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STABILITY FOR STATIC WALLS IN FERROMAGNETIC NANOWIRES

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Abstract. The goal of this article is to analyze the time asymptotic stability of one dimensional Bloch walls in ferromagnetic materials. The equation involved in modelling such materials is the Landau-Lifchitz system which is non-linear and parabolic. We demonstrate that the equilibrium states called Bloch walls are asymptotically stable modulo a rotation and a translation transverse to the wall. The linear part of the perturbed equation admits zero as an eigenvalue forbiding a direct proof.

1. **Introduction.**

Over the last decade, the interest for ferromagnetism modelization had grown (see [7]). One of the main goals of these mathematical studies is to understand the behaviour of dynamical structures in ferromagnets [3, 4, 5, 10, 11, 12] to validate models. The obtained results will be exploited to enhance numerical simulations of ferromagnets used by physicists [9] to understand and optimise the magnetic characteristics of ferromagnetic materials. Remembering that the main mean of observation is the microwave resonnance, we understand the importance of studying the stabilty of the magnetization in ferromagnets; this study would validate mathematically and therefore numerically, the use of that mean of observation. Then, one of the key points to understand this stability is to analyse the stability of the microstructures developped by the magnetization: it is to say the walls, separation zones between the domains in which the magnetization is smooth.

No extensive study of wall stability in micromagnetic states has been done yet. The three dimensional structure of these objects is very complex and there are no mathematical description in the three dimensional case and some for the two dimensional one [1, 6].

The three dimensional model is the following : we denote by $u = (u_1, u_2, u_3)$ the magnetic moment defined on $\mathbb{R}_t^+ \times \Omega$ with values in S^2 the unit sphere of $\mathbb{R}^3$, where

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Ω is the ferromagnetic domain. The variations of u are gouverned by the following Landau-Lifschitz equation :

$$\frac{\partial u}{\partial t} = -u \wedge h_{eff}(u) - u \wedge (u \wedge h_{eff}(u))$$

The effective field $h_{eff}(u)$ is given by

$$h(u) = A \, \triangle u + h_d(u),$$

where $A\triangle u$ is the exchange field, and where the demagnetizing field $h_d(u)$ satisfies

$$\begin{cases} \operatorname{rot}(h_d(u)) = 0, \\ \operatorname{div}(h_d(u)) = -\operatorname{div}(u), \\ u = 0 \text{ in } \mathbb{R}^3 \setminus \Omega. \end{cases} \tag{1}$$

This system has solutions for regular finite domain Ω as shown in [5].

In this paper we consider an asymptotic one dimensional model of nanowire obtained and justified by D. Sanchez in [14]. In this case the demagnetizing field writes :

$$h_d(u) = -u_2 e_2 - u_3 e_3 = u_1 \, e_1 - u \tag{2}$$

where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$ and where $u = (u_1, u_2, u_3)$.

Remark 1. *This model is obtained using a BKW method, taking the limit when the diameter of the wire tends to zero (see* [14]*).*

Finally using a space scaling factor to set $A = 1$, for a line along the x-axis we study the following system

$$\begin{cases} u : \mathbb{R}_t^+ \times \mathbb{R}_x \longrightarrow S^2 \\ \\ \dfrac{\partial u}{\partial t} = -u \wedge h(u) - u \wedge (u \wedge h(u)) \\ \\ \text{with } h(u) = \dfrac{\partial^2 u}{\partial x^2} + u_1 \, e_1 \end{cases} \tag{3}$$

Remark 2. *The demagnetizing field $h_d(u)$ given by Formula* (2) *only appears in Landau-Lifschitz Equation in the expression $u \wedge h_d(u) = u \wedge (u_1 e_1 - u) = u \wedge (u_1 e_1)$. It is the reason why we can work with the expression of $h(u)$ given in* (3)*.*

The aim of the paper is to study the stability of a static wall profile which separates the domain in which $u = -e_1$ (in the neighborhood of $-\infty$) and the domain in which $u = e_1$ (in the neighborhood of $+\infty$).

This profile is given by

$$M_0 = \begin{pmatrix} \operatorname{th} x \\ 0 \\ \frac{1}{\operatorname{ch} x} \end{pmatrix}.$$

We remark that Landau-Lifschitz equation (3) is invariant by translation in the variable x and by rotation around e_1. Hence for all $\Lambda = (\theta, \sigma) \in \mathbb{R} \times \mathbb{R}$, the profile

$x \mapsto M_\Lambda(x) = R_\theta(M_0(x-\sigma))$ is a static solution of (3) satisfying $\lim\limits_{-\infty} u = -e_1$ and $\lim\limits_{+\infty} u = e_1$, where we denote by R_θ the rotation of angle θ around e_1:

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

.

Our main result is the following

Theorem 1. *Let $\varepsilon > 0$, there exists $\eta > 0$ such that, for all v_0 in $H^2(\mathbb{R})$ with $|v_0| = 1$ for all x in $\mathbb{R}$ and such that $\|v_0 - M_0\|_{H^2(\mathbb{R})} < \eta$, if we denote by v the solution of (3) with v_0 as the initial data then, for all t in $\mathbb{R}^+$, $\|v(t) - M_0\|_{H^2(\mathbb{R})} < \varepsilon$. Furthermore there exists $\Lambda = (\theta, \sigma) \in \mathbb{R}^2$ such that v tends to M_Λ when t tends to infinity for the norm $H^1(\mathbb{R})$.*

The invariance of (3) by rotation-translation implies that the linearized equation in the neighborhood of M_0 has zero as an eigen-value, wich is a major obstruction to obtain strightly the stability result. In addition all the known results about the stability of travelling waves are proved for semilinear equation (see [8]). Here the considered Landau-Lifschitz equation is quasilinear and we have to combine variational estimates with the methods used in [8].

The paper is organized as follows : in Section 2 we describe the perturbations of M_0 in the mobile frame $(M_0(x), M_1(x), M_2)$, where $M_1(x) = (\frac{1}{\operatorname{ch} x}, 0, -\operatorname{th} x)$ and $M_2 = (0, 1, 0)$, writing

$$u(t,x) = r_1(t,x)M_1(x) + r_2(t,x)M_2 + \sqrt{1 - r_1^2 - r_2^2}\,M_0(x).$$

We obtain then an equivalent formulation of Equation (3) where the unknown is $r = (r_1, r_2)$, of the form:

$$\frac{\partial r}{\partial t} = \mathcal{L}r + F(x, r, \frac{\partial r}{\partial x}, \frac{\partial^2 r}{\partial x^2}) \tag{4}$$

where $\mathcal{L}r$ denotes the linear part.

The stability of M_0 for Equation (3) is then equivalent to the stability of the zero solution for Equation (4).

The two parameters family of static solutions M_Λ for Equation (3) induces in the new coordinates a two parameters family R_Λ of statics solutions for Equation (4). In Section 3, we decompose the solution r of (4) in

$$r(t,x) = R_{\Lambda(t)}(x) + W(x)$$

where $W \in (\mathrm{Ker}\ \mathcal{L})^\perp$. This decomposition is rather classical for the study of static solution stability for semilinear parabolic equations (see [8]). This technique has also been used in [2] to demonstrate the stability of travelling waves in thin films or in [13] in the case of the radially symmetric travelling waves in reaction-diffusion equations.

The main difficulty here is that Equation (4) is quasilinear and then the non linear term F depends also on $\frac{\partial^2 r}{\partial x^2}$. We then use Section 5 variational estimates for the non linear part combined with more classical linear estimates on the operator $\mathcal{L}$ (proved in Section 4).

In the following we denote by $\cdot$ the scalar product in $\mathbb{R}^3$, and by $(\,|\,)$ the scalar product in $L^2(\mathbb{R})$.

