Regular Solutions for Landau-Lifschitz Equation in $I\!R^3$

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ABSTRACT - In this paper we prove local existence, global existence with small data and uniqueness of regular solutions for Landau-Lifschitz equations. Furthermore we establish local existence and uniqueness for a system coupling Maxwell and Landau-Lifschitz equations arising from Micromagnetism theory.

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1 Introduction

In Micromagnetism theory (see [2]), the behaviour of a ferromagnet is represented by an unitary vector field u called magnetic moment. The variations of u are gouverned by Landau-Lifshitz equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = u \wedge \Delta u - u \wedge (u \wedge \Delta u) \\ u(0, \cdot) = u_0(\cdot) \end{cases} \tag{1.1}$$

where we assume that $|u_0| = 1$.

Existence and non uniqueness for weak solutions of (1.1) are proved by F. Alouges and A. Soyeur in [1]. Furthermore, P.L. Sulem, C. Sulem and C. Bardos establish in [6] local existence of regular solutions for the equation

$$\frac{\partial u}{\partial t} = u \wedge \Delta u. \tag{1.2}$$

Concerning equation (1.1) we prove the following theorems.

Theorem 1. *Assume that*

$$|u_0| = 1, \;\; \nabla u_0 \in I\!H^1(I\!R^3).$$

Then there exists $T > 0$, there exists an unique u such that

(i) $\quad u \in I\!L^\infty((0,T) \times I\!R^3), \;\; |u| = 1,$

(ii) $\quad \nabla u \in L^\infty((0,T); I\!H^1(I\!R^3)) \cap L^2((0,T); I\!H^2(I\!R^3)),$

(iii) $\quad$ *u satisfies (1.1).*

Theorem 2. *There exists $\delta > 0$ such that if $|u_0| = 1$ and if $\nabla u_0 \in I\!H^1(I\!R^3)$ with $\|\nabla u_0\|_{I\!H^1(I\!R^3)} < \delta$, then the solution u given by Theorem 1 exists for $T = +\infty$.*

The term $-u \wedge (u \wedge \Delta u)$ in (1.1) is in fact a dissipation term. For this reason, global existence with small data is valid for (1.1) and not for (1.2) (see [6]).

On the other hand, the propagation of electromagnetic waves in the ferromagnet is gouverned by a system coupling Landau-Lifschitz and Maxwell equations.

$$\frac{\partial u}{\partial t} = u \wedge (\Delta u + H) - u \wedge (u \wedge (\Delta u + H)),$$

$$\frac{\partial B}{\partial t} + \text{curl } E = 0,$$

$$\frac{\partial E}{\partial t} - \text{curl } H = 0,$$

$$B = H + u,$$

$$u(0, \cdot) = u_0(\cdot), \quad B(0, \cdot) = B_0(\cdot), \quad E(0, \cdot) = E_0(\cdot),$$

where we assume that $|u_0| = 1$ and div $B_0 = 0$.

Instead of working with (u, H, E), we will write the system with the unknowns (u, B, E).

$$\frac{\partial u}{\partial t} = u \wedge (\Delta u + B) - u \wedge (u \wedge (\Delta u + B)) \tag{1.3}$$

$$\frac{\partial B}{\partial t} = -\text{curl } E \tag{1.4}$$

$$\frac{\partial E}{\partial t} = \text{curl } B - \text{curl } u \tag{1.5}$$

$$u(0, \cdot) = u_0(\cdot), \quad B(0, \cdot) = B_0(\cdot), \quad E(0, \cdot) = E_0(\cdot) \tag{1.6}$$

and we still assume that

$$|u_0| = 1 \text{ and div } B_0 = 0 \text{ in } I\!R^3. \tag{1.7}$$

We will establish the following theorem.

Theorem 3. Let (u_0, E_0, B_0) such that

$$E_0 \in I\!H^1(I\!R^3), \quad B_0 \in I\!H^1(I\!R^3), \quad \text{div } B_0 = 0,$$
$$|u_0| = 1, \quad \nabla u_0 \in I\!H^1(I\!R^3).$$

Then there exists $T > 0$, there exists an unique (u, E, B) such that

(i) $\quad |u| = 1, \quad \nabla u \in L^\infty((0, T); I\!H^1(I\!R^3)) \cap L^2((0, T); I\!H^2(I\!R^3))$,

(ii) $\quad E$ and B belong to $L^\infty((0, T); I\!H^1(I\!R^3))$,

(iii) $\quad (u, E, B)$ satisfies (1.3)-(1.6).

The existence of weak solutions for the system (1.3)-(1.6) is proved in [7] and in [3] in the case of a bounded domain.

In [4], J.L. Joly, G. Métivier and J. Rauch prove the existence of solutions for a system similar to (1.3)-(1.6) but without Δu in (1.3).

The asymptotic behaviour of weak solutions of (1.3)-(1.6) in a bounded domain is studied in [3].

The proof of Theorems 1, 2 and 3 is based on a semi-discretization used in [1] and [6].

In Part 2, we describe the discretization process. Part 3 is devoted to the proof of Theorems 1 and 2. Theorem 3 is proved in the last part.

