RELAXATION APPROXIMATION OF SOME INITIAL-BOUNDARY VALUE PROBLEM FOR P-SYSTEMS

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Abstract. We consider the Suliciu model which is a relaxation approximation of the p-system. In the case of the Dirichlet boundary condition we prove that the local smooth solution of the p-system is the zero limit of the Suliciu model solutions.

Key words. Zero relaxation limit, p-system, Suliciu model, boundary conditions.

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1. Introduction

We study a relaxation approximation of the following p-system

\[
\begin{align*}
\partial_t u_1 - \partial_x u_2 &= 0, \\
\partial_t u_2 - \partial_x p(u_1) &= 0.
\end{align*}
\]  (1.1)

For the viscoelastic case, Suliciu introduces in [19] the following approximation

\[
\begin{align*}
\partial_t u_1 - \partial_x u_2 &= 0, \\
\partial_t u_2 - \partial_x v &= 0, \\
\partial_t v - \mu \partial_x u_2 &= \frac{1}{\varepsilon} (p(u_1) - v),
\end{align*}
\]  (1.2)

where \( \varepsilon \) and \( \mu \) are positive.

The aim of this paper is to prove convergence results for the initial-boundary value problem when the relaxation coefficient \( \varepsilon \) tends to zero.

Under the classical assumption

\[
\forall \xi \in \mathbb{R}, p'((\xi)) > 0,
\]  (1.3)

the p-system is strictly hyperbolic with eigenvalues

\[
\lambda_1(u_1) = -\sqrt{p''(u_1)}, \quad \lambda_2(u_1) = \sqrt{p''(u_1)}. 
\]  (1.4)

The semi-linear approximation system (1.2) is strictly hyperbolic with 3 constant eigenvalues

\[
\mu_1 = -\sqrt{\mu} < \mu_2 = 0 < \mu_3 = \sqrt{\mu}. 
\]  (1.5)

In all the paper we assume that \( \mu \) is chosen great enough so that the subcharacteristic-type condition holds

\[
\mu > p'(u_1) 
\]  (1.6)
for all the values of $u_1$ under consideration.

Formally, when $\varepsilon$ tends to zero, the behaviour of the solution $w^\varepsilon = (u^\varepsilon, v^\varepsilon) = ((u_1^\varepsilon, u_2^\varepsilon), v^\varepsilon)$ for the relaxation system (1.2) is the following: $p(u_1^\varepsilon) - v^\varepsilon$ tends to zero, so that $u^\varepsilon$ tends to a solution $u = (u_1, u_2)$ of the p-system (1.1).

Recent papers are devoted to the zero relaxation limit in the case of the Cauchy problem. In [22] Wen-An Yong establishes a general framework to study the strong convergence for the smooth solutions. This convergence result is obtained describing the boundary layer which appears at $t = 0$. We can apply Yong’s tools for the Suliciu approximation

$$\begin{cases}
\partial_t u_1^\varepsilon - \partial_x u_2^\varepsilon = 0, \\
\partial_t u_2^\varepsilon - \partial_x v^\varepsilon = 0, \\
\partial_t v^\varepsilon - \mu \partial_x u_2^\varepsilon = \frac{1}{\varepsilon} (p(u_1^\varepsilon) - v^\varepsilon),
\end{cases}
$$

for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, with the smooth initial data:

$$w^\varepsilon(0, x) = w_0(x), x \in \mathbb{R}. \quad (1.8)$$

We give more details about this question in the annex at the end of this paper.

Since the lifespan for a smooth solution $u$ of the Cauchy problem for the p-system is generally finite (see [12]), the strong convergence of the solution $u^\varepsilon$ to $u$ can only be obtained locally in time. Nevertheless, under the assumption

$$\forall \xi \in \mathbb{R}, p'(\xi) \leq \Gamma < \mu, \quad (1.9)$$

if $w_0$ is smooth, the solution for the semi-linear Cauchy problem (1.7)-(1.8) is global and smooth. In this case, the question is: what about the global convergence?

Under further additional assumptions (in particular $p'(\xi) \geq \gamma > 0$) the weak convergence to a global weak solution of the p-system is obtained by Tzavaras in [21] using the compactness methods of [17].

Other convergence results in some particular cases can be found in [8] and [10]. For other connected papers see also [13, 16, 20]...

In this paper we study the zero relaxation limit for the initial-boundary value problem. To our knowledge general convergence results are not available for hyperbolic relaxation systems in domains with boundary in the literature. A special well investigated problem is the semi-linear relaxation approximation to the boundary value problem for a scalar quasilinear equation, see [11, 15, 9, 14], and [5, 1] for related numerical considerations.

A first example of convergence result for a particular p-system (1.1) is obtained in [4]. In that paper the p-system is the one-dimensional Kerr model, so $p$ is the inverse function of $\xi \mapsto (1 + \xi^2)\xi$. The relaxation approximation is given by the Kerr-Debye model which is the following quasilinear hyperbolic system

$$\begin{cases}
\partial_t u_1^\varepsilon - \partial_x u_2^\varepsilon = 0, \\
\partial_t u_2^\varepsilon - \partial_x \left( (1 + v^\varepsilon)^{-1} u_1^\varepsilon \right) = 0, \\
\partial_t v^\varepsilon = \frac{1}{\varepsilon} \left( (1 + v^\varepsilon)^{-2} (u_1^\varepsilon)^2 - v^\varepsilon \right).
\end{cases}$$
For these two models we consider the ingoing wave boundary condition. In the case of the smooth solutions we obtained a local strong convergence result. The main tool of the proof is the use of the entropic variables as proposed in [7]. In these variables, the system is symmetrized and the equilibrium manifold is linearized.