2. **Equation for the perturbations of the wall.**

2.1. **Moving frame.** We consider the following moving frame $(M_0(x), M_1(x), M_2)$ with

$$M_0 = \begin{pmatrix} \operatorname{th} x \\ \\ 0 \\ \\ \dfrac{1}{\operatorname{ch} x} \end{pmatrix}, \quad M_1 = \begin{pmatrix} \dfrac{1}{\operatorname{ch} x} \\ \\ 0 \\ \\ -\operatorname{th} x \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 0 \\ \\ 1 \\ \\ 0 \end{pmatrix}$$

We consider u as a little perturbation of M_0 and we write u on the form

$$u(t,x) = r_1(t,x)M_1(x) + r_2(t,x)M_2(x) + \sqrt{1-(r_1(t,x))^2-(r_2(t,x))^2}\,M_0(x). \quad (5)$$

We denote $\lambda = \sqrt{1-r_1^2-r_2^2}$. In order to ensure the regularity of λ, we assume that $\|u-M_0\|_{L^\infty(\mathbb{R})} \leq \dfrac{1}{2}$. This asumption is correct since we study little perturbations of M_0.

We have

- $\dfrac{dM_0}{dx} = \dfrac{1}{\operatorname{ch} x} M_1,$
- $\dfrac{dM_1}{dx} = -\dfrac{1}{\operatorname{ch} x} M_0,$
- $\dfrac{d^2M_0}{dx^2} = -\dfrac{\operatorname{sh} x}{\operatorname{ch}^2 x} M_1 - \dfrac{1}{\operatorname{ch}^2 x} M_0$
- $e_1 = \operatorname{th} x M_0 + \dfrac{1}{\operatorname{ch} x} M_1$
- $h(M_0) = fM_0$ where $f(x) = 2\operatorname{th}^2 x - 1.$

Furthermore

$$h(u) = a_0M_0 + a_1M_1 + a_2M_2$$

with

$$a_0 = \frac{\partial^2\lambda}{\partial x^2} + \lambda f(x) + 2r_1\frac{\operatorname{sh} x}{\operatorname{ch}^2 x} - 2\frac{\partial r_1}{\partial x}\frac{1}{\operatorname{ch} x}$$

$$a_1 = \frac{\partial^2 r_1}{\partial x^2} + 2\frac{1}{\operatorname{ch} x}\frac{\partial\lambda}{\partial x}$$

$$a_2 = \frac{\partial^2 r_2}{\partial x^2}$$

We replace u by its expression (5) in Equation (3), and we obtain that

$$\begin{aligned}\frac{\partial \lambda}{\partial t}M_0+\frac{\partial r_1}{\partial t}M_1+\frac{\partial r_2}{\partial t}M_2=&-(r_1a_2-r_2a_1)M_0-(r_2a_0-\lambda a_2)M_1-(\lambda a_1-r_1a_0)M_2\\&-\lambda(r_2a_0-\lambda a_2)M_2+\lambda(\lambda a_1-r_1a_0)M_1+r_1(r_1a_2-r_2a_1)M_2\\&-r_1(\lambda a_1-r_1a_0)M_0-r_2(r_1a_2-r_2a_1)M_1+r_2(r_2a_0-\lambda a_2)M_0\end{aligned} \tag{6}$$

Projecting Equation (6) in the directions M_1 and M_2 we obtain that if u is solution of (3) then

$$\begin{aligned}\frac{\partial r_1}{\partial t}&=-r_2a_0+\lambda a_2+\lambda(\lambda a_1-r_1a_0)-r_2(r_1a_2-r_2a_1)\\\frac{\partial r_2}{\partial t}&=-(\lambda a_1-r_1a_0)-\lambda(r_2a_0-\lambda a_2)+r_1(r_1a_2-r_2a_1)\end{aligned} \tag{7}$$

Remark 3. *Equation* (7) *is equivalent to Equation* (3). *Indeed we write Equation (3) on the form :*

$$\frac{\partial u}{\partial t}=F(u)$$

where $F(u)(x)$is orthogonal to $u(x)$ for all $x\in\mathbb{R}$.

Equation (7) is the projection of (3) on the directions M_1 and M_2, that is if (r_1,r_2) satisfies Equation (7), then $u=r_1M_1+r_2M_2+\sqrt{1-r_1^2-r_2^2}M_0$ satisfies $(\frac{\partial u}{\partial t}-F(u))\cdot M_1=(\frac{\partial u}{\partial t}-F(u))\cdot M_2=0$.

We remark that $u=r_1M_1+r_2M_2+\sqrt{1-r_1^2-r_2^2}M_0$ is a normed vector field, thus $\frac{\partial u}{\partial t}\cdot u=0$. Furthermore, $u\cdot F(u)=0$. Thus, if (r_1,r_2) satisfies Equation (7), then $(\frac{\partial u}{\partial t}-F(u))\cdot\lambda M_0=0$ and since $\lambda\neq 0$ (since we consider little perturbations of M_0) we obtain that the third composant of $\frac{\partial u}{\partial t}-F(u)$ is zero.

Thus for little perturbations of M_0, Equation (3) is equivalent to (7).

We detail Equation (7) replacing the a_i's by their values. We obtain that Landau-Lifschitz equation is equivalent for little perturbations of M_0 to the following system:

$$\begin{aligned}
\frac{\partial r_1}{\partial t} = \quad & -r_2\frac{\partial^2\lambda}{\partial x^2} - r_2\lambda f(x) - 2r_2r_1\frac{\operatorname{sh} x}{\operatorname{ch}^2 x} + 2r_2\frac{\partial r_1}{\partial x}\frac{1}{\operatorname{ch} x} \\
& +\lambda\frac{\partial^2 r_2}{\partial x^2} - r_1r_2\frac{\partial^2 r_2}{\partial x^2} + \frac{\partial^2 r_1}{\partial x^2} + 2\frac{1}{\operatorname{ch} x}\frac{\partial\lambda}{\partial x} - r_1^2\frac{\partial^2 r_1}{\partial x^2} \\
& -2r_1^2\frac{1}{\operatorname{ch} x}\frac{\partial\lambda}{\partial x} - \lambda r_1\frac{\partial^2\lambda}{\partial x^2} - \lambda^2 r_1 f(x) - 2\lambda r_1^2\frac{\operatorname{sh} x}{\operatorname{ch}^2 x} + 2\lambda r_1\frac{\partial r_1}{\partial x}\frac{1}{\operatorname{ch} x} \\
\frac{\partial r_2}{\partial t} = \quad & -\lambda\frac{\partial^2 r_1}{\partial x^2} - 2\lambda\frac{1}{\operatorname{ch} x}\frac{\partial\lambda}{\partial x} + r_1\frac{\partial^2\lambda}{\partial x^2} + r_1\lambda f(x) + 2r_1^2\frac{\operatorname{sh} x}{\operatorname{ch}^2 x} - 2r_1\frac{\partial r_1}{\partial x}\frac{1}{\operatorname{ch} x} \\
& +\frac{\partial^2 r_2}{\partial x^2} - r_2^2\frac{\partial^2 r_2}{\partial x^2} - r_1r_2\frac{\partial^2 r_1}{\partial x^2} - 2r_1r_2\frac{1}{\operatorname{ch} x}\frac{\partial\lambda}{\partial x} - \lambda r_2\frac{\partial^2\lambda}{\partial x^2} \\
& -\lambda^2 r_2 f(x) - 2\lambda r_1r_2\frac{\operatorname{sh} x}{\operatorname{ch}^2 x} + 2\lambda r_2\frac{\partial r_1}{\partial x}\frac{1}{\operatorname{ch} x}
\end{aligned} \tag{8}$$

We denote $r = (r_1, r_2)$, and we define $\mu : B(0, \frac{1}{2}) \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $\mu(\xi) = \sqrt{1-|\xi|^2} - 1$ (that is $\lambda = 1 + \mu(r)$). We then write Equation (8) on the condensed form detailed in the following proposition:

