

# REVISITING KNESER'S THEOREM FOR FIELD EXTENSIONS

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ABSTRACT. A Theorem of Hou, Leung and Xiang generalised Kneser's addition Theorem to field extensions. This theorem was known to be valid only in separable extensions, and it was a conjecture of Hou that it should be valid for all extensions. We give an alternative proof of the theorem that also holds in the non-separable case, thus solving Hou's conjecture. This result is a consequence of a strengthening of Hou et al.'s theorem that is a transposition to extension fields of an addition theorem of Balandraud.

## 1. INTRODUCTION

Let  $F$  be a field and let  $L$  be an extension field of  $F$ . If  $S$  and  $T$  are  $F$ -vector subspaces of  $L$ , we shall denote by  $ST$  the  $F$ -linear span of the set of products  $st$ ,  $s \in S$ ,  $t \in T$ .

The following Theorem was obtained by Hou, Leung and Xiang [9], as a transposition to field extensions of Kneser's classical addition Theorem.

**Theorem 1** (Hou, Leung and Xiang). *Let  $F$  be a field,  $L/F$  a field extension, and let  $S, T$  be  $F$ -subvectorspaces of  $L$  of finite dimension. Suppose that every algebraic element in  $L$  is separable over  $F$ . Then one of the following holds:*

- $\dim ST \geq \dim S + \dim T - 1$ ,
- *there exists a subfield  $K$ ,  $F \subsetneq K \subset L$ , such that  $STK = ST$ .*

The conclusion of the Theorem of Hou et al. given in [9] is that we have

$$\dim ST \geq \dim S + \dim T - \dim H(ST),$$

where  $H(ST) = \{x \in L : xST = ST\}$  denotes the stabilizer of  $ST$  in  $L$ . As is explained in [9], the above formulation is easily seen to be equivalent to the conclusion of Theorem 1. Hou et al.'s Theorem 1 can also be seen as a generalisation of the original Theorem of Kneser [12] (see e.g. [15] or [19]), since the latter can be recovered from the former.

Theorem 1 was initially motivated by a problem on difference sets [9], but has since become part of a wider effort to transpose some classical theorems of additive combinatorics to a linear algebra or extension field setting. In particular Eliahou and Lecouvey [6] obtained

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linear analogues of some classical additive theorems including theorems of Olson [16] and Kemperman [11] in nonabelian groups. Lecouvey [14] pursued this direction by obtaining, among other extensions, linear versions of the Plünecké–Ruzsa [18] inequalities. The present authors recently derived a linear analogue of Vosper’s Theorem in [1]. Somewhat more generally, additive combinatorics have had some spectacular successes by lifting purely additive problems into various algebras where the additional structure has provided the key to the original problems, e.g. [5, 10]. This provides in part additional motivation for linear extensions of classical addition theorems.

Going back to Hou et al.’s Theorem 1, a natural question is whether the separability assumption in Theorem 1 is actually necessary. Hou makes an attempt in [8] to work at Theorem 1 without the separability assumption, but only manages a partial result where the involved spaces are assumed to have small dimension. Hou goes on to conjecture [8] that Theorem 1 always holds, i.e. holds without the separability assumption. Recently, Beck and Lecouvey [3] extended Theorem 1 to algebras other than a field extension over  $F$ , but again, their approach breaks down when the algebra contains an infinity of subalgebras, so that the case of non-separable field extensions is not covered.

In the present work, we prove Hou’s conjecture and remove the separability assumption in Theorem 1. We actually prove the slightly stronger statement below.

**Theorem 2.** *Let  $L/F$  be a field extension, and let  $S \subset L$  be an  $F$ -subspace of  $L$  of finite positive dimension. Then*

- *either for every finite dimensional subspace  $T$  of  $L$  we have*

$$\dim ST \geq \dim S + \dim T - 1,$$

- *or there exists a subfield  $K$  of  $L$ ,  $F \subsetneq K \subset L$ , such that for every finite-dimensional subspace  $T$  of  $L$  satisfying*

$$\dim ST < \dim S + \dim T - 1,$$

*we have  $STK = ST$ .*

Besides the removal of the separability condition, the additional strength of Theorem 2 with respect to Theorem 1 lies in the fact that the subfield  $K$  that stabilises  $ST$  seems to depend on both spaces  $S$  and  $T$  in Theorem 1 but actually can be seen to depend only on one of the factors in Theorem 2. Theorem 2 is a transposition to the extension field setting of a theorem of Balandraud [2] which is a similarly stronger form of Kneser’s Addition Theorem and can be stated as:

