ON THE THETA NUMBER OF POWERS OF CYCLE GRAPHS

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ABSTRACT. We give a closed formula for Lovász's theta number of the powers of cycle graphs C_k^d and of their complements, the circular complete graphs $K_{k/d}$. As a consequence, we establish that the circular chromatic number of a circular perfect graph is computable in polynomial time. We also derive an asymptotic estimate for the theta number of C_k^d.

1. INTRODUCTION

Let $G = (V, E)$ be a finite graph with vertex set V and edge set E. The *clique number* $\omega(G)$ and the *chromatic number* $\chi(G)$ are classical invariants of G which can be defined in terms of graph homomorphisms.

A *homomorphism* from a graph $G = (V, E)$ to a graph $G' = (V', E')$ is a mapping $f : V \to V'$ which preserves adjacency: if ij is an edge of G then $f(i)f(j)$ is an edge of G'. If there is a homomorphism from G to G', we write $G \to G'$. Then, the chromatic number of G is the smallest number k such that $G \to K_k$, where K_k denotes the complete graph with k vertices. Similarly, the clique number is the largest number k such that $K_k \to G$.

In the seminal paper [7], Lovász introduced the so-called *theta number* $\vartheta(G)$ of a graph. On one hand, this number provides an approximation of $\omega(G)$ and of $\chi(G)$ since (this is the celebrated *Sandwich Theorem*)

$$\omega(G) \leq \vartheta(\overline{G}) \leq \chi(G),$$

where $\overline{G}$ stands for the complement of G. On the other hand, this number is the optimal value of a *semidefinite program* [11], and, as such, is computable in polynomial time with polynomial space encoding accuracy [11], [5]. In contrast, the computation of either of the clique number or the chromatic number is known to be NP-hard.

By definition, a graph G is *perfect*, if for every induced subgraph H, $\omega(H) = \chi(H)$ [1]. As $\vartheta(\overline{G}) = \chi(G)$ and $\chi(G)$ is an integer, we have

Theorem 1.1. *(Grötschel, Lovász and Schrijver)* [5] *For every perfect graph, the chromatic number is computable in polynomial time.*

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There are very few families of graphs for which an explicit formula for the theta number is known. In [7], the theta numbers of the cycles C_k and of the Kneser graphs $K(n,r)$ are explicitly computed. In particular, it is shown that, if k is an odd number,

$$\vartheta(C_k) = \frac{k\cos\left(\frac{\pi}{k}\right)}{1+\cos\left(\frac{\pi}{k}\right)}. \tag{1}$$

In this paper, we give a closed formula for the theta number of the *circular complete graphs* $K_{k/d}$ and of their complements $\overline{K_{k/d}}$. For $k \geq 2d$, the graph $K_{k/d}$ has k vertices $\{0,1,\ldots,k-1\}$ and two vertices i and j are connected by an edge if $d \leq |i-j| \leq k-d$. We have $K_{k/1} = K_k$ and $\overline{K_{k/2}} = C_k$. More generally, the graph $\overline{K_{k/d}}$ is the $(d-1)^{\text{th}}$ power of the cycle graph C_k. Because the automorphism group of $K_{k/d}$ is vertex transitive (it contains the cyclic permutation $(0,1,\ldots,k-1)$), we have (see [7, Theorem 8])

$$\vartheta(K_{k/d})\vartheta(\overline{K_{k/d}}) = k. \tag{2}$$

Besides the case $d = 2$ demonstrated by Lovász, the only case previously known was due to Brimkov et al. who proved in [4] that, for $d = 3$ and k odd,

$$\vartheta\left(\overline{K_{k/3}}\right) = k\left(1 - \frac{\frac{1}{2} - \cos\left(\frac{2\pi}{k}\lfloor\frac{k}{3}\rfloor\right) - \cos\left(\frac{2\pi}{k}\left(\lfloor\frac{k}{3}\rfloor+1\right)\right)}{\left(\cos\left(\frac{2\pi}{k}\lfloor\frac{k}{3}\rfloor\right)-1\right)\left(\cos\left(\frac{2\pi}{k}\left(\lfloor\frac{k}{3}\rfloor+1\right)\right)-1\right)}\right). \tag{3}$$

In Section 3, we prove the following:

Theorem 1.2. *Let* $d \geq 2$, $k \geq 2d$, *with* $\gcd(k,d) = 1$. *Let, for* $0 \leq n \leq d-1$,

$$c_n := \cos\left(\frac{2n\pi}{d}\right), \quad a_n := \cos\left(\left\lfloor\frac{nk}{d}\right\rfloor\frac{2\pi}{k}\right).$$

Then

$$\vartheta(\overline{K_{k/d}}) = \frac{k}{d}\sum_{n=0}^{d-1}\prod_{s=1}^{d-1}\left(\frac{c_n - a_s}{1-a_s}\right). \tag{4}$$

The notions of clique and chromatic numbers and of perfect graphs have been refined using circular complete graphs. The *circular chromatic number* $\chi_c(G)$ of a graph G was first introduced by Vince in [12]. It is the minimum of the fractions k/d for which $G \to K_{k/d}$. Later, Zhu defined the *circular clique number* $\omega_c(G)$ of G to be the maximum of the k/d for which $K_{k/d} \to G$ and introduced the notion of a *circular perfect graph*, a graph with the property that every induced subgraph H satisfies $\omega_c(H) = \chi_c(H)$ (see Section 7 in [14] for a survey on this notion). The class of circular perfect graphs extends in a natural way the one of perfect graphs. So one can ask for the properties of perfect graphs that generalize to this larger class. In this paper, we prove that Theorem 1.1 still holds for circular perfect graphs:

Theorem 1.3. *For every circular perfect graph, the circular chromatic number is computable in polynomial time.*

In previous works, the polynomial time computability of the chromatic number of circular perfect graphs was established in [8] and of the circular chromatic number of strongly circular perfect graphs (i.e. circular perfect graphs such that the complementary graphs are also circular perfect) was proved in [9].

In contrary to perfect graphs, $\vartheta(\overline{G})$ does not give directly the result, as $\vartheta(G)$ is not always sandwiched between $\omega_c(G)$ and $\chi_c(G)$: for instance, $\vartheta(\overline{C_5}) = \sqrt{5}$ and $\omega_c(\overline{C_5}) = \chi_c(\overline{C_5}) = 5/2$. To bypass this difficulty, we make use of this following basic observation: by definition, for every graph G with n vertices such that $\omega_c(G) = \chi_c(G) = k/d$, we have $\vartheta(\overline{G}) = \vartheta(\overline{K_{k/d}})$, where $k, d \leq n$ (see the next section for more details). Hence, to ensure the polynomial time computability of $\chi_c(G)$, it is sufficient to prove that the values $\vartheta(\overline{K_{k/d}})$ with $k, d \leq n$ are all distinct and separated by at least ϵ for some ϵ with polynomial space encoding.