Proposition 1. *The function $u \in \mathcal{C}^1(\mathbb{R}^+; H^2(\mathbb{R}; S^2))$ such that $\|u - M_0\|_{L^\infty} \leq \frac{1}{2}$ satisfies Landau-Lifschitz equation* (3) *if and only if $u = r_1M_1 + r_2M_2 + \sqrt{1-r_1^2-r_2^2}M_0$ where $r = (r_1, r_2)$ satisfies:*

$$\frac{\partial r}{\partial t} = \mathcal{L}r + G(r)(\frac{\partial^2 r}{\partial x^2}) + H_1(x,r)(\frac{\partial r}{\partial x}) + H_2(r)(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial x}) + P(x,r) \tag{9}$$

with

- *$\mathcal{L} = JL$ with $J = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$ and $L = -\dfrac{\partial^2}{\partial x^2} + f$ (we recall that $f(x) = 2th^2 x - 1$),*
- *$G(r)$ is the matrix defined by:*

$$G(r) = \begin{pmatrix} \dfrac{r_1r_2}{\sqrt{1-r_1^2-r_2^2}} & \dfrac{r_2^2}{\sqrt{1-r_1^2-r_2^2}} + \mu(r) \\ -\mu(r) - \dfrac{r_1^2}{\sqrt{1-r_1^2-r_2^2}} & -\dfrac{r_1r_2}{\sqrt{1-r_1^2-r_2^2}} \end{pmatrix}$$

- *$H_1(x,r)$ is the matrix defined by:*

$$H_1(x,r) = \frac{2}{\sqrt{1-r_1^2-r_2^2}\,ch\,x}\begin{pmatrix} r_2\sqrt{1-r_1^2-r_2^2} - r_1r_2^2 & -r_2 - r_2r_1^2 \\ r_2 - r_2^3 & \sqrt{1-r_1^2-r_2^2}\,r_2 + r_1r_2^2 \end{pmatrix}$$

- $H_2(r) \in \mathcal{L}_2(\mathbb{R}^2)$ *is a symetric bilinear form defined by*

$$H_2(r)(\xi_1,\xi_2) = \begin{pmatrix} \dfrac{\sqrt{1-r_1^2-r_2^2}r_1 + r_2}{\sqrt{1-r_1^2-r_2^2}^3} \\ \\ \dfrac{\sqrt{1-r_1^2-r_2^2}r_2 - r_1}{\sqrt{1-r_1^2-r_2^2}^3} \end{pmatrix} \big((1-r_1^2-r_2^2)(\xi_1\cdot\xi_2) + (r\cdot\xi_1)(r\cdot\xi_2)\big)$$

- P *is defined by*

$$P(x,r) = \begin{pmatrix} -r_2\mu(r)f(x) - 2r_2r_1\dfrac{sh\,x}{ch^2x} + (r_1^2+r_2^2)r_1f(x) - 2\sqrt{1-r_1^2-r_2^2}r_1^2\dfrac{sh\,x}{ch^2x} \\ \\ r_1\mu(r)f(x) + 2r_1^2\dfrac{sh\,x}{ch^2x} + (r_1^2+r_2^2)r_2f(x) - 2\sqrt{1-r_1^2-r_2^2}r_1r_2\dfrac{sh\,x}{ch^2x} \end{pmatrix}$$

The properties concerning G, H_1, H_2 and P are summarized in the following proposition:

Proposition 2.
- $G \in \mathcal{C}^\infty(B(0,1/2);\mathcal{M}^2(\mathbb{R}))$ *and* $G(\xi) = \mathcal{O}(|\xi|^2)$
- $H_1 \in \mathcal{C}^\infty(\mathbb{R}\times B(0,1/2);\mathcal{M}^2(\mathbb{R}))$ *and* $H_1(x,r) = \mathcal{O}(|r|)$
- $H_2 \in \mathcal{C}^\infty(B(0,1/2);\mathcal{L}_2(\mathbb{R}^2))$, *with* $H_2(x,r) = \mathcal{O}(|r|)$
- $P \in \mathcal{C}^\infty(\mathbb{R}\times B(0,1/2);\mathbb{R}^2)$ *with* $P(x,r) = \mathcal{O}(|r|^2)$ *uniformly in* $x\in\mathbb{R}$

3. **A new system of coordinates.** We remark that L is a self adjoint operator on $L^2(\mathbb{R})$, with domain $H^2(\mathbb{R})$. Furthermore, L is positive since we can write $L = l^*\circ l$ with $l = \dfrac{\partial}{\partial x} + \operatorname{th} x$, and Ker L is the one dimensional space generated by $\dfrac{1}{\operatorname{ch} x}$.

The matrix J being invertible, Ker $\mathcal{L}$ is the two dimensional space generated by e_1 and e_2 with

$$e_1(x) = \begin{pmatrix} 0 \\ 1 \\ \dfrac{1}{\operatorname{ch} x} \end{pmatrix}, \quad e_2(x) = \begin{pmatrix} \dfrac{1}{\operatorname{ch} x} \\ 0 \end{pmatrix} \tag{10}$$

We introduce $\mathcal{E} = (\text{Ker } \mathcal{L})^\perp$. We denote by Q the orthogonal projection onto $\mathcal{E}$ for the $L^2(\mathbb{R})$ scalar product.

Landau-Lifschitz equation (3) is invariant by translation in the variable x and by rotation about the axis e_1. This two parameters family of invariance explains the presence of the eigenvalue zero (of multiplicity 2) for the linearized operator $\mathcal{L}$. We will write the solution u as a rotation-translation of M_0 plus a term in $\mathcal{E}$.

For $\Lambda = (\theta,\sigma)$ fixed in $\mathbb{R}^2$ we know that the profile M_0 rotated of the angle θ and translated of σ is a solution of Landau-Lifschitz equation. We denote by M_Λ this solution:

$$M_\Lambda(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} M_0(x-\sigma)$$

and we introduce $R_\Lambda(x)$ the coordinates of $M_\Lambda(x)$ in the basis $(M_1(x), M_2(x))$:

$$R_\Lambda(x) = \begin{pmatrix} M_\Lambda(x)\cdot M_1(x) \\ M_\Lambda(x)\cdot M_2 \end{pmatrix}$$

In a neighborhood of zero (which represents the wall profile M_0 in the frame (M_1, M_2)), we use a coordinate system given by

$$r(x) = R_\Lambda(x) + W(x). \tag{11}$$

with $(\Lambda, W) \in \mathbb{R}^2 \times \mathcal{E}$.

The map $r \mapsto (\Lambda, W)$ is a diffeomorphism from a neighborhood of zero in $H^2(\mathbb{R})$ to a neighborhood of zero in $\mathbb{R}^2 \times \mathcal{E}$. Indeed let $r \in H^2(\mathbb{R})$. In order to use the coordinate system (11), there must exist a unique pair $(\Lambda, W) \in \mathbb{R}^2 \times \mathcal{E}$ such that $r(x) = R_\Lambda(x) + W(x)$.

If $r = R_\Lambda + W$ then taking the scalar product of r with e_1 and e_2, since $W \in \mathcal{E} = (\text{Ker } \mathcal{L})^\perp$ and since (e_1, e_2) defined by (10) is a basis of Ker $\mathcal{L}$, we have

$$(r|e_1) = (R_\Lambda|e_1) \text{ and } (r|e_2) = (R_\Lambda|e_2) \tag{12}$$

Furthermore, if $\Lambda \in \mathbb{R}^2$ satisfies (12) then $W = r - R_\Lambda \in \mathcal{E}$

We define $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$\psi(\Lambda) = \begin{pmatrix} (R_\Lambda|e_1) \\ \\ (R_\Lambda|e_2) \end{pmatrix}$$

Therefore (11) defines a system of coordinates in a neighborhood of 0 if ψ is a local diffeomorphism in a neighborhood of zero. This is the case since ψ is $\mathcal{C}^\infty$ and since $\psi'(0) = Id$.

We compute now the equation of the perturbation in the coordinates (Λ, W).

We write the solution r of Equation (9) on the form :

$$r(t, x) = R_{\Lambda(t)}(x) + W(t, x)$$

where for all t, $W(t) \in \mathcal{E}$ and where $\Lambda : \mathbb{R}_t^+ \mapsto \mathbb{R}^2$.

We will rewrite Equation (9) in the coordinates (Λ, W). The equation on Λ is obtained by taking the scalar product of (9) with e_1 and e_2. The equation on W is obtained using Q the orthogonal projection onto $\mathcal{E}$.