**Theorem 3.** *Let  $G$  be an abelian group, and let  $S \subset G$  be a finite subset of elements of  $G$ . Then*

- *Either for every finite subset  $T$  of  $G$  we have*

$$|S + T| \geq |S| + |T| - 1,$$

- or there exists a subgroup  $H$  of  $G$  such that, for every finite subset  $T$  of  $G$  satisfying

$$|S + T| < |S| + |T| - 1,$$

we have  $S + T + H = S + T$ .

Balandraud proved his theorem through an in-depth study of a phenomenon that he called saturation. For a given set  $S$ , a set  $T$  is saturated with respect to  $S$  if there does not exist a set  $T'$  strictly containing  $T$  such that the sums  $S + T$  and  $S + T'$  are equal. He showed that when  $T$  is a subset of smallest cardinality among all saturated subsets for which the quantity  $|S + T| - |T|$  is a given constant, then  $T$  must be a coset of some subgroup of  $G$ . Furthermore, the subgroups that appear in this way form a chain of nested subgroups, and the smallest non-trivial subgroup of this chain is the subgroup  $H$  of Theorem 3.

Balandraud's approach is very combinatorial in nature and is inspired by Hamidoune's isoperimetric (or atomic) method in additive combinatorics [7, 17]. In the present paper we prove Theorem 2 by exporting Balandraud's method to the extension field setting. This can also be seen as a follow-up to the linear isoperimetric method initiated in Section 3 of [1]. We note that this strategy deviates significantly from Hou et al.'s approach in [9] which relied on a linear variant of the additive  $e$ -transform and required crucially the separability of the field extensions.

We also note that Theorem 2 can be seen as a generalisation of Balandraud's Theorem 3 in groups, since the latter may be derived from the former by exactly the same Galois group argument as that of [9, Section 3].

The paper is organised as follows. Section 2 is devoted to setting up some basic tools and deriving the combinatorics of saturation. Section 3 introduces *kernels*, which are finite dimensional subspaces of minimum dimension among finite-dimensional subspaces  $T$  for which  $\dim ST - \dim T$  is a fixed integer less than  $\dim S - 1$ . A structural theory of kernels is derived, whose core features are Propositions 17, 18 and 19. Finally, Section 4 derives the proof of Theorem 2 from the structure of kernels and concludes the paper.

## 2. PRELIMINARY DEFINITIONS AND PROPERTIES

We assume that  $L/F$  is a field extension and that  $S$  is a finite-dimensional  $F$ -subspace of  $L$  such that  $1 \in S$ . We suppose furthermore that  $F(S) = L$ , where  $F(S)$  denotes the subfield of  $L$  generated by  $S$ .

**2.1. Boundary operator and submodularity.** For every subspace  $X$  of  $L$  we define

$$\partial_S X = \dim XS/X.$$

the increment of dimension of  $X$  when multiplied by  $S$ . We omit the subscript  $S$  in  $\partial_S$  whenever  $S$  is clear from the context. Note that we may have  $\partial X = \infty$ . The essential property of the "boundary" operator  $\partial$  is the submodularity relation:

**Proposition 4.** *Let  $X, Y$  be subspaces of  $L$ . We have*

$$\partial(X + Y) + \partial(X \cap Y) \leq \partial X + \partial Y.$$

A short proof of Proposition 4 is given in [1] when  $L$  is finite-dimensional over  $F$ . In the general case we invoke the following Lemma:

**Lemma 5.** *Let  $A, B, A', B'$  be subspaces of some ambient vector space  $E$ , such that  $A \subset A'$  and  $B \subset B'$ . There is an exact sequence of vector spaces*

$$0 \rightarrow (A' \cap B')/(A \cap B) \rightarrow A'/A \times B'/B \rightarrow (A' + B')/(A + B) \rightarrow 0.$$