Here we study the zero relaxation limit for the Suliciu approximation

\[
\begin{align*}
\partial_t u_1^\varepsilon - \partial_x u_2^\varepsilon &= 0, \\
\partial_t u_2^\varepsilon - \partial_x v^\varepsilon &= 0, \\
\partial_t v^\varepsilon - \mu \partial_x u_2^\varepsilon &= \frac{1}{\varepsilon} (p(u_1^\varepsilon) - v^\varepsilon),
\end{align*}
\]

for \((t,x) \in \mathbb{R}^+ \times \mathbb{R}^+\), with the null initial data

\[
w^\varepsilon(0,x) = 0, x \in \mathbb{R}^+,
\]

and with the Dirichlet boundary condition

\[
u_2^\varepsilon(t,0) = \varphi(t), t \in \mathbb{R}^+.
\]

For the null initial data to be in equilibrium we assume that \(p(0) = 0\). We prove the strong convergence of \(u^\varepsilon\) to the smooth solution of the initial-boundary value problem for the p-system

\[
\begin{align*}
\partial_t u_1 - \partial_x u_2 &= 0, \\
\partial_t u_2 - \partial_x p(u_1) &= 0,
\end{align*}
\]

for \((t,x) \in \mathbb{R}^+ \times \mathbb{R}^+\), with the initial-boundary conditions

\[
u(0,x) = 0, x \in \mathbb{R}^+, \\
u_2(t,0) = \varphi(t), t \in \mathbb{R}^+.
\]

2. Main Results

Let us specify the assumptions on the source term \(\varphi\) in the boundary condition (1.12) or (1.15). In order to simplify we choose \(\varphi\) smooth enough on \(\mathbb{R}\) and such that \(\text{supp} \ \varphi \subset [0,b]\), with \(b > 0\). In this case the boundary conditions and the null initial data (1.11) and (1.14) match each other so both initial-boundary value problem (1.10)-(1.11)-(1.12) and (1.13)-(1.14)-(1.15) admit local smooth solutions.

First we consider the solutions for the second problem (1.13)-(1.14)-(1.15) and using the methods of [12] we establish that the lifespan \(T^*\) is generally finite with formation of shock waves.

**Theorem 2.1.** Assume the property (1.3). Let \(\varphi \in C^\infty(\mathbb{R})\) with \(\text{supp} \ \varphi \subset [0,b]\), \(b > 0\), \(\varphi \neq 0\). Let \(g\) the function defined by

\[
g(\xi) = \int_0^\xi \sqrt{p'(s)} ds.
\]

We assume that

\[
p'' \text{ does not vanish on the interval } g^{-1}(-\varphi(\mathbb{R})).
\]
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Then the local smooth solution of (1.13)-(1.14)-(1.15) exhibits a shock wave at the time $T^* < +\infty$ and we have

$$\|u\|_{L^\infty([0,T^*] \times \mathbb{R}^+)} \leq C\|\varphi\|_{L^\infty(\mathbb{R})}. \quad (2.2)$$

We now investigate the smooth solutions of the initial-boundary value problem (1.10)-(1.11)-(1.12) for a fixed $\varepsilon > 0$. The system is semi-linear strictly hyperbolic and the boundary $\{x = 0\}$ is characteristic. It is easy to prove that the local smooth solution $w$ exists and, if the lifespan $T^*$ is finite, we have

$$\|w\|_{L^\infty([0,T^*] \times \mathbb{R}^+)} = +\infty \quad (2.3)$$

(for general semi-linear hyperbolic systems, see [18]).

If we assume that $p$ is globally lipschitz we establish that the smooth solutions are global.

**Theorem 2.2.** Assume the properties (1.3) and (1.9). Let $\varphi \in H^3(\mathbb{R})$ with $\text{supp} \varphi \subset \mathbb{R}^+$. Then the solution of (1.10)-(1.11)-(1.12) is global and

$$w \in C^0(\mathbb{R}^+; H^1(\mathbb{R})), \partial_tw \in C^0(\mathbb{R}^+; L^2(\mathbb{R})). \quad (2.4)$$

Finally, let us describe the convergence result.

**Theorem 2.3.** We suppose (1.3). Let $\varphi \in H^3(\mathbb{R})$ with $\text{supp} \varphi \subset \mathbb{R}^+$. We consider a smooth solution $u = (u_0^1, u_0^2)$ of (1.13)-(1.14)-(1.15) defined on $[0,T^*]$. We suppose that

$$\mu > \sup_{(t,x)\in [0,T^*] \times \mathbb{R}^+} p'(u_1^0(t,x)). \quad (2.5)$$

Let $T < T^*$. For $\varepsilon$ small enough, the relaxation problem (1.10)-(1.11)-(1.12) admits a solution $w^\varepsilon = (u^\varepsilon, v^\varepsilon)$ defined on $[0,T]$ such that

$$u^\varepsilon = u_0^0 + \varepsilon u_1^1,$$

and there exists a constant $K$ such that

$$\|u_1^1\|_{L^\infty(0,T; H^1(\mathbb{R}^+))} + \|\partial_t u_1^1\|_{L^\infty(0,T; L^2(\mathbb{R}^+))} \leq K. \quad (2.6)$$

In this result we can remark that no boundary layer appears in the time variable because the null initial data belongs to the equilibrium manifold $\mathcal{V} = \{v = p(u_1^0)\}$. For the space variable, we have the same boundary condition for both systems, so no space boundary layer appears.

To prove Theorem 2.3 we don’t use the method in [4]: as observed in [7], with the entropic variables, we lose the semi-linear character of the system (1.10). We prefer write the following expansion of $w^\varepsilon$

$$w^\varepsilon = w_0^0 + \varepsilon w_1^1 = ((u_1^0, u_2^0), p(u_1^0)) + \varepsilon w_1^1$$

so that the rest term $w_1^1$ satisfies a semi-linear hyperbolic system. In order to estimate $w_1^1$, we use the conservative-dissipative variables introduced in [2]. With these variables the system is symmetrized and its semi-linear character is preserved. Furthermore by this method we obtain a more precise result : for $\varepsilon$ small enough the lifespan $T^*$ is greater that the lifespan $T^*$ of the limit system solution and the convergence is proved on all compact subset of $[0,T^*]$. 