If $\Lambda = (\theta, \sigma)$ is fixed, we know that $x \mapsto R_\Lambda(x)$ satisfies (9) that is we have:

$$\mathcal{L}R_\Lambda + G(R_\Lambda)(\frac{d^2 R_\Lambda}{dx^2}) + H_1(x, R_\Lambda)(\frac{dR_\Lambda}{dx}) + H_2(R_\Lambda)(\frac{dR_\Lambda}{dx}, \frac{dR_\Lambda}{dx}) + P(x, R_\Lambda) = 0$$

In order to isolate the linear part in W we perform the Taylor expansion for G, H_1, H_2 and K, and we have at order 1:

$$G(R_\Lambda + W) = G(R_\Lambda) + \widehat{G}(R_\Lambda, W)(W)$$

with

$$\widehat{G}(v_1, v_2)(\xi) = \int_0^1 G'(v_1 + sv_2)(\xi) ds$$

and at order 2:

$$G(R_\Lambda + W) = G(R_\Lambda) + G'(R_\Lambda)(W) + \widetilde{G}(R_\Lambda, W)(W^{(2)})$$

where

$$\widetilde{G}(v_1, v_2)(\xi^{(2)}) = \int_0^1 (1 - s) G''(v_1 + sv_2)(\xi, \xi) ds$$

We will use the same notations for H_1, H_2 and K.

We have

$$\frac{d\theta}{dt}\partial_\theta R_\Lambda + \frac{d\sigma}{dt}\partial_\sigma R_\Lambda + \frac{\partial W}{\partial t} = \mathcal{L}W + T_1 + \dots T_5 \tag{13}$$

where

$$T_1 = \mathcal{K}^1_\Lambda W := G(R_\Lambda)\frac{\partial^2 W}{\partial x^2}$$

$$T_2 = \mathcal{K}^2_\Lambda W := H_1(x, R_\Lambda)\frac{\partial W}{\partial x} + 2H_2(R_\Lambda)(\frac{dR_\Lambda}{dx}, \frac{\partial W}{\partial x}) + G'(R_\Lambda)(W)\frac{\partial^2 R_\Lambda}{\partial x^2}$$

$$+ H_1'(x, R_\Lambda)(W)\frac{\partial R_\Lambda}{\partial x} + H_2'(R_\Lambda)(W)(\frac{dR_\Lambda}{dx}, \frac{dR_\Lambda}{dx}) + P'(x, R_\Lambda)(W)$$

$$T_3 = \mathcal{R}_1(x, \Lambda, W)(\frac{\partial^2 W}{\partial x^2}) := \widehat{G}(R_\Lambda, W)(W)\frac{\partial^2 W}{\partial x^2}$$

$$T_4 = \mathcal{R}^1_2(x, \Lambda, W, \frac{\partial W}{\partial x}) := H_2(R_\Lambda + W)(\frac{\partial W}{\partial x}, \frac{\partial W}{\partial x}) + \widehat{H_1}(x, R_\Lambda, W)(W)(\frac{\partial W}{\partial x})$$

$$+ 2H_2'(R_\Lambda, W)(W)(\frac{dR_\Lambda}{dx}, \frac{\partial W}{\partial x})$$

$$T_5 = \mathcal{R}_3(x, \Lambda, W) := \widetilde{G}(R_\Lambda, W)(W^{(2)})(\frac{\partial^2 R_\Lambda}{\partial x^2}) + \widetilde{H_1}(x, R_\Lambda, W)(W^{(2)})(\frac{dR_\Lambda}{dx})$$

$$+ \widetilde{H_2}(R_\Lambda, W)(W^{(2)})(\frac{dR_\Lambda}{dx}, \frac{dR_\Lambda}{dx}) + \widetilde{P}(x, R_\Lambda, W)(W^{(2)}) \tag{14}$$

We take the scalar product in $L^2(R)$ of (13) with e_1 and e_2. Since $(e_i|\frac{\partial W}{\partial t}) = (\mathcal{L}W|e_i) = 0$, we obtain that:

$$A(\Lambda)\frac{d\Lambda}{dt} = \sum_{i=1}^{5} T_i' \tag{15}$$

where

$$A(\Lambda) = \begin{pmatrix} (e_1|\partial_\theta R_\Lambda) & (e_1|\partial_\sigma R_\Lambda) \\ (e_2|\partial_\theta R_\Lambda) & (e_2|\partial_\sigma R_\Lambda) \end{pmatrix} \tag{16}$$

and

$$T_i' = \begin{pmatrix} (T_i|e_1) \\ (T_i|e_2) \end{pmatrix}$$

We remark that $A(0) = Id$, thus for Λ little enough, we can inverse the matrix $A(\Lambda)$ and we can write the equation satisfied by Λ on the form:

$$\frac{d\Lambda}{dt} = \mathcal{M}_1(\Lambda)(W) + \mathcal{M}_2(W, \frac{\partial W}{\partial x}, \Lambda) \tag{17}$$

where

$$\begin{aligned} \mathcal{M}_1(\Lambda)(W) &= A(\Lambda)^{-1}(T_1' + T_2') \\ \mathcal{M}_2(W, \frac{\partial W}{\partial x}, \Lambda) &= A(\Lambda)^{-1}(T_3' + T_4' + T_5') \end{aligned} \tag{18}$$

Applying the projection operator Q to (13) yields to the following evolution equation for W:

$$\frac{\partial W}{\partial t} = \mathcal{L}W + Q\mathcal{K}_\Lambda W + Q\mathcal{R}_1(x,\Lambda,W)(\frac{\partial^2 W}{\partial x^2}) + Q\mathcal{R}_2(x,\Lambda,W,\frac{\partial W}{\partial x}) + Q\mathcal{R}_3(x,\Lambda,W) \tag{19}$$

where the linear operator $\mathcal{K}_\Lambda$ is defined by

$$\mathcal{K}_\Lambda W = \mathcal{K}_\Lambda^1 W + \mathcal{K}_\Lambda^2 W + \mathcal{K}_\Lambda^3 W \tag{20}$$

with

$$\mathcal{K}_\Lambda^3 W = -\mathcal{M}_1^1(\Lambda)(W)\partial_\theta R_\Lambda - \mathcal{M}_1^2(\Lambda)(W)\partial_\sigma R_\Lambda \tag{21}$$

and where the nonlinear term $\mathcal{R}_2(x,\Lambda,W,\frac{\partial W}{\partial x})$ is given by:

$$\mathcal{R}_2(x,\Lambda,W,\frac{\partial W}{\partial x}) = \mathcal{R}_2'(x,\Lambda,W,\frac{\partial W}{\partial x}) - \mathcal{M}_2^1(\Lambda,W,\frac{\partial W}{\partial x})\partial_\theta R_\Lambda - \mathcal{M}_2^2(\Lambda,W,\frac{\partial W}{\partial x})\partial_\sigma R_\Lambda \tag{22}$$

Remark 4. *In the projection of Equation* (13) *we have replaced* $\frac{d\theta}{dt}$ *and* $\frac{d\sigma}{dt}$ *by their expressions given by Equation* (17)*. In the previous equations,* $\mathcal{M}_i^1$ *and* $\mathcal{M}_i^2$ *are respectively the first and the second component of* $\mathcal{M}_i$*.*

We have thus proved the following proposition:

Proposition 3. *If* $r:(t,x)\mapsto r(t,x)$ *is small enough for the norm* $L^\infty(\mathbb{R}_t^+;H^2(\mathbb{R}))$*, then we can write* r *on the form*

$$r(t,x) = R_{\Lambda(t)}(x) + W(t,x)$$

with $\Lambda \in L^\infty(\mathbb{R}_t^+;\mathbb{R}^2)$ *and* $W \in L^\infty(\mathbb{R}_t^+;\mathcal{E})$*. This decomposition is unique.*

Furthermore r *is solution for Equation* (9) *if and only if* (Λ, W) *satisfies the system coupling Equation* (19) *and Equation* (17)*.*

4. **Linear Estimates.**

4.1. **Study of the operator** L.

The self-adjoint operator L is a compact perturbation of $-\frac{\partial^2}{\partial x^2}+1$, thus its essential spectrum is $[1,+\infty[$. Furthermore, we can write $L = l^* \circ l$ with $l = -\frac{\partial}{\partial x} + \operatorname{th} x$, thus L is positive and 0 is a simple eigenvalue associated with the eigenvector $\frac{1}{\operatorname{ch} x}$.