*Proof.* We may identify the subspace  $A \cap B$  with the subspace of  $A \times B$  consisting of the elements  $(x, -x)$ ,  $x \in A \cap B$ . With the similar identification for  $A' \cap B'$ , we get the isomorphisms

$$(A \times B)/(A \cap B) \xrightarrow{\sim} A + B \quad \text{and} \quad (A' \times B')/(A' \cap B') \xrightarrow{\sim} A' + B' \quad (1)$$

and the following commutative diagram with the rows being exact and  $\gamma$  corresponding to the natural mapping of  $A + B$  into  $A' + B'$ .

$$\begin{array}{ccccccc} A \cap B & \longrightarrow & A \times B & \longrightarrow & (A \times B)/(A \cap B) & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & A' \cap B' & \longrightarrow & A' \times B' & \longrightarrow & (A' \times B')/(A' \cap B') \end{array}$$

The snake lemma (Lang, [13, Ch 3, Section 9]) therefore gives the exact sequence

$$0 \rightarrow \text{coker } \alpha \rightarrow \text{coker } \beta \rightarrow \text{coker } \gamma \rightarrow 0$$

which yields the result after identification of  $A' \times B'/A \times B$  with  $A'/A \times B'/B$  and the identifications (1).  $\square$

Lemma 5 immediately gives:

**Corollary 6.** *If  $A'/A$  and  $B'/B$  have finite dimension, then*

$$\dim (A' + B')/(A + B) = \dim A'/A + \dim B'/B - \dim (A' \cap B')/(A \cap B).$$

*Proof of Proposition 4.* If  $\partial X$  or  $\partial Y$  is  $\infty$  there is nothing to prove, so we may set  $X' = XS$  and  $Y' = YS$  and suppose that  $X'/X$  and  $Y'/Y$  are finite-dimensional. We have  $(X+Y)S \subset X' + Y'$  and  $(X \cap Y)S \subset X' \cap Y'$ , therefore

$$\partial(X + Y) + \partial(X \cap Y) \leq \dim (X' + Y')/(X + Y) + \dim (X' \cap Y')/(X \cap Y)$$

and the conclusion follows from Corollary 6.  $\square$

**2.2. Duality.** Recall that every non-zero linear form  $\sigma : L \rightarrow F$  induces a nondegenerate symmetric bilinear (Frobenius) form defined as  $(x | y)_\sigma = \sigma(xy)$ , with the property:

$$(xy | z)_\sigma = (x | yz)_\sigma \text{ for all } x, y, z \in L. \quad (2)$$

Fix such a bilinear form  $(\cdot | \cdot)$ . For a subspace  $X$  we denote by

$$X^\perp = \{y \in L : \forall x \in X, (x | y) = 0\}.$$

We call the *dual subspace* of the subspace  $X$  the subspace

$$X^* = (XS)^\perp.$$

We will use the notation  $X^{**} = (X^*)^*$  and  $X^{***} = (X^{**})^* = (X^*)^{**}$ .

We shall require the following lemma which is a straightforward consequence of Bourbaki [4, Ch. 9, §1, n. 6, Proposition 4]:

**Lemma 7.** *If  $A$  and  $B$  are subspaces such that the quotient  $A/(B^\perp \cap A)$  is finite dimensional, then  $\dim A/(B^\perp \cap A) = \dim B/(A^\perp \cap B)$ .*

The following elementary properties hold for subspaces and their duals:

**Lemma 8.** *For every  $F$ -subspace  $X$  of  $L$  we have*

- (i)  $X \subseteq X^{**}$ .
- (ii)  $X^* = X^{***}$
- (iii)  $\partial X^* \leq \partial X$ .