3. Proof of Theorem 2.1
We use the methods proposed by Majda in [12] for the Cauchy problem. We denote by \( l \) and \( r \) the left and right Riemann invariants of the system (1.1):
\[
\begin{align*}
  l &= \frac{1}{2}(u_2 + g(u_1)), \\
  r &= \frac{1}{2}(u_2 - g(u_1)).
\end{align*}
\]
These variables define a diffeomorphism which inverse is given by
\[
\begin{align*}
  u_1 &= g^{-1}(l - r), \\
  u_2 &= l + r.
\end{align*}
\]
These invariants \((l, r)\) satisfy the diagonal system
\[
\begin{align*}
  \partial_t l - \nu(l-r) \partial_x l &= 0, \\
  \partial_t r + \nu(l-r) \partial_x r &= 0, \\
  l(0, x) &= r(0, x) = 0, x > 0, \\
  (l+r)(t, 0) &= \varphi(t), t > 0,
\end{align*}
\]
where \( \nu(l-r) = \sqrt{p'(g^{-1}(l-r))} \). The smooth solution of (3.1) is \((0, r)\) where \( r \) is the solution of the scalar equation
\[
\begin{align*}
  \partial_t r + \nu(-r) \partial_x r &= 0, \\
  r(0, x) &= 0, x > 0, \\
  r(t, 0) &= \varphi(t), t > 0.
\end{align*}
\]
Under the assumptions (1.3) and (2.1) we will prove that the lifespan \( T^* \) of the solution of the problem (3.2) is finite and that this solution exhibits shock waves in \( T^* \).

For solving (3.2) we can use the method of characteristics. The function \( r \) is constant on the characteristic curves which are the straight lines \( t = T + \frac{1}{\nu(-\varphi(T))} x, T \in \mathbb{R} \).

Denoting \( \alpha(s) = \frac{1}{\nu(-s)} \) we obtain then that
\[
  r(T, 0) = \varphi(T) = r(T + \alpha(\varphi(T)) x, x).
\]
Let us introduce the mapping
\[
(T, X) \mapsto \Phi(T, X) = (t, x) = (T + \alpha(\varphi(T)) X, X).
\]
This map is a diffeomorphism for \( X < \tilde{X} \) with
\[
\tilde{X} = \left[ \max_{T \in [0, h]} \frac{d}{dT} \alpha(\varphi(T)) \right]^{-1}.
\]
Under assumption (2.1) we have $0 < \dot{X} < +\infty$ and we have
\[ \|r\|_{L^\infty(\mathbb{R}^+ \times [0, \dot{X}])} \leq \|\varphi\|_{L^\infty(\mathbb{R})}. \]
The characteristic curves through $(0, 0)$ and $(b, 0)$ cut the straight line $\{x = \dot{X}\}$ at times $T_1 = \sqrt{p'(0)}^{-1} \dot{X}$ and $T_2 = b + \sqrt{p'(0)}^{-1} \dot{X}$ so $T^* \in [T_1, T_2]$.

4. Proof of Theorem 2.2
In this section $\varepsilon > 0$ and $\mu > 0$ are fixed. We rewrite system (1.10)
\[ \partial_t w + A \partial_x w = h(w) \]
where
\[ A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & -\mu & 0 \end{pmatrix} \quad \text{and} \quad h(w) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\varepsilon} (p(u_1) - v) \end{pmatrix} \]
and by (1.3) and (1.9) $p$ is globally lipschitz. As zero is an eigenvalue of the matrix $A$, the boundary $\{x = 0\}$ is characteristic, so for completeness we give the proof of the global existence. Using (2.3) it is sufficient to prove that the solution $w$ is bounded on any domain $[0, T] \times \mathbb{R}^+$. In a first step we lift the boundary condition (1.12). We set $\omega(t, x) = \varphi(t) \eta(x)$ where $\eta$ is a smooth function compactly supported with $\eta(0) = 1$. We replace $u_2$ by $u_2 - \omega$ and we obtain the following initial-boundary value problem
\begin{equation}
\begin{cases}
\partial_t w + A \partial_x w = h(w) + \begin{pmatrix} \partial_x \omega \\ -\partial_t \omega \\ \mu \partial_x \omega \end{pmatrix}, \\
w(0, x) = 0, x \in \mathbb{R}^+, \\
u_2(t, 0) = 0, t \in \mathbb{R}^+.
\end{cases}
\end{equation}

We diagonalize the matrix $A$ by the matrix $P$: $w = PW$ with
\[ P = \begin{pmatrix} 1 & 1 & 1 \\ \sqrt{\mu} & 0 & -\sqrt{\mu} \\ \mu & 0 & \mu \end{pmatrix}. \]
We obtain
\begin{equation}
\begin{cases}
\partial_t W - \begin{pmatrix} \nu_0 & 0 & 0 \\ 0 & \nu_0 & 0 \\ 0 & 0 & \nu_0 \end{pmatrix} \partial_x W = H(W) + \Phi, \\
W(0, x) = 0, x \in \mathbb{R}^+, \\
W_1(t, 0) - W_3(t, 0) = 0, t \in \mathbb{R}^+.
\end{cases}
\end{equation}
We have $H(W) = P^{-1} h(PW)$ so $H$ is globally lipschitz
\[ \exists K > 0, |\partial_W H| \leq K. \]
In addition, $\Phi$ is given by

$$\Phi = P^{-1} \begin{pmatrix} \partial_x \omega \\ -\partial_t \omega \\ \mu \partial_x \omega \end{pmatrix}.$$  

We denote by $T^*$ the lifespan of the solution $W$ for system (4.2) and we assume that $T^* < +\infty$. We will prove that $\|W\|_{L^\infty([0,T^*] \times \mathbb{R}^+)} < +\infty$ so that by (2.3) we obtain a contradiction.

$L^2$ estimate

We take the inner product of the first equation in (4.2) by $W$ and we obtain

$$\frac{1}{2} \frac{d}{dt} \|W\|_{L^2(\mathbb{R}^+)}^2 + \int_{\mathbb{R}^+} \sqrt{\mu} (-W_1 \partial_x W_1 + W_3 \partial_x W_3) dx = \int_{\mathbb{R}^+} H(W) W dx + \int_{\mathbb{R}^+} \Phi W dx.$$  

Using the third equation in (4.2) and (4.3) we obtain

$$\frac{1}{2} \frac{d}{dt} \|W\|_{L^2(\mathbb{R}^+)}^2 \leq C(1 + \|W\|_{L^2(\mathbb{R}^+)}^2). \quad (4.4)$$  

$H^1$ estimate

We derive system (4.2) with respect to $t$ and with similar computations we obtain that

$$\frac{1}{2} \frac{d}{dt} \|\partial_t W\|_{L^2(\mathbb{R}^+)}^2 \leq C(1 + \|\partial_t W\|_{L^2(\mathbb{R}^+)}^2). \quad (4.5)$$  

By Gronwall lemma we obtain from (4.4) and (4.5) that

$$\|W\|_{L^\infty([0,T^*];L^2(\mathbb{R}^+))} + \|\partial_t W\|_{L^\infty([0,T^*];L^2(\mathbb{R}^+))} \leq C(T^*). \quad (4.6)$$  