This paper is organized as follows: Section 2 gathers the needed definitions and properties of Lovász theta number and of circular numbers. Section 3 proves Theorem 1.2, while Section 4 proves Theorem 1.3. In Section 5, the asymptotic estimate $\vartheta(\overline{K_{k/d}}) = \frac{k}{d} + \mathrm{O}\left(\frac{1}{k}\right)$ is obtained (Theorem 5.1).

2. Preliminaries

The theta number $\vartheta(G)$ of a graph $G = (V, E)$ was introduced in [7], where many equivalent formulations are given. The one of [7, Theorem 4] has the form of a semidefinite program:

$$\begin{aligned} \vartheta(G) = \max\Big\{ \sum_{(x,y)\in V^2} B(x,y) \quad &: \ B \in \mathbb{R}^{V\times V},\ B \succeq 0, \\ &\sum_{x\in V} B(x,x) = 1, \\ &B(x,y) = 0 \quad xy \in E \Big\} \end{aligned} \tag{5}$$

where $B \succeq 0$ stands for: B is a symmetric, positive semidefinite matrix. For a survey on semidefinite programming, we refer to [11]. The dual program gives another formulation for $\vartheta(G)$ (there is no duality gap here because the identity matrix is a strictly feasible solution of (5) so the Slater condition is fulfilled):

$$\begin{aligned} \vartheta(G) = \inf\Big\{ t \quad &: \ B \in \mathbb{R}^{V\times V},\ B \succeq 0, \\ &B(x,x) = t - 1, \\ &B(x,y) = -1 \quad xy \notin E \Big\} \end{aligned} \tag{6}$$

From (6) one can easily derive that, if $G \to G'$, then $\vartheta(\overline{G}) \leq \vartheta(\overline{G'})$. Indeed, if B' is an optimal solution of the dual program defining $\vartheta(\overline{G'})$, then the matrix B defined by $B(x,y) := B'(f(x), f(y))$ is feasible for $\vartheta(\overline{G})$.

The circular complete graphs $K_{k/d}$ have the property that $K_{k/d} \to K_{k'/d'}$ if and only if $k/d \leq k'/d'$ (see [3]). Thus the theta number $\vartheta(\overline{K_{k/d}})$ only depends on the quotient k/d, and we later conveniently assume that k and d are coprime.

From the definition, it follows that

$$\omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G).$$

Moreover, $\omega(G) = \lfloor \omega_c(G) \rfloor$, $\chi(G) = \lceil \chi_c(G) \rceil$, and $\omega_c(G)$ and $\chi_c(G)$ are attained for pairs (k, d) such that $k \geq 2d$ and $k \leq |V|$ (see [3], [13]).

If G is a circular perfect graph, let k, d be such that $\gcd(k, d) = 1$ and $\omega_c(G) = \chi_c(G) = k/d$. Because G and $K_{k/d}$ are homomorphically equivalent, $\vartheta(\overline{G}) = \vartheta(\overline{K_{k/d}})$. Summarizing, we have

Proposition 2.1. *Let G be a circular perfect graph with n vertices. Then,*

1. $\omega_c(G) = \chi_c(G) = k/d$ *for some* (k, d) *such that* $k \geq 2d$*,* $k \leq n$*, and* $\gcd(k, d) = 1$.
2. $\vartheta(\overline{G}) = \vartheta(\overline{K_{k/d}})$.

3. An explicit formula for the theta number of circular complete graphs

In this section we prove Theorem 1.2. We start with an overview of our proof: first of all we show that $\vartheta(\overline{K_{k/d}})$ is the optimal value of a linear program (Proposition 3.1). This step is a standard simplification of a semidefinite program using its symmetries. In a second step, a candidate for an optimal solution of the resulting linear program is defined (Definition 3.2) as the unique solution of a certain linear system. We give an interpretation of this element, in terms of the coefficients of Lagrange interpolation polynomials on the basis of Chebyshev polynomials. Then, playing with the dual linear program, it is easy to prove that this element, *if feasible*, is indeed optimal (Lemma 3.4). The last step, which is also the most technical, amounts to prove that this elements is indeed feasible, i.e. essentially that its coordinates are non negative (Lemma 3.3). To that end, we boil down to prove that a certain polynomial $L_0(y)$ has non negative coefficients when expanded as a linear combination of the Chebyshev polynomials (Lemma 3.6).

3.1. A linear program defining the theta number.

The vertex set of $G = \overline{K_{k/d}}$ can be identified with the additive group $\mathbb{Z}/k\mathbb{Z}$, and the additive action of this group defines automorphisms of this graph. This action allows to transform the semidefinite program (5) into a linear program, as follows:

Proposition 3.1. *Let $k_0 := \lfloor k/2 \rfloor$. We have:*

$$\vartheta(\overline{K_{k/d}}) = \max \Big\{ kf_0 \ : \quad f_j \geq 0, \quad \sum_{j=0}^{k_0} f_j = 1, \quad \sum_{j=0}^{k_0} f_j \cos\Big(\frac{2j\ell\pi}{k}\Big) = 0, \quad 1 \leq \ell \leq d-1 \Big\} \tag{7}$$

and also:

$$\vartheta(\overline{K_{k/d}}) = \min\Big\{ kg_0 \ :\ \sum_{\ell=0}^{d-1} g_\ell \geq 1, \quad \sum_{\ell=0}^{d-1} g_\ell \cos\Big(\frac{2\ell j\pi}{k}\Big) \geq 0, \quad 1 \leq j \leq k_0 \Big\} \tag{8}$$