2 Discretization space, notations

We fix $h > 0$ and we set $x_j^h = jh$ for $j \in \mathbb{Z}$.
For $\alpha = (i, j, k) \in \mathbb{Z}^3$, we note $X_\alpha^h = (x_i^h, x_j^h, x_k^h)$ and

$$C_\alpha^h = \left\{ (x, y, z), x_i^h \le x < x_i^h + h,\ x_j^h \le y < x_j^h + h,\ x_k^h \le z < x_k^h + h \right\}.$$

We denote

$$Z_h^3 = \left\{ X_\alpha^h \in \mathbb{R}^3,\ \alpha \in \mathbb{Z}^3 \right\}.$$

In the sequel of this part, in order to simplify the notations, we will omit the exponent h.

We consider the following operators defined for $u : Z^3 \to \mathbb{R}^3$:

$$\tau_1^+ u(x_i, x_j, x_k) = u(x_{i+1}, x_j, x_k)$$

$$D_1^+ u = \frac{1}{h}(\tau_1^+ u - u)$$

$$\tau_1^- = (\tau_1^+)^{-1},\ \ D_1^- = \tau_1^- \circ D_1^+$$

In the same way, we denote τ_2^+, D_2^+, τ_2^-, D_2^- the same operations concerning the second variable, and τ_3^+, D_3^+, τ_3^-, D_3^- for the third variable.

We set

$$\tilde{\Delta} = \sum_{i=1}^{3} D_i^- D_i^+,$$

$$\widetilde{\operatorname{div}}\ u = \sum_{i=1}^{3} D_i^+ u^i$$

$$\widetilde{\operatorname{curl}}\ u = (D_2^+ u^3 - D_3^+ u^2, D_3^+ u^1 - D_1^+ u^3, D_1^+ u^2 - D_2^+ u^1).$$

We denote

$$\int_{Z^3} u = \sum_{\alpha \in \mathbb{Z}^3} h^3 u(X_\alpha).$$

and we use the following classical notations

$$\|u\|_{l^p} = \left(\int_{Z^3} |u|^p \right)^{\frac{1}{p}},\quad \|D^+ u\|_{l^p} = \left(\sum_{i=1}^{3} \|D_i^+ u\|_{l^p}^p \right)^{\frac{1}{p}},$$

$$\|u\|_{w^{1,p}} = \left(\|u\|_{l^p}^p + \|D^+ u\|_{l^p}^p\right)^{\frac{1}{p}}, \|u\|_{h^1} = \|u\|_{w^{1,2}},$$

$$\|u\|_{h^2} = \left(\|u\|_{h^1}^2 + \sum_{ij} \|D_i^+ D_j^+ u\|_{l^2}^2 \right)^{\frac{1}{2}},\quad \|u\|_{l^\infty} = \sup_{\alpha \in \mathbb{Z}^3} |u(X_\alpha)|.$$

We remark that

$$\int_{Z^3} D_i^+ u \cdot v = - \int_{Z^3} u \cdot D_i^- v,$$

furthermore,

$$D_i^+(uv) = D_i^+ u\ \tau_i^+ v + u D_i^+ v.$$

We recall now the following discrete Sobolev inequalities.

Lemma 1. *There exists a constant C independant of h such that for all $u : Z^3 \to I\!R^3$,*

$$\text{if } \|u\|_{h^1} < +\infty, \text{ then } \|u\|_{l^6(Z^3)} \leq C\|D^+u\|_{l^2(Z^3)}, \tag{2.1}$$

Discrete versions of Sobolev inequalities are established in [5] using interpolation procedures. The same interpolation process is used in this paper (see 3.3 and 4.3) and is presented in Section 3.3.

3 Proof of theorems 1 and 2

1. Discretization

For $h > 0$, let u_0^h defined on the mesh Z_h^3 such that

- $|u_0^h| = 1$ on Z_h^3,
- $r_h u_0^h$ tends to u_0 in $L^2_{loc}(I\!R^3)$
- $\alpha\|D^+u_0^h\|_{h^1(Z_h^3)} \leq \|\nabla u_0\|_{H^1(I\!R^3)} \leq \frac{1}{\alpha}\|D^+u_0^h\|_{h^1(Z_h^3)}$

where r_h is the interpolating operator defined in [5] and in Section 3.3 of this paper, and where α does not depend on h.

Now we fix $h > 0$ and we solve

$$\begin{cases} \dfrac{du^h}{dt} = u^h \wedge \tilde{\Delta}u^h - u^h \wedge (u^h \wedge \tilde{\Delta}u^h) \\ \\ u^h(t=0) = u_0^h \end{cases} \tag{3.1}$$

The map $u \mapsto u \wedge \tilde{\Delta}u - u \wedge (u \wedge \tilde{\Delta}u)$ is locally Lipschitz in $l^\infty(Z_h^3)$, so there exists an unique solution of (3.1) with Cauchy-Lipschitz Theorem.

In order to simplify the notation, we will omit the exponent h in the computations of the following subsection.

2. Estimates

We multiply (3.1) by u and we obtain that

$$\frac{d}{dt}|u|^2 = 0,$$

hence $|u| = 1$ on Z^3, as it is the case for u_0.