We denote $E = (\text{Ker } L)^\perp$. The restriction of L on E is a symmetric definite positive operator. We denote by $\alpha > 0$ its smallest eigenvalue.

Proposition 4. *There exists constants* K_1 *and* K_2 *such that for all* $u \in E$

$$K_1\|L^{\frac12}u\|_{L^2} \le \|u\|_{H^1} \le K_2\|L^{\frac12}u\|_{L^2}$$

$$K_1\|Lu\|_{L^2} \le \|u\|_{H^2} \le K_2\|Lu\|_{L^2}$$

$$K_1\|L^{\frac32}u\|_{L^2} \le \|u\|_{H^3} \le K_2\|L^{\frac32}u\|_{L^2}$$

Proof. Since α is the smaller eigenvalue of L on E, we have:

$$\forall\, u \in H, \|u\|_{L^2} \le \frac{1}{\alpha}\|Lu\|_{L^2}. \tag{23}$$

Furthermore,

$$\|u''\|_{L^2} = \|u'' - fu + fu\|_{L^2} \le \|L(u)\|_{L^2} + \|f\|_{L^\infty}\|u\|_{L^2}$$

thus with the previous inequality, we obtain that there exists a constant K such that for all u in E

$$\|u\|_{H^2} \le K\|Lu\|_{L^2} \tag{24}$$

Since the domination of the L^2 norm of Lu by the H^2 norm of u is obvious, we conclude the proof of the H^2 estimate.

Now we have $L^2u = u^{(4)} - 2fu'' - 2f'u' - f''u + f^2u$ that is

$$\begin{aligned}
\|u^{(4)}\|_{L^2(\mathbb{R})} \le\;& \|L^2u\|_{L^2(\mathbb{R})} + C_1\|u\|_{H^2(\mathbb{R})}\\
& \text{since } f,\ f' \text{ and } f'' \text{ are bounded on } \mathbb{R}\\
\le\;& \|L^2u\|_{L^2(\mathbb{R})} + C_1K\|Lu\|_{L^2(\mathbb{R})}\\
& \text{with Estimate (24)}\\
\le\;& (1 + \frac{C_1K}{\alpha})\|L^2u\|_{L^2(\mathbb{R})}\\
& \text{with Estimate (23) applied on } Lu
\end{aligned}$$

thus we obtain that there exists a constant C_2 such that

$$\|u\|_{H^4(\mathbb{R})} \le C_2\|L^2u\|_{L^2(\mathbb{R})}$$

Since the opposite bound is obvious, we obtain an estimate about the H^4 norm.

By interpolation result, we deduce the intermediate estimates and we conclude the proof of Proposition 4. □

4.2. **Estimates for the perturbed operator $\mathcal{L} + Q\mathcal{K}_\Lambda$.** We recall that $\mathcal{K}_\Lambda$ is defined by (20).

We remark that since $\Lambda \mapsto R_\Lambda$ is regular and since $R_{\Lambda=0} = 0$, there exists a constant C_3 such that

$$\|R_\Lambda\|_{L^\infty(\mathbb{R})} + \|\frac{\partial R_\Lambda}{\partial x}\|_{L^\infty(\mathbb{R})} \le C_3|\Lambda| \tag{25}$$

Therefore by properties of G, H_1, H_2 and P, there exists then a constant C_4 such that

$$\|\mathcal{K}_\Lambda^1 W + \mathcal{K}_\Lambda^2 W\|_{L^2(\mathbb{R})} \le C_4|\Lambda|\ \|W\|_{H^2}$$

Furthermore by properties of $\mathcal{M}_1$ and Proposition 4, since Q is an orthogonal projection in L^2, there exists a constant C_5 such that

$$\|Q\mathcal{K}_\Lambda W\|_{L^2(\mathbb{R})} \le C_5|\Lambda|\ \|LW\|_{L^2} \tag{26}$$

In the same way, we prove that there exists a constant C_5' such that

$$\|L^{\frac{1}{2}}Q\mathcal{K}_\Lambda W\|_{L^2(\mathbb{R})} \le C_5'|\Lambda|\ \|L^{\frac{3}{2}}W\|_{L^2(\mathbb{R})} \tag{27}$$

In addition, for $W \in \mathcal{E}$

$$
\begin{array}{rl}
(Q\mathcal{K}^1_\Lambda W|W) = & (\mathcal{K}^1_\Lambda W|W) \\
& \text{since } QW = W \\
= & \displaystyle\int_{\mathbb{R}} G(R_\Lambda)\frac{\partial^2 W}{\partial x^2} W \\
= & \displaystyle -\int_{\mathbb{R}} G'(R_\Lambda)\frac{\partial R_\Lambda}{\partial x}\frac{\partial W}{\partial x} W - \int_{\mathbb{R}} G(R_\Lambda)\frac{\partial W}{\partial x}\frac{\partial W}{\partial x} \\
& \text{by integration by parts} \\
\left|(Q\mathcal{K}^1_\Lambda W|W)\right| \leq & C_6|\Lambda|\|W\|^2_{H^1(\mathbb{R})} \\
& \text{with estimate (25)}
\end{array}
$$

where the constant C_6 does not depend on Λ nor on W.

Writing that

$$
\begin{array}{rl}
|(Q\mathcal{K}^2_\Lambda W + Q\mathcal{K}^3_\Lambda W|W)| \leq & \|Q\mathcal{K}^2_\Lambda W + Q\mathcal{K}^3_\Lambda W\|_{L^2}\|W\|_{L^2} \\
\leq & C_7|\Lambda|\|W\|^2_{H^1}
\end{array}
$$

we obtain then that there exists a constant C_8 suct that

$$
|(Q\mathcal{K}_\Lambda W|W)| \leq C_8|\Lambda|\|L^{\frac{1}{2}}W\|^2_{L^2(\mathbb{R})} \tag{28}
$$

We denote by $\mathcal{S}_\Lambda(t)$ the semigroup generated by the linear operator $\mathcal{L} + Q\mathcal{K}_\Lambda$. We have the following proposition:

Proposition 5. *There exists $\beta > 0$, there exists $\eta_1 > 0$, there exists a constant K_3 such that if $|\Lambda(t)| \leq \eta_1$ for all $t \geq 0$ then for $t > 0$*

$$
\|\mathcal{S}_\Lambda(t)W_0\|_{H^1} \leq K_3 e^{-\beta t}\|W_0\|_{H^1}
$$

$$
\|\mathcal{S}_\Lambda(t)W_0\|_{H^1} \leq K_3 \frac{e^{-\beta t}}{\sqrt{t}}\|W_0\|_{L^2}
$$

for $W_0 \in E$

Proof. We fix $W_0 \in \mathcal{E}$ and we denote by W the solution of the Cauchy problem

$$
\begin{cases}
\dfrac{\partial W}{\partial t} = \mathcal{L}W + Q\mathcal{K}_\Lambda W \\
W(t=0) = W_0
\end{cases}
$$

We set $A(t) = \|L^{\frac{1}{2}}W(t)\|^2_{L^2(\mathbb{R})}$.

$$\begin{aligned}
\frac{dA}{dt} &= 2(L^{\frac{1}{2}}\frac{\partial W}{\partial t}|L^{\frac{1}{2}}W)\\
&= 2(\frac{\partial W}{\partial t}|LW)\\
&= 2(JLW|LW) + (Q\mathcal{K}_\Lambda W|LW)\\
&= -2\|LW\|^2_{L^2(\mathbb{R})} + 2(Q\mathcal{K}_\Lambda W|LW)\\
&\leq -2\|LW\|^2_{L^2(\mathbb{R})} + 2C_5|\Lambda|\ \|LW\|^2_{L^2(\mathbb{R})}
\end{aligned}$$

with Estimate (26).