So using the first equation in (4.2) we have

$$\|\partial_x W_1\|_{L^\infty([0,T^*];L^2(\mathbb{R}^+))} + \|\partial_x W_3\|_{L^\infty([0,T^*];L^2(\mathbb{R}^+))} \leq C(T^*), \quad (4.7)$$  

In addition we have

$$\partial_t \partial_x W_2 - \partial W_2 H_2(W) \partial_x W_2 = \mathcal{H}(t,x),$$  

where

$$\mathcal{H} = \partial W_1 H_2(W) \partial_x W_1 + \partial W_3 H_2(W) \partial_x W_3 + \partial_x \Phi_2.$$  

By (4.3) and (4.7) we have

$$\|\mathcal{H}\|_{L^\infty([0,T^*];L^2(\mathbb{R}^+))} \leq C(T^*),$$  

and since

$$\partial_t W_2(t,x) = \int_0^t \left( \exp \int_s^t \partial W_2 H_2(W(\tau,x)) d\tau \right) \mathcal{H}(s,x) ds,$$

we conclude that

$$\|\partial_x W_2\|_{L^\infty([0,T^*];L^2(\mathbb{R}^+))} \leq C(T^*).$$  

By Sobolev injections we can apply the continuation principle and we conclude the proof of Theorem 2.2.
5. Proof of Theorem 2.3

We denote by $T^*$ the lifespan of the smooth solution $u^0 = (u^0_1, u^0_2)$ of system (1.13)-(1.14)-(1.15). Since the boundary data $\varphi$ belongs to $H^3(\mathbb{R})$ we have

$$\partial_t^i u^0 \in C^0([0,T^*]; H^{3-i}(\mathbb{R}^+)), \ i = 0, 1, 2, 3. \quad (5.1)$$

We define the profile $w^0$ by

$$w^0 = (u^0, v^0) = ((u^0_1, u^0_2), p(u^0_1)). \quad (5.2)$$

We denote

$$\gamma(t,x) = p'(u^0_1(t,x)), t < T^*, x > 0, \quad (5.3)$$

$$\Gamma = \sup_{(t,x) \in [0,T^*] \times \mathbb{R}^+} \gamma(t,x), \quad (5.4)$$

and by (2.2), $\Gamma < +\infty$. We fix $\mu$ such that

$$\mu > \Gamma. \quad (5.5)$$

We will construct the solution $w^\varepsilon$ of the relaxation problem (1.10)-(1.11)-(1.12) writing

$$w^\varepsilon = w^0 + \varepsilon \begin{pmatrix} 0 \\ 0 \\ v^1 \end{pmatrix} + \varepsilon r, \quad (5.6)$$

where

$$v^1 = -\partial_t v^0 + \mu \partial_x u^0_2, \quad (5.7)$$

so that $r$ satisfies the following system

$$\begin{cases}
\partial_t r_1 - \partial_x r_2 = 0, \\
\partial_t r_2 - \partial_x r_3 = \partial_x v^1, \\
\partial_t r_3 - \mu \partial_x r_2 = \frac{1}{\varepsilon} (\varepsilon (u^0_1) r_1 - r_3) + F(t,x,\varepsilon r_1)(r_1)^2 - \partial_t v^1,
\end{cases} \quad (5.8)$$

for $(t,x) \in [0,T^*] \times \mathbb{R}^+$, with the initial-boundary conditions

$$\begin{cases}
r(0,x) = 0, x \in \mathbb{R}^+, \\
r_2(t,0) = 0, 0 \leq t < T^*.
\end{cases} \quad (5.9)$$

The function $F$ is defined by

$$F(t,x,\xi) = \int_0^1 (1-s) p''(u^0_1(t,x) + s\xi) ds. \quad (5.10)$$
First step: we want to construct a suitable symmetrization for system (5.8). We denote by $A$ and $B$ the matrices

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & -\mu & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma(t, x) & 0 & -1 \end{pmatrix}. $$

With this object, we will use the conservative-dissipative form introduced in [2]. We first need a symmetric positive definite matrix $A_0$ such that $AA_0$ is a symmetric matrix, and such that

$$BA_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -d \end{pmatrix} \quad \text{with} \quad d > 0. $$

Following [7], such a matrix can be constructed using the entropic variables. For the special case of the Suliciu model we have

$$A_0(t, x) = \begin{pmatrix} (\gamma(t, x))^{-1} & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & \mu \end{pmatrix} = \begin{pmatrix} A_{0,11} & A_{0,12} \\ A_{0,21} & A_{0,22} \end{pmatrix}. $$

We obtain

$$AA_0 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -\mu \\ 0 & -\mu & 0 \end{pmatrix}, \quad BA_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma - \mu \end{pmatrix},$$

and we remark that with (5.5), we have $\mu - \gamma \geq \mu - \Gamma > 0$. Finally we can apply Proposition 2.7 in [2]: the conservative-dissipative variables $\rho$ is defined by $\rho = P(t, x)\sigma$ with

$$P(t, x) = \begin{pmatrix} (A_{0,11})^{-\frac{1}{2}} & 0 & 0 \\ 0 & \gamma^{-\frac{1}{2}} & 0 \\ (A_{0,21})^{-\frac{1}{2}} & (A_{0,22})^{-\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \gamma^{-\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ \gamma^{-\frac{1}{2}} & 0 & (\mu - \gamma)^{-\frac{1}{2}} \end{pmatrix}. $$

In these variables, system (5.8) is equivalent to

$$\partial_t \rho + A_1 \partial_x \rho + L \rho = -\frac{1}{\varepsilon} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & F_1(t, x, \varepsilon \rho_1) \rho_2^2 \end{pmatrix} + H, \quad (5.11)$$

for $(t, x) \in [0, T^*] \times \mathbb{R}^+$, with the initial-boundary conditions

$$\rho(0, x) = 0 \quad \text{for} \quad x \in \mathbb{R}^+ \quad \text{and} \quad \rho_2(t, 0) = 0 \quad \text{for} \quad t \in [0, T^*]. \quad (5.12)$$

