STABILITY FOR STATIC WALLS IN FERROMAGNETIC NANOWIRES

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Abstract. The goal of this article is to analyze the time asymptotic stability of one dimensional Bloch walls in ferromagnetic materials. The equation involved in modelling such materials is the Landau-Lifchitz system which is non-linear and parabolic. We demonstrate that the equilibrium states called Bloch walls are asymptotically stable modulo a rotation and a translation transverse to the wall. The linear part of the perturbed equation admits zero as an eigenvalue forbidding a direct proof.

1. Introduction. Over the last decade, the interest for ferromagnetism modelization has grown (see [7]). One of the main goals of these mathematical studies is to understand the behaviour of dynamical structures in ferromagnets [3, 4, 5, 10, 11, 12] to validate models. The obtained results will be exploited to enhance numerical simulations of ferromagnets used by physicists [9] to understand and optimise the magnetic characteristics of ferromagnetic materials. Remembering that the main mean of observation is the microwave resonance, we understand the importance of studying the stability of the magnetization in ferromagnets; this study would validate mathematically and therefore numerically, the use of that mean of observation. Then, one of the key points to understand this stability is to analyse the stability of the microstructures developed by the magnetization: it is to say the walls, separation zones between the domains in which the magnetization is smooth.

No extensive study of wall stability in micromagnetic states has been done yet. The three dimensional structure of these objects is very complex and there are no mathematical description in the three dimensional case and some for the two dimensional one [1, 6].

The three dimensional model is the following : we denote by $u = (u_1, u_2, u_3)$ the magnetic moment defined on $\mathbb{R}_+^3 \times \Omega$ with values in $S^2$ the unit sphere of $\mathbb{R}^3$, where

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Ω is the ferromagnetic domain. The variations of \( u \) are governed by the following Landau-Lifschitz equation:

\[
\frac{\partial u}{\partial t} = -u \wedge h_{\text{eff}}(u) - u \wedge (u \wedge h_{\text{eff}}(u))
\]

The effective field \( h_{\text{eff}}(u) \) is given by

\[
h(u) = A \Delta u + h_d(u),
\]

where \( A \Delta u \) is the exchange field, and where the demagnetizing field \( h_d(u) \) satisfies

\[
\begin{align*}
\text{rot}(h_d(u)) &= 0, \\
\text{div}(h_d(u)) &= -\text{div}(u), \\
u &= 0 \text{ in } \mathbb{R}^3 \setminus \Omega.
\end{align*}
\]

This system has solutions for regular finite domain \( \Omega \) as shown in [5].

In this paper we consider an asymptotic one dimensional model of nanowire obtained and justified by D. Sanchez in [14]. In this case the demagnetizing field writes:

\[
h_d(u) = -u_2 e_2 - u_3 e_3 = u_1 e_1 - u
\]

where \((e_1, e_2, e_3)\) is the canonical basis of \( \mathbb{R}^3 \) and where \( u = (u_1, u_2, u_3) \).

**Remark 1.** This model is obtained using a BKW method, taking the limit when the diameter of the wire tends to zero (see [14]).

Finally using a space scaling factor to set \( A = 1 \), for a line along the \( x \)-axis we study the following system

\[
\begin{align*}
u : \mathbb{R}_+^T \times \mathbb{R}_x &\longrightarrow S^2 \\
\frac{\partial u}{\partial t} &= -u \wedge h(u) - u \wedge (u \wedge h(u)) \\
\text{with } h(u) &= \frac{\partial^2 u}{\partial x^2} + u_1 e_1
\end{align*}
\]

**Remark 2.** The demagnetizing field \( h_d(u) \) given by Formula (2) only appears in Landau-Lifschitz Equation in the expression \( u \wedge h_d(u) = u \wedge (u_1 e_1 - u) = u \wedge (u_1 e_1) \). It is the reason why we can work with the expression of \( h(u) \) given in (3).

The aim of the paper is to study the stability of a static wall profile which separates the domain in which \( u = -e_1 \) (in the neighborhood of \( -\infty \)) and the domain in which \( u = e_1 \) (in the neighborhood of \( +\infty \)).

This profile is given by

\[
M_0 = \begin{pmatrix}
th x \\
0 \\
\frac{1}{\text{ch} x}
\end{pmatrix}.
\]

We remark that Landau-Lifschitz equation (3) is invariant by translation in the variable \( x \) and by rotation around \( e_1 \). Hence for all \( \Lambda = (\theta, \sigma) \in \mathbb{R} \times \mathbb{R} \), the profile
$x \mapsto M_\Lambda(x) = R_\theta(M_0(x - \sigma))$ is a static solution of (3) satisfying $\lim u = -e_1$ and $\lim u = e_1$, where we denote by $R_\theta$ the rotation of angle $\theta$ around $e_1$:

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$$

Our main result is the following

**Theorem 1.** Let $\varepsilon > 0$, there exists $\eta > 0$ such that, for all $v_0$ in $H^2(\mathbb{R})$ with $|v_0| = 1$ for all $x$ in $\mathbb{R}$ and such that $\|v_0 - M_0\|_{H^2(\mathbb{R})} < \eta$, if we denote by $v$ the solution of (3) with $v_0$ as the initial data then, for all $t$ in $\mathbb{R}^+$, $\|v(t) - M_0\|_{H^2(\mathbb{R})} < \varepsilon$. Furthermore there exists $\Lambda = (\theta, \sigma) \in \mathbb{R}^2$ such that $v$ tends to $M_\Lambda$ when $t$ tends to infinity for the norm $H^1(\mathbb{R})$.

The invariance of (3) by rotation-translation implies that the linearized equation in the neighborhood of $M_0$ has zero as an eigen-value, which is a major obstruction to obtain strictly the stability result. In addition all the known results about the stability of travelling waves are proved for semilinear equation (see [8]). Here the considered Landau-Lifschitz equation is quasilinear and we have to combine variational estimates with the methods used in [8].

The paper is organized as follows : in Section 2 we describe the perturbations of $M_0$ in the mobile frame $(M_0(x), M_1(x), M_2)$, where $M_1(x) = (\frac{1}{\cosh x}, 0, -\sinh x)$ and $M_2 = (0, 1, 0)$, writing

$$u(t, x) = r_1(t, x)M_1(x) + r_2(t, x)M_2 + \sqrt{1 - r_1^2 - r_2^2}M_0(x).$$

We obtain then an equivalent formulation of Equation (3) where the unknown is $r = (r_1, r_2)$, of the form:

$$\frac{\partial r}{\partial t} = \mathcal{L} r + F(x, r, \frac{\partial r}{\partial x}, \frac{\partial^2 r}{\partial x^2})$$

where $\mathcal{L} r$ denotes the linear part.

The stability of $M_0$ for Equation (3) is then equivalent to the stability of the zero solution for Equation (4).

The two parameters family of static solutions $M_\Lambda$ for Equation (3) induces in the new coordinates a two parameters family $R_\Lambda$ of statics solutions for Equation (4). In Section 3, we decompose the solution $r$ of (4) in

$$r(t, x) = R_\Lambda(t)(x) + W(x)$$

where $W \in (\text{Ker } \mathcal{L})^\perp$. This decomposition is rather classical for the study of static solution stability for semilinear parabolic equations (see [8]). This technique has also been used in [2] to demonstrate the stability of travelling waves in thin films or in [13] in the case of the radially symmetric travelling waves in reaction-diffusion equations.

The main difficulty here is that Equation (4) is quasilinear and then the non linear term $F$ depends also on $\frac{\partial^2 r}{\partial x^2}$. We then use Section 5 variational estimates for
the non linear part combined with more classical linear estimates on the operator $\mathcal{L}$ (proved in Section 4).

In the following we denote by $\cdot$ the scalar product in $\mathbb{R}^3$, and by $( | )$ the scalar product in $L^2(\mathbb{R})$.