Proof. Taking the average over the translations by the elements of $\mathbb{Z}/k\mathbb{Z}$, one constructs from a matrix B which is optimal for (5), another optimal matrix which is translation invariant, i.e. which satisfies $B(x+z, y+z) = B(x,y)$ for all $x, y, z \in \mathbb{Z}/k\mathbb{Z}$. Thus one can restrict in (5) to the matrices B which are translation invariant. In other words, we can assume that $B(x,y) = F(x-y)$ for some $F : \mathbb{Z}/k\mathbb{Z} \to \mathbb{R}$. Then we can use the Fourier transform over $\mathbb{Z}/k\mathbb{Z}$ to express F as

$$F(z) = \sum_{j=0}^{k-1} f_j e^{2ijz\pi/k}.$$

Then $B \succeq 0$ if and only if $f_j = f_{k-j}$ and $f_j \geq 0$ for all $j = 0, \dots, k-1$. After a change from f_j to $2f_j$ for $j \neq 0, k/2$, we can rewrite

$$B(x,y) = \sum_{j=0}^{k_0} f_j \cos\Big(\frac{2j(x-y)\pi}{k}\Big).$$

Then, it remains to transfer to $(f_0, \dots, f_{k_0})$ the constraints on B that stand in (5). We have $\sum_{(x,y)\in(\mathbb{Z}/k\mathbb{Z})^2} B(x,y) = k^2 f_0$, and $\sum_{x\in\mathbb{Z}/k\mathbb{Z}} B(x,x) = k\sum_{j=0}^{k_0} f_j$. The edges of $K_{k/d}$ are the pairs (x,y) with $1 \leq |x-y| \leq d-1$ so the condition that $B(x,y) = 0$ for all edges (x,y) translates to

$$\sum_{j=0}^{k_0} f_j \cos\Big(\frac{2j\ell\pi}{k}\Big) = 0, \quad 1 \leq \ell \leq d-1.$$

Changing f_j to f_j/k leads to (7). The linear program (8) is the dual formulation of (7). □

3.2. **A candidate for an optimal solution of** (7)**.** In order to understand the construction of this solution, it is worth to take a look at the case when d divides k. Indeed, in this case, the system of linear equations

$$\sum_{j=0}^{k_0} f_j \cos\Big(\frac{2j\ell\pi}{k}\Big) = \delta_{0,l}, \quad 0 \leq \ell \leq d-1$$

which is equivalent to

$$\sum_{j=0}^{k-1} f'_j e^{2ij\ell\pi/k} = \delta_{0,l}, \quad 0 \leq \ell \leq d-1$$

where $f'_j = f'_{k-j} = f_j/2$ for $j \neq 0, k/2$, otherwise $f'_j = f_j$, has an obvious solution $f = (f'_0, \dots, f'_{k-1})$ defined as follows: take $f_j = 1/d$ for the indices j which are multiples of k/d, i.e. for $j = nk/d$, $n = 0, \dots, d-1$. Take $f_j = 0$ for other indices. Then f has exactly d non zero coefficients, is feasible because $f_j \geq 0$, and its objective value equals k/d, which is also the optimal value of the linear program.

In the case when $\gcd(k, d) = 1$, none of the rational numbers nk/d, for $n = 0, \dots, d-1$, are integers. Instead, we choose indices which are as close as possible, namely we choose the indices of the form $\lfloor \frac{nk}{d} \rfloor$ for $0 \leq n \leq d-1$ and set all other coefficients to zero. Then, the d-tuple of coefficients which are not set to zero, satisfies a linear system with d equations, and this linear system has a unique solution. We shall prove that in this way an optimal solution of (7) is obtained.

Now we introduce some additional notations. The Chebyshev polynomials ([10]), denoted $(T_\ell)_{\ell \geq 0}$ are defined by the characteristic property: $T_\ell(\cos(\theta)) = \cos(\ell\theta)$. They can be iteratively computed by the relation $T_{\ell+1}(x) = 2xT_\ell(x) - T_{\ell-1}(x)$ and the first terms $T_0 = 1$, $T_1 = x$. These polynomials are orthogonal for the measure $dx/\sqrt{1-x^2}$ supported on the interval $[-1, 1]$.

The numbers a_n, $0 \leq n \leq d-1$, introduced in Theorem 1.2, come into play now. Recall that

$$a_n = \cos\Big(\Big\lfloor \frac{nk}{d} \Big\rfloor \frac{2\pi}{k}\Big).$$

We remark that the coefficients in the linear constraints of (7) associated to the indices $\lfloor \frac{nk}{d} \rfloor$ are precisely equal to $T_\ell(a_n)$.

We assume for the rest of this section that $\gcd(k, d) = 1$. Then the real numbers a_n are pairwise distinct. We introduce the Lagrange polynomials ([10]) associated to $(a_0, \dots, a_{d-1})$:

$$L_n(y) := \prod_{\substack{s=0 \\ s \neq n}}^{d-1} \Big(\frac{y - a_s}{a_n - a_s}\Big). \tag{9}$$

Now we have two basis for the space of polynomials of degree at most equal to $d-1$: the Chebyshev basis $\{T_0, \dots, T_{d-1}\}$ and the Lagrange basis $\{L_0, \dots, L_{d-1}\}$. We introduce the two $d \times d$ matrices $T = (\tau_{\ell,n})$ and $L = (\lambda_{n,\ell})$ such that

$$T_\ell(y) = \tau_{\ell,0}L_0(y) + \tau_{\ell,1}L_1(y) + \dots + \tau_{\ell,d-1}L_{d-1}(y) \quad 0 \leq \ell \leq d-1 \tag{10}$$

and

$$L_n(y) = \lambda_{n,0}T_0(y) + \lambda_{n,1}T_1(y) + \dots + \lambda_{n,d-1}T_{d-1}(y) \quad 0 \leq n \leq d-1. \tag{11}$$

Obviously we have

$$\tau_{\ell,n} = T_\ell(a_n) \tag{12}$$

and

$$TL = LT = I_d.$$

In particular, the d-tuple $(\lambda_{0,0}, \lambda_{1,0}, \dots, \lambda_{d-1,0})$ satisfies the equations:

$$\sum_{n=0}^{d-1} \lambda_{n,0} T_\ell(a_n) = \delta_{\ell,0}, \quad 0 \le \ell \le d-1. \tag{13}$$

Now we can define our candidate for an optimal solution of (7):

Definition 3.2. *With the above notations, let* $f^* = (f_0^*, \dots, f_{k_0}^*)$ *be defined by:*

$$\begin{cases} f_j^* = \lambda_{n,0} & \text{if } j = \pm\lfloor \frac{nk}{d} \rfloor \mod k \\ f_j^* = 0 & \text{otherwise.} \end{cases}$$

It remains to prove that f^* is indeed optimal for (7). It will result from the two following lemmas:

Lemma 3.3. *For all* j*,* $0 \le j \le k_0$*,* $f_j^* \ge 0$*.*

We postpone the proof of Lemma 3.3 to the next subsection.