The matrix $A_1 = P A_0^{-1}$ is symmetric

$$A_1(t, x) = \begin{pmatrix} 0 & -\gamma^{-\frac{1}{2}} & 0 \\ -\gamma^{-\frac{1}{2}} & 0 & -(\mu - \gamma)^{\frac{1}{2}} \\ 0 & -(\mu - \gamma)^{\frac{1}{2}} & 0 \end{pmatrix}. $$

The matrix $L$ is given by $L(t, x) = P \partial_t P^{-1} + PA_0^{-1}P^{-1}$. In addition, $F_1$ and $H$ are given by

$$F_1(t, x, \xi) = (\mu - \gamma)^{-\frac{1}{2}} F(t, x, \gamma^{-\frac{1}{2}} \xi), \quad (5.13)$$
\[ H(t, x) = \begin{pmatrix} 0 \\ \partial_x v^1 \\ -(\mu - \gamma)^{-\frac{1}{2}} \partial_t v^1 \end{pmatrix}. \]

From (5.1) we have

\[ \partial_t^i \gamma \in C^0([0, T^*]; H^{3-i}(\mathbb{R}^+)), i = 0, 1, 2, 3, \tag{5.14} \]

and using (2.2) there exists \( \alpha > 0 \) such that

\[ \gamma(t, x) \geq \alpha \text{ for } (t, x) \in [0, T^*] \times \mathbb{R}^+. \tag{5.15} \]

Using (5.14), (5.15) and (5.5) we have

\[ A_1, \partial_t A_1, \partial_x A_1 \in C^0([0, T^*]; L^\infty(\mathbb{R}^+)), \tag{5.16} \]

\[ L, \partial_t L, \partial_x L \in C^0([0, T^*]; L^\infty(\mathbb{R}^+)). \tag{5.17} \]

Using (5.1) and (5.7) we have

\[ \partial_t^i H \in C^0([0, T^*]; H^{1-i}(\mathbb{R}^+)), i = 0, 1. \tag{5.18} \]

We recall that by (5.10) and (5.13) we have

\[ F_1(t, x, \xi) = \gamma^{-1}(t, x)(\mu - \gamma(t, x))^{-\frac{1}{2}} \int_0^1 (1 - s)\rho''(u^0_1(t, x) + s\gamma^{-\frac{1}{2}}(t, x)\xi)ds, \]

so, by (5.14), (5.15) and (5.5) we have

\[ F_1, \partial_t F_1, \partial_x F_1, \partial_\xi F_1 \in C^0([0, T^*]; L^\infty(\mathbb{R}^+ \times [-1, 1])). \tag{5.19} \]

Now we fix \( T < T^* \) and we introduce \( T_\varepsilon \) defined by

\[ T_\varepsilon = \sup \left\{ t \leq T, \| \rho \|_{L^\infty([0, t] \times \mathbb{R}^+)} \leq \frac{1}{\varepsilon} \right\}. \tag{5.20} \]

We will prove that, for \( \varepsilon \) small enough, \( T_\varepsilon = T \) and that there exists \( K \) such that for all \( \varepsilon \) small enough,

\[ \| \rho \|_{L^\infty([0, T]; H^1(\mathbb{R}^+))} + \| \partial_t \rho \|_{L^\infty([0, T]; L^2(\mathbb{R}^+))} \leq K. \tag{5.21} \]

First, by variational methods, we obtain \( L^2 \)-estimates on \( \rho \) and \( \partial_t \rho \). To obtain \( L^2 \)-estimates on \( \partial_x \rho \) we use the equations taking into account that the boundary \( \{ x = 0 \} \) is characteristic.

**Second step: variational estimates**

We take the inner product of system (5.11) by \( \rho \) and we obtain that

\[ \frac{1}{2} \frac{d}{dt} \| \rho \|_{L^2(\mathbb{R}^+)}^2 + \int_{\mathbb{R}^+} A_1 \partial_x \rho \cdot \rho dx + \int_{\mathbb{R}^+} L \rho \cdot \rho dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^+} \rho_3^2 dx = \int_{\mathbb{R}^+} F_1(t, x, \varepsilon \rho_1) \rho_1^2 \rho_3 + \int_{\mathbb{R}^+} H \cdot \rho dx. \]
Using (5.12) we obtain that
\[
\int_{\mathbb{R}^+} A_1 \partial_x \rho \cdot \rho \, dx = -\frac{1}{2} \int_{\mathbb{R}^+} (\partial_x A_1) \rho \cdot \rho \, dx.
\]

With the estimates (5.16),(5.19) and since \(\varepsilon|\rho| \leq 1\) on \([0,T_{\varepsilon}] \times \mathbb{R}^+\), there exists a constant \(C>0\) such that, for \(t \leq T_{\varepsilon}\),
\[
\frac{1}{2} \frac{d}{dt} \| \rho \|^2_{L^2(\mathbb{R}^+)} + \frac{1}{\varepsilon} \int_{\mathbb{R}^+} \rho_x^2 \, dx \leq C(1 + \| \rho \|^2_{L^2(\mathbb{R}^+)} + \| \rho_1 \|_{L^\infty(\mathbb{R}^+)} \| \rho_1 \|_{L^2(\mathbb{R}^+)} \| \rho_3 \|_{L^2(\mathbb{R}^+)}).
\]

Therefore we obtain that for \(t \leq T_{\varepsilon}\),
\[
\frac{d}{dt} \| \rho \|^2_{L^2(\mathbb{R}^+)} + \frac{1}{\varepsilon} \int_{\mathbb{R}^+} \rho_x^2 \, dx \leq C(1 + \| \rho \|^2_{L^2(\mathbb{R}^+)} + \varepsilon \| \rho_1 \|^2_{L^\infty(\mathbb{R}^+)} \| \rho_1 \|^2_{L^2(\mathbb{R}^+)}). \tag{5.22}
\]

We can derivatize (5.11)-(5.12) with respect to \(t\)
\[
\partial_t \partial_t \rho + A_1 \partial_x \partial_t \rho + L \partial_t \rho + \frac{1}{\varepsilon} \begin{pmatrix} 0 \\ \partial_t \rho_3 \end{pmatrix} = -\partial_t A_1 \partial_x \rho - \partial_t L \rho + \begin{pmatrix} 0 \\ \partial_t F_1(t,x,\varepsilon \rho_1) \rho_1^2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \partial_x F_1(t,x,\varepsilon \rho_1) \partial_t \rho_1 \rho_1^2 \\ \partial_t H \end{pmatrix}.
\]