2. Equation for the perturbations of the wall.

2.1. Moving frame. We consider the following moving frame $(M_0(x), M_1(x), M_2)$ with

\[
M_0 = \begin{pmatrix} \text{th} x \\ 0 \\ \frac{1}{\text{ch} x} \end{pmatrix}, \quad M_1 = \begin{pmatrix} \frac{1}{\text{ch} x} \\ 0 \\ -\text{th} x \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
\]

We consider $u$ as a little perturbation of $M_0$ and we write $u$ on the form

\[
u(t, x) = r_1(t, x)M_1(x) + r_2(t, x)M_2(x) + \sqrt{1 - (r_1(t, x))^2 - (r_2(t, x))^2} M_0(x). \quad (5)
\]

We denote $\lambda = \sqrt{1 - r_1^2 - r_2^2}$. In order to ensure the regularity of $\lambda$, we assume that $\|u - M_0\|_{L^\infty(\mathbb{R})} \leq \frac{1}{2}$. This assumption is correct since we study little perturbations of $M_0$.

We have

- $\frac{dM_0}{dx} = \frac{1}{\text{ch} x} M_1$,
- $\frac{dM_1}{dx} = -\frac{1}{\text{ch} x} M_0$,
- $\frac{d^2M_0}{dx^2} = -\frac{\text{sh} x}{\text{ch}^2 x} M_1 - \frac{1}{\text{ch}^2 x} M_0$,
- $e_1 = \text{th} x M_0 + \frac{1}{\text{ch} x} M_1$,
- $h(M_0) = fM_0$ where $f(x) = 2\text{th}^2 x - 1$.

Furthermore

\[ h(u) = a_0 M_0 + a_1 M_1 + a_2 M_2 \]

with

\[
\begin{align*}
a_0 &= \frac{\partial^2 \lambda}{\partial x^2} + \lambda f(x) + 2r_1 \frac{\text{sh} x}{\text{ch}^2 x} - 2 \frac{\partial r_1}{\partial x} \frac{1}{\text{ch} x} \\
a_1 &= \frac{\partial^2 r_1}{\partial x^2} + 2 \frac{1}{\text{ch} x} \frac{\partial \lambda}{\partial x} \\
a_2 &= \frac{\partial^2 r_2}{\partial x^2}
\end{align*}
\]
We replace $u$ by its expression (5) in Equation (3), and we obtain that

\[
\frac{\partial \lambda}{\partial t} M_0 + \frac{\partial r_1}{\partial t} M_1 + \frac{\partial r_2}{\partial t} M_2 = -(r_1 a_2 - r_2 a_1) M_0 - (r_2 a_0 - \lambda a_2) M_1 - (\lambda a_1 - r_1 a_0) M_2 \\
- \lambda (r_2 a_0 - \lambda a_2) M_2 + \lambda (\lambda a_1 - r_1 a_0) M_1 + r_1 (r_1 a_2 - r_2 a_1) M_2 \\
- r_1 (\lambda a_1 - r_1 a_0) M_0 - r_2 (r_1 a_2 - r_2 a_1) M_1 + r_2 (r_2 a_0 - \lambda a_2) M_0
\]

(6)

Projecting Equation (6) in the directions $M_1$ and $M_2$ we obtain that if $u$ is solution of (3) then

\[
\frac{\partial r_1}{\partial t} = -r_2 a_0 + \lambda a_2 + \lambda (\lambda a_1 - r_1 a_0) - r_2 (r_1 a_2 - r_2 a_1) \\
\frac{\partial r_2}{\partial t} = -(\lambda a_1 - r_1 a_0) - \lambda (r_2 a_0 - \lambda a_2) + r_1 (r_1 a_2 - r_2 a_1)
\]

(7)

Remark 3. Equation (7) is equivalent to Equation (3). Indeed we write Equation (3) on the form :

\[
\frac{\partial u}{\partial t} = F(u)
\]

where $F(u)(x)$ is orthogonal to $u(x)$ for all $x \in \mathbb{R}$.

Equation (7) is the projection of (3) on the directions $M_1$ and $M_2$, that is if $(r_1, r_2)$ satisfies Equation (7), then $u = r_1 M_1 + r_2 M_2 + \sqrt{1 - r_1^2 - r_2^2} M_0$ satisfies

\[
\left(\frac{\partial u}{\partial t} - F(u)\right) \cdot M_1 = (\frac{\partial u}{\partial t} - F(u)) \cdot M_2 = 0.
\]

We remark that $u = r_1 M_1 + r_2 M_2 + \sqrt{1 - r_1^2 - r_2^2} M_0$ is a normed vector field, thus $\frac{\partial u}{\partial t} \cdot u = 0$. Furthermore, $u \cdot F(u) = 0$. Thus, if $(r_1, r_2)$ satisfies Equation (7), then $(\frac{\partial u}{\partial t} - F(u)) \cdot \lambda M_0 = 0$ and since $\lambda \neq 0$ (since we consider little perturbations of $M_0$) we obtain that the third component of $\frac{\partial u}{\partial t} - F(u)$ is zero.

Thus for little perturbations of $M_0$, Equation (3) is equivalent to (7).

We detail Equation (7) replacing the $a_i$'s by their values. We obtain that Landau-Lifschitz equation is equivalent for little perturbations of $M_0$ to the following system:
\[
\frac{\partial r_1}{\partial t} = -r_2 \frac{\partial^2 \lambda}{\partial x^2} - r_2 \lambda f(x) - 2r_2 r_1 \frac{\text{sh} x}{\text{ch}^2 x} + 2r_2 \frac{\partial r_1}{\partial x} \frac{1}{\text{ch} x} + 2 \frac{1}{\text{ch} x} \frac{\partial r_1}{\partial x} - r_2 \frac{\partial^2 r_1}{\partial x^2}
\]
\[
+ \lambda \frac{\partial^2 r_2}{\partial x^2} - r_1 r_2 \frac{\partial^2 r_2}{\partial x^2} + \frac{1}{\text{ch} x} \frac{\partial r_1}{\partial x} + 2r_2 \lambda \frac{\partial \lambda}{\partial x} + 2r_2 \lambda \frac{\partial r_2}{\partial x} - r_1 r_2 \frac{\partial^2 r_1}{\partial x^2}
\]
\[
-2r_2 \frac{1}{\text{ch} x} \frac{\partial}{\partial x} - \lambda r_1 \frac{\partial^2 \lambda}{\partial x^2} - \lambda^2 r_1 f(x) - 2\lambda r_1 \frac{\text{sh} x}{\text{ch}^2 x} + 2\lambda r_1 \frac{\partial r_1}{\partial x} \frac{1}{\text{ch} x}
\]
\[
\frac{\partial r_2}{\partial t} = -\lambda \frac{\partial^2 r_2}{\partial x^2} - 2\lambda \frac{1}{\text{ch} x} \frac{\partial}{\partial x} + r_1 \frac{\partial^2 \lambda}{\partial x^2} + r_1 \lambda f(x) + 2r_2 \lambda \frac{\text{sh} x}{\text{ch}^2 x} - 2r_1 \frac{1}{\text{ch} x} \frac{\partial r_1}{\partial x}
\]
\[
+ \frac{\partial^2 r_2}{\partial x^2} - r_1 \frac{\partial^2 r_2}{\partial x^2} - r_1 r_2 \frac{\partial^2 r_1}{\partial x^2} - 2r_1 r_2 \frac{1}{\text{ch} x} \frac{\partial}{\partial x} - \lambda \frac{\partial^2 \lambda}{\partial x^2}
\]
\[
-\lambda^2 r_2 f(x) - 2\lambda r_1 r_2 \frac{\text{sh} x}{\text{ch}^2 x} + 2\lambda r_2 \frac{\partial r_1}{\partial x} \frac{1}{\text{ch} x}
\]
(8)