We fix $\eta_1' = \dfrac{1}{2C_5}$ and for $|\Lambda| \leq \eta_1'$ we obtain that

$$\begin{aligned}
\frac{dA}{dt} &\leq -\|LW\|^2_{L^2(\mathbb{R})}\\
&\leq -\frac{1}{K_2^2}\|W\|^2_{H^2(\mathbb{R})} \leq -\frac{1}{K_2^2}\|W\|^2_{H^1(\mathbb{R})}\\
&\quad \text{with Proposition 4}\\
&\leq -\frac{K_1^2}{K_2^2}\|L^{\frac{1}{2}}W\|^2_{L^2(\mathbb{R})}\\
&\quad \text{with Proposition 4}\\
&\leq -\frac{K_1^2}{K_2^2}A
\end{aligned}$$

thus $A(t) \leq A(0)e^{-\frac{K_1^2}{K_2^2}t}$ and then with Proposition 4 there exists a constant K_3' such that

$$\|W(t)\|_{H^1(\mathbb{R})} \leq K_3' e^{-\beta' t}\|W_0\|_{H^1(\mathbb{R})}$$

with $\beta' = \dfrac{K_1^2}{2K_2^2}$.

We set now $B(t) = \|W(t)\|^2_{L^2(\mathbb{R})} + t\|L^{\frac{1}{2}}W(t)\|^2_{L^2(\mathbb{R})}$.

$$\begin{aligned}
\frac{dB}{dt} &= 2(W|\frac{\partial W}{\partial t}) + (L^{\frac{1}{2}}W|L^{\frac{1}{2}}W) + 2t(L^{\frac{1}{2}}\frac{\partial W}{\partial t}|L^{\frac{1}{2}}W)\\
&= 2(W|\mathcal{L}W) + 2(W|Q\mathcal{K}_\Lambda W) + (W|LW) + 2t(\frac{\partial W}{\partial t}|LW)\\
&= -(W|LW) + 2(W|Q\mathcal{K}_\Lambda W) + 2t\left[-\|LW\|^2_{L^2(\mathbb{R})} + (Q\mathcal{K}_\lambda W|LW)\right]\\
&\leq -\|L^{\frac{1}{2}}W\|^2_{L^2(\mathbb{R})} + 2C_8|\Lambda|\ \|L^{\frac{1}{2}}W\|^2_{L^2(\mathbb{R})} - 2t\|LW\|^2_{L^2(\mathbb{R})} + 2tC_5|\Lambda|\ \|LW\|^2_{L^2(\mathbb{R})}\\
&\quad \text{with Estimates (26) and (28)}
\end{aligned}$$

We set $\eta_1'' = \min(\frac{1}{4C_8}, \frac{1}{2C_5})$ and if $|\Lambda| \leq \eta_1''$ we obtain that

$$\begin{aligned}
\frac{dB}{dt} \leq & \quad -\frac{1}{2}\|L^{\frac{1}{2}}W\|^2_{L^2(\mathbb{R})} - t\|LW\|^2_{L^2(\mathbb{R})} \\
\leq & \quad -\frac{1}{2K_2^2}\|W\|^2_{H^1(\mathbb{R})} - t\frac{1}{K_2^2}\|W\|_{H^2(\mathbb{R})} \\
& \quad \text{with Proposition 4} \\
\leq & \quad -\frac{1}{2K_2^2}\|W\|^2_{L^2(\mathbb{R})} - t\frac{K_1^2}{K_2^2}\||L^{\frac{1}{2}}W\|_{L^2(\mathbb{R})} \\
& \quad \text{with Proposition 4} \\
\leq & \quad -\frac{K_1^2}{2K_2^2}B
\end{aligned}$$

Therefore $B(t) \leq B(0)e^{-\frac{K_1^2}{2K_2^2}t}$. We remark that $B(0) = \|W_0\|_{L^2(\mathbb{R})}$, thus if we denote $\beta'' = -\frac{K_1^2}{4K_2^2}$, we obtain that

$$\|W(t)\|^2_{L^2(\mathbb{R})} + t\|L^{\frac{1}{2}}W\|^2_{L^2(\mathbb{R})} \leq \|W_0\|^2_{L^2(\mathbb{R})}e^{-2\beta'' t}$$

and so using Proposition 4 there exists a constant K_3'' such that

$$\|W(t)\|_{H^1(\mathbb{R})} \leq \frac{K_3''}{\sqrt{t}}\|W_0\|_{L^2(\mathbb{R})}e^{-\beta'' t}$$

Setting $\eta_1 = \min(\eta_1', \eta_1'')$, $\beta = \min(\beta', \beta'')$ and $K_3 = \max(K_3', K_3'')$, we conclude the proof of Proposition 5.

□

5. **Stability.** We consider (Λ, W) the solution of System (17)-(19) with initial data $(\Lambda_0, W_0) \in \mathbb{R}^2 \times (H^2(\mathbb{R}))^2$.

In a first step, under Hypothesis **H**:"$\Lambda(t)$ remains little", we prove that if W_0 is small, then $W(t)$ remains closed to zero for the H^2 norm.

In a second step, under Hypothesis **H**, we show that in addition, $(1+t)^2 W(t)$ remains bounded for the H^1 norm.

As a conclusion, we establish that Hypothesis **H** is justified when Λ_0 and W_0 are small.

In the following subsection, we prove preliminar estimates on the non linear terms.

5.1. Preliminar nonlinear estimates.

Lemma 1. *There exists a constant K_4 such that for all $\lambda \in \mathbb{R}^2$ such that $|\lambda| \leq \eta_1$ and all $w \in \mathcal{E}$,*

$$|\mathcal{M}_1(\lambda)(w)| \leq K_4|\lambda|\|w\|_{H^1(\mathbb{R})}$$

$$|\mathcal{M}_2(w, \frac{dw}{dx}, \lambda)| \leq K_4\|w\|^2_{H^1(\mathbb{R})}$$

Proof. We recall that $\mathcal{M}_1$ and $\mathcal{M}_2$ are defined by (18).

We have for $k = 1..2$

$$\begin{aligned}(T_1|e_k) = & \int_{\mathbb{R}} G(R_\lambda(x))\frac{d^2 w}{dx^2}(x)\cdot e_k(x)dx \\ = & -\int_{\mathbb{R}}\left(G'(R_\lambda)(\frac{dR_\lambda}{dx})e_k + G(R_\lambda)\frac{de_k}{dx}\right)\frac{dw}{dx}dx \\ & \text{by integration by parts} \\ |(T_1|e_k)| \leq & \ C|\Lambda|\|w\|_{H^1(\mathbb{R})}\end{aligned}$$

thus

$$|T_1'| \leq C|\Lambda|\|w\|_{H^1(\mathbb{R})}$$

Furthermore, with the definition of T_2 (cf. Equation (14)) there exists a constant C such that

$$|T_2'| \leq C\|w\|_{H^1(\mathbb{R})}|\lambda|$$

Since the matrix $A(\lambda)$ is invertible for $|\lambda| \leq \eta_1$, we obtain the estimation on $\mathcal{M}_1$.

Concerning $\mathcal{M}_2$ we remark that for $k = 1..2$

$$\begin{aligned}(T_3|e_k) = & \int_{\mathbb{R}} \widehat{G}(R_\lambda, w)(w)\frac{d^2 w}{dx^2}(x)\cdot e_k(x)dx \\ = & -\int_{\mathbb{R}}\left[\frac{d}{dx}\left(\widehat{G}(R_\lambda, w)(w)\right)\frac{dw}{dx}e_k(x) + \widehat{G}(R_\lambda, w)(w)\frac{dw}{dx}(x)\cdot\frac{de_k}{dx}(x)\right] \\ & \text{by integration by parts}\end{aligned}$$

that is there exists a constant C such that

$$|T_4'| \leq C\|w\|^2_{H^1(\mathbb{R})}$$

A straightforward estimate on T_4 and T_5 gives that there exists a constant C such that $|T_4'| + |T_5'| \leq C\|w\|^2_{H^1(\mathbb{R})}$, therefore since $A(\lambda)$ is invertible, we conclude the proof of Lemma 1. □