*Proof.*

- (i) Let  $x \in X$  and let  $x^* \in X^*$  and  $s \in S$ . By definition of  $X^*$  we have  $(xs | x^*) = 0$ , hence  $(x | x^*s) = 0$ , therefore  $x \in (X^*S)^\perp = X^{**}$ .
- (ii) Applying (i), we have  $X^* \subset (X^*)^{**}$ . We also have that if  $Y \subset Z$  then  $Z^* \subset Y^*$  which yields  $(X^{**})^* \subset X^*$ .
- (ii) If  $\partial X = \infty$  there is nothing to prove, so assume  $\partial X < \infty$ . From (i) and  $X \subset XS$  we have  $X \subset (X^{**} \cap XS)$ , hence

$$\dim XS/(X^{**} \cap XS) \leq \dim XS/X < \infty.$$

Applying Lemma 7 we therefore have:

$$\begin{aligned} \partial X^* &= \dim X^*S/X^* = \dim X^*S/(XS)^\perp = \dim XS/((X^*S)^\perp \cap XS) \\ &= \dim XS/(X^{**} \cap XS) \leq \dim XS/X = \partial X. \end{aligned}$$

□

One would expect the stronger properties  $X^{**} = X$  and  $\partial X^* = \partial X$  to hold. Unfortunately this is not true for all subspaces, only for those who are *saturated*, a notion that we introduce below.

**2.3. Saturated spaces.** For a subspace  $X$  let us define the subspace  $\tilde{X}$  to be the set of all  $x \in L$  such that

$$xS \subset XS.$$

Clearly we have  $\tilde{X}S = XS$ ,  $X \subseteq \tilde{X}$ , and  $\partial\tilde{X} \leq \partial X$ . A subspace  $X$  is said to be *saturated* if

$$\tilde{X} = X.$$

**Lemma 9.** *For every  $F$ -subspace  $X$  of  $L$ , we have*

- (i)  $X^*$  is saturated.
- (ii) *If  $X$  is finite-dimensional then  $X^{**} = \tilde{X}$ . In particular a finite-dimensional subspace  $X$  is saturated if and only if  $X = X^{**}$ .*

*Proof.*

- (i) Let  $y \in L$  such that  $yS \subset X^*S$ , and let us prove that  $y \in X^* = (XS)^\perp$ . Since  $ys \in X^*S$ , we have  $ys = \sum x_i^* s_i$  where  $x_i^* \in X^*$  and  $s_i \in S$ . Therefore, for any  $x \in X$ ,  $s \in S$ , we have

$$(y | xs) = (ys | x) = \sum_i (x_i^* s_i | x) = \sum_i (x_i^* | s_i x) = 0$$

which means that  $y \in X^*$ .

- (ii) We recall that, for a finite-dimensional subspace  $A$ , we have  $(A^\perp)^\perp = A$  (Bourbaki, [4, Ch. 9, §1, n. 6, cor. 1]).

$$\begin{aligned} y \in X^{**} &\Leftrightarrow y \in (X^*S)^\perp \\ &\Leftrightarrow \forall x^* \in X^*, \forall s \in S, (y | x^*s) = 0 \\ &\Leftrightarrow \forall x^* \in X^*, \forall s \in S, (ys | x^*) = 0 \\ &\Leftrightarrow yS \subset ((XS)^\perp)^\perp = XS \\ &\Leftrightarrow y \in \tilde{X}. \end{aligned}$$

□

We denote by  $\mathcal{S}$  the family of finite-dimensional saturated subspaces  $X$  of  $L$  together with their duals  $X^*$ . The next lemma summarizes the properties of the elements of  $\mathcal{S}$  that we will need in the proof of Theorem 2.

**Lemma 10.** *For every  $X \in \mathcal{S}$ ,  $Y \in \mathcal{S}$ ,*

- (i)  $X$  is saturated and  $X^* \in \mathcal{S}$ .
- (ii)  $X^{**} = X$ .
- (iii)  $\partial X = \partial X^*$ .
- (iv)  $X \cap Y \in \mathcal{S}$ .
- (v)  $(X + Y)^{**} \in \mathcal{S}$ .

*Proof.*

- (i) (ii)  $X$  is saturated by Lemma 8(i). If  $X \in \mathcal{S}$  has finite dimension,  $X^*$  belongs to  $\mathcal{S}$  by definition, and  $X^{**} = X$  by Lemma 9(ii). Otherwise,  $X = X_1^*$  where  $X_1$  is saturated and of finite dimension, and  $X^* = X_1^{**} = X_1$  by Lemma 9(ii), so  $X^*$  belongs to  $\mathcal{S}$  and  $X^{**} = X$  applying Lemma 8(ii).
- (iii) From Lemma 8(iii), we have  $\partial X^* \leq \partial X$ . Applying again this inequality to  $X^*$  and combining with  $X^{**} = X$  leads to  $\partial X^* = \partial X$ .
- (iv) We have

$$X^* \cap Y^* = (XS)^\perp \cap (YS)^\perp = (XS + YS)^\perp = ((X + Y)S)^\perp = (X + Y)^*.$$