With the above remark we can modify the form of the equation. We first note that

$$u \wedge (u \wedge \tilde{\Delta}u) = (u \cdot \tilde{\Delta}u)u - \tilde{\Delta}u.$$

Furthermore, writting that $\tilde{\Delta}|u^2| = 0$ as $|u| = 1$, we obtain that

$$2u \cdot \tilde{\Delta}u + |D^+u|^2 + |D^-u|^2 = 0.$$

Hence, Equation (3.1) takes the form

$$\frac{du}{dt} = u \wedge \tilde{\Delta} u + \tilde{\Delta} u + \frac{1}{2}(|D^+ u|^2 + |D^- u|^2) u \tag{3.2}$$

First estimate. We multiply (3.2) by $\tilde{\Delta} u$ and after summation on Z^3, we get

$$-\frac{1}{2}\frac{d}{dt}\|D^+ u\|_{l^2}^2 = \|\tilde{\Delta} u\|_{l^2}^2 + \frac{1}{2}\int_{Z^3}\left(|D^- u|^2 + |D^+ u|^2\right) u \cdot \tilde{\Delta} u,$$

so we obtain

$$\frac{d}{dt}\|D^+ u\|_{l^2}^2 + 2\|\tilde{\Delta} u\|_{l^2}^2 \leq 2\|D^+ u\|_{l^4}^2 \|\tilde{\Delta} u\|_{l^2} \tag{3.3}$$

Second estimate. We multiply now (3.2) by $\tilde{\Delta}^2 u$ and after summation on Z^3 we obtain

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\Delta} u\|_{l^2}^2 = \int_{Z^3} u \wedge \tilde{\Delta} u \cdot \tilde{\Delta}^2 u - \|D^+ \tilde{\Delta} u\|_{l^2}^2$$
$$+\frac{1}{2}\int_{Z^3}\left(|D^+ u|^2 + |D^- u|^2\right) u \cdot \tilde{\Delta}^2 u.$$

We remark now that

$$\begin{aligned}\left|\int_{Z^3} u \wedge \tilde{\Delta} u \cdot \tilde{\Delta}^2 u\right| &= \left|\sum_i \int_{Z^3} D_i^+(u \wedge \tilde{\Delta} u) \cdot D_i^+ \tilde{\Delta} u\right| \\ &= \left|\sum_i \int_{Z^3} D_i^+ u \wedge \tilde{\Delta} u \cdot D_i^+ \tilde{\Delta} u\right| \\ &\leq \|D^+ \tilde{\Delta} u\|_{l^2} \|\tilde{\Delta} u\|_{l^3} \|D^+ u\|_{l^6} \end{aligned} \tag{3.4}$$

Furthermore,

$$\left|\frac{1}{2}\int_{Z^3}\left(|D^- u|^2 + |D^+ u|^2\right) u \cdot \tilde{\Delta}^2 u\right| = \left|\frac{1}{2}\sum_i \int_{Z^3} D_i^+\left((|D^- u|^2 + |D^+ u|^2) u\right) \cdot D_i^+ \tilde{\Delta} u\right|$$
$$\leq \frac{1}{2}\|D^+\left((|D^- u|^2 + |D^+ u|^2) u\right)\|_{l^2} \|D^+ \tilde{\Delta} u\|_{l^2}.$$

Now we compute $D_i^+(|D^+ u|^2)$ and we obtain that

$$\|D^+\left((|D^- u|^2 + |D^+ u|^2) u\right)\|_{l^2} \leq C\left(\sum_{ij}\|D_i^+ D_j^+ u \cdot D_j^+ u\|_{l^2} + \||D^+ u|^3\|_{l^2}\right)$$

as $|u| = 1$.

Thus there exists a constant K independant of h such that

$$\frac{d}{dt}\|\tilde{\Delta} u\|_{l^2}^2 \ + \ 2\|D^+ \tilde{\Delta} u\|_{l^2}^2 \leq +K\left(\|\tilde{\Delta} u\|_{l^3} \ \|D^+ u\|_{l^6} \ + \|D^+ u\|_{l^6}^3\right)\|D^+ \tilde{\Delta} u\|_{l^2} \tag{3.5}$$

With Lemma 2.1 and by interpolation there exists a universal constant C such that

$$\begin{aligned}&\|D^+ u\|_{l^6} \leq C\|\tilde{\Delta} u\|_{l^2} \\ &\|\tilde{\Delta} u\|_{l^3} \leq C\|\tilde{\Delta} u\|_{l^2}^{\frac{1}{2}} \ \|D^+ \tilde{\Delta} u\|_{l^2}^{\frac{1}{2}}\end{aligned} \tag{3.6}$$

From (3.5) and (3.6) we deduce

$$\frac{d}{dt}\|\tilde{\Delta} u\|_{l^2}^2 \ + \ 2\|D^+\tilde{\Delta} u\|_{l^2}^2 \leq K\left(\|D^+\tilde{\Delta} u\|_{l^2}^{\frac{3}{2}} \ \|\tilde{\Delta} u\|_{l^2}^{\frac{3}{2}} + \|D^+\tilde{\Delta} u\|_{l^2} \ \|\tilde{\Delta} u\|_{l^2}^{3}\right) \tag{3.7}$$

Estimate for Theorem 1.
We absorb $\|D^+\tilde{\Delta} u\|_{l^2}$ in (3.7) and we obtain

$$\frac{d}{dt}\|\tilde{\Delta} u\|_{l^2}^2 \ + \ \|D^+\tilde{\Delta} u\|_{l^2}^2 \leq K\|\tilde{\Delta} u\|_{l^2}^6 \tag{3.8}$$

