{(-1)^i}) \otimes \left(\prod_{\mathfrak{p}} \log(N(\mathfrak{p}))\right)} & \left(\bigotimes_i \Delta_f(h^i(X))_{\mathbb{R}}^{(-1)^i}\right) \otimes \mathbb{R} & \xrightarrow{\mathrm{Id}} & \bigotimes_i \Delta_f(h^i(X))_{\mathbb{R}}^{(-1)^i}
\end{array}$$

Indeed, under the canonical identification $\mathbb{R} \otimes \mathbb{R} = \mathbb{R}$, the composition of the top horizontal maps (resp., of the lower horizontal maps) is $\lambda_{\mathcal{X}_U}$ (resp., $\bigotimes_i (\tilde{\vartheta}_\infty^i)^{(-1)^i}$). Here Id is the obvious identification, η is induced by (73), and b is induced by (68). The right-hand side square is commutative by the definition of ϑ_W. The left-hand side square is the tensor product of two squares; it is therefore enough to check the commutativity of both factors separately. The commutativity of the square corresponding to the first factor follows from the fact that (73) is compatible with $\lambda_{\mathcal{X}}$ and $\bigotimes_i (\vartheta^i_\infty)^{(-1)^i}$. The commutativity of the square corresponding to the second factor boils down to the identity

$$\lambda^{-1}_{\mathcal{X}_\mathfrak{p}} = \log\big(N(\mathfrak{p})\big) \cdot \mu^{-1}_{\mathcal{X}_\mathfrak{p},\mathbb{R}} : \det_\mathbb{R} R\Gamma(\mathcal{X}_{\mathfrak{p},W}, \mathbb{Z})_\mathbb{R} \to \mathbb{R}. \tag{79}$$

Identity (79) follows from the fact that $\mu_{\mathcal{X}_\mathfrak{p},\mathbb{R}}$ is obtained by cup product with the class $e \in H^1(W_{\mathbb{F}_\mathfrak{p}}, \mathbb{R}) = \mathrm{Hom}(W_{\mathbb{F}_\mathfrak{p}}, \mathbb{R})$ sending $\mathrm{Fr}_\mathfrak{p} \in W_{\mathbb{F}_\mathfrak{p}}$ to 1, while $\lambda_{\mathcal{X}_\mathfrak{p}}$ is obtained by cup product with the class $\theta \in H^1(W_{\mathbb{F}_\mathfrak{p}}, \mathbb{R})$ sending $\mathrm{Fr}_\mathfrak{p}$ to $\log(N(\mathfrak{p}))$. This is an easy computation in view of the fact that both complexes $[\cdots \to H^i(\mathcal{X}_{\mathfrak{p},W}, \tilde{\mathbb{R}}) \to H^i(\mathcal{X}_{\mathfrak{p},W}, \tilde{\mathbb{R}}) \to \cdots]$ are isomorphic to $[\mathbb{R} \to \mathrm{Hom}(W_{\mathbb{F}_\mathfrak{p}}, \mathbb{R})]$ put in degrees $0, 1$, because $\mathcal{X}_\mathfrak{p}/\mathbb{F}_\mathfrak{p}$ is geometrically connected. Identity (79) is of course compatible with $Z^*(\mathcal{X}_\mathfrak{p}, 1) = \log(N(\mathfrak{p})) \cdot \zeta^*(\mathcal{X}_\mathfrak{p}, 0)$. This concludes the proof of the theorem. □

4.4. Proven cases

Let $\mathbb{F}_q$ be a finite field, and let $A(\mathbb{F}_q)$ be the class of smooth projective varieties over $\mathbb{F}_q$ defined in Section 5.2.

THEOREM 4.12
Let $\mathcal{X}$ be a smooth projective variety over the finite field $\mathbb{F}_q$. If $\mathcal{X}$ lies in $A(\mathbb{F}_q)$, then Conjecture 4.2 holds for $\mathcal{X}$.

Proof
The variety $\mathcal{X}$ lies in $A(\mathbb{F}_q)$; hence $\mathbf{L}(\mathcal{X}_{\text{ét}}, d) \Leftrightarrow \mathbf{L}(\mathcal{X}_W, d)$ holds (see Proposition 5.7). In view of Theorem 2.13 the result follows from [34, Theorem 7.4]. □

THEOREM 4.13
If $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$ is the spectrum of a number ring, then Conjecture 4.2 holds for $\mathcal{X}$.

Proof
The result follows from the explicit computation of the Weil-étale cohomology in this case (see Section 4.5.1) and from the analytic class number formula

$$\mathrm{ord}_{s=0}\, \zeta(\mathcal{X}, s) = \sharp\mathcal{X}_\infty - 1 \qquad \text{and} \qquad \zeta^*_F(0) = -hR/w,$$

where h (resp., w) is on the order of $\mathrm{Cl}(F)$ (resp., of μ_F) and R is the regulator of the number field F. □

THEOREM 4.14
Let $\mathcal{X}$ be a smooth projective scheme over the number ring $\mathcal{O}_\mathcal{X}(\mathcal{X}) = \mathcal{O}_F$, where F

is an abelian number field. Assume that $\mathcal{X}_F$ admits a smooth cellular decomposition (see Definition 5.13), and assume that $\mathcal{X}_\mathfrak{p} \in \mathcal{L}(\mathbb{Z})$ for any finite prime $\mathfrak{p}$ of F. Then Conjecture 4.2 holds for $\mathcal{X}$.

Proof

The scheme $\mathcal{X}$ is connected and we set $d = \dim(\mathcal{X})$. The scheme $\mathcal{X}$ satisfies Conjecture $\mathbf{L}(\mathcal{X}_{\text{ét}}, d)$ (resp., conjecture $\mathbf{B}(\mathcal{X}, d)$) by Proposition 5.14 (resp., by Proposition 5.15). Moreover, $\mathcal{X}_F$ is a cellular variety; hence $h(\mathcal{X}_F)$ is of the form $\bigoplus_k h^0(F)(r_k)$ (in the category of Chow motives, see [4, Theorem 3.1]). Hence $L(h^i(\mathcal{X}_F), s)$ is a product of shifts of the Dedekind zeta function $\zeta_F(s)$; in particular, $L(h^i(\mathcal{X}_F), s)$ satisfies meromorphic continuation and the functional equation. By Theorem 3.6 and [13, Theorem 1.1], Conjecture 4.2(a) for $\mathcal{X}$ follows. Moreover, the Galois representations $H^i(\mathcal{X}_{\overline{F}}, \mathbb{Q}_l)$ are a sum of $\mathbb{Q}_l(r)$ (with $r \leq 0$), and hence satisfy Conjecture 4.8 (see [1, Lemme 4.3.1]). Finally, Conjecture 4.9 is known for $h^i(\mathcal{X}_F)$ (which is a sum of $h^0(F)(r)$) since $F/\mathbb{Q}$ is abelian (see [11]). Hence Conjecture 4.2(b) for $\mathcal{X}$ follows from Theorem 4.11. □

The simplest nontrivial example of a scheme satisfying the assumptions of the previous theorem is $\mathbb{P}^n_{\mathcal{O}_F}$ where $F/\mathbb{Q}$ is abelian. The result then gives a cohomological interpretation of $\zeta_F^*(n)$ for $n \leq 0$ (see Section 4.5.2).

4.5. Examples

4.5.1. Number rings

Let $\mathcal{O}_F$ be a number ring, and let $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$. Note that $\mathbf{L}(\mathcal{X}_{\text{ét}}, \mathbb{Z}(1))$ holds (see Theorem 5.1). The cohomology $H^*(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})$ is computed in [38, Proposition 6.6]. By Proposition 2.10 one has

$$H^i_W(\overline{\mathcal{X}}, \mathbb{Z}) = 0 \quad \text{for } i < 0 \text{ and } i > 3.$$

By Proposition 2.10 again one has $H^0_W(\overline{\mathcal{X}}, \mathbb{Z}) = \mathbb{Z}$, $H^1_W(\overline{\mathcal{X}}, \mathbb{Z}) = 0$, an exact sequence

$$0 \to H^2(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})_{\text{codiv}} \to H^2_W(\overline{\mathcal{X}}, \mathbb{Z}) \to \mathrm{Hom}\big(H^1\big(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}(1)\big), \mathbb{Z}\big) \to 0, \tag{80}$$

and an isomorphism

$$H^3_W(\overline{\mathcal{X}}, \mathbb{Z}) \simeq H^3(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z})_{\text{codiv}} = \mathrm{Hom}(\mathcal{O}_F^\times, \mathbb{Q}/\mathbb{Z})_{\text{codiv}} = \mu_F^D,$$

since $H^0(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}(1)) = 0$. The sequence (80) reads as follows (see [38, Proposition 6.6]):

$$0 \to \mathrm{Cl}(F)^D \to H^2_W(\overline{\mathcal{X}}, \mathbb{Z}) \to \mathrm{Hom}(\mathcal{O}_F^\times, \mathbb{Z}) \to 0,$$

where $\mathrm{Cl}(F)$ is the class group of F, $\mathcal{O}_F^\times$ is the unit group, and $\mu_F := (\mathcal{O}_F^\times)_{\mathrm{tor}}$. The map

$$H^i_{W,c}(\mathcal{X},\mathbb{Z})\otimes\mathbb{R}\xrightarrow{\sim} H^i_{W,c}(\mathcal{X},\tilde{\mathbb{R}})$$

is trivial for $i\neq 1,2$, the obvious isomorphism for $i=1$, and the inverse of the dual of the classical regulator map

$$H^2_{W,c}(\mathcal{X},\mathbb{Z})\otimes\mathbb{R}\simeq \mathrm{Hom}(\mathcal{O}_F^\times,\mathbb{R})\to\Big(\prod_{\mathcal{X}_\infty}\mathbb{R}\Big)/\mathbb{R}$$

for $i=2$. The acyclic complex $(H^*_{W,c}(\mathcal{X},\tilde{\mathbb{R}}),\cup\theta)$ is reduced to the identity map

$$H^1_{W,c}(\mathcal{X},\tilde{\mathbb{R}})=\Big(\prod_{\mathcal{X}_\infty}\mathbb{R}\Big)/\mathbb{R}\xrightarrow{\mathrm{Id}}\Big(\prod_{\mathcal{X}_\infty}\mathbb{R}\Big)/\mathbb{R}=H^2_{W,c}(\mathcal{X},\tilde{\mathbb{R}}).$$

We obtain (see [35, Section 7])

$$\lambda_{\mathcal{X}}^{-1}\big(\det_{\mathbb{Z}}R\Gamma_{W,c}(\mathcal{X},\mathbb{Z})\big)=(w/hR)\cdot\mathbb{Z}.$$

4.5.2. Projective spaces over number rings

Let $\mathcal{O}_F$ be the ring of integers in a totally imaginary number field F, and let $n\geq 1$. We set $\mathcal{X}=\mathbb{P}^n_{\mathcal{O}_F}$ and $d=\dim(\mathcal{X})=n+1$. It follows easily from Proposition 5.14 (or directly from (81) below and Theorem 5.1) that $\mathbf{L}(\mathcal{X}_{\text{ét}},d)$ holds.