In particular, if  $X$  and  $Y$  belong to  $\mathcal{S}$ ,  $X \cap Y = X^{**} \cap Y^{**} = (X^* + Y^*)^*$  so, by Lemma 9(i),  $X \cap Y$  is saturated. If, moreover,  $X$  or  $Y$  is of finite dimension, we can conclude that  $X \cap Y \in \mathcal{S}$ . Otherwise,  $X = X_1^*$  and  $Y = Y_1^*$ , where  $X_1$  and  $Y_1$  are both of finite dimension, and  $X \cap Y = (X_1 + Y_1)^*$ . We remark that  $(X_1 + Y_1)^* = (X_1 + Y_1)^{***} = \widetilde{(X_1 + Y_1)^*}$ , applying Lemma 8(ii) and Lemma 9(ii), so  $X \cap Y \in \mathcal{S}$ .

- (v) If the dimensions of  $X$  and  $Y$  are finite, then, by Lemma 9(ii),  $(X + Y)^{**} = \widetilde{X + Y}$  belongs to  $\mathcal{S}$  (we note that, because  $\tilde{Z} \subset \tilde{Z}S = ZS$ , if  $Z$  is finite dimensional, it is also the case for  $\tilde{Z}$ ). Otherwise, without loss of generality we may assume that  $X = X_1^*$  where  $X_1$  is saturated and of finite dimension. Let  $Z := (X + Y)^*$ . By Lemma 9(i),  $Z$  is saturated, and by Lemma 9(ii),  $Z \subset X^* = X_1^{**} = X_1$ , so  $Z$  is of finite dimension and we can conclude that  $Z^* = (X + Y)^{**}$  belongs to  $\mathcal{S}$ .

□

In the proof of Theorem 2, we will apply many times the submodularity inequality of Proposition 4 to certain subspaces  $X, Y$  belonging to  $\mathcal{S}$ . We will have  $X \cap Y \in \mathcal{S}$  (Lemma 10 (iv)), but we will have to deal with the issue that in general  $X + Y \notin \mathcal{S}$ . Lemma 10 (v) will allow us to replace  $X + Y$  by the larger  $(X + Y)^{**}$  since  $\partial(X + Y)^{**} \leq \partial(X + Y)$ . The following Lemma will be used several times in order to ensure that  $(X + Y)^{**} \neq L$  holds and that we do not have  $\partial(X + Y)^{**} = 0$ .

**Lemma 11.** *Let  $X, Y$  be subspaces of  $L$  such that  $\dim X < \infty$ ,  $\dim XS \leq \dim X + \dim S - 1$ ,  $\dim(X \cap Y) \geq 1$  and  $\dim X \leq \dim Y^*$ . Then,*

$$(X + Y)^{**} \neq L.$$

*Proof.* We will prove that  $(X + Y)^* = (X + Y)^{***} \neq L^* = \{0\}$ . Since  $Y \subset (X + Y)$ , we have  $(X + Y)^* \subset Y^*$ . We will show that  $\dim Y^*/(X + Y)^*$  is finite and less than  $\dim(Y^*)$ , which will imply  $(X + Y)^* \neq \{0\}$ . Note that

$$YS \subset (YS^\perp)^\perp \cap (X + Y)S \subset (X + Y)S \subset XS + YS,$$

therefore

$$\dim(X + Y)S / ((YS^\perp)^\perp \cap (X + Y)S) \leq \dim(XS + YS) / YS.$$

The right-hand side is finite, therefore so is the left-hand side, and by Lemma 7 it equals  $\dim Y^*/(X+Y)^*$ : we therefore have:

$$\begin{aligned} \dim Y^*/(X+Y)^* &\leq \dim(XS+YS)/YS = \dim XS/(XS \cap YS) \\ &= \dim XS - \dim(XS \cap YS). \end{aligned}$$

From the hypothesis we have  $\dim XS \leq \dim X + \dim S - 1$  and from  $\dim(X \cap Y) \geq 1$  we have  $\dim(XS \cap YS) \geq \dim S$ , hence

$$\dim Y^*/(X+Y)^* \leq \dim X + \dim S - 1 - \dim S = \dim X - 1 < \dim Y^*.$$

□

When trying to prove that a saturated subspace  $X$  has a non-trivial stabilizer, it will be useful to consider its dual subspace instead. The last Lemma of this section states that a subfield stabilizes a saturated subspace if and only if it stabilizes its dual subspace.