Lemma 3.4. f^* *is an optimal solution of* (7)*.*

Proof. Lemma 3.3, joined with (13) shows that f^* is feasible. Thus we can derive the inequality:

$$k\lambda_{0,0} \le \vartheta(\overline{K_{k/d}}).$$

Now we claim that the element $g^* = (\lambda_{0,0}, \lambda_{0,1}, \dots, \lambda_{0,d-1})$ is a feasible solution of the dual program (8). For that we need to prove that

$$\sum_{\ell=0}^{d-1} \lambda_{0,\ell} \cos\Big(\frac{2\ell j\pi}{k}\Big) \ge \delta_{j,0}, \quad 0 \le j \le k_0$$

which can be rewritten as

$$\sum_{\ell=0}^{d-1} \lambda_{0,\ell} T_\ell\Big(\cos\Big(\frac{2j\pi}{k}\Big)\Big) \ge \delta_{j,0}, \quad 0 \le j \le k_0$$

or, taking account of (11),

$$L_0\Big(\cos\Big(\frac{2j\pi}{k}\Big)\Big) \ge \delta_{j,0}, \quad 0 \le j \le k_0. \tag{14}$$

For $j = 0$, (14) holds because $L_0(1) = L_0(a_0) = 1$. For $j \ge 1$, we take a look at the position of $\cos(2j\pi/k)$ with respect to the roots $a_1, \dots, a_{d-1}$ of L_0. Indeed, these roots belong to the set $\{\cos(2j\pi/k), j = 1, \dots, k-1\}$, but it should be noticed that they go in successive pairs. More precisely, a_n and a_{d-n} are equal to the first coordinate of neighbor vertices of the regular k-gone. So either $\cos(2j\pi/k)$ is equal to one of the a_n, or there is an even number of roots a_n, $n \ge 1$, which are greater than $\cos(2j\pi/k)$. In the later case, $L_0(\cos(2j\pi/k))$ and $L_0(1)$ have the same sign. Since $L_0(1) = 1$, we are done.

Since g^* is a feasible solution of (8), its objective value, which is equal to $k\lambda_{0,0}$, upper bounds $\vartheta(\overline{K_{k/d}})$. So we conclude that

$$\vartheta(\overline{K_{k/d}}) = k\lambda_{0,0}$$

and that f^* is an optimal solution of (7). □

3.3. **The proof of Lemma 3.3.** We want to prove that $\lambda_{n,0} \geq 0$ for all $n = 0, \dots, d-1$. We first prove that this condition is equivalent to: $\lambda_{0,\ell} \geq 0$ for all $0 \leq \ell \leq d-1$.

Lemma 3.5. *The tuples* $(\lambda_{n,0}, 0 \leq n \leq d-1)$ *and* $(\lambda_{0,\ell}, 0 \leq \ell \leq d-1)$ *are equal up to a permutation of their coordinates.*

Proof. It turns out that, up to a permutation of the a_n, the matrix T is symmetric. Since $\gcd(k, d) = 1$, one can find v, $1 \leq v \leq d-1$, and $t \geq 0$, such that $kv = 1 + td$. By definition,

$$a_n := \cos\Big(\Big\lfloor \frac{nk}{d} \Big\rfloor \frac{2\pi}{k}\Big)$$

only depends on $n \mod d$. Let us compute $a_{n'}$ where $n' = vn \mod d$. Since $vnk/d = n/d + nt$, we have

$$a_{n'} = \cos\Big(\Big\lfloor \frac{vnk}{d} \Big\rfloor \frac{2\pi}{k}\Big) = \cos\Big(\frac{2nt\pi}{k}\Big).$$

If we set

$$x = x(k,d) := \cos\Big(\frac{2t\pi}{k}\Big) = \cos\Big(\Big(\frac{v}{d} - \frac{1}{kd}\Big)2\pi\Big), \tag{15}$$

we have

$$a_{n'} = T_n(x).$$

If we reorder the a_n according to the permutation $n \mapsto vn \mod d$, which fixes 0, the coefficients of the corresponding matrix T are equal to:

$$\tau_{\ell,n} = T_\ell(a_{n'}) = T_\ell(T_n(x)) = T_{\ell n}(x) = T_{n\ell}(x) = \tau_{n,\ell}.$$

Thus the new matrix T is symmetric. This reordering of the a_n, permutes accordingly the coordinates of $(\lambda_{n,0}, 0 \leq n \leq d-1)$. Also the matrix $L = T^{-1}$ has become symmetric, so the permuted $\lambda_{n,0}$ are equal to $\lambda_{0,n}$ (who have not changed in the procedure because the polynomial $L_0(y)$ is not affected by the reordering of the a_n). □

The next lemma ends the proof of Lemma 3.3:

Lemma 3.6. *For all* $0 \leq \ell \leq d-1$, $\lambda_{0,\ell} \geq 0$.

Proof. Since $\prod_{s=1}^{d-1}(1 - a_s) \geq 0$, we can replace $L_0(y)$ by

$$\prod_{s=1}^{d-1}(y - a_s) = \prod_{s=1}^{d-1}(y - T_s(x))$$

where x is defined in (15). The right hand side becomes a polynomial in the variables x and y, depending only on d. This polynomial has an expansion in the Chebyshev basis:

$$\prod_{s=1}^{d-1}(y - T_s(x)) = \sum_{\ell=0}^{d-1} Q_\ell(x) T_\ell(y). \tag{16}$$

We introduce complex variables X and Y, such that $2x = X + 1/X$ and $2y = Y + 1/Y$. Then, (16) becomes:

$$\prod_{s=1}^{d-1}(Y - X^s)(Y - X^{-s}) = Y^{d-1} \sum_{\ell=-(d-1)}^{d-1} Q'_\ell(x) Y^\ell \tag{17}$$

where $Q'_0 = 2^{d-1}Q_0$ and, for $\ell = 1, \dots, d-1$, $Q'_{-\ell} = Q'_\ell = 2^{d-2}Q_\ell$. We want to prove that $Q'_\ell(x) \geq 0$ when x is given by (15). To that end, we will prove that this sequence of numbers is decreasing:

$$Q'_0(x) \geq Q'_1(x) \geq \cdots \geq Q'_{d-1}(x) \tag{18}$$

and since $Q'_{d-1}(x) = 1$, we will be done. Now the idea is to multiply the equation (17) by $(Y - 1)$, so that the successive differences $Q'_{\ell-1}(x) - Q'_\ell(x)$ appear in the right hand side as the coefficients of Y^ℓ. We obtain, setting $Q'_{-d} = Q'_d := 0$:

$$\prod_{s=-(d-1)}^{d-1} (Y - X^s) = Y^{d-1} \sum_{\ell=-(d-1)}^{d} (Q'_{\ell-1}(x) - Q'_\ell(x)) Y^\ell. \tag{19}$$