On the other hand, from (3.3), by interpolation in $\mathbb{L}^4$, we derive

$$\frac{d}{dt}\|D^+ u\|_{l^2}^2 + \|\tilde{\Delta} u\|_{l^2}^2 \leq K\|D^+ u\|_{l^2} \ \|D^+ u\|_{l^6}^3 \leq K\|D^+ u\|_{l^2} \ \|\tilde{\Delta} u\|_{l^2}^3 \tag{3.9}$$

Combining (3.8) and (3.9) we obtain

$$\frac{d}{dt}\|D^+ u\|_{h^1}^2 + \|\tilde{\Delta} u\|_{h^1}^2 \leq K\left(1 + \|D^+ u\|_{h^1}^6\right) \tag{3.10}$$

We set now $g(t) = \|D^+ u\|_{h^1}^2$ and we have

$$\frac{dg}{dt} \leq K(1+g^3),$$

hence there exist $T > 0$ and K independant of h such that

$$\begin{cases} \|D^+ u\|_{L^\infty(0,T;h^1)} \leq K, \\ \\ \|\tilde{\Delta} u\|_{L^2(0,T;h^1)} \leq K, \\ \\ \|\dfrac{du}{dt}\|_{L^\infty(0,T;l^2)} \leq K. \end{cases} \tag{3.11}$$

The last estimate is obtained using (3.1) and the previous estimate concerning $D^+ u$.

Estimate for Theorem 2.
We absorb $\|D^+\tilde{\Delta} u\|_{l^2}$ only in the first term of the right hand-side of (3.7) writting

$$\|D^+\tilde{\Delta} u\|_{l^2}^{\frac{3}{2}} \ \|\tilde{\Delta} u\|_{l^2}^{\frac{3}{2}} \leq \frac{1}{2}\left(\|D^+\tilde{\Delta} u\|_{l^2}^2 + \|D^+\tilde{\Delta} u\|_{l^2} \ \|\tilde{\Delta} u\|_{l^2}^3\right),$$

and we obtain

$$\frac{d}{dt}\|\tilde{\Delta} u\|_{l^2}^2 + \|D^+\tilde{\Delta} u\|_{l^2}^2 \leq K\|\tilde{\Delta} u\|_{l^2}^3 \ \|D^+\tilde{\Delta} u\|_{l^2} \tag{3.12}$$

Combining (3.9) and (3.12) we derive

$$\frac{d}{dt}\|D^+ u\|_{h^1}^2 + \|\tilde{\Delta} u\|_{h^1}^2 \leq K\left(\|\tilde{\Delta} u\|_{l^2}^3 \ \|D^+\tilde{\Delta} u\|_{l^2} + \|D^+ u\|_{l^2}^2\|\tilde{\Delta} u\|_{l^2}^2\right)$$

Hence there exists a constant K independant of h such that

$$\frac{d}{dt}\|D^+ u\|_{h^1}^2 + \|\tilde{\Delta} u\|_{h^1}^2 \leq K\|D^+ u\|_{h^1}^2 \ \|\tilde{\Delta} u\|_{h^1}^2$$

thus

$$\frac{d}{dt}\|D^+u\|_{h^1}^2 + \|\tilde{\Delta}u\|_{h^1}^2(1 - K\|D^+u\|_{h^1}^2) \leq 0 \tag{3.13}$$

We set now $\delta = \frac{1}{\sqrt{K}}$ and we suppose that $\|D^+u_0\|_{h^1} < \delta$.

We claim that for all $t \geq 0$, $\|D^+u(t)\|_{h^1} < \delta$.

If it is not the case, then let t_1 be the first $t > 0$ such that $\|D^+u(t)\|_{h^1} \geq \delta$.

For all $t < t_1$,

$$1 - K\|D^+u\|_{h^1}^2 \geq 0,$$

hence, for all $t < t_1$,

$$\frac{d}{dt}\|D^+u\|_{h^1}^2 \leq 0, \text{ so } \|D^+u\|_{h^1}^2(t_1) \leq \|D^+u(0)\|_{h^1}^2 < \delta$$

which leads to a contradiction.

Therefore, if $\|D^+u_0\| < \delta$,

$$\|D^+u\|_{L^\infty(0,+\infty;h^1)} < \delta \tag{3.14}$$

and from (3.13) we deduce that there exists K such that

$$\|\tilde{\Delta}u\|_{L^2(0,+\infty;h^1)} \leq K \tag{3.15}$$

3. Limit when h goes to zero

Let us prove Theorem 1.

In the preceding subsection, for all $h > 0$ we have constructed a solution u^h of (3.1) defined on the mesh Z_h^3 which satisfies (3.1) and (3.11).

We extend u^h to the whole space using an interpolation process described in [5], p. 224.

We introduce the following interpolating operators :

For $X = (x_1, x_2, x_3) \in C_\alpha^h$, if we note $X_\alpha^h = (x_1^h, x_2^h, x_3^h)$, we set

- $r_h u^h(X) = u^h(X_\alpha^h)$,

- $p_h u^h(X) = u^h(X_\alpha^h) + \sum_{i=1}^3 D_i^+ u^h(X_\alpha^h)(x_i - x_i^h)$

  $+\sum_{1\leq i<j\leq 3} D_i^+ D_j^+(X_\alpha^h)(x_i - x_i^h)(x_j - x_j^h) + D_1^+ D_2^+ D_3^+ u^h(X_\alpha^h) \prod_{i=1}^3 (x_i - x_i^h)$,

- $q_h^k u^h(X) = u^h(X_\alpha^h) + \sum_{i\neq k} D_i^+ u^h(X_\alpha^h)(x_i - x_i^h)$

  $+\sum_{\substack{1\leq i<j\leq 3\\ i,j\neq k}} D_i^+ D_j^+(X_\alpha^h)(x_i - x_i^h)(x_j - x_j^h)$.