We denote \( r = (r_1, r_2), \) and we define \( \mu : B(0, \frac{1}{2}) \subset \mathbb{R}^2 \to \mathbb{R} \) by \( \mu(\xi) = \sqrt{1 - |\xi|^2} - 1 \) (that is \( \lambda = 1 + \mu(r) \)). We then write Equation (8) on the condensed form detailed in the following proposition:

**Proposition 1.** The function \( u \in C^1(\mathbb{R}^+; H^2(\mathbb{R}; S^2)) \) such that \( \|u - M_0\|_{L^\infty} \leq \frac{1}{2} \) satisfies Landau-Lifschitz equation (3) if and only if \( u = r_1 M_1 + r_2 M_2 + \sqrt{1 - r_1^2 - r_2^2} M_0 \) where \( r = (r_1, r_2) \) satisfies:

\[
\frac{\partial r}{\partial t} = Lr + G(r) \frac{\partial^2 r}{\partial x^2} + H_1(x, r) \frac{\partial r}{\partial x} + H_2(r) \frac{\partial r}{\partial x} + P(x, r)
\]
(9)

with

- \( L = JL \) with \( J = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \) and \( L = -\frac{\partial^2}{\partial x^2} + f \) (we recall that \( f(x) = 2th^2 x - 1 \)),
- \( G(r) \) is the matrix defined by:

\[
G(r) = \begin{pmatrix} \frac{r_1 r_2}{\sqrt{1 - r_1^2 - r_2^2}} & \frac{r_2^2}{\sqrt{1 - r_1^2 - r_2^2}} + \mu(r) \\ -\mu(r) - \frac{r_1^2}{\sqrt{1 - r_1^2 - r_2^2}} & -\frac{r_1 r_2}{\sqrt{1 - r_1^2 - r_2^2}} \end{pmatrix}
\]

- \( H_1(x, r) \) is the matrix defined by:

\[
H_1(x, r) = \frac{2}{\sqrt{1 - r_1^2 - r_2^2} \text{ch} x} \begin{pmatrix} r_2 \sqrt{1 - r_1^2 - r_2^2} - r_1 r_2^2 & -r_2 - r_2 r_1^2 \\ r_2 - r_1 r_2^2 & -\sqrt{1 - r_1^2 - r_2^2 r_2 + r_1 r_2^2} \end{pmatrix}
\]
• $H_2(r) \in L_2(\mathbb{R}^2)$ is a symmetric bilinear form defined by

$$H_2(r)(\xi_1, \xi_2) = \begin{pmatrix}
\sqrt{1 - r_1^2 - r_2^2} & r_1 (1 - r_2^2 - r_1^2) \\
\sqrt{1 - r_1^2 - r_2^2} & r_2 (1 - r_2^2 - r_1^2) \\
\sqrt{1 - r_1^2 - r_2^2} & r_1 (1 - r_2^2 - r_1^2) \\
\sqrt{1 - r_1^2 - r_2^2} & r_2 (1 - r_2^2 - r_1^2)
\end{pmatrix} \begin{pmatrix}
(1 - r_1^2 - r_2^2)(\xi_1 \cdot \xi_2) + (r_1 \cdot \xi_1)(r_2 \cdot \xi_2)
\end{pmatrix}$$

• $P$ is defined by

$$P(x, r) = \begin{pmatrix}
-r_2 \mu(r) f(x) - 2r_2 r_1 \frac{sh x}{ch^2 x} + (r_1^2 + r_2^2) r_1 f(x) - 2 \sqrt{1 - r_1^2 - r_2^2} \frac{sh x}{ch^2 x} \\
r_1 \mu(r) f(x) + 2r_1^2 \frac{sh x}{ch^2 x} + (r_1^2 + r_2^2) r_2 f(x) - 2 \sqrt{1 - r_1^2 - r_2^2} r_1 \frac{sh x}{ch^2 x}
\end{pmatrix}$$

The properties concerning $G$, $H_1$, $H_2$ and $P$ are summarized in the following proposition:

**Proposition 2.**

• $G \in C^\infty(B(0, 1/2); \mathcal{M}^2(\mathbb{R}))$ and $G(\xi) = O(|\xi|^2)$
• $H_1 \in C^\infty(\mathbb{R} \times B(0, 1/2); \mathcal{M}^2(\mathbb{R}))$ and $H_1(x, r) = O(|r|)$
• $H_2 \in C^\infty(B(0, 1/2); L_2(\mathbb{R}^2))$, with $H_2(x, r) = O(|r|)$
• $P \in C^\infty(\mathbb{R} \times B(0, 1/2); \mathbb{R}^2)$ with $P(x, r) = O(|r|^2)$ uniformly in $x \in \mathbb{R}$

3. A new system of coordinates. We remark that $L$ is a self adjoint operator on $L^2(\mathbb{R})$, with domain $H^2(\mathbb{R})$. Furthermore, $L$ is positive since we can write $L = l^* \circ l$ with $l = \frac{\partial}{\partial x} + th x$, and Ker $L$ is the one dimensional space generated by $\frac{1}{\text{ch } x}$.

The matrix $J$ being invertible, Ker $\mathcal{L}$ is the two dimensional space generated by $e_1$ and $e_2$ with

$$e_1(x) = \begin{pmatrix} 0 \\ 1 / \text{ch } x \end{pmatrix}, \quad e_2(x) = \begin{pmatrix} 1 / \text{ch } x \\ 0 \end{pmatrix} \quad (10)$$

We introduce $\mathcal{E} = (\text{Ker } \mathcal{L})^\perp$. We denote by $Q$ the orthogonal projection onto $\mathcal{E}$ for the $L^2(\mathbb{R})$ scalar product.

Landau-Lifschitz equation (3) is invariant by translation in the variable $x$ and by rotation about the axis $e_1$. This two parameters family of invariance explains the presence of the eigenvalue zero (of multiplicity 2) for the linearized operator $\mathcal{L}$. We will write the solution $u$ as a rotation-translation of $M_0$ plus a term in $\mathcal{E}$.

For $\Lambda = (\theta, \sigma)$ fixed in $\mathbb{R}^2$ we know that the profile $M_0$ rotated of the angle $\theta$ and translated of $\sigma$ is a solution of Landau-Lifschitz equation. We denote by $M_\Lambda$ this solution:

$$M_\Lambda(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} M_0(x - \sigma)$$

and we introduce $R_\Lambda(x)$ the coordinates of $M_\Lambda(x)$ in the basis $(M_1(x), M_2(x))$:

$$R_\Lambda(x) = \begin{pmatrix} M_\Lambda(x) \cdot M_1(x) \\ M_\Lambda(x) \cdot M_2 \end{pmatrix}$$
In a neighborhood of zero (which represents the wall profile $M_0$ in the frame $(M_1, M_2)$), we use a coordinate system given by

$$r(x) = R_\Lambda(x) + W(x).$$

(11)

with $(\Lambda, W) \in \mathbb{R}^2 \times \mathcal{E}$.

The map $r \mapsto (\Lambda, W)$ is a diffeomorphism from a neighborhood of zero in $H^2(\mathbb{R})$ to a neighborhood of zero in $\mathbb{R}^2 \times \mathcal{E}$. Indeed let $r \in H^2(\mathbb{R})$. In order to use the coordinate system (11), there must exist a unique pair $(\Lambda, W) \in \mathbb{R}^2 \times \mathcal{E}$ such that $r(x) = R_\Lambda(x) + W(x)$.