We let:

$$P(Y) := \prod_{s=-(d-1)}^{d-1} (Y - X^s) := \sum_{j=0}^{2d-1} C_j(X) Y^j. \tag{20}$$

We have:

$$\begin{aligned} P(XY) &= \prod_{j=-(d-1)}^{d-1} (XY - X^j) \\ &= X^{2d-1} \prod_{j=-(d-1)}^{d-1} (Y - X^{j-1}) \\ &= X^{2d-1} \frac{Y - X^{-d}}{Y - X^{d-1}} P(Y). \end{aligned}$$

This equation leads to:

$$(Y - X^{d-1}) \sum_{j=0}^{2d-1} C_j(X) X^j Y^j = (X^{2d-1}Y - X^{d-1}) \sum_{j=0}^{2d-1} C_j(X) Y^j.$$

Comparing the coefficients of Y^j in both sides, we obtain the formula:

$$C_j(X) = C_{j-1}(X) \frac{X^{j-d} - X^d}{X^j - 1}, \quad 1 \leq j \leq 2d-1. \tag{21}$$

If $X = e^{i\theta}$, we obtain in (21)

$$C_j(X) = C_{j-1}(X) \frac{\sin((\frac{j}{2} - d)\theta)}{\sin(\frac{j\theta}{2})}, \quad 1 \leq j \leq 2d-1. \tag{22}$$

Thus, taking account of (15), (19), (20) and (22), the inequalities (18) that we want to establish, are equivalent to the non negativity of $\frac{\sin((\frac{j}{2}-d)\theta)}{\sin(\frac{j\theta}{2})}$ when $\theta = \big(\frac{v}{d} - \frac{1}{kd}\big)2\pi$, $1 \le j \le d-1$, and $k \ge 2d$. Let us prove it now: let

$$\begin{cases} N = N(k,d) := \lfloor \frac{jv}{d} \rfloor \\ \varepsilon = \varepsilon(k,d) := \frac{jv}{d} - N. \end{cases}$$

Since v and d are coprime and $1 \le j < d$, we have $\varepsilon \in \{\frac{1}{d}, \frac{2}{d}, \dots, \frac{d-1}{d}\}$. We first study the sign of $\sin(\frac{j\theta}{2})$: since $\frac{j\theta}{2} = \pi(N + \varepsilon - \frac{j}{kd})$, this number belongs to $]N\pi, (N+1)\pi[$, which means that the sign of $\sin(\frac{j\theta}{2})$ is $(-1)^N$.

Now we determine the sign of $\sin((\frac{j}{2}-d)\theta)$: we have $(\frac{j}{2}-d)\theta = \pi(N-2v+\varepsilon-\frac{j}{kd}+\frac{2}{k})$, from which we obtain that $(\frac{j}{2}-d)\theta$ belongs to $](N-2v)\pi, (N-2v+1)\pi[$, thus the sign of $\sin((\frac{j}{2}-d)\theta)$ equals $(-1)^{N+2v}$.

□

3.4. **The end of the proof of Theorem 1.2.** We have obtained an optimal solution f^* of (7), given in Definition 3.2, with objective value equal to $k\lambda_{0,0}$. So we have

$$\vartheta(\overline{K_{k/d}}) = k\lambda_{0,0}. \tag{23}$$

We recall that:

$$L_0(y) = \lambda_{0,0}T_0(y) + \lambda_{0,1}T_1(y) + \cdots + \lambda_{0,d-1}T_{d-1}(y). \tag{24}$$

If we plug in (24) the value $y = c_n$ and sum up for $n = 0, \dots, d-1$, taking account of $T_0 = 1$ and $\sum_{n=0}^{d-1} T_j(c_n) = \sum_{n=0}^{d-1} \cos(2jn\pi/d) = 0$, we obtain the formula (4).

3.5. **Other expressions for $\vartheta(\overline{K_{k/d}})$.** Alternatively, we can integrate (24) for the measure $dy/\sqrt{1-y^2}$, for which the Chebyshev polynomials are orthogonal, leading to different expressions for $\vartheta(\overline{K_{k/d}})$:

Theorem 3.7. *We have, with the notations of Theorem 1.2:*

$$\vartheta(\overline{K_{k/d}}) = \frac{k}{\pi}\int_{-1}^{1} L_0(y)\frac{dy}{\sqrt{1-y^2}} \tag{25}$$

$$= \frac{(-1)^{d-1}k}{\prod_{n=1}^{d-1}(1-a_n)} \sum_{j=0}^{\lfloor (d-1)/2 \rfloor} \frac{1}{2^{2j}}\binom{2j}{j}\sigma_{d-1-2j}(a_1,\dots,a_{d-1}) \tag{26}$$

where $\sigma_0, \dots, \sigma_{d-1}$ denote the elementary symmetric polynomials in $d-1$ variables.

Proof. Integrating (24) for the measure $dy/\sqrt{1-y^2}$ over the interval $[-1,1]$ leads to (25) because the Chebyshev polynomials T_n satisfy:

$$\frac{1}{\pi}\int_{-1}^{1} T_n(y)\frac{dy}{\sqrt{1-y^2}} = \delta_{n,0}.$$

Then, (26) is obtained from (25) with the monomial expansion of $L_0(y)$ and the formula (ref ??)

$$\frac{1}{\pi}\int_{-1}^{1} y^j \frac{dy}{\sqrt{1-y^2}} = \begin{cases} 0 & \text{if } j \text{ is odd} \\ \frac{1}{2^j}\binom{j}{j/2} & \text{otherwise .} \end{cases}$$

□

Remark 3.8. *The expression* (4) *specializes, when* $d = 2$ *and* $d = 3$*, to the expressions given respectively in* [7] *and* [4]*. Indeed, in the case* $d = 2$*, we have* $c_0 = 1$*,* $c_1 = -1$*, and* $a_1 = -\cos(\pi/k)$*. Replacing in* (4)*, we recover* (1)*.*

For $d = 3$*, we have* $c_1 = c_2 = -1/2$*,* $a_1 = \cos(\lfloor \frac{k}{3} \rfloor \frac{2\pi}{k})$ *and* $a_2 = \cos((\lfloor \frac{k}{3} \rfloor + 1)\frac{2\pi}{k})$*. We obtain in* (4)

$$\begin{aligned}\vartheta(\overline{K_{k/3}}) &= \frac{k}{3}\Big(1 + \frac{(c_1-a_1)(c_1-a_2)}{(1-a_1)(1-a_2)} + \frac{(c_2-a_1)(c_2-a_2)}{(1-a_1)(1-a_2)}\Big) \\ &= \frac{k(1/2 + a_1 a_2)}{(1-a_1)(1-a_2)}\end{aligned}$$

which agrees with the expression (3) *given in* [4, Theorem 1].