With the same arguments as before we obtain that there exists \(C>0\) such that for \(t \leq T_{\varepsilon}\),
\[
\frac{d}{dt} \| \partial_t \rho \|^2_{L^2(\mathbb{R}^+)} + \frac{1}{\varepsilon} \int_{\mathbb{R}^+} (\partial_t \rho_3)^2 \, dx \leq C(1 + \| \rho \|^2_{L^2(\mathbb{R}^+)} + \| \partial_t \rho \|^2_{L^2(\mathbb{R}^+)} + \| \partial_x \rho \|^2_{L^2(\mathbb{R}^+)} + \varepsilon \| \rho_1 \|^2_{L^\infty(\mathbb{R}^+)} (\| \rho_1 \|^2_{L^2(\mathbb{R}^+)} + \| \partial_t \rho_1 \|^2_{L^2(\mathbb{R}^+)})). \tag{5.23}
\]

We define \(\psi\) by
\[
\psi(t) = \left( \| \rho(t) \|^2_{L^2(\mathbb{R}^+)} + \| \partial_t \rho(t) \|^2_{L^2(\mathbb{R}^+)} \right)^{\frac{1}{2}}, \tag{5.24}
\]
so we obtain by (5.22) and (5.23) the \(L^2\)-estimate: there exists \(C>0\) such that for \(t \leq T_{\varepsilon}\),
\[
\frac{d}{dt} (\psi(t))^2 + \frac{1}{\varepsilon} (\| \rho_3 \|^2_{L^2(\mathbb{R}^+)} + \| \partial_t \rho_3 \|^2_{L^2(\mathbb{R}^+)}) \leq C(1 + (\psi(t))^2 + \varepsilon \| \rho_1 \|^2_{L^\infty(\mathbb{R}^+)} (\psi(t))^2 + \| \partial_x \rho \|^2_{L^2(\mathbb{R}^+)}) \tag{5.25}
\]

**Third step**

We now estimate \(\partial_x \rho\) using the equations
\[
\begin{cases}
\partial_t \rho_1 - \gamma \frac{1}{2} \partial_x \rho_2 + (L \rho)_1 = 0, \\
\partial_t \rho_2 - \gamma \frac{1}{2} \partial_x \rho_1 - (\mu - \gamma) \frac{1}{2} \partial_x \rho_3 + (L \rho)_2 = H_2, \\
\partial_t \rho_3 - (\mu - \gamma) \frac{1}{2} \partial_x \rho_2 + (L \rho)_3 + \frac{1}{\varepsilon} \rho_3 = F_1(t,x,\varepsilon \rho_1) \rho_1^2 + H_3.
\end{cases} \tag{5.26}
\]

From the first equation in (5.26), and with (5.15) and (5.17) we have for \(t \in [0,T_{\varepsilon}]\)
\[
\| \partial_x \rho_2 \|_{L^2(\mathbb{R}^+)} \leq C \psi. \tag{5.27}
\]
Let us introduce \( \tilde{\rho}_1 = \rho_1 + \gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}\rho_3 \). From the second equation in (5.26) we have

\[
\partial_t \rho_2 - \gamma^{\frac{1}{2}}\partial_x \tilde{\rho}_1 + \gamma^{\frac{1}{2}}\partial_x (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}})\rho_3 + (L\rho)_2 = H_2,
\]

so, by (5.15), (5.14), (5.17) and (5.18) we obtain that

\[
\|\partial_x \tilde{\rho}_1\|_{L^2(\mathbb{R}^+)} \leq C(1 + \psi).
\]  

(5.28)

We cannot estimate \( \partial_x \rho_1 \) or \( \partial_x \rho_3 \) by the same method because the boundary \( \{x = 0\} \) is characteristic. We rewrite the third equation in (5.26)

\[
\partial_t \rho_3 + \frac{1}{\varepsilon} \rho_3 = \gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}(\partial_t \rho_1 + (L\rho)_1) - (L\rho)_3 + F_1(t,x,\varepsilon \rho_1) \rho_1^2 + H_3.
\]

So eliminating \( \rho_1 \) we obtain

\[
\mu^{-1}\gamma \partial_t \rho_3 + \frac{1}{\varepsilon} \rho_3 = \gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}[\partial_t \tilde{\rho}_1 - \partial_x (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}})\rho_3] + M_1(t,x) \tilde{\rho}_1 + M_2(t,x)\rho_2 + M_3(t,x)\rho_3 + H_3 + F_1(t,x,\varepsilon \rho_1) \rho_1^2.
\]

(5.29)

with \( \rho_1 = \tilde{\rho}_1 - \gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}\rho_3 \). We derivate (5.29) with respect to \( x \) and we obtain the equation satisfied by \( \partial_x \rho_3 \)

\[
\partial_t \partial_x \rho_3 + \tau(t,x) \partial_x \rho_3 = \sum_{i=1}^{6} T_i,
\]  

(5.30)

with

\[
\tau = \mu^{-1}\gamma \left( \frac{1}{\varepsilon} + \gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}} \partial_x (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}) + \varepsilon \partial_x F_1(t,x,\varepsilon \rho_1) \gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}} \rho_1^2 + 2F_1(t,x,\varepsilon \rho_1) \rho_1^2 + M_3(t,x) \right),
\]

\[
T_1 = \mu^{-1}\gamma \left( \partial_x (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}) \partial_t \tilde{\rho}_1 - \partial_x (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}) \partial_t \rho_3 - \partial_x (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}) \partial_t (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}) \right),
\]

\[
T_2 = \mu^{-1}\gamma \left( \partial_x (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}) \partial_t \tilde{\rho}_1 - \partial_x (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}) \partial_t \rho_3 - \partial_x (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}) \partial_t (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}) \right),
\]

\[
T_3 = \mu^{-1}\gamma \partial_x H_3,
\]

\[
T_4 = \mu^{-1}\gamma (M_1 \partial_x \tilde{\rho}_1 + M_2 \partial_x \rho_2),
\]

\[
T_5 = \mu^{-1}\gamma \left( \partial_x F_1(t,x,\varepsilon \rho_1) \rho_1^2 - \varepsilon \partial_x F_1(t,x,\varepsilon \rho_1) \partial_x (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}) \rho_1^2 \rho_3 - 2F_1(t,x,\varepsilon \rho_1) \partial_x (\gamma^{-\frac{1}{2}}(\mu - \gamma)^{\frac{1}{2}}) \rho_1 \rho_3 \right),
\]

\[
T_6 = \mu^{-1}\gamma (\varepsilon \partial_x F_1(t,x,\varepsilon \rho_1) \rho_1^2 \partial_x \tilde{\rho}_1 + 2F_1(t,x,\varepsilon \rho_1) \rho_1 \partial_x \tilde{\rho}_1).
\]