If $r = R_\Lambda + W$ then taking the scalar product of $r$ with $e_1$ and $e_2$, since $W \in \mathcal{E} = (\text{Ker } L)^\perp$ and since $(e_1, e_2)$ defined by (10) is a basis of Ker $L$, we have

$$(r|e_1) = (R_\Lambda|e_1) \quad \text{and} \quad (r|e_2) = (R_\Lambda|e_2)$$

(12)

Furthermore, if $\Lambda \in \mathbb{R}^2$ satisfies (12) then $W = r - R_\Lambda \in \mathcal{E}$

We define $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$\psi(\Lambda) = \begin{pmatrix} (R_\Lambda|e_1) \\ (R_\Lambda|e_2) \end{pmatrix}$$

Therefore (11) defines a system of coordinates in a neighborhood of 0 if $\psi$ is a local diffeomorphism in a neighborhood of zero. This is the case since $\psi$ is $C^\infty$ and since $\psi'(0) = \text{Id}$.

We compute now the equation of the perturbation in the coordinates $(\Lambda, W)$.

We write the solution $r$ of Equation (9) on the form :

$$r(t, x) = R_{\Lambda(t)}(x) + W(t, x)$$

where for all $t$, $W(t) \in \mathcal{E}$ and where $\Lambda : \mathbb{R}_t^+ \rightarrow \mathbb{R}^2$.

We will rewrite Equation (9) in the coordinates $(\Lambda, W)$. The equation on $\Lambda$ is obtained by taking the scalar product of (9) with $e_1$ and $e_2$. The equation on $W$ is obtained using $Q$ the orthogonal projection onto $\mathcal{E}$.

If $\Lambda = (\theta, \sigma)$ is fixed, we know that $x \mapsto R_\Lambda(x)$ satisfies (9) that is we have:

$$\mathcal{L}R_\Lambda + G(R_\Lambda) \left( \frac{d^2 R_\Lambda}{dx^2} \right) + H_1(x, R_\Lambda) \left( \frac{dR_\Lambda}{dx} \right) + H_2(R_\Lambda) \left( \frac{d^2 R_\Lambda}{dx^2} \right) + P(x, R_\Lambda) = 0$$

In order to isolate the linear part in $W$ we perform the Taylor expansion for $G$, $H_1$, $H_2$ and $K$, and we have at order 1:

$$G(R_\Lambda + W) = G(R_\Lambda) + \tilde{G}(R_\Lambda, W)(W)$$

with

$$\tilde{G}(v_1, v_2)(\xi) = \int_0^1 G'(v_1 + sv_2)(s \xi)ds$$

and at order 2:

$$G(R_\Lambda + W) = G(R_\Lambda) + G'(R_\Lambda)(W) + \tilde{G}(R_\Lambda, W)(W^{(2)})$$

where

$$\tilde{G}(v_1, v_2)(\xi^{(2)}) = \int_0^1 (1 - s)G''(v_1 + sv_2)(s \xi, \xi)ds$$

We will use the same notations for $H_1$, $H_2$ and $K$. 

We have
\[
\frac{d\theta}{dt} \partial_\theta R_\Lambda + \frac{d\sigma}{dt} \partial_\sigma R_\Lambda + \frac{\partial W}{\partial t} = LW + T_1 + \ldots T_5 \tag{13}
\]
where
\[
T_1 = K_1^2 W := G(R_\Lambda) \frac{\partial^2 W}{\partial x^2}
\]
\[
T_2 = K_2^2 W := H_1(x, R_\Lambda) \frac{\partial W}{\partial x} + 2H_2(R_\Lambda) \left( \frac{dR_\Lambda}{dx}, \frac{\partial W}{\partial x} \right) + G'(R_\Lambda)(W) \frac{\partial^2 R_\Lambda}{\partial x^2}
\]
\[
+ H'_1(x, R_\Lambda)(W) \frac{\partial R_\Lambda}{\partial x} + H'_2(R_\Lambda)(W) \left( \frac{dR_\Lambda}{dx}, \frac{dR_\Lambda}{dx} \right) + P'(x, R_\Lambda)(W)
\]
\[
T_3 = R_1(x, \Lambda, W)(\frac{\partial^2 W}{\partial x^2}) := \tilde{G}(R_\Lambda, W)(W) \frac{\partial^2 W}{\partial x^2}
\]
\[
T_4 = R_2'(x, \Lambda, W)(\frac{\partial W}{\partial x}) := H_2(R_\Lambda + W) \left( \frac{\partial W}{\partial x}, \frac{\partial W}{\partial x} \right) + \tilde{H}_1(x, R_\Lambda, W)(W) \left( \frac{\partial W}{\partial x} \right)
\]
\[
+ 2H'_2(R_\Lambda, W)(W) \left( \frac{dR_\Lambda}{dx}, \frac{\partial W}{\partial x} \right)
\]
\[
T_5 = R_3(x, \Lambda, W) := \tilde{G}(R_\Lambda, W)(W^{(2)}) \left( \frac{\partial^2 R_\Lambda}{\partial x^2} \right) + \tilde{H}_1(x, R_\Lambda, W)(W^{(2)}) \left( \frac{dR_\Lambda}{dx} \right)
\]
\[
+ \tilde{H}_2(R_\Lambda, W)(W^{(2)}) \left( \frac{dR_\Lambda}{dx}, \frac{dR_\Lambda}{dx} \right) + \tilde{P}(x, R_\Lambda, W)(W^{(2)})
\]
We take the scalar product in $L^2(R)$ of (13) with $e_1$ and $e_2$. Since $(e_i, \frac{\partial W}{\partial t}) = (\mathcal{L}W|e_i) = 0$, we obtain that:
\[
A(\Lambda) \frac{d\Lambda}{dt} = \sum_{i=1}^5 T'_i \tag{15}
\]
where
\[
A(\Lambda) = \begin{pmatrix}
(e_1 | \partial_\theta R_\Lambda) & (e_1 | \partial_\sigma R_\Lambda) \\
(e_2 | \partial_\theta R_\Lambda) & (e_2 | \partial_\sigma R_\Lambda)
\end{pmatrix} \tag{16}
\]
and
\[
T'_i = \begin{pmatrix}
(T_i | e_1) \\
(T_i | e_2)
\end{pmatrix}
\]
We remark that $A(0) = Id$, thus for $\Lambda$ little enough, we can inverse the matrix $A(\Lambda)$ and we can write the equation satisfied by $\Lambda$ on the form:
\[
\frac{d\Lambda}{dt} = \mathcal{M}_1(\Lambda)(W) + \mathcal{M}_2(W, \frac{\partial W}{\partial x}, \Lambda) \tag{17}
\]
where
\[
\mathcal{M}_1(\Lambda)(W) = A(\Lambda)^{-1} (T'_1 + T'_2)
\]
\[
\mathcal{M}_2(W, \frac{\partial W}{\partial x}, \Lambda) = A(\Lambda)^{-1} (T'_3 + T'_4 + T'_5) \tag{18}
\]
Applying the projection operator $Q$ to (13) yields to the following evolution equation for $W$:

$$
\frac{\partial W}{\partial t} = LW + QK_\Lambda W + QR_1(x, \Lambda, W)(\frac{\partial^2 W}{\partial x^2}) + QR_2(x, \Lambda, W, \frac{\partial W}{\partial x}) + QR_3(x, \Lambda, W)
$$

(19)

where the linear operator $K_\Lambda$ is defined by

$$
K_\Lambda W = K_1^\Lambda W + K_2^\Lambda W + K_3^\Lambda W
$$

(20)

with

$$
K_3^\Lambda W = M_1^1(\Lambda)(W)\partial_0 R_\Lambda - M_1^2(\Lambda)(W)\partial_0 R_\Lambda
$$

(21)

and where the nonlinear term $R_2(x, \Lambda, W, \frac{\partial W}{\partial x})$ is given by:

$$
R_2(x, \Lambda, W, \frac{\partial W}{\partial x}) = R'_2(x, \Lambda, W, \frac{\partial W}{\partial x}) - M_1^1(\Lambda, W, \frac{\partial W}{\partial x})\partial_0 R_\Lambda - M_2^2(\Lambda, W, \frac{\partial W}{\partial x})\partial_2 R_\Lambda
$$

(22)

**Remark 4.** In the projection of Equation (13) we have replaced $\frac{d\theta}{dt}$ and $\frac{d\sigma}{dt}$ by their expressions given by Equation (17). In the previous equations, $M_1^1$ and $M_2^2$ are respectively the first and the second component of $M_i$.