PROPOSITION 4.15

For $2\leq i\leq 2d+1$, *we have an exact sequence*

$$0\to\big(K_{i-2}(\mathcal{O}_F)_{\mathrm{tor}}\big)^D\to H^i_W(\overline{\mathcal{X}},\mathbb{Z})\to\mathrm{Hom}_{\mathbb{Z}}\big(K_{i-1}(\mathcal{O}_F),\mathbb{Z}\big)\to 0.$$

Proof

The localization sequence (see [19, Corollary 7.2(a)])

$$\cdots\to H^{i-2}\big(\mathbb{P}^{n-1}_{\mathcal{O}_F,\text{ét}},\mathbb{Z}(d-1)\big)\to H^i\big(\mathbb{P}^n_{\mathcal{O}_F,\text{ét}},\mathbb{Z}(d)\big)\to H^i\big(\mathbb{A}^n_{\mathcal{O}_F,\text{ét}},\mathbb{Z}(d)\big)\to\cdots$$

is split by $H^i(\mathbb{A}^n_{\mathcal{O}_F,\text{ét}},\mathbb{Z}(d))\xleftarrow{\sim}H^i(\mathrm{Spec}(\mathcal{O}_F)_{\text{ét}},\mathbb{Z}(d))\to H^i(\mathbb{P}^n_{\mathcal{O}_F,\text{ét}},\mathbb{Z}(d))$ (see Lemma 5.11). By induction, this yields the projective bundle formula

$$H^i\big(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d)\big)\simeq H^i\big(\mathcal{X}_{\text{ét}},\mathbb{Z}(d)\big)\simeq\bigoplus_{j=0}^{n}H^{i-2j}\big(\mathrm{Spec}(\mathcal{O}_F)_{\text{ét}},\mathbb{Z}(d-j)\big).\qquad(81)$$

If $0\leq j\leq n-1$, then $H^{i-2j}(\mathrm{Spec}(\mathcal{O}_F)_{\text{ét}},\mathbb{Z}(d-j))=0$ for $i-2j\neq 1,2$ (by Lemma 5.4) and

$$K_{2d-i}(\mathcal{O}_F)=K_{2(d-j)-(i-2j)}(\mathcal{O}_F)\simeq H^{i-2j}\big(\mathrm{Spec}(\mathcal{O}_F),\mathbb{Z}(d-j)\big)$$
$$\simeq H^{i-2j}\big(\mathrm{Spec}(\mathcal{O}_F)_{\text{ét}},\mathbb{Z}(d-j)\big)$$

for $i-2j=1,2$ by Theorem 5.1(f) and (83).

We obtain $H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d))\simeq K_{2d-i}(\mathcal{O}_F)$ for $1\leq i\leq 2d-1$ and $H^{2d}(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d))\simeq \mathrm{Cl}(F)=K_0(\mathcal{O}_F)_{\text{tor}}$. The result then follows from the exact sequence (see Proposition 2.10)

$$0\to H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})_{\text{codiv}}\to H^i_W(\overline{\mathcal{X}},\mathbb{Z})\to \mathrm{Hom}_{\mathbb{Z}}\big(H^{2d+2-(i+1)}\big(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d)\big)_{\geq 0},\mathbb{Z}\big)\to 0$$

and from $H^i(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z})_{\text{codiv}}\simeq (H^{2d+2-i}(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(d))_{\geq 0,\text{tor}})^D$ for $i\geq 1$ (see Lemma 2.5). □

Recall from [3] that $K_i(\mathcal{O}_F)$ is finitely generated for any $i\geq 0$ and finite for $i\neq 0$ even. By Propositions 4.15 and 2.10, $H^*_W(\overline{\mathcal{X}},\mathbb{Z})$ is given by the following identifications and exact sequences:

$$\begin{aligned}
H^i_W(\overline{\mathcal{X}},\mathbb{Z}) &= \mathbb{Z}\quad \text{for } i=0,\\
&= 0\quad \text{for } i=1,\\
0\to \mathrm{Cl}(F)^D\to H^2_W(\overline{\mathcal{X}},\mathbb{Z}) &\to \mathrm{Hom}(\mathcal{O}_F^\times,\mathbb{Z})\to 0,\\
&= \mu_F^D\quad \text{for } i=3,\\
0\to K_2(\mathcal{O}_F)^D\to H^4_W(\overline{\mathcal{X}},\mathbb{Z}) &\to \mathrm{Hom}\big(K_3(\mathcal{O}_F),\mathbb{Z}\big)\to 0,\\
&= \big(K_3(\mathcal{O}_F)_{\text{tor}}\big)^D\quad \text{for } i=5,\\
&\;\;\vdots\\
0\to K_{2n}(\mathcal{O}_F)^D\to H^{2n+2}_W(\overline{\mathcal{X}},\mathbb{Z}) &\to \mathrm{Hom}\big(K_{2n+1}(\mathcal{O}_F),\mathbb{Z}\big)\to 0,\\
&= \big(K_{2n+1}(\mathcal{O}_F)_{\text{tor}}\big)^D\quad \text{for } i=2n+3,\\
&= 0\quad \text{for } i>2n+3.
\end{aligned}$$

The long exact sequence

$$\cdots\to H^i_{W,c}(\mathcal{X},\mathbb{Z})\otimes\mathbb{R}\to H^i_W(\overline{\mathcal{X}},\mathbb{Z})\otimes\mathbb{R}\to H^i(\mathcal{X}_\infty,\mathbb{Z})\otimes\mathbb{R}\to\cdots$$

and Proposition 3.3 show that, for $i\geq 2$, $H^i_{W,c}(\mathcal{X},\mathbb{Z})\otimes\mathbb{R}$ is canonically isomorphic to either $H^i_W(\overline{\mathcal{X}},\mathbb{Z})\otimes\mathbb{R}$ or $H^{i-1}(\mathcal{X}_\infty,\mathbb{Z})\otimes\mathbb{R}$ depending on the parity of i. Hence the acyclic complex

$$\cdots\xrightarrow{\cup\theta} H^i_{W,c}(\mathcal{X},\mathbb{Z})\otimes\mathbb{R}\xrightarrow{\cup\theta} H^{i+1}_{W,c}(\mathcal{X},\mathbb{Z})\otimes\mathbb{R}\xrightarrow{\cup\theta}\cdots$$

is canonically isomorphic to

$$0 \longrightarrow \Big(\prod_{F\hookrightarrow\mathbb{C}} \mathbb{R}\Big)^{G_{\mathbb{R}}}/\mathbb{R} \xrightarrow{R_0^*} \mathrm{Hom}(\mathcal{O}_F^{\times},\mathbb{R}) \xrightarrow{0} \Big(\prod_{F\hookrightarrow\mathbb{C}} (2i\pi)^{-1}\mathbb{R}\Big)^{G_{\mathbb{R}}}$$

$$\xrightarrow{R_1^*} \mathrm{Hom}\big(K_3(\mathcal{O}_F),\mathbb{R}\big) \xrightarrow{0} \cdots \xrightarrow{0} \Big(\prod_{F\hookrightarrow\mathbb{C}} (2i\pi)^{-n}\mathbb{R}\Big)^{G_{\mathbb{R}}}$$

$$\xrightarrow{R_n^*} \mathrm{Hom}\big(K_{2n+1}(\mathcal{O}_F),\mathbb{R}\big) \xrightarrow{0} 0,$$

where the isomorphisms R_m^* are dual to the regulator maps. It follows (see [35, Section 7]) that $\lambda_{\mathcal{X}}^{-1}$ maps $\det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X},\mathbb{Z})$ to

$$(w/hR_0)\cdot\big(\sharp K_3(\mathcal{O}_F)_{\mathrm{tor}}/\sharp K_2(\mathcal{O}_F)\cdot R_1\big)\cdot\ldots\cdot\big(\sharp K_{2n+1}(\mathcal{O}_F)_{\mathrm{tor}}/\sharp K_{2n}(\mathcal{O}_F)\cdot R_n\big)\cdot\mathbb{Z},$$

where R_m is the Beilinson regulator. (We follow the indexing of [32].) In view of

$$\zeta^*(\mathbb{P}^n_{\mathcal{O}_F},0)=\zeta_F^*(0)\cdot\zeta_F^*(-1)\cdot\ldots\cdot\zeta_F^*(-n)$$

we see that Conjecture 4.2 for $\mathcal{X}=\mathbb{P}^n_{\mathcal{O}_F}$ gives a cohomological reformulation of the classical version of Lichtenbaum's conjecture (see [32, Question 4.2]). Note that Weil-étale cohomology gives (conjecturally) the right 2-torsion for any number field, that is, possibly with some real places (see [32, Section 2.6]). For example, if $F/\mathbb{Q}$ is abelian, then Conjecture 4.2 holds for $\mathcal{X}=\mathbb{P}^n_{\mathcal{O}_F}$ and any $n\geq 0$ by Theorem 4.14.

5. The class $\mathcal{L}(\mathbb{Z})$

The goal of this section is to establish simple cases of conjectures $\mathbf{L}(\mathcal{X}_{\text{ét}},d)$ and $\mathbf{B}(\mathcal{X},d)$, which are used in Section 4.4.