4. Separating the $\vartheta(\overline{K_{k/d}})$

In this section, we prove Theorem 1.3. Jointly with Proposition 2.1, and following the discussion in the Introduction, it will be an immediate consequence of the following theorem:

Theorem 4.1. *There exists an absolute and effective constant* c *such that for all* $N \in \mathbb{N}, k \le N, k' \le N, k \ge 2d, k' \ge 2d'$ *with* $\gcd(k,d) = \gcd(k',d') = 1$*, and* $k/d \neq k'/d'$*,*

$$|\vartheta(\overline{K_{k/d}}) - \vartheta(\overline{K_{k'/d'}})| \ge \frac{1}{c^{N^5}}.$$

We start with a proof of the weaker property that $\vartheta(\overline{K_{k/d}}) \neq \vartheta(\overline{K_{k'/d'}})$ if $k/d \neq k'/d'$.

Theorem 4.2. *If* $\vartheta(\overline{K_{k/d}}) = \vartheta(\overline{K_{k'/d'}})$ *then* $k/d = k'/d'$*.*

Proof. Assume that $\vartheta(\overline{K_{k/d}}) = \vartheta(\overline{K_{k'/d'}})$ for $k/d < k'/d'$. Since $\vartheta(\overline{K_{p/q}})$ is an increasing function of p/q (see Section 2), it implies that $\vartheta(\overline{K_{p/q}})$ is constant for all $p/q \in [k/d, k'/d']$. This constant will be denoted ϑ for simplicity.

Claim 4.3. *The number* ϑ *is rational.*

Proof. Let $q \ge 5$ be a prime such that $1/q < \frac{1}{4}(k'/d' - k/d)$. Then there exists r such that $r/q, (r+1)/q, (r+2)/q, (r+3)/q \in [k/d, k'/d']$. Since $q \ge 5$, it divides at most one of the four numbers $r, r+1, r+2, r+3$. Hence one can find p such that $p/q, (p+1)/q \in [k/d, k'/d']$ and q is prime to p and $p+1$.

For any positive integer a, denote $\zeta_a = \exp(2i\pi/a)$. We refer to [6] for the basic notions of algebraic number theory that will be involved next. For a number field K, we let $\mathrm{Gal}(K)$ denote its Galois group over $\mathbb{Q}$. For number fields $K \subset L$, and $x \in L$, $\mathrm{Trace}_K^L(x)$ and $\mathrm{Norm}_K^L(x)$ denote respectively the trace and norm of x in the extension L/K.

It is well-known (see [6]) that

$$\begin{array}{rcl} \Psi_a \ : \ (\mathbb{Z}/a\mathbb{Z})^\times & \longrightarrow & \mathrm{Gal}(\mathbb{Q}(\zeta_a)) \\ n & \mapsto & \sigma_n \text{ such that } \sigma_n(\zeta_a) = \zeta_a^n \end{array}$$

is an isomorphism. Furthermore, if a and b are coprime,

$$(\mathbb{Z}/ab\mathbb{Z})^\times = (\mathbb{Z}/a\mathbb{Z})^\times \times (\mathbb{Z}/b\mathbb{Z})^\times$$

by Chinese Remainder Theorem. It implies immediately that

$$\mathrm{Gal}(\mathbb{Q}(\zeta_{ab})) = \mathrm{Gal}(\mathbb{Q}(\zeta_a)) \times \mathrm{Gal}(\mathbb{Q}(\zeta_b)),$$

hence the fields $\mathbb{Q}(\zeta_a)$ and $\mathbb{Q}(\zeta_b)$ are linearly disjoint over $\mathbb{Q}$.

We now compute $\vartheta = \vartheta(\overline{K_{p/q}})$ using formula (4). By definition, we have $c_n = \cos(2n\pi/q) = \frac{1}{2}(\zeta_q^n + \zeta_q^{-n}) = \sigma_n(c_1)$ for $1 \leq n \leq q-1$. It follows that

$$\vartheta = \frac{p}{q}(1 + \mathrm{Trace}_{\mathbb{Q}(\zeta_p)}^{\mathbb{Q}(\zeta_{pq})}(L_0(c_1))).$$

It gives immediately that $\vartheta \in \mathbb{Q}(\zeta_p)$. The same result using $(p+1)/q$ leads to $\vartheta \in \mathbb{Q}(\zeta_{p+1})$. Since the fields $\mathbb{Q}(\zeta_p)$ and $\mathbb{Q}(\zeta_{p+1})$ are linearly disjoint, this proves the result. ☐

Claim 4.4. *The number ϑ is an integer.*

Proof. Let $\vartheta = \frac{a}{b}$ with $a, b \in \mathbb{N}$ coprime. Using the same arguments as in the previous lemma, for any prime p such that $1/p < \frac{1}{4}(d/k - d'/k')$, one can find q, with p coprime to q and $q+1$, such that $q/p, (q+1)/p \in [d/k, d'/k']$. It means that $p/q, p/(q+1) \in [k/d, k'/d']$.

Using formula (4) for p/q, one sees that $x = q\prod_{n=1}^{q-1}(2 - 2a_n)\vartheta$ is an algebraic integer, hence $\mathrm{Norm}_{\mathbb{Q}}^{\mathbb{Q}(\zeta pq)}(x) \in \mathbb{Z}$. We now compute this norm.

Since $q\vartheta$ is rational, $\mathrm{Norm}_{\mathbb{Q}}^{\mathbb{Q}(\zeta pq)}(q\vartheta) = (q\vartheta)^{\phi(pq)}$ where ϕ is the Euler function. Since p is a prime, a_n is a conjugate of a_1 for all $1 \leq n \leq q-1$, hence $\mathrm{Norm}_{\mathbb{Q}}^{\mathbb{Q}(\zeta pq)}(\prod_{n=1}^{q-1}(2 - 2a_n)) = (\mathrm{Norm}_{\mathbb{Q}}^{\mathbb{Q}(\zeta pq)}(2 - 2a_1))^{q-1}$. We also have

$$2 - 2a_1 = 2 - 2\cos(\lfloor \frac{p}{q} \rfloor \frac{2\pi}{p}) = (1 - \zeta_p^{\lfloor \frac{p}{q} \rfloor})(1 - \zeta_p^{-\lfloor \frac{p}{q} \rfloor}).$$