**Lemma 12.** *If  $X \in \mathcal{S}$ , then the stabilizer field  $H(X)$  of  $X$  satisfies:  $H(X) = H(X^*)$ .*

*Proof.* Let  $k \in H(X)$ . For  $x^* \in X^*$ ,  $x \in X$  and  $s \in S$ , we have

$$(kx^* | xs) = (x^* | kxs)$$

from which we get that if  $k$  stabilizes  $X$  then  $(kx^* | xs) = 0$  for every  $x, x^*, s$ , and therefore  $kx^*$  is in  $X^*$ : if  $k$  stabilizes  $X^*$  then we have just proved that  $k$  stabilizes  $X^{**}$ , and  $X^{**} = X$  by Lemma 10 (i). □

### 3. STRUCTURE OF CELLS AND KERNELS OF A SUBSPACE

We assume, like in the previous section, that  $S$  is a finite dimensional  $F$ -subspace of  $L$  containing 1, and that  $L = F(S)$ . We will moreover assume that  $S$  is not a field, i.e.  $S \neq L$  (in which case we would have  $\mathcal{S} = \{\{0\}, L\}$ ). We will also assume that there exists a non-zero finite dimensional subspace  $T \subset L$  such that  $ST \neq L$  and with

$$\dim ST < \dim S + \dim T - 1.$$

Equivalently,  $\partial T < \dim S - 1$ .

Let

$$\Lambda = \{\partial(X) : X \in \mathcal{S}\}.$$

We denote the elements of  $\Lambda$  by

$$\Lambda = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots\}$$

and by

$$\mathcal{S}_i = \{X \in \mathcal{S} : \partial X = \lambda_i\}.$$

Spaces belonging to a set  $\mathcal{S}_i$  will be called  $i$ -cells. By Lemma 10 (i) (iii), the dual of an  $i$ -cell is an  $i$ -cell. An  $i$ -cell of smallest dimension will be said to be an  $i$ -kernel.

We note that  $\mathcal{S}_0 = \{\{0\}, L\}$ , and that from the existence of  $T$ ,  $\lambda_1 < \dim S - 1$ . Let  $n$  be the largest integer such that  $\lambda_n < \dim S$ . We note that we have  $\lambda_n = \dim S - 1$  and that  $F$  is an  $n$ -kernel since  $\partial F = \dim S - 1$  and  $F$  is clearly saturated.

If  $N$  is an  $i$ -kernel, then clearly so is  $xN$  for any non-zero  $x \in L$ . Therefore, when an  $i$ -kernel exists there exists an  $i$ -kernel containing  $F$ . Let  $F_1, F_2, \dots, F_n$  be  $1, 2, \dots, n$ -kernels containing  $F$ , which implies in particular  $F_n = F$ .

Our core result is the following theorem.

**Theorem 13.** *We have*

$$F_1 \supset F_2 \supset \dots \supset F_n.$$

*Furthermore the  $F_i$  are all subfields of  $L$ , and every space  $X \in \mathcal{S}_i$  is stabilized by  $F_i$ .*

Note that this last statement implies in particular that the  $i$ -kernel containing  $F$  is unique.

We shall prove Theorem 13 in several steps. First we prove the result for  $F_1$ .