5.1. Motivic cohomology of number rings

Let F be a number field, and let $\mathcal{O}_F$ be its ring of integers. In order to ease the notations, in this section we denote by $H^i(\mathcal{O}_F,\mathbb{Z}(n)) := H^p(\mathrm{Spec}(\mathcal{O}_F)_{\mathrm{Zar}},\mathbb{Z}(n))$ and $H^i_{\text{ét}}(\mathcal{O}_F,\mathbb{Z}(n)) := H^p(\mathrm{Spec}(\mathcal{O}_F)_{\text{ét}},\mathbb{Z}(n))$ the Zariski and étale hypercohomologies of the cycle complex $\mathbb{Z}(n)$ over $\mathrm{Spec}(\mathcal{O}_F)$. We consider the spectral sequence constructed by Levine (see [30, Spectral Sequence (8.8)])

$$E_2^{p,q}=H^p\big(\mathcal{O}_F,\mathbb{Z}(-q/2)\big)\Rightarrow K_{-p-q}(\mathcal{O}_F). \tag{82}$$

The aim of this section is to prove Theorem 5.1. It is well known (see, e.g., [29, Proposition 2.1]) that this result follows one way or another from the Bloch–Kato conjecture (relating Milnor K-theory to Galois cohomology), which is proven in [47]. However, we could not find a proof of Theorem 5.1 in the literature.

THEOREM 5.1
The following assertions are true.

(a) *For $i \leq 2$, $H^i_{\text{ét}}(\mathcal{O}_F, \mathbb{Z}(1))$ is finitely generated.*
(b) *For any $n \geq 0$ and any $i \in \mathbb{Z}$, $H^i(\mathcal{O}_F, \mathbb{Z}(n))$ is finitely generated.*
(c) *For any $n \geq 2$ and any $i \in \mathbb{Z}$, $H^i_{\text{ét}}(\mathcal{O}_F, \mathbb{Z}(n))$ is finitely generated.*
(d) *The edge morphisms from the spectral sequence (82) yield maps*

$$c_{i,n} : K_{2n-i}(\mathcal{O}_F) \longrightarrow H^i\big(\mathcal{O}_F, \mathbb{Z}(n)\big)$$

for $n \geq 2$ and $i = 1, 2$.
(e) *The kernel and the cokernel of $c_{i,n}$ are both finite and 2-primary torsion.*
(f) *If F is totally imaginary, then the maps $c_{i,n}$ are isomorphisms.*

Proof
The proof of Theorem 5.1 requires the following lemmas.

LEMMA 5.2
Let $n \geq 2$. If $i \neq 1, 2$, then

$$H^i_{\text{ét}}\big(\mathcal{O}_F, \mathbb{Z}(n)\big) \simeq H^{i-1}_{\text{ét}}\big(\mathcal{O}_F, \mathbb{Q}/\mathbb{Z}(n)\big).$$

Proof
Applying the exact functor $- \otimes_{\mathbb{Z}} \mathbb{Q}$ to the spectral sequence (82), we obtain a spectral sequence with $\mathbb{Q}$-coefficients. By [31, Theorem 11.7], (82) degenerates with $\mathbb{Q}$-coefficients and yields

$$H^i\big(\mathcal{O}_F, \mathbb{Q}(n)\big) \simeq K_{2n-i}(\mathcal{O}_F)^{(n)}_{\mathbb{Q}}.$$

By Borel's theorem [3, Proposition 12.2], $K_{2n-i}(\mathcal{O}_F)_{\mathbb{Q}} = 0$ for i even, $i \neq 2n$. Moreover, one has $K_{2n-i}(\mathcal{O}_F)^{(n)}_{\mathbb{Q}} = 0$ for i odd, $i \neq 1$ (see [48, Theorem 47]). Since $K_0(\mathcal{O}_F)_{\mathbb{Q}} = K_0(\mathcal{O}_F)^{(0)}_{\mathbb{Q}}$, this gives $H^i(\mathcal{O}_F, \mathbb{Q}(n)) = 0$ for i even, except for $(i,n) = (0,0)$, and $H^i(\mathcal{O}_F, \mathbb{Q}(n)) = 0$ for i odd, except for $i = 1$. Lemma 5.2 then follows from the long exact sequence

$$\cdots \to H^i_{\text{ét}}\big(\mathcal{O}_F, \mathbb{Z}(n)\big) \to H^i_{\text{ét}}\big(\mathcal{O}_F, \mathbb{Q}(n)\big) \to H^i_{\text{ét}}\big(\mathcal{O}_F, \mathbb{Q}/\mathbb{Z}(n)\big) \to \cdots$$

and from the isomorphism $H^i(\mathcal{O}_F, \mathbb{Q}(n)) \simeq H^i_{\text{ét}}(\mathcal{O}_F, \mathbb{Q}(n))$ (see [16, Proposition 3.6]). □

For a prime number p, we write $j_p : \mathrm{Spec}(\mathcal{O}_F[1/p]) \to \mathrm{Spec}(\mathcal{O}_F)$ for the obvious open embedding, $j_{p,*}$ for the direct image functor with respect to the étale topology, and $Rj_{p,*}$ for its total derived functor.

LEMMA 5.3

For $n \geq 2$ one has an isomorphism in the derived category of étale sheaves on $\operatorname{Spec}(\mathcal{O}_F)$

$$\mathbb{Q}/\mathbb{Z}(n) \simeq \bigoplus_p \varinjlim Rj_{p,*}\mu_{p^r}^{\otimes n},$$

where μ_{p^r} is the étale sheaf of p^rth roots of unity and the direct sum (resp., the colimit) is taken over all prime numbers (resp., over r).

Proof

It is enough to show that $\mathbb{Z}/p^r\mathbb{Z}(n) = Rj_{p,*}\mu_{p^r}^{\otimes n}$ for any prime number p. Using [19, Corollary 7.2], [20, Theorem 8.5], and [16, Theorem 1.2(4)], we obtain an exact triangle

$$i_{p,*}(\nu_r^{n-1})\big[-(n-1)-2\big] \to \mathbb{Z}/p^r\mathbb{Z}(n) \to Rj_{p,*}(\mu_{p^r}^{\otimes n}),$$

where $\nu_r^n = \nu_{p^r}^n = W\Omega^n_{r,\log}$ is the logarithmic de Rham–Witt sheaf, and i_p is the closed immersion of the points lying over p. But ν_r^{n-1} is trivial on the finite field $\mathbb{F}_{\mathfrak{p}} := \mathcal{O}_F/\mathfrak{p}$ (for $\mathfrak{p} \mid p$) because $n-1 \geq 1$. The result then follows from $\mathbb{Q}/\mathbb{Z}(n) \simeq \bigoplus_p \varinjlim \mathbb{Z}/p^r\mathbb{Z}(n)$. □

LEMMA 5.4

The following assertions are true.

- *If $n \geq 1$ and $i \leq 0$, then we have $H^i_{\text{ét}}(\mathcal{O}_F, \mathbb{Z}(n)) = 0$.*
- *If $n \geq 2$ and $i \geq 3$, then $H^i_{\text{ét}}(\mathcal{O}_F, \mathbb{Z}(n))$ is finite and 2-torsion.*
- *Assume that F is totally imaginary. If $n \geq 2$ and $i \geq 3$, then $H^i_{\text{ét}}(\mathcal{O}_F, \mathbb{Z}(n)) = 0$.*

Proof

In view of $\mathbb{Z}(1) \simeq \mathbb{G}_m[-1]$, where $\mathbb{G}_m$ is the multiplicative group (see [19, Lemma 7.4]), the fact that $H^i_{\text{ét}}(\mathcal{O}_F, \mathbb{Z}(n)) = 0$ for $n \geq 1$ and $i \leq 0$ follows immediately from Lemmas 5.2 and 5.3.

We assume now that $n \geq 2$. By [42, Théorème 5], the group $H^2_{\text{ét}}(\mathcal{O}_F[1/p], \varinjlim \mu_{p^r}^{\otimes n})$ is of finite exponent for any $p \neq 2$. (The colimit is taken over r.) But $H^3_{\text{ét}}(\mathcal{O}_F[1/p], \mu_{p^r}^{\otimes n}) = 0$ for $p \neq 2$ by Artin–Verdier duality (see [37, Corollary 3.3]); hence $H^2_{\text{ét}}(\mathcal{O}_F[1/p], \varinjlim \mu_{p^r}^{\otimes n})$ is divisible. This yields $H^2_{\text{ét}}(\mathcal{O}_F[1/p], \varinjlim \mu_{p^r}^{\otimes n}) = 0$ for $p \neq 2$, since this group is both divisible and of finite exponent. Moreover, we have that

$$H^i_{\text{ét}}\big(\mathcal{O}_F[1/p], \varinjlim \mu_{p^r}^{\otimes n}\big) = 0$$

for any $i \geq 3$ and $p \neq 2$, again by Artin–Verdier duality. By [41, Corollary 4.4 and Proposition 4.6], for any $i \geq 2$, we have that $H^i_{\text{ét}}(\mathcal{O}_F[1/2], \varinjlim \mu_{2^r}^{\otimes n}) = (\mathbb{Z}/2\mathbb{Z})^{r_1}$ for $n - i$ odd and $H^i_{\text{ét}}(\mathcal{O}_F[1/2], \varinjlim \mu_{2^r}^{\otimes n}) = 0$ for $n - i$ even, where r_1 is the number of real places of F.

Using $H^i_{\text{ét}}(\mathcal{O}_F, \mathbb{Q}/\mathbb{Z}(n)) \simeq \bigoplus_p \varinjlim H^i_{\text{ét}}(\mathcal{O}_F[1/p], \mu_{p^r}^{\otimes n})$ (see Lemma 5.3), it then follows that $H^i_{\text{ét}}(\mathcal{O}_F, \mathbb{Q}/\mathbb{Z}(n))$ is finite 2-torsion for $i \geq 2$, and $H^i_{\text{ét}}(\mathcal{O}_F, \mathbb{Q}/\mathbb{Z}(n)) = 0$ for F totally imaginary and $i \geq 2$. We obtain the result using Lemma 5.2. □

Proof of Theorem 5.1

The finite generation of $H^i_{\text{ét}}(\mathcal{O}_F, \mathbb{Z}(1))$ for $i \leq 2$ follows from $\mathbb{Z}(1) \simeq \mathbb{G}_m[-1]$ (see [19, Lemma 7.4]). Indeed, $H^i_{\text{ét}}(\mathcal{O}_F, \mathbb{G}_m) = 0$ for $i < 0$, $H^0_{\text{ét}}(\mathcal{O}_F, \mathbb{G}_m) = \mathcal{O}_F^\times$ is finitely generated, and the class group $H^1_{\text{ét}}(\mathcal{O}_F, \mathbb{G}_m) = \mathrm{Cl}(F)$ is finite.