We recall that

$$\frac{\partial}{\partial x_i}(p_h u^h) = q_h^i(D_i^+ u^h).$$

Furthermore we have the following proposition proved in [5].

Proposition 1. *If one of the interpolates $p_h u^h$, $q_h u^h$, or $r_h u^h$ converges strongly (resp. weakly) in L^2 when h goes to zero, then the two others also converge to the same limit in L^2 strongly (resp. weakly).*

The estimate (3.11) gives that there exists $K > 0$ such that for all $h > 0$,

$$\|r_h(D_i^+ D_j^+ D_k^+ u^h)\|_{L^2((0,T)\times I\!R^3)} \leq K$$

$$\|r_h(D_i^+ D_j^+ u^h)\|_{L^2((0,T)\times I\!R^3)} \leq K$$

$$\|r_h(D_i^+ u^h)\|_{L^2((0,T)\times I\!R^3)} \leq K$$

Furthermore $r_h u^h$ is bounded in L^2_{loc} independentely of $h > 0$.
Thus, up to subsequences, we deduce that when h goes to zero,

$$r_h u^h \rightharpoonup u \text{ in } L^2_{loc} \text{ weakly,}$$

$$r_h(D_i^+ u^h) \rightharpoonup v_i \text{ in } L^2 \text{ weakly,}$$

$$r_h(D_i^+ D_j^+ u^h) \rightharpoonup w_{ij} \text{ in } L^2 \text{ weakly,}$$

$$r_h(D_i^+ D_j^+ D_k^+ u^h) \rightharpoonup \omega_{ijk} \text{ in } L^2 \text{ weakly,}$$

$$r_h(\frac{du^h}{dt}) \rightharpoonup f \text{ in } L^2 \text{ weakly.}$$

Now with Proposition 1., $q_h^i(D_i^+(D_j^+ D_k^+ u^h))$ and $r_h(D_i^+ D_j^+ D_k^+ u^h)$ have the same limit ω_{ijk}, and since $q_h^i(D_i^+(D_j^+ D_k^+ u^h)) = \dfrac{\partial}{\partial x_i}(p_h(D_j^+ D^+ k u^h))$, as $p_h(D_j^+ D^+ k u^h)$ tends to w_{jk} in L^2 weak, we deduce by uniqueness of the limit in $\mathcal{D}'$ that

$$\omega_{ijk} = \frac{\partial}{\partial x_i} w_{jk}.$$

With the same reasonnement we deduce that

$$v_i = \frac{\partial u}{\partial x_i}, \quad w_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad \omega_{ijk} = \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}.$$

In addition, we have $f = \dfrac{\partial u}{\partial t}$ and, since $\dfrac{\partial}{\partial t} p_h u^h \rightharpoonup \dfrac{\partial u}{\partial t}$ and $\dfrac{\partial}{\partial x_i} p_h u^h \rightharpoonup \dfrac{\partial u}{\partial x_i}$ in L^2 weakly, since $p_h u^h \rightharpoonup u$ in L^2_{loc} weakly, we deduce that

$$p_h u^h \to u \text{ in } L^p_{loc} \text{ strongly for } 2 \leq p < 6,$$

using the compactness of Sobolev embeddings in bounded domains.

In order to prove that u satisfies (1.1), we take $\varphi \in \mathcal{D}((0,T) \times I\!R^3)$ and we introduce Ω such that φ is zero outside of Ω. We have

$$\int_\Omega r_h(\frac{du^h}{dt}) \cdot \varphi = \int_\Omega r_h u^h \wedge r^h \tilde{\Delta} u^h \cdot \varphi - \int_\Omega r^h u^h \wedge (r^h u^h \wedge r^h \tilde{\Delta} u^h) \cdot \varphi \tag{3.16}$$

Now,

$$r_h \frac{du^h}{dt} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in } L^2(I\!R^3) \text{ weakly,}$$

$$r_h \tilde{\Delta} u^h \rightharpoonup \Delta u \quad \text{in } L^2(I\!R^3) \text{ weakly,}$$

$$r_h u^h \to u \quad \text{in } L^2(\Omega) \text{ strongly (for the first term),}$$

$$r_h u^h \to u \quad \text{in } L^4(\Omega) \text{ strongly (for the second term).}$$

Thus we can take the limit in (3.16) and we obtain that u satisfies (1.1) in $\mathcal{D}'(I\!R^3)$.

Furthermore, by lower semicontinuity of the different norms, we obtain from (3.11) that u satisfies

$$\nabla u \in L^\infty((0,T); I\!H^1(I\!R^3)) \cap L^2((0,T); I\!H^2(I\!R^3)).$$

Finaly, since $r_h u^h \to u$ in L^2_{loc} strongly, by extracting a subsequence, $r_h u^h \to u$ a.e., hence $|u| = 1$ as it is the case for $r_h u^h$.

Let us prove now the uniqueness of the solution of (1.1) satisfying (i) and (ii) in Theorem 1.