Lemma 2. *There exists a constant K_5 such that for all λ such that $|\lambda| \leq \eta_1$ and all $w \in \mathcal{E}$,*

$$\|Q\mathcal{R}_1(x,\lambda,w)(\frac{d^2 w}{dx^2})\|_{L^2(\mathbb{R})} \leq K_5\|w\|_{H^1(\mathbb{R})}\|w\|_{H^2(\mathbb{R})}$$

$$\|Q\mathcal{R}_1(x,\lambda,w)(\frac{d^2 w}{dx^2})\|_{H^1(\mathbb{R})} \leq K_5\|w\|_{H^2(\mathbb{R})}\|w\|_{H^3(\mathbb{R})}$$

$$\|Q\mathcal{R}_2(x,\lambda,w,\frac{dw}{dx})\|_{H^1(\mathbb{R})} \leq K_5\|w\|_{H^1(\mathbb{R})}\|w\|_{H^2(\mathbb{R})}$$

$$\|Q\mathcal{R}_3(x,\lambda,w)\|_{H^1(\mathbb{R})} \leq K_5\|w\|^2_{H^1(\mathbb{R})}$$

Proof. It is a straightforward application of the definitions of $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, of the properties of G, H_1, H_2, and P, and of Proposition 4. □

5.2. **First step: variational estimate on W.**

Proposition 6. *There exists $\eta_2 > 0$ (with $\eta_2 < \eta_1$) such that if $|\Lambda(t)| \leq \eta_2$ for all t, then, there exists a constant γ_1 such that if $\|LW(t=0)\|_{L^2} \leq \gamma_1$, then $t \mapsto \|LW\|_{L^2}$ is decreasing and there exists K_6 such that*

$$\forall\ t,\ \ \|W(t)\|_{H^2(\mathbb{R})} \leq K_6 \|W_0\|_{H^2(\mathbb{R})}$$

Proof. We take the scalar product on Equation (19) with $J^2\mathcal{L}^2 W$. We remark that:

- $\left(\dfrac{\partial W}{\partial t} \mid J^2\mathcal{L}^2 W\right) = \left(\dfrac{\partial W}{\partial t} \mid J^4 L^2 W\right) = -4\left(\dfrac{\partial W}{\partial t} \mid L^2 W\right) = -2\dfrac{d}{dt}\|LW\|^2_{L^2}$
- $\left(\mathcal{L}W \mid J^2\mathcal{L}^2 W\right) = -4\left(JLW \mid L^2 W\right) = -4\left(JL^{\frac{3}{2}}W \mid L^{\frac{3}{2}}W\right) = 4\|L^{\frac{3}{2}}W\|^2_{L^2}$
- $\left(Q\mathcal{K}_\Lambda W \mid J^2\mathcal{L}^2 W\right) = \left(L^{\frac{1}{2}}Q\mathcal{K}_\Lambda W \mid J^4 L^{\frac{3}{2}} W\right)$ thus, since $J^4 = -4Id$, with Estimate (27)
$$\left|\left(Q\mathcal{K}_\Lambda W \mid J^2\mathcal{L}^2 W\right)\right| \leq 4C_5'|\Lambda|\|L^{\frac{3}{2}}W\|^2_{L^2}$$
- $\left(Q\mathcal{R}_1(x,\Lambda,W)(\dfrac{\partial^2 W}{\partial x^2}) \mid J^2\mathcal{L}^2 W\right) = -4\left(L^{\frac{1}{2}}[Q\mathcal{R}_1(x,\Lambda,W)(\dfrac{\partial^2 W}{\partial x^2})] \mid L^{\frac{3}{2}}W\right)$
thus with Lemma 2

$$\begin{aligned}
\left|\left(Q\mathcal{R}_1(x,\Lambda,W)(\frac{\partial^2 W}{\partial x^2}) \mid J^2\mathcal{L}^2 W\right)\right| &\leq 4\left\|L^{\frac{1}{2}}[Q\mathcal{R}_1(x,\Lambda,W)(\frac{\partial^2 W}{\partial x^2})]\right\|_{L^2(\mathbb{R})}\|L^{\frac{3}{2}}W\|_{L^2(\mathbb{R})} \\
&\leq \frac{4}{K_1}\left\|Q\mathcal{R}_1(x,\Lambda,W)(\frac{\partial^2 W}{\partial x^2})]\right\|_{H^1(\mathbb{R})}\|L^{\frac{3}{2}}W\|_{L^2(\mathbb{R})} \\
&\quad \text{with Proposition 4} \\
&\leq \frac{4K_5}{K_1}\|W\|_{H^2(\mathbb{R})}\|W\|_{H^3(\mathbb{R})}\|L^{\frac{3}{2}}W\|_{L^2(\mathbb{R})} \\
&\quad \text{with Lemma 2} \\
&\leq \frac{4K_5K_2^2}{K_1}\|LW\|_{L^2}\|L^{\frac{3}{2}}W\|^2_{L^2} \\
&\quad \text{with Proposition 4}
\end{aligned}$$

In the same way, we prove that

$$\left|\left(Q\mathcal{R}_2(x,\Lambda,W,\frac{\partial W}{\partial x}) \mid J^2\mathcal{L}^2 W\right)\right| \leq \frac{4K_5K_2^2}{K_1}\|LW\|_{L^2}\|L^{\frac{3}{2}}W\|^2_{L^2}$$

and that

$$\left|\left(Q\mathcal{R}_3(x,\Lambda,W)|J^2\mathcal{L}^2 W\right)\right| \leq \frac{4K_5K_2^2}{K_1}\|LW\|_{L^2}\|L^{\frac{3}{2}}W\|^2_{L^2}$$

Therefore we obtain that if $|\Lambda| \leq \eta_2$, then

$$\frac{d}{dt}\|LW\|^2_{L^2} + 2\|L^{\frac{3}{2}}W\|^2_{L^2} \leq 2C_5'\eta_2\|L^{\frac{3}{2}}W\|^2_{L^2} + \frac{6K_5K_2^2}{K_1}\|LW\|_{L^2}\|L^{\frac{3}{2}}W\|^2_{L^2}$$

that is:

$$\frac{d}{dt}\|LW\|^2_{L^2} + \|L^{\frac{3}{2}}W\|^2_{L^2}\left(2 - 2C_5'\eta_2 - \frac{6K_5K_2^2}{K_1}\|LW\|_{L^2}\right) \leq 0 \tag{29}$$

We fix $\eta_2 < \eta_1$ such that $2C_5'\eta_2 < 1$, and we set $\gamma_1 = \dfrac{K_1}{6K_5K_2^2}$. If $|\Lambda| \leq \eta_2$ then while $\|LW(t)\|_{L^2(\mathbb{R})} \leq \gamma_1$ this quantity remains decreasing with Equation (29), and thus remains less than γ_1.

Therefore, with Proposition 4, we have:

$$\forall\, t \geq 0, \|W(t)\|_{H^2(\mathbb{R})} \leq K_2\|LW(t)\|_{L^2(\mathbb{R})} \leq K_2\|LW_0\|_{L^2(\mathbb{R})} \leq \frac{K_2}{K_1}\|W_0\|_{H^2}$$

and we conclude the proof setting $K_6 = \dfrac{K_2}{K_1}$.

□

5.3. **Second step: parabolic estimates on W.** Using Equation (19) we have:

$$\begin{aligned} W(t) = \quad & \mathcal{S}_\Lambda(t)W_0 + \int_0^t \mathcal{S}_\Lambda(t-s)Q\mathcal{R}_1(x,\Lambda,W)(\frac{\partial^2 W}{\partial x^2})(s)\,ds \\ & + \int_0^t \mathcal{S}_\Lambda(t-s)Q\mathcal{R}_2(x,\Lambda,W,\frac{\partial W}{\partial x})(s)\,ds \\ & + \int_0^t \mathcal{S}_\Lambda(t-s)Q\mathcal{R}_3(x,\Lambda,W)(s)\,ds \end{aligned}$$

and with Proposition 5, we know that while $|\Lambda(t)| \leq \eta_1$ there exists a constant K_3 such that

$$\begin{aligned} \|W(t)\|_{H^1(\mathbb{R})} \leq \quad & K_3e^{-\beta t}\|W_0\|_{H^1(\mathbb{R})} + \int_0^t K_3\frac{e^{-\beta(t-s)}}{\sqrt{t-s}}\|Q\mathcal{R}_1(x,\Lambda,W)(\frac{\partial^2 W}{\partial x^2})(s)\|_{L^2(\mathbb{R})}ds \\ & + \int_0^t K_3e^{-\beta(t-s)}\|Q\mathcal{R}_2(x,\Lambda,W,\frac{\partial W}{\partial x})(s)\|_{H^1(\mathbb{R})} \\ & + \int_0^t K_3e^{-\beta(t-s)}\|Q\mathcal{R}_3(x,\Lambda,W)(s)\|_{H^1(\mathbb{R})} \end{aligned}$$