By [47, Theorem 6.16] and by [16, Theorem 1.2(2)], the map

$$H^i\big(\mathcal{O}_F, \mathbb{Z}(n)\big) \xrightarrow{\sim} H^i_{\text{ét}}\big(\mathcal{O}_F, \mathbb{Z}(n)\big), \tag{83}$$

induced by the morphism from the étale site to the Zariski site, is an isomorphism for $i \leq n+1$. Note that, for dimension reasons, we have $H^i(\mathcal{O}_F, \mathbb{Z}(n)) = 0$ for $i > n+1$. By Lemma 5.4, we have $H^i(\mathcal{O}_F, \mathbb{Z}(n)) = 0$ for $n \geq 1$ and $i \leq 0$. It follows that the edge morphisms from the spectral sequence (82) give maps

$$c_{i,n} : K_{2n-i}(\mathcal{O}_F) \longrightarrow H^i\big(\mathcal{O}_F, \mathbb{Z}(n)\big)$$

for $n \geq 2$ and $i = 1, 2$. This yields Theorem 5.1(d).

Let us prove Theorem 5.1(b). The codomain of any nontrivial differential $d_r^{p,q}$ of the spectral sequence (82) at the E_r-page for $r \geq 2$ is a finite 2-torsion abelian group. Moreover, for (p, q) fixed, the differential $d_r^{p,q}$ is zero for $r \gg 0$ big enough. Since $E_\infty^{p,q}$ is a subquotient of some K-group, it is finitely generated by [3]. It follows that $E_2^{p,q}$ is finitely generated for any p and any q. Hence $H^i(\mathcal{O}_F, \mathbb{Z}(n))$ is finitely generated for any i and any n.

If F is totally imaginary, the spectral sequence (82) degenerates by Lemma 5.4 (any differential at the E_2-page has either trivial domain or trivial codomain), and the maps $c_{i,n}$ are isomorphisms. This proves Theorem 5.1(f). For an arbitrary number field F, the spectral sequence (82) degenerates with $\mathbb{Z}[1/2]$-coefficients by Lemma 5.4, and yields isomorphisms

$$c_{i,n} \otimes \mathbb{Z}[1/2] : K_{2n-i}(\mathcal{O}_F) \otimes \mathbb{Z}[1/2] \xrightarrow{\sim} H^i\big(\mathcal{O}_F, \mathbb{Z}(n)\big) \otimes \mathbb{Z}[1/2]$$

for $n \geq 2$ and $i = 1, 2$. This proves Theorem 5.1(e).

Finally, $H^i_{\text{ét}}(\mathcal{O}_F, \mathbb{Z}(n))$ is finitely generated for $n \geq 2$ and any $i \in \mathbb{Z}$. Indeed, (83) is an isomorphism for $i \leq n+1$ and $H^i_{\text{ét}}(X, \mathbb{Z}(n))$ is finite for $i \geq n+2 \geq 4$ by Lemma 5.4. This gives Theorem 5.1(c). □

Remark 5.5

Using arguments of Levine [31], one can determine most of the differentials of the spectral sequence (82) (in particular, (82) degenerates at E_4), and obtain precise information on the kernel and the cokernel of the map $c_{i,n}$ (see [31, Theorem 14.10]).

5.2. Smooth projective varieties over finite fields

For a smooth projective scheme Y over a finite field $\mathbb{F}_q$, we consider the cohomology $H^i(Y_W, \mathbb{Z}(n))$ of the Weil-étale topos Y_W with coefficients in Bloch's cycle complex (see [34] and [17]). The following conjecture is due to Lichtenbaum and Geisser.

CONJECTURE 5.6 ($\mathbf{L}(Y_W, n)$)

For any $i \in \mathbb{Z}$, $H^i(Y_W, \mathbb{Z}(n))$ is finitely generated.

Following [43] and [17], we consider the full subcategory $A(\mathbb{F}_q)$ of the category of smooth projective varieties over $\mathbb{F}_q$ generated by products of curves and the following operations:

(1) If X and Y are in $A(\mathbb{F}_q)$, then $X \coprod Y$ is in $A(\mathbb{F}_q)$.

(2) If Y is in $A(\mathbb{F}_q)$ and there are morphisms $c : X \to Y$ and $c' : Y \to X$ in the category of Chow motives such that $c' \circ c : X \to X$ is multiplication by a constant, then X is in $A(\mathbb{F}_q)$.

(3) If $\mathbb{F}_{q^m}/\mathbb{F}_q$ is a finite extension and $X \times_{\mathbb{F}_q} \mathbb{F}_{q^m}$ is in $A(\mathbb{F}_{q^m})$, then X is in $A(\mathbb{F}_q)$.

(4) If Y is a closed subscheme of X with X and Y in $A(\mathbb{F}_q)$, then the blowup X' of X along Y is in $A(\mathbb{F}_q)$.

PROPOSITION 5.7

Let Y be a connected smooth projective scheme over a finite field $\mathbb{F}_q$ of dimension d. The following assertions are true.

- *We have $\mathbf{L}(Y_W, d) \Leftrightarrow \mathbf{L}(Y_{\text{ét}}, d)$.*
- *If Y belongs to $A(\mathbb{F}_q)$, then $\mathbf{L}(Y_{\text{ét}}, d)$ holds.*
- *If Y belongs to $A(\mathbb{F}_q)$ and $n > d$, then $H^i(Y_{\text{ét}}, \mathbb{Z}(n))$ is finitely generated for any $i \in \mathbb{Z}$.*

Proof

By [17, Theorem 7.1], there is an exact sequence (for any n)

$$\cdots \to H^i\big(Y_{\text{ét}}, \mathbb{Z}(n)\big) \to H^i\big(Y_W, \mathbb{Z}(n)\big) \to H^{i-1}\big(Y_{\text{ét}}, \mathbb{Q}(n)\big)$$

$$\to H^{i+1}\big(Y_{\text{ét}}, \mathbb{Z}(n)\big) \to \cdots, \tag{84}$$

which yields isomorphisms

$$H^i\big(Y_W, \mathbb{Z}(n)\big) \otimes \mathbb{Q} \simeq H^i\big(Y_W, \mathbb{Q}(n)\big) \simeq H^i\big(Y_{\text{ét}}, \mathbb{Q}(n)\big) \oplus H^{i-1}\big(Y_{\text{ét}}, \mathbb{Q}(n)\big). \tag{85}$$

Assume now that conjecture $\mathbf{L}(Y_W, d)$ holds. By [17, Theorem 8.4], it follows from $\mathbf{L}(Y_W, d)$ that we have an isomorphism

$$H^i\big(Y_W, \mathbb{Z}(d)\big) \otimes \mathbb{Z}_l \simeq H^i_{\text{cont}}\big(Y, \mathbb{Z}_l(d)\big)$$

for any prime number l and any i. But for $i \neq 2d, 2d+1$, $H^i_{\text{cont}}(Y, \mathbb{Z}_l(d))$ is finite for any l and zero for almost all l (see the proof of [25, Corollaire 3.8] for references). Hence $H^i(Y_W, \mathbb{Z}(d))$ is finite for $i \neq 2d, 2d+1$. Then (85) gives $H^i(Y_{\text{ét}}, \mathbb{Q}(d)) = 0$ for $i < 2d$; hence $H^i(Y_{\text{ét}}, \mathbb{Q}(d)) \simeq H^i(Y, \mathbb{Q}(d)) = 0$ for $i \neq 2d$. The exact sequence (84) then shows that $H^i(Y_{\text{ét}}, \mathbb{Z}(d)) \to H^i(Y_W, \mathbb{Z}(d))$ is injective for $i \leq 2d+1$. Hence $H^i(Y_{\text{ét}}, \mathbb{Z}(d))$ is finitely generated for $i \leq 2d+1$. This yields $\mathbf{L}(Y_W, d) \Rightarrow \mathbf{L}(Y_{\text{ét}}, d)$. Conversely, we have $\mathbf{L}(Y_{\text{ét}}, d) \Rightarrow \mathbf{L}(Y_W, d)$ by Theorem 2.13.

Consider now a variety $Y \in A(\mathbb{F}_q)$ of pure dimension d. The fact that $\mathbf{L}(Y_W, d)$ holds is given by [17, Theorem 9.5]. Let $n > d$. It follows from the proofs of [17, Theorems 9.4 and 9.5] that $H^i(Y, \mathbb{Q}(n)) = 0$ for any $i < 2n$. Moreover, $H^i(Y, \mathbb{Q}(n)) = 0$ for any $i \geq 2n > n + d$. The last claim of the proposition then follows from (84) since $H^i(Y_W, \mathbb{Z}(n))$ is known to be finitely generated for any $i \in \mathbb{Z}$ by [17, Theorem 9.5]. □

5.3. *The class $\mathcal{L}(\mathbb{Z})$*

In this section we follow Geisser's notation (see [19, Section 2]) for the cycle complex $\mathbb{Z}^c(n)$. If $\mathcal{X}$ is a d-dimensional connected scheme which is proper over $\mathbb{Z}$, then we have

$$\mathbb{Z}^c(n) = \mathbb{Z}(d-n)[2d]. \tag{86}$$

Definition 5.8

Let $\mathcal{X}$ be a separated scheme of finite type over $\text{Spec}(\mathbb{Z})$. We say that $\mathcal{X}$ satisfies $\mathbf{L}^c(\mathcal{X}_{\text{ét}})$ if one has the following:

- $H^i(\mathcal{X}_{\text{ét}}, \mathbb{Z}^c(0))$ is finitely generated for any $i \leq 0$;
- $H^i(\mathcal{X}_{\text{ét}}, \mathbb{Z}^c(n))$ is finitely generated for any $i \in \mathbb{Z}$ and any $n < 0$.

Note that if $\mathcal{X}$ is a regular connected scheme of dimension d which is proper over $\mathbb{Z}$, then $\mathbf{L}^c(\mathcal{X}_{\text{ét}}) \Rightarrow \mathbf{L}(\mathcal{X}_{\text{ét}}, d)$ (see (86) above). We define below a class of (simple) arithmetic schemes satisfying $\mathbf{L}^c(\mathcal{X}_{\text{ét}})$. Let $\text{SFT}(\mathbb{Z})$ be the category of separated schemes of finite type over $\text{Spec}(\mathbb{Z})$.