We have thus proved the following proposition:

**Proposition 3.** If $r : (t, x) \mapsto r(t, x)$ is small enough for the norm $L^\infty(\mathbb{R}_t^+; H^2(\mathbb{R}))$, then we can write $r$ on the form

$$
r(t, x) = R_{\Lambda(t)}(x) + W(t, x)
$$

with $\Lambda \in L^\infty(\mathbb{R}_t^+; \mathbb{R}^2)$ and $W \in L^\infty(\mathbb{R}_t^+; \mathcal{E})$. This decomposition is unique.

Furthermore $r$ is solution for Equation (9) if and only if $(\Lambda, W)$ satisfies the system coupling Equation (19) and Equation (17).

4. Linear Estimates.

4.1. **Study of the operator $L$.** The self-adjoint operator $L$ is a compact perturbation of $-\frac{\partial^2}{\partial x^2} + 1$, thus its essential spectrum is $[1, +\infty]$. Furthermore, we can write $L = l^* \circ l$ with $l = -\frac{\partial}{\partial x} + th x$, thus $L$ is positive and 0 is a simple eigenvalue associated with the eigenvector $\frac{1}{ch x}$.

We denote $E = (\text{Ker } L)^\perp$. The restriction of $L$ on $E$ is a symmetric definite positive operator. We denote by $\alpha > 0$ its smallest eigenvalue.

**Proposition 4.** There exist constants $K_1$ and $K_2$ such that for all $u \in E$

$$
K_1||L^* u||_{L^2} \leq ||u||_{H^1} \leq K_2||L u||_{L^2}
$$

$$
K_1||L u||_{L^2} \leq ||u||_{H^2} \leq K_2||L u||_{L^2}
$$

$$
K_1||L^2 u||_{L^2} \leq ||u||_{H^3} \leq K_2||L^2 u||_{L^2}
$$
Proof. Since $\alpha$ is the smaller eigenvalue of $L$ on $E$, we have:

$$\forall u \in H, \|u\|_{L^2} \leq \frac{1}{\alpha} \|Lu\|_{L^2}. \quad (23)$$

Furthermore,

$$\|u''\|_{L^2} = \|u'' - fu + fu\|_{L^2} \leq \|L(u)\|_{L^2} + \|f\|_{L^\infty} \|u\|_{L^2}$$

thus with the previous inequality, we obtain that there exists a constant $K$ such that for all $u$ in $E$

$$\|u\|_{H^2} \leq K \|Lu\|_{L^2} \quad (24)$$

Since the domination of the $L^2$ norm of $Lu$ by the $H^2$ norm of $u$ is obvious, we conclude the proof of the $H^2$ estimate.

Now we have $L^2 u = u^{(4)} - 2fu'' - 2f'u' - f''u + f^2u$ that is

$$\|u^{(4)}\|_{L^2(\mathbb{R})} \leq \|L^2 u\|_{L^2(\mathbb{R})} + C_1 \|u\|_{H^2(\mathbb{R})}$$

since $f$, $f'$ and $f''$ are bounded on $\mathbb{R}$

$$\leq \|L^2 u\|_{L^2(\mathbb{R})} + C_2 K \|Lu\|_{L^2(\mathbb{R})}$$

with Estimate (24)

$$\leq (1 + \frac{C_1 K}{\alpha}) \|L^2 u\|_{L^2(\mathbb{R})}$$

with Estimate (23) applied on $Lu$

thus we obtain that there exists a constant $C_2$ such that

$$\|u\|_{H^4(\mathbb{R})} \leq C_2 \|L^2 u\|_{L^2(\mathbb{R})}$$

Since the opposite bound is obvious, we obtain an estimate about the $H^4$ norm.

By interpolation result, we deduce the intermediate estimates and we conclude the proof of Proposition 4. \qed

4.2. Estimates for the perturbed operator $\mathcal{L} + \mathcal{K}_\Lambda$. We recall that $\mathcal{K}_\Lambda$ is defined by (20).

We remark that since $\Lambda \mapsto R_\Lambda$ is regular and since $R_{\Lambda=0} = 0$, there exists a constant $C_3$ such that

$$\|R_\Lambda\|_{L^\infty(\mathbb{R})} + \|\frac{\partial R_\Lambda}{\partial x}\|_{L^\infty(\mathbb{R})} \leq C_3 |\Lambda| \quad (25)$$

Therefore by properties of $G$, $H_1$, $H_2$ and $P$, there exists then a constant $C_4$ such that

$$\|\mathcal{K}_\Lambda^1 W + \mathcal{K}_\Lambda^2 W\|_{L^2(\mathbb{R})} \leq C_4 |\Lambda| \|W\|_{H^2}$$

Furthermore by properties of $\mathcal{M}_1$ and Proposition 4, since $Q$ is an orthogonal projection in $L^2$, there exists a constant $C_5$ such that

$$\|Q\mathcal{K}_\Lambda W\|_{L^2(\mathbb{R})} \leq C_5 |\Lambda| \|LW\|_{L^2} \quad (26)$$

In the same way, we prove that there exists a constant $C'_5$ such that

$$\|L^{\frac{3}{2}}Q\mathcal{K}_\Lambda W\|_{L^2(\mathbb{R})} \leq C'_5 |\Lambda| \|L^{\frac{3}{2}}W\|_{L^2(\mathbb{R})} \quad (27)$$
In addition, for $W \in \mathcal{E}$

$$(QK_A^2 W|W) = (K_A^2 W|W)$$

since $QW = W$

$$= \int_{\mathbb{R}} G(R_{\Lambda}) \frac{\partial^2 W}{\partial x^2} W$$

$$= - \int_{\mathbb{R}} G'(R_{\Lambda}) \frac{\partial R_{\Lambda}}{\partial x} \frac{\partial W}{\partial x} - \int_{\mathbb{R}} G(R_{\Lambda}) \frac{\partial W}{\partial x} \frac{\partial W}{\partial x}$$

by integration by parts

$$|(QK_A^1 W|W)| \leq C_6 |\Lambda| \|W\|_{H^1(\mathbb{R})}^2$$

with estimate (25)

where the constant $C_6$ does not depend on $\Lambda$ nor on $W$.

Writing that

$$|(QK_A^2 W + QK_A^3 W|W)| \leq \|QK_A^2 W + QK_A^3 W\|_{L^2} \|W\|_{L^2}$$

$$\leq C_7 |\Lambda| \|W\|_{H^1}^2$$

we obtain then that there exists a constant $C_8$ such that

$$|(QK_A W|W)| \leq C_8 |\Lambda| \ell^2 \|W\|_{L^2(\mathbb{R})}^2$$

(28)

We denote by $S_\Lambda(t)$ the semigroup generated by the linear operator $\mathcal{L} + QK_A$.