Using Lemma 2 we obtain that

$$\begin{aligned} \|W(t)\|_{H^1(\mathbb{R})} \leq \quad & K_3e^{-\beta t}\|W_0\|_{H^1(\mathbb{R})} + \int_0^t K_3\frac{e^{-\beta(t-s)}}{\sqrt{t-s}}K_5\|W(s)\|_{H^1(\mathbb{R})}\|W(s)\|_{H^2(\mathbb{R})}ds \\ & + \int_0^t K_3e^{-\beta(t-s)}K_5\|W(s)\|_{H^1(\mathbb{R})}\|W(s)\|_{H^2(\mathbb{R})}ds \\ & + \int_0^t K_3e^{-\beta(t-s)}K_5\|W(s)\|^2_{H^1(\mathbb{R})} \end{aligned}$$

Using Proposition 6 we know that if $\|W_0\|_{H^2(\mathbb{R})} \leq \gamma_1$ and if $|\Lambda(t)|$ remains less than η_2 then $\|W(s)\|_{H^2(\mathbb{R})} \leq K_6\|W_0\|_{H^2(\mathbb{R})}$ for all s.

We define $G(t)$ by

$$G(t) = \sup_{s\in[0,t]}(1+s)^2\|W(s)\|_{H^1}$$

We obtain then that

$$\|W(t)\|_{H^1} \leq \quad K_3 e^{-\beta t}\|W_0\|_{H^1} + K_3K_5\left(\int_0^t (1+s)^{-4}e^{-\beta(t-s)}ds\right)[G(t)]^2$$
$$+K_3K_5K_6\|W_0\|_{H^2(\mathbb{R})}G(t)\left(\int_0^t \frac{e^{-\beta(t-s)}}{\sqrt{t-s}}(1+s)^{-2}ds + \int_0^t e^{-\beta(t-s)}(1+s)^{-2}\right)$$

Now there exists a constant K_7 such that for all t we have

$$\frac{e^{-\beta t}}{(1+t)^2} \leq K_7$$

$$\int_0^t \frac{e^{-\beta(t-s)}}{\sqrt{t-s}}(1+s)^{-2}ds + \int_0^t e^{-\beta(t-s)}(1+s)^{-2}ds \leq \frac{K_7}{(1+t)^2}$$

$$\int_0^t (1+s)^{-4}e^{-\beta(t-s)}ds \leq \frac{K_7}{(1+t)^2}$$

Thus we obtain that there exists a constant C such that

$$(1+t)^k\|W(t)\|_{H^1} \leq K_3K_7\|W_0\|_{H^1} + K_3K_5K_6K_7\|W_0\|_{H^2}G(t) + K_3K_5K_7(G(t))^2$$

and since G is a non decreasing map, denoting α_1 =max$\{K_3K_7, \frac{1}{4}\}$, $\alpha_2 = K_3K_5K_6K_7$ and $\alpha_3 = K_3K_5K_7$ we obtain that

$$G(t) \leq \alpha_1\|W_0\|_{H^1} + \alpha_2\|W_0\|_{H^2}G(t) + \alpha_3(G(t))^2 \tag{30}$$

We have then the following result:

Proposition 7. *Let η_2 and γ_1 being given by Proposition 6. There exists γ_2 with $0 < \gamma_2 < \gamma_1$ such that for all $\delta > 0$ there exists $\tau > 0$ such that if the following assumptions are satisfied:*

(i) for all t, $|\Lambda(t)| \leq \eta_2$,
(ii) $\|LW_0\|_{L^2(\mathbb{R})} \leq \gamma_2$,
(iii) $\|W_0\|_{H^1(\mathbb{R})} \leq \tau$,

then for all $t > 0$ we have

$$\|W(t)\|_{H^1(\mathbb{R})} \leq \frac{\delta}{(1+t)^2}$$

Proof. Under Hypothesis (i) and if $\|LW_0\|_{L^2(\mathbb{R})} \leq \gamma_1$ we have proved Estimate (30).

We set γ_2 =min$(\frac{1}{2\alpha_2}, \gamma_1)$. Under Hypothesis (i) and (ii) we have that for all t

$$\alpha_3(G(t))^2 - \frac{1}{2}G(t) + \alpha_1\|W_0\|_{H^1(\mathbb{R})} \geq 0 \tag{31}$$

Let us study the polynomial map $P_\nu : \xi \mapsto \alpha_3\xi^2 - \frac{1}{2}\xi + \alpha_1\nu$. If $\nu < \gamma_2 := \frac{1}{16\alpha_1\alpha_3}$ then this polynomial map has two positive zeros. The smaller one is $\xi_1(\nu) = \frac{1}{4\alpha_3}(1 - \sqrt{1 - 16\alpha_1\alpha_3\nu})$. We remark that since $\alpha_1 \geq \frac{1}{4}$ then $\xi_1(\nu) \geq \nu$.

Let $\delta > 0$ be fixed.

The map $\nu \mapsto \xi_1(\nu)$ tends to zero when ν tends to zero, so we can fix $\tau > 0$ such that for $\nu \in [0, \tau]$, $\xi(\nu) \leq \delta$. Even if it means reducing τ we can assume that $\tau \leq \delta$ and $\tau \leq \gamma_2/2$.

Under Hypothesis (i), (ii) and (iii), the map $G(t)$ satisfies (31) and $G(0) = \|W_0\|_{H^1(\mathbb{R})} \leq \xi_1(\|W_0\|_{H^1(\mathbb{R})})$. Thus for all t, $G(t) \leq \xi_1(\|W_0\|_{H^1(\mathbb{R})}) \leq \delta$.

This conludes the proof of Proposition 7. $\square$

5.4. **Estimates for Λ.** We integrate Equation (17) between $t = 0$ and t. We obtain that

$$|\Lambda(t)| \leq |\Lambda_0| + \int_0^t |\mathcal{M}_1(\Lambda(s))(W(s))| ds + \int_0^t |\mathcal{M}_2(W(s), \frac{\partial W}{\partial x}(s), \Lambda(s))| ds \quad (32)$$

We assume that $|\Lambda_0| \leq \frac{\eta_2}{2}$ and that $\|LW_0\|_{L^2(\mathbb{R})} \leq \gamma_2$. We fix an arbitrary δ. with Proposition 7, while $|\Lambda(t)|$ remains less that η_2 we have, if $\|W_0\|_{H^1(\mathbb{R})} \leq \tau$ we have: $\|W(t)\|_{H^1(\mathbb{R})} \leq \dfrac{\delta}{(1+t)^2}$.

Using this estimate in Equation (32) and using Lemma 1 we obtain that while $|\Lambda(t)| \leq \eta_2$ we have:

$$|\Lambda(t)| \leq \frac{\eta_2}{2} + \int_0^t K_4 \eta_2 \delta \frac{1}{(1+s)^2} ds + \int_0^t K_4 \delta^2 \frac{1}{(1+s)^4} ds. \quad (33)$$

We fix $\delta > 0$ such that

$$K_4 \eta_2 \delta \int_0^{+\infty} \frac{1}{(1+s)^2} ds + K_4 \delta^2 \int_0^{+\infty} \frac{1}{(1+s)^4} ds \leq \frac{\eta_2}{2}$$

With Proposition 6 we find $\tau_0 > 0$ and if $|\Lambda_0| \leq \dfrac{\eta_2}{2}$, $\|LW_0\|_{H^2(\mathbb{R})} \leq \gamma_2$ and if $\|W_0\|_{H^1(\mathbb{R})} \leq \tau_0$ then with Estimate (33), $|\Lambda(t)|$ remains less than η_2 for all time, and all the estimates are true for all time, which concludes the proof of our theorem.

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