Definition 5.9

We denote by $\mathcal{L}(\mathbb{Z})$ the class of schemes of $\mathrm{SFT}(\mathbb{Z})$ generated by the following objects:

- the empty scheme $\emptyset$;
- varieties $Y \in A(\mathbb{F}_q)$ for any finite field $\mathbb{F}_q$;
- spectra of number rings $\mathrm{Spec}(\mathcal{O}_F)$;

and the following operations:

($\mathcal{L}0$) Let $Z \hookrightarrow X$ be a closed immersion with open complement U such that Z is regular and proper. If two objects of (Z, X, U) belong to $\mathcal{L}(\mathbb{Z})$, then so does the third.

($\mathcal{L}1$) Let $Z \hookrightarrow X$ be a closed immersion with open complement $U \in \mathcal{L}(\mathbb{Z})$. Then $X \in \mathcal{L}(\mathbb{Z})$ if and only if $Z \in \mathcal{L}(\mathbb{Z})$.

($\mathcal{L}2$) We have $X_i \in \mathcal{L}(\mathbb{Z})$ for $0 \leq i \leq p$ if and only if $\coprod_{0 \leq i \leq p} X_i \in \mathcal{L}(\mathbb{Z})$.

($\mathcal{L}3$) If $V \to U$ is an affine bundle and U belongs to $\mathcal{L}(\mathbb{Z})$, then so does V.

($\mathcal{L}4$) Let $\{U_i \to X, i \in I\}$ be a finite surjective family of étale morphisms. If $U_{i_0,\dots,i_p}$ belongs to $\mathcal{L}(\mathbb{Z})$ for any $(i_0, \dots, i_p) \in I^{p+1}$ and any $p \geq 0$, then so does X.

In the statement of ($\mathcal{L}4$), we write $U_{i_0,\dots,i_p} := U_{i_0} \times_X \cdots \times_X U_{i_p}$ as usual. In practice, we use ($\mathcal{L}4$) for a finite étale Galois cover $U \to X$, in which case it is enough to check that $U \in \mathcal{L}(\mathbb{Z})$.

PROPOSITION 5.10

Any object $\mathcal{X}$ in the class $\mathcal{L}(\mathbb{Z})$ satisfies $\mathbf{L}^c(\mathcal{X}_{\text{ét}})$.

We say that the property $\mathbf{L}^c$ is stable under operation ($\mathcal{L}i$) if any scheme $X \in \mathrm{SFT}(\mathbb{Z})$ constructed out of schemes X_α satisfying $\mathbf{L}^c(X_{\alpha,\text{ét}})$ by operation ($\mathcal{L}i$) also satisfies $\mathbf{L}^c(X_{\text{ét}})$.

Proof

By Theorem 5.1 (see also (86)), any number ring $\mathrm{Spec}(\mathcal{O}_F)$ satisfies $\mathbf{L}^c(\mathrm{Spec}(\mathcal{O}_F)_{\text{ét}})$. By Proposition 5.7, any variety $Y \in A(\mathbb{F}_q)$ satisfies $\mathbf{L}^c(Y_{\text{ét}})$. It remains to check that the property $\mathbf{L}^c$ is stable under operations ($\mathcal{L}i$) for $i = 0, \dots, 4$. By purity (see [19, Corollary 7.2]), we have a long exact sequence

$$\begin{aligned} \cdots \to H^i\big(Z_{\text{ét}}, \mathbb{Z}^c(n)\big) \to H^i\big(X_{\text{ét}}, \mathbb{Z}^c(n)\big) \to H^i\big(U_{\text{ét}}, \mathbb{Z}^c(n)\big) \\ \to H^{i+1}\big(Z_{\text{ét}}, \mathbb{Z}^c(n)\big) \to \cdots \end{aligned}$$

for any open-closed decomposition $U \hookrightarrow X \hookleftarrow Z$ and any $n \leq 0$. Moreover, if Z is regular proper, then $H^1(Z_{\text{ét}}, \mathbb{Z}^c(0))$ is finitely generated. Indeed, $H^1(Z_{\text{ét}}, \mathbb{Z}^c(0)) = 0$

if $Z(\mathbb{R}) = \emptyset$ and $H^1(Z_{\text{ét}}, \mathbb{Z}^c(0))$ is a finite-dimensional $\mathbb{Z}/2\mathbb{Z}$-vector space otherwise (see Lemma 2.3). It follows that the property $\mathbf{L}^c$ is stable under operations $(\mathcal{L}0)$ and $(\mathcal{L}1)$.

The fact that $\mathbf{L}^c$ is stable under operation $(\mathcal{L}2)$ is obvious since cohomology respects finite direct sums. It is stable under operation $(\mathcal{L}3)$ by Lemma 5.11.

Let $\{U_i \to X, i \in I\}$ be a finite étale covering family. We write $X_p = \coprod_{(i_0,\dots,i_p)\in I^{p+1}} U_{i_0,\dots,i_p}$ for $p \geq 0$. The Cartan–Leray spectral sequence

$$E_1^{p,q} = H^q\big(X_{p,\text{ét}}, \mathbb{Z}^c(n)\big) \Longrightarrow H^{p+q}\big(X_{\text{ét}}, \mathbb{Z}^c(n)\big) \tag{87}$$

converges by Lemma 5.12. Indeed, Lemma 5.12 implies that $H^q(X_{p,\text{ét}}, \mathbb{Z}^c(n))$ is a $\mathbb{Q}$-vector space for $q < -2 \cdot \dim(X)$, which must be trivial since $H^q(X_{p,\text{ét}}, \mathbb{Z}^c(n))$ is assumed to be a finitely generated abelian group. The spectral sequence (87) then shows that $\mathbf{L}^c$ is stable under operation $(\mathcal{L}4)$. □

LEMMA 5.11
Let X be separated of finite type over Spec($\mathbb{Z}$)*, and let $f : \mathbb{A}^r_{X,\text{ét}} \to X_{\text{ét}}$ be the natural map. Then one has $Rf_*\mathbb{Z}^c(n) \simeq \mathbb{Z}^c(n-r)[2r]$ for any $n \leq 0$.*

Proof
Since $\mathbb{Z}^c(n)$ satisfies étale cohomological descent for $n \leq 0$ (see [19, Theorem 7.1]), one is reduced to showing the analogous statement for the Zariski topology. By using [16, Corollary 3.4], the result follows from the homotopy formula $Rp_*\mathbb{Z}^c(n) \simeq \mathbb{Z}^c(n-1)[2]$, where $p : \mathbb{A}^1_{Y,\text{Zar}} \to Y_{\text{Zar}}$ is the natural map and Y is defined over a field (see [16, Corollary 3.5]). □

One defines $\mathbb{Z}/m\mathbb{Z}^c(n) = \mathbb{Z}^c(n) \otimes \mathbb{Z}/m\mathbb{Z} \simeq \mathbb{Z}^c(n) \otimes^L \mathbb{Z}/m\mathbb{Z}$ (since $\mathbb{Z}^c(n)$ is a complex of flat sheaves) and $\mathbb{Q}/\mathbb{Z}^c(n) = \varinjlim \mathbb{Z}/m\mathbb{Z}^c(n)$. Note that in the situation of Lemma 5.11 one has

$$Rf_*\mathbb{Q}/\mathbb{Z}^c(n) \simeq \mathbb{Q}/\mathbb{Z}^c(n-r)[2r]. \tag{88}$$

Indeed, applying Lemma 5.11 and the functor $- \otimes^L \mathbb{Z}/m\mathbb{Z}$ and passing to the limit, we obtain (88). Here we use the fact that Rf_* commutes with filtered inductive limits (since X is separated of finite type over Spec($\mathbb{Z}$)).

LEMMA 5.12
Let X be separated of finite type over Spec($\mathbb{Z}$)*, and let $n \leq 0$. Then*

$$H^i\big(X_{\text{ét}}, \mathbb{Q}/\mathbb{Z}^c(n)\big) = 0$$

for $i < -2 \cdot \dim(X)$.

Proof

Replacing X with $\mathbb{A}_X^{-n}$, we may assume $n = 0$ by (88). There exists a finite étale covering family $\{V_i \to \mathrm{Spec}(\mathbb{Z})\}$ such that $V_i(\mathbb{R}) = \emptyset$, and hence there exists a finite étale covering family $\{U_i \to X\}$ such that U_i is defined over $\mathrm{Spec}(\mathcal{O}_{K_i})$ for some totally imaginary number field K_i. If the result is known for the $U_{i_0,\dots,i_p}$'s, then it follows for X by the spectral sequence

$$E_1^{p,q} = H^q\big(X_{p,\text{ét}}, \mathbb{Q}/\mathbb{Z}^c(n)\big) \Longrightarrow H^{p+q}\big(X_{\text{ét}}, \mathbb{Q}/\mathbb{Z}^c(n)\big),$$

where $X_p = \coprod_{(i_0,\dots,i_p)\in I^{p+1}} U_{i_0,\dots,i_p}$ and $U_{i_0,\dots,i_p} = U_{i_0} \times_X \cdots \times_X U_{i_p}$ is of dimension less than or equal to $\dim(X)$. So we may assume that X is defined over $\mathrm{Spec}(\mathcal{O}_K)$ for some totally imaginary number field K. We have an isomorphism of finite groups (see [19, Theorem 7.8])

$$H^{1-i}\big(X_{\text{ét}}, \mathbb{Z}/m\mathbb{Z}^c(0)\big) = H_c^i(X_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})^D, \tag{89}$$

where the right-hand side is the Pontryagin dual of $H_c^i(X_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})$. (Here $H_c^i(X_{\text{ét}}, -)$ denotes usual the étale cohomology with compact support.) Take a Nagata compactification of X over $\mathrm{Spec}(\mathcal{O}_K)$, that is, an open immersion $j : X \hookrightarrow X'$ with dense image such that X' is proper over $\mathrm{Spec}(\mathcal{O}_K)$. Note that $X'(\mathbb{R}) = \emptyset$; hence X' is of l-cohomological dimension $2 \cdot \dim(X) + 1$ for any prime number l (see [23, Exposé X, Théorème 6.2]). We obtain

$$H_c^i(X_{\text{ét}}, \mathbb{Z}/m\mathbb{Z}) = H^i(X'_{\text{ét}}, j_!\mathbb{Z}/m\mathbb{Z}) = 0 \quad \text{for } i > 2\cdot\dim(X) + 1;$$

hence $H^i(X_{\text{ét}}, \mathbb{Z}/m\mathbb{Z}^c(0)) = 0$ for $i < -2\cdot\dim(X)$ by (89). Now the result follows from

$$H^i\big(X_{\text{ét}}, \mathbb{Q}/\mathbb{Z}^c(0)\big) = \varinjlim H^i\big(X_{\text{ét}}, \mathbb{Z}/m\mathbb{Z}^c(0)\big),$$

which is valid since the étale site of X is Noetherian. (We have that X is separated of finite type.) □

The class $\mathcal{L}(\mathbb{Z})$ contains singular schemes. For example, it is easy to see that any proper curve (possibly singular) over a finite field lies in $\mathcal{L}(\mathbb{Z})$.