For \( t \in [0,T]\), using (5.5), (5.14), (5.15) and (5.19) we obtain that

\[
\left| \tau(t,x) - \frac{\mu^{-1}\gamma}{\varepsilon} \right| \leq C + C_0 \|\rho_1\|_{L^\infty(\mathbb{R}^+)}. 
\]
We define $T^1_\varepsilon \leq T_\varepsilon$ by

$$T^1_\varepsilon = \max \left\{ t \leq T_\varepsilon, \| \rho_1 \|_{L^\infty([0,t] \times \mathbb{R}^+)} \leq \frac{1}{2C_0\varepsilon} \right\},$$

(5.31)

so there exists $\tau_1 > 0$ and $\tau_2 > 0$ such that

$$\forall t \leq T^1_\varepsilon, \forall x > 0 \frac{\tau_1}{\varepsilon} \leq \tau(t,x) \leq \frac{\tau_2}{\varepsilon}. \quad (5.32)$$

We solve Equation (5.30) by Duhamel formula

$$\partial_x \rho_3 = \sum_{i=1}^{6} T_i,$$  

(5.33)

with

$$T_i(t,x) = \int_0^t \exp(- \int_s^t \tau(\sigma,x)d\sigma)T_i(s,x)ds.$$  

We define $\Psi$ by

$$\Psi(t) = \sup_{[0,t]} \psi(s), \quad (5.34)$$

where $\psi$ is given by (5.24). Integrating by parts in $T_1$ we obtain

$$T_1(t,x) = -\int_0^t \mu^{-1} \gamma^{\frac{2}{3}}(\mu - \gamma)^{\frac{1}{3}} \tau(s,x) \exp(- \int_s^t \tau(\sigma,x)d\sigma)\partial_x \rho_1(s,x)ds$$

$$+ \int_0^t \exp(- \int_s^t \tau(\sigma,x)d\sigma)\partial_x (\mu^{-1} \gamma^{\frac{2}{3}}(\mu - \gamma)^{\frac{1}{3}})(s,x)\partial_x \rho_1(s,x)ds$$

$$+ \mu^{-1} \gamma^{\frac{2}{3}}(\mu - \gamma)^{\frac{1}{3}} \partial_x \rho_1(t,x).$$

Using (5.32), (5.5), (5.14), (5.15) and (5.28) we have

$$\| T_i(t,\cdot) \|_{L^2(\mathbb{R}^+)} \leq \int_0^t \exp(- \frac{\tau_1}{\varepsilon}(t-s))C(\psi(s)+1)(1 + \frac{\tau_2}{\varepsilon})ds + C(\psi(t)+1),$$

and we obtain that

$$\forall t \leq T^1_\varepsilon, \| T_i \|_{L^2(\mathbb{R}^+)} \leq C(1 + \Psi(t)).$$

(5.35)

Using (5.5) (5.14) (5.15) (5.24) (5.34) and also (5.18) for $T_3$ and (5.27) and (5.28) for $T_4$, we obtain

$$\forall t \leq T^1_\varepsilon, \| T_2 \|_{L^2(\mathbb{R}^+)} + \| T_3 \|_{L^2(\mathbb{R}^+)} + \| T_4 \|_{L^2(\mathbb{R}^+)} \leq C\varepsilon(1 + \Psi(t)).$$

(5.36)

For the nonlinear terms $T_5$ and $T_6$ we use in addition (5.19) (5.20) and we obtain

$$\forall t \leq T^1_\varepsilon, \| T_5 \|_{L^2(\mathbb{R}^+)} + \| T_6 \|_{L^2(\mathbb{R}^+)} \leq C(1 + \Psi(t)).$$

(5.37)

Therefore we obtain the following estimation for $\partial_x \rho$ using (5.27) (5.28) (5.33) (5.35) (5.36) (5.37)

$$\forall t \leq T^1_\varepsilon, \| \partial_x \rho \|_{L^2(\mathbb{R}^+)} \leq C(1 + \Psi(t)).$$

(5.38)
so we have
\[ \forall t \leq T_{\varepsilon}^1, \|\rho\|_{L^\infty(\mathbb{R}^+)} \leq C_1(1 + \Psi(t)). \]  
(5.39)

**Fourth step**

By a comparison method we estimate \( \Psi \). For \( t \leq T_{\varepsilon}^1 \), integrating (5.25) from 0 to \( t \), using (5.38) and (5.39) we obtain that
\[ (\Psi(t))^2 \leq C_2 \int_0^t (1 + (\Psi(s))^2 + \varepsilon(\Psi(s))^4) ds. \]  
(5.40)

We introduce the differential equation
\[ y'_\varepsilon = C_2(1 + y_\varepsilon + \varepsilon y_\varepsilon^2), \ y_\varepsilon(0) = 0. \]  
(5.41)

There exists \( \varepsilon_0 > 0 \) such that, for \( \varepsilon \leq \varepsilon_0 \), the lifespan of \( y_\varepsilon \) is greater than \( T \). So we have
\[ \forall \varepsilon \leq \varepsilon_0, \forall t \leq T, y_\varepsilon(t) \leq y_{\varepsilon_0}(t) \leq y_{\varepsilon_0}(T) = C_3. \]

By comparison principle we deduce from (5.40) that
\[ \forall \varepsilon \leq \varepsilon_0, \forall t \leq T_{\varepsilon}^1, (\Psi(t))^2 \leq C_3, \]
and from (5.39),
\[ \forall \varepsilon \leq \varepsilon_0, \forall t \leq T_{\varepsilon}^1, \|\rho\|_{L^\infty(\mathbb{R}^+)} \leq C_1(1 + \sqrt{C_3}). \]

Let \( \varepsilon_1 > 0 \) such that \( \varepsilon_1 \leq \varepsilon_0 \) such that
\[ \forall \varepsilon \leq \varepsilon_1, C_1(1 + \sqrt{C_3}) \leq \frac{1}{2C_0 \varepsilon}. \]