We have the following proposition:

**Proposition 5.** There exists $\beta > 0$, there exists $\eta_1 > 0$, there exists a constant $K_3$ such that if $|\Lambda(t)| \leq \eta_1$ for all $t \geq 0$ then for $t > 0$

$$\|S_\Lambda(t)W_0\|_{H^1} \leq K_3 e^{-\beta t} \|W_0\|_{H^1}$$

$$\|S_\Lambda(t)W_0\|_{H^1} \leq K_3 e^{-\beta t} \sqrt{t} \|W_0\|_{L^2}$$

for $W_0 \in E$

**Proof.** We fix $W_0 \in \mathcal{E}$ and we denote by $W$ the solution of the Cauchy problem

$$\begin{cases}
\frac{\partial W}{\partial t} = \mathcal{L} W + QK_A W \\
W(t = 0) = W_0
\end{cases}$$

We set $A(t) = \|L^2 W(t)\|_{L^2(\mathbb{R})}^2$.
\[
\frac{dA}{dt} = 2(L \frac{\partial W}{\partial t}|L \frac{\partial W}{\partial t})
\]
\[
= 2(\frac{\partial W}{\partial t}|LW)
\]
\[
= 2(JLW|LW) + (QK_A|LW)
\]
\[
= -2\|LW\|_{L^2(\mathbb{R})}^2 + 2(QK_A|LW)
\]
\[
\leq -2\|LW\|_{L^2(\mathbb{R})}^2 + 2C_5|\Lambda| \|LW\|_{L^2(\mathbb{R})}^2
\]
with Estimate (26).

We fix \( \eta' = \frac{1}{2C_5} \) and for \(|\Lambda| \leq \eta'\) we obtain that
\[
\frac{dA}{dt} \leq -\|LW\|_{L^2(\mathbb{R})}^2
\]
\[
\leq -\frac{1}{K_2^2}\|W\|_{H^2(\mathbb{R})}^2 \leq -\frac{1}{K_2^2}\|W\|_{H^1(\mathbb{R})}^2
\]
with Proposition 4
\[
\leq -\frac{K_3^2}{K_2^2}\|L \frac{\partial W}{\partial t}\|_{L^2(\mathbb{R})}^2
\]
with Proposition 4
\[
\leq \frac{K_3^2}{K_2^2}A
\]
thus \( A(t) \leq A(0)e^{-\frac{K_3^2}{K_2^2}t} \) and then with Proposition 4 there exists a constant \( K'_3 \) such that
\[
\|W(t)\|_{H^1(\mathbb{R})} \leq K'_3e^{-\beta't}\|W_0\|_{H^1(\mathbb{R})}
\]
with \( \beta' = \frac{K_3^2}{2K_2^2} \).

We set now \( B(t) = \|W(t)\|_{L^2(\mathbb{R})}^2 + t\|L \frac{\partial W}{\partial t}\|_{L^2(\mathbb{R})}^2 \).
\[
\frac{dB}{dt} = 2(W|\frac{\partial W}{\partial t}) + (L \frac{\partial W}{\partial t}|L \frac{\partial W}{\partial t}) + 2t(L \frac{\partial W}{\partial t}|L \frac{\partial W}{\partial t})
\]
\[
= 2(W|\mathcal{L}W) + 2(W|QK_AW) + (W|LW) + 2t(\frac{\partial W}{\partial t}|LW)
\]
\[
= -(W|LW) + 2(W|QK_AW) + 2t \left[-\|LW\|_{L^2(\mathbb{R})}^2 + (QK_AW|LW)\right]
\]
\[
\leq -\|L \frac{\partial W}{\partial t}\|_{L^2(\mathbb{R})}^2 + 2C_5|\Lambda| \|L \frac{\partial W}{\partial t}\|_{L^2(\mathbb{R})}^2 - 2t\|LW\|_{L^2(\mathbb{R})}^2 + 2tC_5|\Lambda| \|LW\|_{L^2(\mathbb{R})}^2
\]
with Estimates (26) and (28).
We set \( \eta'' = \min\left(\frac{1}{4C_8}, \frac{1}{2C_9}\right) \) and if \( |\Lambda| \leq \eta'' \) we obtain that
\[
\frac{dB}{dt} \leq -\frac{1}{2} \|L^+W\|_{L^2(\mathbb{R})}^2 + t \|LW\|_{L^2(\mathbb{R})}^2
\]
\[
\leq -\frac{1}{2K_2^2} \|W\|_{H^1(\mathbb{R})}^2 - t \frac{1}{K_2} \|W\|_{H^2(\mathbb{R})}
\]
with Proposition 4
\[
\leq -\frac{1}{2K_2^2} \|W\|_{L^2(\mathbb{R})}^2 - t \frac{K_1^2}{K_2^2} \|L^+W\|_{L^2(\mathbb{R})}
\]
with Proposition 4
\[
\leq -\frac{K_1^2}{2K_2^2} B
\]
Therefore \( B(t) \leq B(0)e^{-\frac{K_1^2}{2K_2^2} t} \). We remark that \( B(0) = \|W_0\|_{L^2(\mathbb{R})} \), thus if we denote \( \beta'' = -\frac{K_1^2}{4K_2^2} \), we obtain that
\[
\|W(t)\|_{L^2(\mathbb{R})}^2 + t \|L^+W\|_{L^2(\mathbb{R})}^2 \leq \|W_0\|_{L^2(\mathbb{R})}^2 e^{-2\beta'' t}
\]
and so using Proposition 4 there exists a constant \( K_3'' \) such that
\[
\|W(t)\|_{H^1(\mathbb{R})} \leq \frac{K_3''}{\sqrt{t}} \|W_0\|_{L^2(\mathbb{R})} e^{-\beta'' t}
\]
Setting \( \eta_1 = \min(\eta', \eta'') \), \( \beta = \min(\beta', \beta'') \) and \( K_3 = \max(K_3', K_3'') \), we conclude the proof of Proposition 5.

5. **Stability.** We consider \((\Lambda, W)\) the solution of System (17)-(19) with initial data \((\Lambda_0, W_0) \in \mathbb{R}^2 \times (H^2(\mathbb{R}))^2\).

In a first step, under Hypothesis \( H: \Lambda(t) \) remains little, we prove that if \( W_0 \) is small, then \( W(t) \) remains close to zero for the \( H^2 \) norm.

In a second step, under Hypothesis \( H \), we show that in addition, \((1 + t)^2 W(t)\) remains bounded for the \( H^1 \) norm.

As a conclusion, we establish that Hypothesis \( H \) is justified when \( \Lambda_0 \) and \( W_0 \) are small.

In the following subsection, we prove preliminar estimates on the non linear terms.

5.1. **Preliminary nonlinear estimates.**

**Lemma 1.** There exists a constant \( K_4 \) such that for all \( \lambda \in \mathbb{R}^2 \) such that \( |\lambda| \leq \eta_1 \) and all \( w \in \mathcal{E} \),
\[
|\mathcal{M}_1(\lambda)(w)| \leq K_4 |\lambda| \|w\|_{H^1(\mathbb{R})}
\]
\[
|\mathcal{M}_2(w, \frac{dw}{dx}, \lambda)| \leq K_4 \|w\|_{L^2(\mathbb{R})}^2
\]
Proof. We recall that \( M_1 \) and \( M_2 \) are defined by (18). We have for \( k = 1, 2 \)
\[
(T_1 e_k) = \int_{\mathbb{R}} G(R_\lambda)(x) \frac{d^2 w}{dx^2}(x) \cdot e_k(x) \, dx
\]
\[
= - \int_{\mathbb{R}} \left( G'(R_\lambda) \frac{dR_\lambda}{dx} e_k + G(R_\lambda) \frac{de_k}{dx} \right) \frac{dw}{dx} \, dx
\]
by integration by parts.

Thus
\[
|T_1 e_k| \leq C|\lambda| \|w\|_{H^1(\mathbb{R})}
\]

Furthermore, with the definition of \( T_2 \) (cf. Equation (14)) there exists a constant \( C \) such that
\[
|T_2| \leq C\|w\|_{H^1(\mathbb{R})}|\lambda|
\]
Since the matrix \( A(\lambda) \) is invertible for \(|\lambda| \leq \eta_1 \), we obtain the estimation on \( M_1 \).