5.4. *Geometrically cellular schemes*

Definition 5.13

Let k be a field, and let Y be a scheme separated and of finite type over k. We say that the k-scheme Y has a *cellular decomposition* if there exists a filtration of Y by reduced closed subschemes

$$Y^{\mathrm{red}} = Y_N \supseteq Y_{N-1} \supseteq \cdots \supseteq Y_{-1} = \emptyset \tag{90}$$

such that $Y_i \setminus Y_{i-1} \simeq \mathbb{A}_k^{a_i}$ is k-isomorphic to an affine space over k. We say that (90) is a *smooth cellular decomposition* if Y_i is moreover smooth over k for any $i \geq 0$.

The k-scheme Y is *geometrically cellular* if $Y \otimes_k \bar{k}$ has a cellular decomposition, where $\bar{k}$ is a separable closure of k.

An S-scheme $\mathcal{X} \to S$ separated and of finite type is *geometrically cellular* if the fiber $\mathcal{X}_s$ is geometrically cellular for any $s \in S$.

One can easily show that a k-scheme Y is geometrically cellular if and only if there exists a finite Galois extension k'/k such that $Y \otimes_k k'$ is cellular. It follows from the proof of Proposition 5.14 below that any geometrically cellular scheme over a finite field belongs to $\mathcal{L}(\mathbb{Z})$. More generally, any geometrically cellular scheme over a number ring belongs to $\mathcal{L}(\mathbb{Z})$.

PROPOSITION 5.14
Let $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_F)$ be flat, separated, and of finite type over a number ring $\mathcal{O}_F$ such that $\mathcal{X}_F$ is geometrically cellular. The following conditions are equivalent.

- *For any finite prime $\mathfrak{p}$ of F, $\mathcal{X}_{\mathfrak{p}} \in \mathcal{L}(\mathbb{Z})$.*
- $\mathcal{X} \in \mathcal{L}(\mathbb{Z})$.

Here we set $\mathcal{X}_F := \mathcal{X} \otimes_{\mathcal{O}_F} F$ and $\mathcal{X}_{\mathfrak{p}} := \mathcal{X} \otimes_{\mathcal{O}_F} (\mathcal{O}_F/\mathfrak{p})$.

Proof
By assumption there exists a finite Galois extension K/F such that $\mathcal{X}_K$ is cellular. We write

$$(\mathcal{X} \otimes_{\mathcal{O}_F} K)^{\mathrm{red}} = Y_N \supseteq Y_{N-1} \supseteq \cdots \supseteq Y_{-1} = \emptyset$$

such that $Y_i \setminus Y_{i-1} \simeq \mathbb{A}_K^{a_i}$ and we consider the closure $\overline{Y_i}$ of Y_i in $\mathcal{X} \otimes_{\mathcal{O}_F} \mathcal{O}_K$, where $\overline{Y_i}$ is endowed with its reduced closed subscheme structure. We obtain an isomorphism

$$(\overline{Y_i} \setminus \overline{Y_{i-1}}) \otimes_{\mathcal{O}_K} K \simeq Y_i \setminus Y_{i-1} \simeq \mathbb{A}_K^{a_i}$$

and it follows that there exist an open $\mathrm{Spec}(\mathcal{O}_{K,S_i}) \subseteq \mathrm{Spec}(\mathcal{O}_K)$ and an isomorphism over $\mathrm{Spec}(\mathcal{O}_{K,S_i})$

$$(\overline{Y_i} \setminus \overline{Y_{i-1}}) \otimes_{\mathcal{O}_K} \mathcal{O}_{K,S_i} \simeq \mathbb{A}^{r_i}_{\mathcal{O}_{K,S_i}}.$$

Now we take a finite set $S \supseteq \bigcup S_i$ big enough so that $\mathrm{Spec}(\mathcal{O}_{K,S}) \to \mathrm{Spec}(\mathcal{O}_{F,S})$ is an étale Galois cover of group G. Here S also denotes its image in $\mathrm{Spec}(\mathcal{O}_F)$. Then we set $\mathcal{Y}_i = \overline{Y_i} \otimes_{\mathcal{O}_K} \mathcal{O}_{K,S}$ and we obtain a filtration by (reduced) closed subschemes

$$(\mathcal{X} \otimes_{\mathcal{O}_F} \mathcal{O}_{K,S})^{\mathrm{red}} = \mathcal{Y}_N \supseteq \mathcal{Y}_{N-1} \supseteq \cdots \supseteq \mathcal{Y}_{-1} = \emptyset$$

such that $\mathcal{Y}_i \setminus \mathcal{Y}_{i-1} \simeq \mathbb{A}^{r_i}_{\mathcal{O}_{K,S}}$. But $\mathrm{Spec}(\mathcal{O}_{K,S})$ belongs to $\mathcal{L}(\mathbb{Z})$ by ($\mathcal{L}0$), hence so does $\mathbb{A}^{r_i}_{\mathcal{O}_{K,S}}$ by ($\mathcal{L}3$), and hence so does $\mathcal{X} \otimes_{\mathcal{O}_F} \mathcal{O}_{K,S}$ by induction and ($\mathcal{L}1$). Notice that a scheme $X \in \mathrm{SFT}(\mathbb{Z})$ belongs to $\mathcal{L}(\mathbb{Z})$ if and only if X^{red} does: this follows from ($\mathcal{L}1$) since $\emptyset \in \mathcal{L}(\mathbb{Z})$.

Moreover,

$$\mathcal{X} \otimes_{\mathcal{O}_F} \mathcal{O}_{K,S} \longrightarrow \mathcal{X} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,S}$$

is a finite étale Galois cover; hence $\mathcal{X} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,S}$ lies in $\mathcal{L}(\mathbb{Z})$ by ($\mathcal{L}4$). Therefore, we have

$$\forall \mathfrak{p} \in S, \mathcal{X}_{\mathfrak{p}} \in \mathcal{L}(\mathbb{Z}) \iff \mathcal{X}_S = \coprod_{\mathfrak{p} \in S} \mathcal{X}_{\mathfrak{p}} \in \mathcal{L}(\mathbb{Z}) \iff \mathcal{X} \in \mathcal{L}(\mathbb{Z})$$

by ($\mathcal{L}2$) and ($\mathcal{L}1$). One may enlarge the finite set S so as to include any given $\mathfrak{p}$. ☐

In order to obtain a more functorial formulation of Conjecture 1.4 (which implies $\mathbf{B}(\mathcal{X}, d)$) we use results and notations of [24]. We denote by $\widehat{H}^n(\mathcal{X}, \mathbb{R}(p))$ the Arakelov motivic cohomology with real coefficients in the sense of [24, Remark 4.7]. These groups are defined following the construction of [24] using the spectrum $H_{\mathrm{B}} \otimes \mathbb{R}$ instead of H_{B} so that there is an exact sequence (see [24, Theorem 4.5(ii)])

$$\cdots \to \widehat{H}^n(\mathcal{X}, \mathbb{R}(p)) \to H^n(\mathcal{X}, \mathbb{Q}(p))_{\mathbb{R}} \to H^n_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(p)) \to \cdots$$

at least for $\mathcal{X}$ a local complete intersection (l.c.i.) scheme over a number ring in the sense of [24, Definition 2.3], where $X = \mathcal{X}_{\mathbb{Q}}$. Here we use the identification $H^n(\mathcal{X}, \mathbb{Q}(p))_{\mathbb{R}} \simeq K_{2p-n}(\mathcal{X})^{(p)}_{\mathbb{R}}$ for $\mathcal{X}$ regular (see [30, Theorem 11.7]). Let $\mathcal{X}$ be an l.c.i. scheme over a number ring. If $\mathcal{X}$ is moreover proper, regular, connected, and d-dimensional, then Conjecture 1.4 for $\mathcal{X}$ is equivalent to

$$\widehat{H}^n(\mathcal{X}, \mathbb{R}(d)) = 0 \quad \text{for } n \neq 2d \qquad \text{and} \qquad \widehat{H}^{2d}(\mathcal{X}, \mathbb{R}(d)) = \mathbb{R}. \tag{91}$$

If the (proper, regular, connected, and d-dimensional) scheme $\mathcal{X}$ lies over a finite field, we say that $\mathcal{X}$ satisfies Conjecture 1.4 if (91) holds, that is, if one has

$$H^n(\mathcal{X}, \mathbb{Q}(d)) = 0 \quad \text{for } n \neq 2d \qquad \text{and} \qquad H^{2d}(\mathcal{X}, \mathbb{Q}(d)) = \mathbb{Q}.$$

Note that for $\mathcal{X}$ (proper, regular, connected, and d-dimensional) over a finite field one has that

$$\mathbf{L}(\mathcal{X}_{\text{ét}}, d) \Rightarrow \text{Conjecture 1.4 for } \mathcal{X}. \tag{92}$$

PROPOSITION 5.15
Let $\mathcal{X}$ be a smooth and projective scheme over the number ring $\mathcal{O}_{\mathcal{X}}(\mathcal{X}) = \mathcal{O}_F$. Assume that $\mathcal{X} \in \mathcal{L}(\mathbb{Z})$, and assume that $\mathcal{X}_F$ admits a smooth cellular decomposition. Then $\mathcal{X}$ satisfies Conjecture 1.4. In particular, $\mathbf{B}(\mathcal{X}, d)$ *holds, where* $d = \dim(\mathcal{X})$.