So, by (5.20) and (5.31), we have for \( \varepsilon \leq \varepsilon_1 \), \( T_{\varepsilon}^1 = T_{\varepsilon} = T \) and we conclude the proof by the estimate
\[ \exists K > 0, \forall \varepsilon \leq \varepsilon_1, \|\rho\|_{L^\infty([0,T];H^1(\mathbb{R}^+))} + \|\partial_t \rho\|_{L^\infty([0,T];L^2(\mathbb{R}^+))} \leq K. \]

**6. Annex**

Using the method in W.A. Yong [22] we show the convergence result for the Cauchy problem
\[
\begin{aligned}
\partial_t u_1^\varepsilon - \partial_x u_2^\varepsilon &= 0, \\
\partial_t u_2^\varepsilon - \partial_x v^\varepsilon &= 0, \\
\partial_t v^\varepsilon - \mu \partial_x u_2^\varepsilon &= \frac{1}{\varepsilon}(p(u_1^\varepsilon) - v^\varepsilon),
\end{aligned}
\]  
(6.1)

for \( (t,x) \in \mathbb{R}^+ \times \mathbb{R} \) with the smooth initial data
\[ w^\varepsilon(0,x) = w_0(x) = (u_0(x), v_0(x)) \text{ for } x \in \mathbb{R}. \]  
(6.2)
Let us introduce \( u^0 \) the smooth solution of the Cauchy problem

\[
\begin{aligned}
\partial_t u^0_1 - \partial_x u^0_2 &= 0, \\
\partial_t u^0_2 - \partial_x p(u^0_1) &= 0,
\end{aligned}
\]

with the initial data

\[ u^0(0,x) = u_0(x). \]

As in Tzavaras [21] we assume that there exists \( \gamma > 0 \) and \( \Gamma > 0 \) such that

\[
\forall \xi \in \mathbb{R}, \gamma \leq p'(\xi) \leq \Gamma < \mu,
\]

so the problem (6.1)-(6.2) admits a global solution \( w^\varepsilon = (u^\varepsilon, v^\varepsilon) \) such that

\[
w^\varepsilon \in C^0(\mathbb{R}^+; H^s(\mathbb{R})) \cap C^1(\mathbb{R}^+; H^{s-1}(\mathbb{R})).
\]

We will prove the following convergence theorem.

**Theorem 6.1.** Under assumption (6.5), if \( w_0 \in H^s(\mathbb{R}) \) with \( s \geq 2 \), then there exists \( T_1 > 0 \) such that when \( \varepsilon \) tends to zero, \( u^\varepsilon \) tends to \( u^0 \) in \( L^1([0,T_1]; H^s(\mathbb{R})) \).

**Remark 6.1.** It would be possible to relax hypothesis (6.5) as in Theorem 2.3; in this case, the lifespan of \( w^\varepsilon \) is uniformly greater than \( T_1 \).

**Remark 6.2.** In fact it appears a boundary layer in time which affects only the third component of \( w^\varepsilon \).

**Sketch of the proof**

**First step:** the stability assumption in [22] are satisfied. As in [21] and [7], we consider the strictly convex entropy function for the system (6.1)

\[
\mathcal{E}(u_1, u_2, v) = \frac{1}{2} u_2^2 + u_1 v - \frac{\mu}{2} u_1^2 - \int_0^{v-\mu u_1} h^{-1}(y) dy,
\]

where \( h(\xi) = p(\xi) - \mu \xi \) which is strictly decreasing by (6.5). So \( A_0(w) = \mathcal{E}''(w) \) is a symmetrizer for the system. Denoting \( a = (h^{-1})'(v-\mu u_1) \) we obtain

\[
A_0(w) = \begin{pmatrix}
-\mu - \mu^2 a & 0 & 1 + \mu a \\
0 & 1 & 0 \\
1 + \mu a & 0 & -a
\end{pmatrix},
\]

and the system (6.1) is equivalent to the quasilinear symmetric system

\[
A_0(w) \partial_t w + \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{pmatrix} \partial_x w = \frac{1}{\varepsilon} (p(u_1) - v) \begin{pmatrix}
1 + \mu a \\
0 \\
-\mu
\end{pmatrix}.
\]

We denote

\[
Q(w) = \begin{pmatrix}
0 \\
0 \\
p(u_1) - v
\end{pmatrix} \quad \text{and} \quad P(w) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-p'(u_1) & 0 & 1
\end{pmatrix},
\]

and we obtain

\[
P(w)Q'(w)P^{-1}(w) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}.
\]
On the equilibrium manifold $V = \{ v = p(u_1) \}$, we have
\[
A_0(w)Q'(w) + Q'(w)A_0(w) = \frac{2}{p'(u_1) - \mu} \begin{pmatrix} (p'(u_1))^2 & 0 & -p'(u_1) \\ 0 & 0 & 0 \\ -p'(u_1) & 0 & 1 \end{pmatrix}.
\] (6.8)

Using (6.6), (6.7) and (6.8) we obtain the stability conditions in [22].

**Second step:** we use Theorems 6.1 and 6.2 in [22]. We introduce the interior profile $w^0 = ((u_0^0, u_2), p(u_1^0))$ and the boundary layer term $I^0 = \tilde{I}^0 - w^0(0, x)$ where $\tilde{I}^0$ is the solution of
\[
d\tilde{I}^0 = Q(\tilde{I}^0), \tilde{I}^0(\tau = 0) = w_0(x).
\]

We have $I_1^0 = I_2^0 = 0$ and
\[
I_3^0(\tau, x) = (v_0(x) - p(u_1, 0)) e^{-\tau},
\]
and we obtain
\[
\begin{aligned}
\tau^\varepsilon(t, x) &= w^0(t, x) + I^0(\frac{t}{\varepsilon}, x) + O(\varepsilon),
\end{aligned}
\]
so we conclude the proof of Theorem 6.1.

**Remark 6.3.** If $w^0$ belongs to the equilibrium manifold then the order zero boundary layer term vanishes.

**Remark 6.4.** In fact using more precisely [22] and the appendix of [3] we can prove that $T_1$ can be arbitrarily close to the lifespan of $u^0$ as in Theorem 2.3.

**Remark 6.5.** In this annexe the matrix $P$ introduced in [22] plays an analogous role as the matrix $P$ in section 5.

REFERENCES