Let $\tilde{u}$ be another solution, and let $\bar{u} = u - \tilde{u}$.

We have

$$\frac{\partial \bar{u}}{\partial t} = \Delta \bar{u} + \bar{u} \wedge \Delta u + \tilde{u} \wedge \Delta \bar{u} + |\nabla u|^2 \bar{u} - \nabla \bar{u} \cdot (\nabla u + \nabla \tilde{u}) \tilde{u} \tag{3.17}$$

We multiply (3.17) by $\bar{u}$ and we obtain

$$\frac{1}{2}\frac{d}{dt}\|\bar{u}\|^2_{L^2} + \|\nabla \bar{u}\|^2_{L^2} \leq \|\nabla \bar{u}\|_{L^2}\|\bar{u}\|_{L^2}\|\nabla \tilde{u}\|_{L^\infty} + \|\bar{u}\|^2_{L^2}\|\nabla u\|^2_{L^\infty}$$

$$+\|\bar{u}\|_{L^2}\|\nabla \bar{u}\|_{L^2}(\|\nabla u\|_{L^\infty} + \|\nabla \tilde{u}\|_{L^\infty}).$$

We absorb $\|\nabla \bar{u}\|_{L^2}$ in the left hand-side of the inequality and we obtain

$$\frac{d}{dt}\|\bar{u}\|^2_{L^2} + \|\nabla \bar{u}\|^2_{L^2} \leq K\|\bar{u}\|_{L^2}(\|\nabla u\|_{L^\infty} + \|\nabla \tilde{u}\|_{L^\infty}).$$

Now since ∇u and $\nabla \tilde{u}$ belong to $L^1(0,T; L^\infty)$ (with Sobolev injections), we can use Gronwall Lemma to conclude that $\bar{u} = 0$.

Therefore Theorem 1 is proved.

In the same way we prove Theorem 2, starting from Estimates (3.14) and (3.15).

4 Proof of Theorem 3

1. Discretization.

For all $h > 0$ we consider (u_0^h, E_0^h, B_0^h) defined on Z_h^3 such that

- $|u_0^h| = 1, \; r_h u_0^h \underset{h\to 0}{\longrightarrow} u_0$ in $L^2_{loc}(I\!R^3)$,
- $\alpha \|D^+ u_0^h\|_{h^1(Z_h^3)} \leq \|\nabla u_0\|_{H^1(I\!R^3)} \leq \frac{1}{\alpha}\|D^+ u_0^h\|_{h^1(Z_h^3)}$,
- $B_0^h \in l^\infty(Z_h^3), \quad r_h B_0^h \underset{h\to 0}{\longrightarrow} B_0$ in $L^2(I\!R^3)$,
- $\alpha \|B_0^h\|_{h^1(Z_h^3)} \leq \|\nabla B_0\|_{H^1(I\!R^3)} \leq \frac{1}{\alpha}\|B_0^h\|_{h^1(Z_h^3)}$,
- $H_0^h \in l^\infty(Z_h^3), \quad r_h H_0^h \underset{h\to 0}{\longrightarrow} H_0$ in $L^2(I\!R^3)$,
- $\alpha \|H_0^h\|_{h^1(Z_h^3)} \leq \|\nabla H_0\|_{H^1(I\!R^3)} \leq \frac{1}{\alpha}\|H_0^h\|_{h^1(Z_h^3)}$,

where α does not depend on h.

We remark that B_0^h and E_0^h are bounded in $l^\infty(Z_h^3)$ but not uniformly in h.

Now, for h fixed, we solve the system

$$\frac{du^h}{dt} = u^h \wedge (\tilde{\Delta} u^h + B^h) - u^h \wedge (u^h \wedge (\tilde{\Delta} u^h + B^h)) \tag{4.1}$$

$$\frac{dB^h}{dt} = -\widetilde{\text{curl}}\, E^h \tag{4.2}$$

$$\frac{dE^h}{dt} = \widetilde{\text{curl}}\, B^h - \widetilde{\text{curl}}\, u^h \tag{4.3}$$

$$u^h(t=0) = u_0^h, \;\; B^h(t=0) = B_0^h, \;\; E^h(t=0) = E_0^h \tag{4.4}$$

Using Cauchy-Lipschitz theorem, there exists a local solution of (4.1)-(4.4).

Multiplying (4.1) by u we prove that $|u^h| = 1$ hence, we can write (4.1) on the form

$$\frac{du^h}{dt} = u^h \wedge (\tilde{\Delta} u^h + B^h) + \tilde{\Delta} u^h + \frac{1}{2}(|D^+ u^h|^2 + |D^- u^h|^2)u^h - u^h \wedge (u^h \wedge B^h) \tag{4.5}$$

Furthermore, we can eliminate E in (4.2)-(4.3) to obtain

$$\frac{d^2 B^h}{dt^2} - \tilde{\Delta} B^h = \widetilde{\text{curl}}\, \widetilde{\text{curl}}\, u^h \tag{4.6}$$

as $\widetilde{\text{div}}\, B^h = 0$.

In order to simplify the notations we will omit the exponent h in the computations of the following section.