Proof
By assumption there exist a filtration

$$\mathcal{X}_F = Y_N \supsetneq Y_{N-1} \supsetneq \cdots \supsetneq Y_0 \supsetneq Y_{-1} = \emptyset$$

by smooth closed subschemes Y_i and isomorphisms $Y_i \setminus Y_{i-1} \simeq \mathbb{A}_F^{a_i}$. As in the proof of Proposition 5.14, one can show that there exist an open subscheme $U \subset \mathrm{Spec}(\mathcal{O}_F)$, a filtration

$$\mathcal{X}_U = \mathcal{Y}_N \supsetneq \mathcal{Y}_{N-1} \supsetneq \cdots \supsetneq \mathcal{Y}_0 \supsetneq \mathcal{Y}_{-1} = \emptyset,$$

and U-isomorphisms $\mathcal{Y}_i \setminus \mathcal{Y}_{i-1} \simeq \mathbb{A}_U^{a_i}$ such that the following hold: $\mathcal{Y}_i \otimes_U F \simeq Y_i$, the scheme $\mathcal{Y}_i$ is a closed subscheme of $\mathcal{Y}_{i+1}$, and $\mathcal{Y}_i$ is smooth over U.

By [36, Corollary 5.3.17], $\mathcal{X}_F$ is smooth and connected, and hence irreducible. This gives $CH^0(\mathcal{X}_F) = \mathbb{Z}$. On the other hand, there is a direct sum decomposition $h(\mathcal{X}_F) = \bigoplus_{0 \le i \le N} h(F)(-a_i)$ in the category of Chow motives (see [4, Theorem 3.1]). It follows that there exists a unique index $0 \le i_0 \le N$ such that $a_{i_0} = 0$. But $\mathcal{X}_F$ is proper (over F), hence so is $Y_0 \simeq \mathbb{A}_F^{a_0}$, and hence $i_0 = 0$.

The (absolute) dimension yields a locally constant map $d_i : \mathcal{Y}_i \to \mathbb{Z}$. We define $c_i : \mathcal{Y}_i \to \mathbb{Z}$ similarly. Let $y \in \mathcal{Y}_i$, and let $\mathcal{Y}_{i,y}$ (resp., $\mathcal{Y}_{i+1,y}$) be the connected component of $\mathcal{Y}_i$ (resp., of $\mathcal{Y}_{i+1}$) containing y. Then we define $c_i(y) = \mathrm{codim}(\mathcal{Y}_{i,y}, \mathcal{Y}_{i+1,y})$ and

$$\widehat{H}^{n-2c_i}\big(\mathcal{Y}_i, \mathbb{R}(d_i)\big) := \bigoplus_{y \in \pi_0(\mathcal{Y}_i)} \widehat{H}^{n-2c_i(y)}\big(\mathcal{Y}_{i,y}, \mathbb{R}\big(d_i(y)\big)\big).$$

For any $0 \le i \le N$ we have a long exact sequence (see [24, Theorem 4.16(iii)])

$$\cdots \to \widehat{H}^{n-2c_{i-1}}\big(\mathcal{Y}_{i-1}, \mathbb{R}(d_{i-1})\big) \to \widehat{H}^n\big(\mathcal{Y}_i, \mathbb{R}(d_i)\big) \to \widehat{H}^n\big(\mathcal{Y}_i \setminus \mathcal{Y}_{i-1}, \mathbb{R}(d_i)\big) \to \cdots.$$

Notice that the locally constant function d_i is constant on $\mathcal{Y}_i \setminus \mathcal{Y}_{i-1} \simeq \mathbb{A}_U^{a_i}$ with constant value $a_i + 1$. But for $i \ge 1$, one has $a_i > 0$ and

$$\widehat{H}^n\big(\mathcal{Y}_i \setminus \mathcal{Y}_{i-1}, \mathbb{R}(d_i)\big) \simeq \widehat{H}^n\big(\mathbb{A}_U^{a_i}, \mathbb{R}(a_i+1)\big) = \widehat{H}^n\big(U, \mathbb{R}(a_i+1)\big) = 0 \quad \text{for all } n.$$

Indeed, the second equality is given by [24, Theorem 4.16(ii)] and the vanishing of $\widehat{H}^n(U, \mathbb{R}(a_i+1))$ for $a_i > 0$ follows from the fact that U is of the form $\mathrm{Spec}(\mathcal{O}_{F,S})$ since the Beilinson regulator

$$H^n\big(\mathrm{Spec}(\mathcal{O}_{F,S}),\mathbb{Q}(a_i+1)\big)_{\mathbb{R}} \xrightarrow{\sim} H^n\big(\mathrm{Spec}(F),\mathbb{Q}(a_i+1)\big)_{\mathbb{R}}$$
$$\xrightarrow{\sim} H^n_{\mathcal{D}}\big(\mathrm{Spec}(F)_{/\mathbb{R}},\mathbb{R}(a_i+1)\big)$$

is an isomorphism for $a_i > 0$ and all n. We obtain an identity

$$\widehat{H}^{n-2c_{i-1}}\big(\mathcal{Y}_{i-1},\mathbb{R}(d_{i-1})\big) \simeq \widehat{H}^n\big(\mathcal{Y}_i,\mathbb{R}(d_i)\big) \quad \text{for all } n.$$

Notice that $\mathcal{Y}_0 \simeq V$, and notice that d_0 and $\sum_{i=0}^{i=N-1} c_i$ are both constant on $\mathcal{Y}_0$ with constant values 1 and $d-1$, respectively. An induction on i yields

$$\widehat{H}^n\big(\mathcal{X}_U,\mathbb{R}(d)\big) \simeq \widehat{H}^{n-2d+2}\big(\mathcal{Y}_0,\mathbb{R}(1)\big) \simeq \widehat{H}^{n-2d+2}\big(U,\mathbb{R}(1)\big) \quad \text{for all } n. \tag{93}$$

Using (93), the fact that U is of the form $\mathrm{Spec}(\mathcal{O}_{F,S})$ (with $S \neq \emptyset$), and well-known facts concerning the Dirichlet regulator, we obtain $\widehat{H}^n(\mathcal{X}_U,\mathbb{R}(d)) = 0$ for $n \neq 2d-1$ and an exact sequence

$$0 \to \widehat{H}^{2d-1}\big(\mathcal{X}_U,\mathbb{R}(d)\big) \to \prod_{\mathfrak{p}\in S}\mathbb{R} \to \mathbb{R} \to 0,$$

where S denotes the closed complement of U in $\mathrm{Spec}(\mathcal{O}_F)$. For any finite prime $\mathfrak{p}$ of F, $\mathcal{X}_{\mathfrak{p}}$ is (geometrically) connected (see [36, Corollary 5.3.17]) of dimension $d-1$. By Propositions 5.14 and 5.10, $\mathcal{X}_{\mathfrak{p}}$ satisfies $\mathbf{L}(\mathcal{X}_{\mathfrak{p},\text{ét}}, d-1)$; hence $\mathcal{X}_{\mathfrak{p}}$ satisfies Conjecture 1.4 by (92). We set $\mathcal{X}_S := \coprod_S \mathcal{X}_{\mathfrak{p}}$. The fact that $\widehat{H}^{n-2}(\mathcal{X}_S,\mathbb{R}(d-1)) = 0$ for $n \neq 2d$ and the long exact sequence

$$\cdots \to \widehat{H}^{n-2}\big(\mathcal{X}_S,\mathbb{R}(d-1)\big) \to \widehat{H}^n\big(\mathcal{X},\mathbb{R}(d)\big) \to \widehat{H}^n\big(\mathcal{X}_U,\mathbb{R}(d)\big) \to \cdots$$

then give $\widehat{H}^n(\mathcal{X},\mathbb{R}(d)) = 0$ for $n \neq 2d, 2d-1$. The result follows from the fact that there is a morphism of exact sequences

$$\begin{array}{ccccccccccc}
0 & \to & \widehat{H}^{2d-1}(\mathcal{X},\mathbb{R}(d)) & \to & \widehat{H}^{2d-1}(\mathcal{X}_U,\mathbb{R}(d)) & \to & \widehat{H}^{2d-2}(\mathcal{X}_S,\mathbb{R}(d-1)) & \to & \widehat{H}^{2d}(\mathcal{X},\mathbb{R}(d)) & \to & 0 \\
 & & \downarrow & & \downarrow{=} & & \downarrow{\simeq} & & \downarrow & & \\
0 & \to & 0 & \to & \widehat{H}^{2d-1}(\mathcal{X}_U,\mathbb{R}(d)) & \to & \prod_{\mathfrak{p}\in S}\mathbb{R} & \to & \mathbb{R} & \to & 0
\end{array}$$

□

Acknowledgments. I am very grateful to C. Deninger for comments and advice and to M. Flach for many discussions related to Weil-étale cohomology. I would also like to thank T. Chinburg, T. Geisser, and S. Lichtenbaum for comments and discussions. This work is based on ideas of S. Lichtenbaum. Finally, I would like thank the referees for many constructive suggestions and comments.

References

[1] D. BENOIS and T. NGUYEN QUANG DO, *Les nombres de Tamagawa locaux et la conjecture de Bloch et Kato pour les motifs* $\mathbb{Q}(m)$ *sur un corps abélien*, Ann. Sci. Éc. Norm. Supér. (4) **35** (2002), 641–672. MR 1951439. DOI 10.1016/S0012-9593(02)01104-7. *(1317)*

[2] S. BLOCH, "Algebraic cycles and the Beilinson conjectures" in *The Lefschetz Centennial Conference, Part I (Mexico City, 1984)*, Contemp. Math. **58**, Amer. Math. Soc., Providence, 1986, 65–79. MR 0860404. DOI 10.1090/conm/058.1/860404. *(1269, 1305)*

[3] A. BOREL, *Stable real cohomology of arithmetic groups*, Ann. Sci. Éc. Norm. Supér. (4) **7** (1974), 235–272. MR 0387496. *(1319, 1321, 1323)*

[4] P. BROSNAN, *On motivic decompositions arising from the method of Białynicki-Birula*, Invent. Math. **161** (2005), 91–111. MR 2178658. DOI 10.1007/s00222-004-0419-7. *(1317, 1331)*

[5] J. I. BURGOS GIL, *Semipurity of tempered Deligne cohomology*, Collect. Math. **59** (2008), 79–102. MR 2384539. DOI 10.1007/BF03191183. *(1302)*

[6] J. I. BURGOS GIL, E. FELIU, and Y. TAKEDA, *On Goncharov's regulator and higher arithmetic chow groups*, Int. Math. Res. Not. IMRN **2011**, no. 1, 40–73. MR 2755482. DOI 10.1093/imrn/rnq066. *(1301)*

[7] D. BURNS, *Perfecting the nearly perfect*, Pure Appl. Math. Q. **4** (2008), 1041–1058. MR 2441692. DOI 10.4310/PAMQ.2008.v4.n4.a3. *(1267)*