Hence $\mathrm{Norm}_{\mathbb{Q}}^{\mathbb{Q}(\zeta pq)}(2 - 2a_1) = (\mathrm{Norm}_{\mathbb{Q}}^{\mathbb{Q}(\zeta pq)}(1 - \zeta_p))^2$. Finally,

$$\begin{aligned} \mathrm{Norm}_{\mathbb{Q}}^{\mathbb{Q}(\zeta pq)}(1 - \zeta_p) &= \mathrm{Norm}_{\mathbb{Q}}^{\mathbb{Q}(\zeta p)}(\mathrm{Norm}_{\mathbb{Q}(\zeta_p)}^{\mathbb{Q}(\zeta pq)}(1 - \zeta_p)) \\ &= (\mathrm{Norm}_{\mathbb{Q}}^{\mathbb{Q}(\zeta p)}(1 - \zeta_p))^{\phi(q)} \\ &= p^{\phi(q)} \end{aligned}$$

(see [6]). Summing up all partial results, one gets

$$(q\frac{a}{b})^{\phi(pq)} p^{2(q-1)\phi(q)} \in \mathbb{Z}.$$

If $l \neq p$ is a prime factor of b, then l divides q by the previous formula. But the same formula holds with $q+1$, hence l divides also $q+1$. It follows that b is a power of p. But this is true for any p large enough. Hence $b = 1$. This proves the result. □

To finish the proof of Theorem 4.2, we use the following result from [8] (see also [5]): if $\vartheta(\overline{K_{k/d}}) \in \mathbb{N}$ then $k/d \in \mathbb{N}$. But every rational number in the interval $[k/d, k'/d']$ cannot be an integer. □

We can now start the proof of Theorem 4.1. It is based on the following obvious lemma.

Lemma 4.5. *Let α be a non zero algebraic integer of degree less than δ and $c \geq 1$ such that the absolute values of the conjugates of α are less than c then*

$$|\alpha| \geq \frac{1}{c^\delta}$$

Proof. Since α is a non zero algebraic integer , $|\operatorname{Norm}_{\mathbb{Q}}^{\mathbb{Q}(\alpha)}(\alpha)| \geq 1$. It follows immediately that

$$|\alpha| c^{\delta-1} \geq 1.$$

□

Let

$$\begin{aligned}
\alpha &= dd' \prod_{n=1}^{d-1}(2-2a_n) \prod_{n=1}^{d'-1}(2-2a'_n)(\vartheta(\overline{K_{k/d}}) - \vartheta(\overline{K_{k'/d'}})) \\
&= kd' \prod_{n=1}^{d'-1}(2-2a'_n) \sum_{n=0}^{d-1} \prod_{m=1}^{d-1}(2c_n - 2a_m) - k'd \prod_{n=1}^{d-1}(2-2a_n) \sum_{n=0}^{d'-1} \prod_{m=1}^{d'-1}(2c'_n - 2a'_m)
\end{aligned}$$

with the obvious notations $c'_n := \cos\left(\frac{2n\pi}{d'}\right)$ and $a'_n := \cos\left(\left\lfloor \frac{nk'}{d'} \right\rfloor \frac{2\pi}{k'}\right)$. The number α is thus an algebraic integer, and it is non zero by Theorem 4.2. Moreover it belongs to $\mathbb{Q}(\zeta_{kdk'd'})$, hence its degree is less than N^4.

Let β be a conjugate of α. Since the absolute values of the conjugates of a_n, a'_n, c_n and c'_n are all less than 1, one gets

$$|\beta| \leq kd'4^{d'-1}d4^{d-1} + k'd4^{d-1}d'4^{d'-1} \leq 2N\frac{N}{2}4^{\frac{N}{2}}\frac{N}{2}4^{\frac{N}{2}} \leq N^3 4^N.$$

It follows from Lemma 4.5 that

$$|\alpha| \geq \frac{1}{(N^3 4^N)^{N^4}}.$$

Furthermore, $|dd' \prod_{n=1}^{d-1}(2-2a_n) \prod_{n=1}^{d'-1}(2-2a'_n)| \leq N^2 4^N$. This implies immediately that

$$|\vartheta(\overline{K_{k/d}}) - \vartheta(\overline{K_{k'/d'}})| \geq \frac{1}{N^2 4^N (N^3 4^N)^{N^4}}.$$

This finishes the proof of Theorem 4.1.

5. THE ASYMPTOTIC BEHAVIOUR OF $\vartheta(\overline{K_{k/d}})$

From Lovász's formula (1), the asymptotic behaviour of the theta number of odd holes C_{2k+1} is (see [2] for instance):

$$\vartheta\left(C_{2k+1}\right) = \frac{2k+1}{2} + O\left(\frac{1}{k}\right). \tag{27}$$

In general, we have $\vartheta(\overline{K_{k/d}}) \leq k/d$. Indeed,

$$\vartheta(\overline{K_{k/d}}) = k/\vartheta(K_{k/d}) \leq k/\omega(\overline{K_{k/d}}) = k/d.$$

In this section, we prove:

Theorem 5.1. *If $d \geq 3$ and $k \geq 4d^3/\pi$ then $\vartheta(\overline{K_{k/d}}) \geq \frac{k}{d} - \frac{4e\pi^2}{3}\frac{d}{k}$. Hence, for d fixed,*

$$\vartheta(\overline{K_{k/d}}) = \frac{k}{d} + \mathrm{O}\left(\frac{1}{k}\right).$$

Notice that for $d = 2$, Theorem 5.1 agrees with Equation (27).

Proof. Let $d \geq 3$ and $k \geq 4d^3/\pi$. For every $0 \leq i \leq d-1$, let $c_i = \cos(2i\pi/d)$, $\sigma_i = \sin(2i\pi/d)$, $a_i = \cos\left(\left\lfloor \frac{ik}{d} \right\rfloor \frac{2\pi}{k}\right)$ and $\delta_i = c_i - a_i$. We have $a_i = \cos(2i\pi/d - 2\pi\epsilon_i)$, with $\epsilon_i = s_i/kd$ and $s_i = ik \mod d$.