2. Estimates.

First Estimate. We multiply (4.5) by $\tilde{\Delta} u$ and after summation on Z^3, we get

$$-\frac{d}{dt}\|D^+u\|_{l^2}^2 = \|\tilde{\Delta} u\|_{l^2}^2 + \int_{Z^3} ((u\wedge B) - u\wedge(u\wedge B))\cdot\tilde{\Delta} u + \frac{1}{2}\int_{Z^3}\left(|D^+u|^2 + |D^-u|^2\right) u\cdot\tilde{\Delta} u.$$

Hence

$$\frac{d}{dt}\|D^+u\|_{l^2}^2 + \|\tilde{\Delta} u\|_{l^2}^2 \le C\left(\|B\|_{l^2}\ \|\tilde{\Delta} u\|_{l^2} + \|D^+u\|_{l^2}\ \|\tilde{\Delta} u\|_{l^2}^2\right) \tag{4.7}$$

Second Estimate. We multiply (4.5) by $\tilde{\Delta}^2 u$ and we obtain

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\Delta} u\|_{l^2}^2 + \|D^+\tilde{\Delta} u\|_{l^2}^2 = \int_{Z^3} u\wedge\tilde{\Delta} u\cdot\tilde{\Delta}^2 u$$

$$+\frac{1}{2}\int_{Z^3}\left(|D^+u|^2 + |D^-u|^2\right) u\cdot\tilde{\Delta}^2 u$$

$$-\int_{Z^3}\sum_i D_i^+(u\wedge B)\cdot D_i^+\tilde{\Delta} u + \int_{Z^3}\sum_i D_i^+(u\wedge(u\wedge B))\cdot D_i^+\tilde{\Delta} u.$$

Now

$$\int_{Z^3}\sum_i D_i^+(u\wedge B)\cdot D_i^+\tilde{\Delta} u = \sum_i\int_{Z^3}\left(D_i^+u\wedge B + \tau_i^+u\wedge D_i^+B\right)\cdot D_i^+\tilde{\Delta} u,$$

thus

$$\left|\int_{Z^3}\sum_i D_i^+(u\wedge B)\cdot D_i^+\tilde{\Delta} u\right| \le K\|D^+\tilde{\Delta} u\|_{l^2}\left(\|D^+u\|_{l^4}\|B\|_{l^4} + \|D^+B\|_{l^2}\right).$$

In the same way,

$$\left|\int_{Z^3}\sum_i D_i^+(u\wedge(u\wedge B))\cdot D_i^+\tilde{\Delta} u\right| \le K\|D^+\tilde{\Delta} u\|_{l^2}\left(\|D^+u\|_{l^4}\|B\|_{l^4} + \|D^+B\|_{l^2}\right).$$

We treat the first two terms as in part 3 and we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\tilde{\Delta} u\|_{l^2}^2 + \|D^+\tilde{\Delta} u\|_{l^2}^2 &\le K\|D^+\tilde{\Delta} u\|_{l^2}\left(\|D^+u\|_{l^4}\|B\|_{l^4} + \|D^+B\|_{l^2}\right)\\ &+K\left(\|D^+\tilde{\Delta} u\|_{l^2}^{\frac{3}{2}}\ \|\tilde{\Delta} u\|_{l^2}^{\frac{3}{2}} + \|D^+\tilde{\Delta} u\|_{l^2}\ \|\tilde{\Delta} u\|_{l^2}^3\right)\end{aligned} \tag{4.8}$$

Now, by interpolation and discrete Sobolev embeddings, and absorbing $\|D^+\tilde{\Delta} u\|_{l^2}$ in the right hand-side of (4.8), we obtain

$$\begin{aligned}\frac{d}{dt}\|\tilde{\Delta} u\|_{l^2}^2 + \|D^+\tilde{\Delta} u\|_{l^2}^2 \le K\Big(&\|D^+u\|_{l^2}\|\tilde{\Delta} u\|_{l^2}\|B\|_{l^2}\|D^+B\|_{l^2}\\ &+\|D^+B\|_{l^2}^2 + \|\tilde{\Delta} u\|_{l^2}^6\Big)\end{aligned} \tag{4.9}$$

Third estimate. We multiply (4.2) by B and (4.3) by E and we obtain, as $\|\widetilde{\mathrm{curl}}\, u\|_{l^2} \le C\|D^+u\|_{l^2}$,

$$\frac{d}{dt}\left(\|B\|_{l^2}^2 + \|E\|_{l^2}^2\right) \le K\|D^+u\|_{l^2}\ \|E\|_{l^2} \tag{4.10}$$

Fourth estimate. We multiply (4.6) by $\dfrac{dB}{dt}$ and we get, since $\|\widetilde{\text{curl}}\ \widetilde{\text{curl}}\ u\|_{l^2} \leq K\|\tilde{\Delta}u\|_{l^2}$,

$$\frac{d}{dt}\left(\|\frac{dB}{dt}\|_{l^2}^2 + \|D^+B\|_{l^2}^2\right) \leq K\|\tilde{\Delta}u\|_{l^2}\ \|\frac{dB}{dt}\|_{l^2} \tag{4.11}$$

Combining (4.7), (4.9), (4.10) and (4.11), if we denote

$$\mathcal{E}(t) = \left(\|D^+u\|_{l^2}^2 + \|\tilde{\Delta}u\|_{l^2}^2 + \|E\|_{l^2}^2 + \|B\|_{l^2}^2 + \|\frac{dB}{dt}\|_{l^2}^2 + \|D^+B\|_{l^2}^2\right)(t),$$

we obtain

$$\frac{d\mathcal{E}}{dt} + \|\tilde{\Delta}u\|_{h^1}^2(t) \leq K\left(1 + \mathcal{E}^3\right).$$