[8] D. BURNS and M. FLACH, *On Galois structure invariants associated to Tate motives*, Amer. J. Math. **120** (1998), 1343–1397. MR 1657186. *(1311)*

[9] P. DELIGNE, *La conjecture de Weil. I*, Inst. Hautes Études Sci. Publ. Math. **43** (1974), 273–307. MR 0340258. *(1298, 1307)*

[10] C. DENINGER, "Analogies between analysis on foliated spaces and arithmetic geometry" in *Groups and Analysis*, London Math. Soc. Lecture Note Ser. **354**, Cambridge Univ. Press, Cambridge, 2008, 174–190. MR 2528467. DOI 10.1017/CBO9780511721410.010. *(1268)*

[11] M. FLACH, "The equivariant Tamagawa number conjecture: a survey" with an appendix by C. Greither in *Stark's Conjectures: Recent Work and New Directions*, Contemp. Math. **358**, Amer. Math. Soc., Providence, 2004, 79–125. MR 2088713. DOI 10.1090/conm/358/06537. *(1317)*

[12] ———, *Cohomology of topological groups with applications to the Weil group*, Compos. Math. **144** (2008), 633–656. MR 2422342. DOI 10.1112/S0010437X07003338. *(1264)*

[13] M. FLACH and B. MORIN, *On the Weil-étale topos of regular arithmetic schemes*, Doc. Math. **17** (2012), 313–399. MR 2946826. *(1264, 1265, 1268, 1269, 1276, 1287, 1293, 1294, 1295, 1304, 1308, 1317)*

[14] J.-M. FONTAINE, *Valeurs spéciales des fonctions* L *des motifs*, Astérisque **206** (1992), 205–249, Séminaire Bourbaki 1991/1992, no. 751. MR 1206069. *(1267, 1268, 1306, 1308, 1310)*

[15] J.-M. FONTAINE and B. PERRIN-RIOU, "Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L" in *Motives (Seattle, WA, 1991)*, Proc. Sympos. Pure Math. **55**, Amer. Math. Soc., Providence, 1994, 599–706. MR 1265546. *(1267, 1268, 1306, 1308, 1310)*

[16] T. GEISSER, *Motivic cohomology over Dedekind rings*, Math. Z. **248** (2004), 773–794. MR 2103541. DOI 10.1007/s00209-004-0680-x. *(1267, 1269, 1271, 1285, 1321, 1322, 1323, 1327)*

[17] ———, *Weil-étale cohomology over finite fields*, Math. Ann. **330** (2004), 665–692. MR 2102307. DOI 10.1007/s00208-004-0564-8. *(1266, 1267, 1287, 1324, 1325)*

[18] ———, "Motivic cohomology, K-theory and topological cyclic homology" in *Handbook of K-Theory, I*, Springer, Berlin, 2005, 193–234. MR 2181824. DOI 10.1007/3-540-27855-9_6. *(1269)*

[19] ———, *Duality via cycle complexes*, Ann. of Math. (2) **172** (2010), 1095–1126. MR 2680487. DOI 10.4007/annals.2010.172.1095. *(1265, 1267, 1271, 1273, 1279, 1286, 1318, 1322, 1323, 1325, 1326, 1327, 1328)*

[20] T. GEISSER and M. LEVINE, *The K-theory of fields in characteristic p*, Invent. Math. **139** (2000), 459–493. MR 1738056. DOI 10.1007/s002220050014. *(1267, 1322)*

[21] ———, *The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky*, J. Reine Angew. Math. **530** (2001), 55–103. MR 1807268. DOI 10.1515/crll.2001.006. *(1267)*

[22] A. B. GONCHAROV, *Polylogarithms, regulators, and Arakelov motivic complexes*, J. Amer. Math. Soc. **18** (2005), 1–60. MR 2114816. DOI 10.1090/S0894-0347-04-00472-2. *(1301)*

[23] A. GROTHENDIECK, M. ARTIN, and J. L. VERDIER, *Théorie des topos et cohomologie étale des schémas, I*, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Lecture Notes in Math. **269**, Springer, Berlin, 1972; *II*, Lecture Notes in Math. **270**; *III*, Lecture Notes in Math. **305**, 1973. MR 0354652. MR 0354653. MR 0354654. *(1271, 1293, 1328)*

[24] A. HOLMSTROM and J. SCHOLBACH, *Arakelov motivic cohomology I*, to appear in J. Algebraic Geom., preprint, arXiv:1012.2523v3 [math.NT]. *(1330, 1331)*

[25] B. KAHN, *Équivalences rationnelle et numérique sur certaines variétés de type abélien sur un corps fini*, Ann. Sci. Éc. Norm. Supér. (4) **36** (2003), 977–1002. MR 2032532. DOI 10.1016/j.ansens.2003.02.002. *(1325)*

[26] ———, "Algebraic K-theory, algebraic cycles and arithmetic geometry" in *Handbook of K-Theory, I*, Springer, Berlin, 2005, 351–428. MR 2181827. DOI 10.1007/3-540-27855-9_9. *(1270)*

[27] K. KATO and S. SAITO, "Global class field theory of arithmetic schemes" in *Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory, Part I, II (Boulder, Colo., 1983)*, Contemp. Math. **55**, Amer. Math. Soc., Providence, 1986, 255–331. MR 0862639. DOI 10.1090/conm/055.1/862639. *(1271, 1272, 1288)*

[28] F. F. KNUDSEN and D. MUMFORD, *The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div,"* Math. Scand. **39** (1976), 19–55. MR 0437541. *(1299, 1300, 1304)*

[29] M. KOLSTER and J. W. SANDS, *Annihilation of motivic cohomology groups in cyclic 2-extensions*, Ann. Sci. Math. Québec **32** (2008), 175–187. MR 2562043. *(1320)*

[30] M. LEVINE, *Techniques of localization in the theory of algebraic cycles*, J. Algebraic Geom. **10** (2001), 299–363. MR 1811558. *(1267, 1269, 1285, 1320, 1330)*

[31] ———, *K-theory and motivic cohomology of schemes*, preprint, 1999, http://www.math.uiuc.edu/K-theory/336. *(1269, 1271, 1305, 1321, 1324)*

[32] S. LICHTENBAUM, "Values of zeta-functions, étale cohomology, and algebraic K-theory" in *Algebraic K-theory, II: "Classical" Algebraic K-Theory and Connections with Arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, Lecture Notes in Math. **342**, Springer, Berlin, 1973, 489–501. MR 0406981. *(1320)*

[33] ———, "Values of zeta-functions at nonnegative integers" in *Number Theory (Noordwijkerhout, 1983)*, Lecture Notes in Math. **1068**, Springer, Berlin, 1984, 127–138. MR 0756089. *(1265)*

[34] ———, *The Weil-étale topology on schemes over finite fields*, Compos. Math. **141** (2005), 689–702. MR 2135283. DOI 10.1112/S0010437X04001150. *(1264, 1266, 1290, 1311, 1312, 1313, 1316, 1324)*

[35] ———, *The Weil-étale topology for number rings*, Ann. of Math. (2) **170** (2009), 657–683. MR 2552104. DOI 10.4007/annals.2009.170.657. *(1263, 1264, 1266, 1268, 1292, 1318, 1320)*

[36] Q. LIU, *Algebraic Geometry and Arithmetic Curves*, Oxf. Grad. Texts Math. **6**, Oxford Univ. Press, Oxford, 2002. MR 1917232. *(1306, 1331, 1332)*

[37] J. S. MILNE, *Arithmetic Duality Theorems*, Perspect. Math. **1**, Academic Press, Boston, 1986. MR 0881804. *(1276, 1279, 1322)*

[38] B. MORIN, *On the Weil-étale cohomology of number fields*, Trans. Amer. Math. Soc. **363** (2011), no. 9, 4877–4927. MR 2806695. DOI 10.1090/S0002-9947-2011-05124-X. *(1267, 1292, 1317)*

[39] ———, *The Weil-étale fundamental group of a number field II*, Selecta Math. (N.S.) **17** (2011), 67–137. MR 2765000. DOI 10.1007/s00029-010-0041-z. *(1292)*

[40] J. NEKOVÁŘ, "Beĭlinson's conjectures" in *Motives (Seattle, Wash., 1991)*, Proc. Sympos. Pure Math. **55**, Amer. Math. Soc., Providence, 1994, 537–570. MR 1265544. *(1267)*

[41] J. ROGNES and C. WEIBEL, *Two-primary algebraic K-theory of rings of integers in number fields*, with Appendix A by M. Kolster, J. Amer. Math. Soc. **13** (2000), 1–54. MR 1697095. DOI 10.1090/S0894-0347-99-00317-3. *(1323)*

[42] C. SOULÉ, *K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale*, Invent. Math. **55** (1979), 251–295. MR 0553999. DOI 10.1007/BF01406843. *(1322)*

[43] ———, *Groupes de Chow et K-théorie de variétés sur un corps fini*, Math. Ann. **268** (1984), 317–345. MR 0751733. DOI 10.1007/BF01457062. *(1265, 1324)*

[44] ———, "K-théorie et zéros aux points entiers de fonctions zêta" in *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, PWN, Warsaw, 1984, 437–445. MR 0804699. *(1267, 1305)*

[45] R. G. SWAN, *A new method in fixed point theory*, Comment. Math. Helv. **34** (1960), 1–16. MR 0115176. *(1277)*

[46] J.-L. VERDIER, *Des catégories dérivées des catégories abéliennes*, Astérisque **239**, Soc. Math. France, Paris, 1996. MR 1453167. *(1274)*

[47] V. VOEVODSKY, *On motivic cohomology with* $\mathbf{Z}/l$*-coefficients*, Ann. of Math. (2) **174** (2011), 401–438. MR 2811603. DOI 10.4007/annals.2011.174.1.11. *(1320, 1323)*

[48] C. WEIBEL, "Algebraic K-theory of rings of integers in local and global fields" in *Handbook of* K*-Theory, I*, Springer, Berlin, 139–190. MR 2181823. DOI 10.1007/3-540-27855-9_5. *(1321)*

[49] G. WIESEND, *Class field theory for arithmetic schemes*, Math. Z. **256** (2007), 717–729. MR 2308885. DOI 10.1007/s00209-006-0095-y. *(1272, 1289)*

CNRS, Université Paul Sabatier, Institut de Mathématiques de Toulouse, 31062 Toulouse Cedex 9, France; baptiste.morin@math.univ-toulouse.fr