Claim 5.2. *For every $1 \leq j \leq d-1$, we have*

$$\prod_{\substack{i=0 \\ i\neq j, d-j}}^{d-1} (c_j - c_i) = \frac{-d^2}{2^d \sigma_j^2} \quad \textit{if } j \neq 0, d/2 \tag{28}$$

$$\prod_{\substack{i=0 \\ i\neq j, d-j}}^{d-1} (c_j - c_i) = \frac{-d^2}{2^{d-1}} \quad \textit{if } j = d/2 \tag{29}$$

$$\sum_{i=1}^{d-1} \frac{1}{1-c_i} = \frac{d^2-1}{6} \tag{30}$$

Proof. The proof of these equalities is a short computation and the details are omitted. Equation (29) (respectively (28), (30)) is obtained by taking the first derivative (respectively the second derivative, the third derivative) with respect to x of the equality $T_k(\cos x) = \cos(kx)$, taking into account the identity $T_k(x) - 1 = 2^{k-1}\prod_{i=0}^{k-1}(x - c_i)$, then by evaluating the resulting identity at π (respectively $2j\pi/q$, respectively 0). □

Claim 5.3. *For every $1 \leq j \leq d-1$, we have*

$$\left|\frac{\delta_j \delta_{d-j}}{\sigma_j^2}\right| \leq \frac{4\pi^2}{k^2}\left(1 + \frac{d\pi}{k}\right) \quad \textit{if } j \neq d/2 \tag{31}$$

$$|\delta_j| \leq \frac{4\pi^2}{k^2} \quad \textit{if } j = d/2 \tag{32}$$

Proof. For every $1 \leq j \leq d-j$, let $z_j \in [2j\pi/d - 2\pi\epsilon_j, 2j\pi/d]$ such that $\delta_j = -2\pi\epsilon_j \sin(z_j)$. As $|\sin(z_j)| \leq |\sigma_j|$ or $|\sin(z_{d-j})| \leq |\sigma_j|$, we get

$$|\delta_j \delta_{d-j}| \leq 4\pi^2 |\sigma_j|(|\sigma_j| + 2\pi/k)/k^2.$$

Taking into account $|\sigma_j| \geq \sin\frac{\pi}{d} \geq 2/d$, we obtain (31).

The inequality (32) is straightforward. □

Claim 5.4. *For every* $1 \leq j \leq d-1$*, we have*

$$\prod_{i=1}^{d-1} |c_j - a_i| \leq \frac{e\pi^2}{2^{d-3}(1-c_j)} \frac{d^2}{k^2}. \tag{33}$$

Proof. Let $m_j = \delta_j \delta_{d-j}$ if $j \neq d/2$, and $m_{d/2} = \delta_{d/2}$. We have

$$\begin{aligned}
\prod_{i=1}^{d-1} |c_j - a_i| &= \left| m_j \prod_{\substack{i=1 \\ i \neq j, d-j}}^{d-1} (c_j - c_i) \right| \left| \prod_{\substack{i=1 \\ i \neq j, d-j}}^{d-1} \left(1 + \frac{\delta_i}{c_j - c_i}\right) \right| \\
&\leq \frac{\pi^2}{2^{d-3}(1-c_j)} \frac{d^2}{k^2} \left| \prod_{\substack{i=1 \\ i \neq j, d-j}}^{d-1} \left(1 + \frac{\delta_i}{c_j - c_i}\right) \right| \text{ due to (28), (29), (31), (32)} \\
&\leq \frac{\pi^2}{2^{d-3}(1-c_j)} \frac{d^2}{k^2} \exp\left(\frac{2\pi}{k} \sum_{\substack{i=1 \\ i \neq j, d-j}}^{d-1} \frac{1}{|c_j - c_i|} \right) \\
&\leq \frac{\pi^2}{2^{d-3}(1-c_j)} \frac{d^2}{k^2} \exp\left(\frac{4}{\pi} \frac{d^3}{k} \right) \text{ since } |c_j - c_i| \geq \frac{\pi^2}{2d^2} \text{ for every } i \neq j, d-j \\
&\leq \frac{\pi^2 e}{2^{d-3}(1-c_j)} \frac{d^2}{k^2} \quad \text{as } k \geq 4d^3/\pi.
\end{aligned}$$

□

Claim 5.5. *We have*

$$\prod_{i=1}^{d-1} (1 - a_i) \geq \frac{d^2}{2^d}.$$

Proof. Indeed,

$$\begin{aligned}
\prod_{i=1}^{d-1} (1 - a_i) &= \prod_{i=1}^{d-1} (1 - c_i) \prod_{i=1}^{d-1} \left(1 + \frac{\delta_i}{1 - c_i}\right) \\
&\geq \frac{d^2}{2^{d-1}} \prod_{i=1}^{d-1} \left(1 - \frac{2\pi}{k(1 - c_i)}\right) \text{ due to (29).}
\end{aligned}$$

If $x_1, \ldots, x_l$ are real numbers belonging to $[0,1]$, then $\prod_{i=1}^{l}(1-x_i) \geq 1 - (x_1 + \ldots + x_l)$. Since for every i, $\frac{2\pi}{k(1-c_i)} \leq 1$, it follows:

$$\begin{aligned}\prod_{i=1}^{d-1}(1-a_i) &\geq \frac{d^2}{2^{d-1}}\left(1 - \frac{2\pi}{k}\sum_{i=1}^{d-1}\frac{1}{1-c_i}\right)\\ &\geq \frac{d^2}{2^d} \quad \text{due to (30) and } k \geq 4d^3/\pi.\end{aligned}$$

□

Now we are ready to prove Theorem 5.1. We have the following chain of inequalities:

$$\begin{aligned}\vartheta\left(\overline{K_{k/d}}\right) &= \frac{k}{d} + \frac{k}{d}\sum_{n=1}^{d-1}\frac{\prod_{i=1}^{d-1}(c_n - a_i)}{\prod_{i=1}^{d-1}(1-a_i)} \quad \text{from (4)}\\ &\geq \frac{k}{d} - \frac{k}{d}\frac{2^d}{d^2}\sum_{n=1}^{d-1}\left|\prod_{i=1}^{d-1}(c_n - a_i)\right| \text{ by Claim 5.5}\\ &\geq \frac{k}{d} - \frac{k}{d}\frac{2^d}{d^2}\frac{e\pi^2}{2^{d-3}}\frac{d^2}{k^2}\sum_{n=1}^{d-1}\frac{1}{1-c_n} \text{ by Claim 5.4}\\ &\geq \frac{k}{d} - \frac{4e\pi^2}{3}\frac{d}{k} \quad \text{due to (30)}\end{aligned}$$

□

Notice that Theorem 5.1 shows that ϑ is close to the circular chromatic number of *dense* circular perfect graphs (where dense means that the clique number is large compared to the stability number):

Corollary 5.6. *For every $\epsilon > 0$, for every positive integer α, there is a positive integer ω such that for every circular-perfect graph G satisfying $\omega(G) \geq \omega$ and $\alpha(G) \leq \alpha$, we have $|\vartheta\left(\overline{G}\right) - \chi_c(G)| \leq \epsilon$.*

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