Therefore, there exist T and K independent of h such that

$$\|D^+u\|_{L^\infty(0,T;h^1)} \leq K,\ \ \|\tilde{\Delta}u\|_{L^2(0,T;h^1)} \leq K$$

$$\|\frac{du}{dt}\|_{L^\infty(0,T;l^2)} \leq K \tag{4.12}$$

$$\|B\|_{L^\infty(0,T;h^1)} \leq K,\ \ \|E\|_{L^\infty(0,T;h^1)} \leq K$$

We note that we can estimate $\|D^+E\|_{l^2}$ since

$$\|D^+E\|_{l^2} = \|\widetilde{\text{div}}\ E\|_{l^2}^2 + \|\widetilde{\text{curl}}\ E\|_{l^2}^2$$

$$= \|\widetilde{\text{div}}\ E_0\|_{l^2}^2 + \|\frac{dB}{dt}\|_{l^2}^2.$$

3. Limit when h goes to zero and uniqueness.

As in Part 3, we extend the discrete solution (u^h, E^h, B^h) to the whole space and with the same arguments, we can take the limit when h goes to zero, using (4.12).

The limit (u, E, B) satisfies the properties (i), (ii), and (iii) anounced in theorem 3.

Now let us prove the uniqueness of the regular solution for (1.3)-(1.6).

Let us consider $(\tilde{u}, \tilde{E}, \tilde{B})$ another regular solution for (1.3)-(1.6). We set

$$\bar{u} = u - \tilde{u},\ \ \bar{E} = E - \tilde{E},\ \ \bar{B} = B - \tilde{B}.$$

We have

$$\frac{\partial \bar{B}}{\partial t} = -\text{curl}\ \bar{E} \tag{4.13}$$

$$\frac{\partial \bar{E}}{\partial t} = \text{curl}\ \bar{B} - \text{curl}\ \bar{u} \tag{4.14}$$

$$\begin{aligned}\frac{\partial \bar{u}}{\partial t} &= \Delta\bar{u} + \bar{u}\wedge\Delta u + \tilde{u}\wedge\Delta\bar{u} + \bar{u}\wedge B - \tilde{u}\wedge\bar{B}\\ &+|\nabla u|^2\bar{u} - \nabla\bar{u}\cdot(\nabla u + \nabla\tilde{u})\tilde{u}\\ &-\bar{u}\wedge(u\wedge B) - \tilde{u}\wedge(\bar{u}\wedge B + \tilde{u}\wedge\bar{B}).\end{aligned} \tag{4.15}$$

We multiply (4.15) by $\bar{u}$ and we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\bar{u}\|_{L^2}^2 + \|\nabla\bar{u}\|_{L^2}^2 &= \int \tilde{u}\wedge\Delta\bar{u}\cdot\bar{u} - \int \tilde{u}\wedge\bar{B}\cdot\bar{u} \\ &+ \int |\nabla u|^2|\bar{u}|^2 - \int \nabla\bar{u}\cdot(\nabla u+\nabla\tilde{u})(\tilde{u}\cdot\bar{u}) \\ &- \int \tilde{u}\wedge(\bar{u}\wedge B)\cdot\bar{u} - \int \tilde{u}\wedge(\tilde{u}\wedge\bar{B})\cdot\bar{u}.\end{aligned}$$

Hence, using that $\|\bar{u}\|_{L^4}^2 \le C\|\bar{u}\|_{L^2}\|\nabla\bar{u}\|_{L^2}$, and absorbing the term $\|\nabla\bar{u}\|_{L^2}$, we obtain that

$$\frac{d}{dt}\|\bar{u}\|_{L^2}^2 + \|\nabla\bar{u}\|_{L^2}^2 \le K\|\bar{u}\|_{L^2}^2\left(\|\nabla\tilde{u}\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty}^2 + \|B\|_{L^2}^2 + 1\right) + \|\bar{B}\|_{L^2}^2 \tag{4.16}$$

Furthermore, multiplying (4.13) by $\bar{B}$ and (4.14) by $\bar{E}$, we obtain

$$\frac{d}{dt}\left(\|\bar{E}\|_{L^2}^2 + \|\bar{B}\|_{L^2}^2\right) \le 2\|\nabla\bar{u}\|_{L^2}\|\bar{E}\|_{L^2} \tag{4.17}$$

We set

$$\mathcal{E}(t) = \left(\|\bar{E}\|_{L^2}^2 + \|\bar{B}\|_{L^2}^2 + \|\bar{u}\|_{L^2}^2\right)(t)$$

and

$$f(t) = \left(1 + \|B\|_{L^2}^2 + \|\nabla u\|_{L^\infty}^2 + \|\nabla\tilde{u}\|_{L^\infty}^2\right)(t).$$

Combining (4.16) and (4.17) and absorbing $\|\nabla\bar{u}\|_{L^2}$, we obtain

$$\frac{d}{dt}\mathcal{E}(t) \le Kf(t)\mathcal{E}(t).$$

We remark that $f(t) \in L^1(0,T)$, and with Gronwall Lemma, we conclude that $\mathcal{E} = 0$.

Therefore, Theorem 3 is proved.

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