Concerning \( M_2 \) we remark that for \( k = 1, 2 \)
\[
(T_3 e_k) = \int_{\mathbb{R}} \hat{G}(R_\lambda, w)(w) \frac{d^2 w}{dx^2}(x) \cdot e_k(x) \, dx
\]
\[
= - \int_{\mathbb{R}} \left[ \frac{d}{dx} \left( \hat{G}(R_\lambda, w)(w) \frac{dw}{dx} + \hat{G}(R_\lambda, w)(w) \frac{d^2 w}{dx^2} \right) \frac{de_k}{dx} \right] \, dx
\]
that is there exists a constant \( C \) such that
\[
|T_3^4| \leq C\|w\|^2_{H^1(\mathbb{R})}
\]
A straightforward estimate on \( T_4 \) and \( T_5 \) gives that there exists a constant \( C \) such that
\[
|T_4| + |T_5| \leq C\|w\|_{H^1(\mathbb{R})}^2,
\]
therefore since \( A(\lambda) \) is invertible, we conclude the proof of Lemma 1.

Lemma 2. There exists a constant \( K_5 \) such that for all \( \lambda \) such that \(|\lambda| \leq \eta_1 \) and all \( w \in \mathcal{E} \),
\[
\|QR_1(x, \lambda, w)\left(\frac{d^2 w}{dx^2}\right)\|_{L^2(\mathbb{R})} \leq K_5 \|w\|_{H^1(\mathbb{R})} \|w\|_{H^2(\mathbb{R})}
\]
\[
\|QR_1(x, \lambda, w)\|_{H^1(\mathbb{R})} \leq K_5 \|w\|_{H^2(\mathbb{R})} \|w\|_{H^3(\mathbb{R})}
\]
\[
\|QR_2(x, \lambda, w)\|_{H^1(\mathbb{R})} \leq K_5 \|w\|_{H^1(\mathbb{R})} \|w\|_{H^2(\mathbb{R})}
\]
\[
\|QR_3(x, \lambda, w)\|_{H^1(\mathbb{R})} \leq K_5 \|w\|_{H^1(\mathbb{R})}^2
\]
Proof. It is a straightforward application of the definitions of \( R_1, R_2, R_3 \), of the properties of \( G, H_1, H_2, \) and \( P \), and of Proposition 4.
5.2. First step: variational estimate on $W$.

**Proposition 6.** There exists $\eta_2 > 0$ (with $\eta_2 < \eta_1$) such that if $|\Lambda(t)| \leq \eta_2$ for all $t$, then, there exists a constant $\gamma_1$ such that if $\|LW(t = 0)\|_{L^2} \leq \gamma_1$, then $t \mapsto \|LW\|_{L^2}$ is decreasing and there exists $K_6$ such that

$$\forall \ t, \ \|W(t)\|_{H^2(\mathbb{R})} \leq K_6\|W_0\|_{H^2(\mathbb{R})}$$

**Proof.** We take the scalar product on Equation (19) with $J^2L^2W$. We remark that:

- $\left(\frac{\partial W}{\partial t} \mid J^2L^2W\right) = \left(\frac{\partial W}{\partial t} \mid J^4L^2W\right) = -4\left(\frac{\partial W}{\partial t} \mid L^2W\right) = -2\frac{d}{dt}\|LW\|_{L^2}^2$

- $(LW \mid J^2L^2W) = -4(JLW \mid L^2W) = -4(JL^2W \mid L^2W) = 4\|L^2W\|_{L^2}^2$

- $(Q\Lambda W \mid J^2L^2W) = \left(L^2Q\Lambda W \mid J^4L^2W\right)$ thus, since $J^4 = -4Id$, with Estimate (27)

$$\left|\left(Q\Lambda W \mid J^2L^2W\right)\right| \leq 4C_5\|\Lambda\|\|L^2W\|_{L^2}^2$$

- $\left|\left(QR_1(x, \Lambda, W)(\frac{\partial^2 W}{\partial x^2}) \mid J^2L^2W\right)\right| \leq 4\left\|L^2\left[QR_1(x, \Lambda, W)(\frac{\partial^2 W}{\partial x^2})\right]\right\|\|L^2W\|_{L^2(\mathbb{R})}^2$

$$\leq \frac{4K_1}{K_1}\left\|QR_1(x, \Lambda, W)(\frac{\partial^2 W}{\partial x^2})\right\|_{H^1(\mathbb{R})}\|L^2W\|_{L^2(\mathbb{R})}^2$$

with Proposition 4

$$\leq \frac{4K_5K_2^2}{K_1}\|LW\|_{L^2}\|L^2W\|_{L^2}^2$$

with Proposition 4

In the same way, we prove that

$$\left|\left(QR_2(x, \Lambda, W, \frac{\partial W}{\partial x}) \mid J^2L^2W\right)\right| \leq \frac{4K_5K_2^2}{K_1}\|LW\|_{L^2}\|L^2W\|_{L^2}^2$$

and that

$$\left|\left(QR_3(x, \Lambda, W)\mid J^2L^2W\right)\right| \leq \frac{4K_5K_2^2}{K_1}\|LW\|_{L^2}\|L^2W\|_{L^2}^2$$

Therefore we obtain that if $|\Lambda| \leq \eta_2$, then

$$\frac{d}{dt}\|LW\|_{L^2}^2 + 2\|L^2W\|_{L^2}^2 \leq 2C_5\eta_2\|L^2W\|_{L^2}^2 + \frac{6K_5K_2^2}{K_1}\|LW\|_{L^2}\|L^2W\|_{L^2}^2$$

that is:

$$\frac{d}{dt}\|LW\|_{L^2}^2 + \|L^2W\|_{L^2}^2 \left(2 - 2C_5\eta_2 - \frac{6K_5K_2^2}{K_1}\|LW\|_{L^2}\right) \leq 0 \quad (29)$$
We fix $\eta_2 < \eta_1$ such that $2C_3^* \eta_2 < 1$, and we set $\gamma_1 = \frac{K_1}{6 K_5 K_2^2}$. If $|A| \leq \eta_2$ then while $\|LW(t)\|_{L^2(\mathbb{R})} \leq \gamma_1$ this quantity remains decreasing with Equation (29), and thus remains less than $\gamma_1$.

Therefore, with Proposition 4, we have:

$$\forall t \geq 0, \|W(t)\|_{H^2(\mathbb{R})} \leq \frac{K_2}{K_1} \|LW(t)\|_{L^2(\mathbb{R})} \leq \frac{K_2}{K_1} \|LW_0\|_{L^2(\mathbb{R})}$$

and we conclude the proof setting $K_6 = \frac{K_2}{K_1}$. 

\[ \square \]

### 5.3. Second step: parabolic estimates on $W$. Using Equation (19) we have:

$$W(t) = S_\Lambda(t)W_0 + \int_0^t S_\Lambda(t-s)QR_1(x, \Lambda, W)(\frac{\partial^2 W}{\partial x^2})(s)\,ds$$

$$+ \int_0^t S_\Lambda(t-s)QR_2(x, \Lambda, W, \frac{\partial W}{\partial x})(s)\,ds$$

$$+ \int_0^t S_\Lambda(t-s)QR_3(x, \Lambda, W)(s)\,ds$$

and with Proposition 5, we know that while $|\Lambda(t)| \leq \eta_1$ there exists a constant $K_3$ such that

$$\|W(t)\|_{H^1(\mathbb{R})} \leq K_3 e^{-\beta t} \|W_0\|_{H^1(\mathbb{R})} + \int_0^t K_3 e^{-\beta(t-s)} \|QR_1(x, \Lambda, W)(\frac{\partial^2 W}{\partial x^2})(s)\|_{L^2(\mathbb{R})} ds$$

$$+ \int_0^t K_3 e^{-\beta(t-s)} \|QR_2(x, \Lambda, W, \frac{\partial W}{\partial x})(s)\|_{H^1(\mathbb{R})} ds$$

$$+ \int_0^t K_3 e^{-\beta(t-s)} \|QR_3(x, \Lambda, W)(s)\|_{H^1(\mathbb{R})} ds$$

Using Lemma 2 we obtain that

$$\|W(t)\|_{H^1(\mathbb{R})} \leq K_3 e^{-\beta t} \|W_0\|_{H^1(\mathbb{R})} + \int_0^t K_3 e^{-\beta(t-s)} K_5 \|W(s)\|_{H^1(\mathbb{R})} W(s)\|_{H^2(\mathbb{R})} ds$$

$$+ \int_0^t K_3 e^{-\beta(t-s)} K_5 \|W(s)\|_{H^1(\mathbb{R})} W(s)\|_{H^2(\mathbb{R})} ds$$

$$+ \int_0^t K_3 e^{-\beta(t-s)} K_5 \|W(s)\|_{H^1(\mathbb{R})}^2 ds$$

Using Proposition 6 we know that if $\|W_0\|_{H^2(\mathbb{R})} \leq \gamma_1$ and if $|\Lambda(t)|$ remains less than $\eta_2$ then $\|W(s)\|_{H^2(\mathbb{R})} \leq K_6 \|W_0\|_{H^2(\mathbb{R})}$ for all $s$.

We define $G(t)$ by

$$G(t) = \sup_{s \in [0,t]} (1 + s)^2 \|W(s)\|_{H^1}$$
We obtain then that
\[ \|W(t)\|_{H^1} \leq K_3 e^{-\beta t} \|W_0\|_{H^1} + K_3 K_5 \left( \int_0^t (1 + s)^{-4} e^{-\beta (t-s)} ds \right) [G(t)]^2 \]
\[ + K_3 K_5 K_6 \|W_0\|_{H^2(R)} G(t) \left( \int_0^t \frac{e^{-\beta (t-s)}}{\sqrt{t-s}} (1 + s)^{-2} ds + \int_0^t e^{-\beta (t-s)} (1 + s)^{-2} ds \right) \]

Now there exists a constant \( K_7 \) such that for all \( t \) we have
\[ \frac{e^{-\beta t}}{(1 + t)^2} \leq K_7 \]
\[ \int_0^t \frac{e^{-\beta (t-s)}}{\sqrt{t-s}} (1 + s)^{-2} ds + \int_0^t e^{-\beta (t-s)} (1 + s)^{-2} ds \leq \frac{K_7}{(1 + t)^2} \]
\[ \int_0^t (1 + s)^{-4} e^{-\beta (t-s)} ds \leq \frac{K_7}{(1 + t)^2} \]

Thus we obtain that there exists a constant \( C \) such that
\( (1 + t)^k \|W(t)\|_{H^1} \leq K_3 K_7 \|W_0\|_{H^1} + K_3 K_5 K_6 K_7 \|W_0\|_{H^2} G(t) + K_3 K_5 K_7 (G(t))^2 \)
and since \( G \) is a non decreasing map, denoting \( \alpha_1 = \max\{K_3 K_7, \frac{1}{4}\} \), \( \alpha_2 = K_3 K_5 K_6 K_7 \)
and \( \alpha_3 = K_3 K_5 K_7 \) we obtain that
\[ G(t) \leq \alpha_1 \|W_0\|_{H^1} + \alpha_2 \|W_0\|_{H^2} G(t) + \alpha_3 (G(t))^2 \] (30)

We have then the following result:

**Proposition 7.** Let \( \eta_2 \) and \( \gamma_1 \) being given by Proposition 6. There exists \( \gamma_2 \) with \( 0 < \gamma_2 < \gamma_1 \) such that for all \( \delta > 0 \) there exists \( \tau > 0 \) such that if the following assumptions are satisfied:

(i) for all \( t \), \( |\Lambda(t)| \leq \eta_2 \),
(ii) \( \|LW_0\|_{L^2(R)} \leq \gamma_2 \),
(iii) \( \|W_0\|_{H^1(R)} \leq \tau \),

then for all \( t > 0 \) we have
\[ \|W(t)\|_{H^1(R)} \leq \frac{\delta}{(1 + t)^2} \]

**Proof.** Under Hypothesis (i) and if \( \|LW_0\|_{L^2(R)} \leq \gamma_1 \) we have proved Estimate (30).

We set \( \gamma_2 = \min\left(\frac{1}{2\alpha_2}, \gamma_1\right) \). Under Hypothesis (i) and (ii) we have that for all \( t \)
\[ \alpha_3 (G(t))^2 - \frac{1}{2} G(t) + \alpha_1 \|W_0\|_{H^1(R)} \geq 0 \] (31)

Let us study the polynomial map \( P_\nu : \xi \mapsto \alpha_3 \xi^2 - \frac{1}{2} \xi + \alpha_1 \nu \). If \( \nu < \gamma_2 := \frac{1}{16 \alpha_1 \alpha_3} \)
then this polynomial map has two positive zeros. The smaller one is \( \xi_1(\nu) = \frac{1}{4 \alpha_3} \sqrt{4 \alpha_3 - 1} \).
We remark that since \( \alpha_1 \geq \frac{1}{4} \) then \( \xi_1(\nu) \geq \nu \).

Let \( \delta > 0 \) be fixed.
The map $\nu \mapsto \xi_1(\nu)$ tends to zero when $\nu$ tends to zero, so we can fix $\tau > 0$ such that for $\nu \in [0, \tau]$, $\xi(\nu) \leq \delta$. Even if it means reducing $\tau$ we can assume that $\tau \leq \delta$ and $\tau \leq \gamma_2/2$.

Under Hypothesis $(i)$, $(ii)$ and $(iii)$, the map $G(t)$ satisfies (31) and $G(0) = \|W_0\|_{H^1(\mathbb{R})} \leq \xi_1(\|W_0\|_{H^1(\mathbb{R})})$. Thus for all $t$, $G(t) \leq \xi_1(\|W_0\|_{H^1(\mathbb{R})}) \leq \delta$.

This concludes the proof of Proposition 7.

5.4. Estimates for $\Lambda$. We integrate Equation (17) between $t = 0$ and $t$. We obtain that

$$|\Lambda(t)| \leq |\Lambda_0| + \int_0^t |M_1(\Lambda(s))(W(s))|ds + \int_0^t |M_2(W(s)) \frac{\partial W}{\partial x}(s), \Lambda(s))|ds$$

(32)

We assume that $|\Lambda_0| \leq \frac{\eta_2}{2}$ and that $\|W_0\|_{L^2(\mathbb{R})} \leq \gamma_2$. We fix an arbitrary $\delta$. with Proposition 7, while $|\Lambda(t)|$ remains less that $\eta_2$ we have, if $\|W_0\|_{H^1(\mathbb{R})} \leq \tau$ we have:

$$\|W(t)\|_{H^1(\mathbb{R})} \leq \frac{\delta}{(1 + t)^2}.$$  

Using this estimate in Equation (32) and using Lemma 1 we obtain that while $|\Lambda(t)| \leq \eta_2$ we have:

$$|\Lambda(t)| \leq \frac{\eta_2}{2} + \int_0^t K_4 \eta_2 \delta \frac{1}{(1 + s)^2} ds + \int_0^t K_4 \delta^2 \frac{1}{(1 + s)^4} ds.$$  

(33)

We fix $\delta > 0$ such that

$$K_4 \eta_2 \delta \int_0^{+\infty} \frac{1}{(1 + s)^2} ds + K_4 \delta^2 \int_0^{+\infty} \frac{1}{(1 + s)^4} ds \leq \frac{\eta_2}{2}.$$  

With Proposition 6 we find $\tau_0 > 0$ and if $|\Lambda_0| \leq \frac{\eta_2}{2}$, $\|LW_0\|_{H^2(\mathbb{R})} \leq \gamma_2$ and if $\|W_0\|_{H^1(\mathbb{R})} \leq \tau_0$ then with Estimate (33), $|\Lambda(t)|$ remains less than $\eta_2$ for all time, and all the estimates are true for all time, which concludes the proof of our theorem.

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