**Proposition 14.**  *$F_1$  is a subfield of  $L$  and any 1-cell  $X$  satisfies  $XF_1 = X$ .*

*Proof.* Let  $X$  be a 1-cell and let  $x$  be a non-zero vector of  $X$ , so that  $X$  has a non-zero intersection with  $xF_1$ . By submodularity we have

$$\partial(xF_1 + X)^{**} + \partial(xF_1 \cap X) \leq \partial(xF_1 + X) + \partial(xF_1 \cap X) \leq 2\lambda_1.$$

Since  $xF_1 \cap X$  is non-zero and of finite dimension, by Lemma 10 (iv) we have  $xF_1 \cap X \in \mathcal{S}$  and therefore  $xF_1 \cap X \in \mathcal{S}_k$  for some  $k \geq 1$ . Since  $xF_1$  is a kernel we have  $\dim xF_1 \leq \dim X^*$  and Lemma 11 implies  $(xF_1 + X)^{**} \in \mathcal{S}_\ell$  for some  $\ell \geq 1$ . It follows that  $k = \ell = 1$ . Therefore, by the minimality of the dimension of 1-kernels, we have  $xF_1 \subset X$ . This is for an arbitrary  $x \in X$ , therefore we have proved  $XF_1 = X$ . Applying this to  $X = F_1$  we obtain that  $F_1$  is a subfield of  $L$ .  $\square$

Now Proposition 14 enables us to define  $i$  to be the largest integer in the range  $2 \leq i \leq n$  such that

- $F_1 \supset \dots \supset F_{i-1}$ ,
- $F_{i-1}$  is a subfield of  $L$ ,
- any  $(i-1)$ -cell is stabilized by  $F_{i-1}$ .

We shall prove that  $F_i \subset F_{i-1}$ , that  $F_i$  is also a subfield and that  $F_i$  stabilizes every  $i$ -cell. This will prove  $i = n$  and therefore prove Theorem 13.

**Lemma 15.** *No  $i$ -cell  $X$  is stabilized by  $F_{i-1}$ .*

*Proof.* Suppose  $F_{i-1}X = X$ . Then  $X$  and  $SX$  are  $F_{i-1}$ -vector spaces and  $\lambda_i = \dim XS/X$  is a multiple of  $\dim F_{i-1}$ . The quantity  $\lambda_{i-1} = \dim F_{i-1}S - \dim F_{i-1}$  is also a multiple of  $\dim F_{i-1}$ , and since  $\lambda_i > \lambda_{i-1}$ ,

$$\lambda_i \geq \lambda_{i-1} + \dim F_{i-1} = \dim F_{i-1}S \geq \dim S,$$

contradicting  $\lambda_i < \dim S$ . □

**Lemma 16.**  $F_i \subset F_1$ .

*Proof.* By Lemma 15, there exists  $x \in F_i$  such that  $F_{i-1}x \not\subset F_i$ . We have

$$\partial(xF_1 + F_i) + \partial(xF_1 \cap F_i) \leq \lambda_1 + \lambda_i.$$

Since  $\dim(xF_1) < \dim(F_i) \leq \dim(F_i^*)$ , Lemma 11 implies  $\partial(xF_1 + F_i) \geq \partial(\widetilde{xF_1 + F_i}) \geq \lambda_1$ , which implies  $\partial(xF_1 \cap F_i) \leq \lambda_i$ . Now  $xF_1 \cap F_i$  is saturated and contains  $x$ , but not  $F_{i-1}x$  and hence not  $F_jx$  either for  $j \leq i-1$ , therefore  $\partial(xF_1 \cap F_i) \neq \lambda_j$  for all  $j \leq i$ . Hence  $\partial(xF_1 \cap F_i) = \lambda_i$  which implies that  $F_i \subset xF_1$ . Since  $1 \in F_i$  we must have  $1 \in xF_1$  which implies  $xF_1 = F_1$ . □

**Proposition 17.** For every  $j < i$  we have  $F_i \subset F_j$ .

*Proof.* We prove this by induction on  $j$ . Lemma 16 gives the result for  $j = 1$ , so suppose we already have  $F_i \subset F_{j-1}$  and let us prove  $F_i \subset F_j$ . Suppose first that  $\dim F_i > \dim F_j$ . Let  $N_j$  be a  $j$ -kernel intersecting  $F_i^*$ . We consider  $Z := (N_j + F_i^*)^{**}$ , which belongs to  $\mathcal{S}$  by Lemma 10 (iv). We have:

$$\partial(Z) + \partial(N_j \cap F_i^*) \leq \partial(N_j + F_i^*) + \partial(N_j \cap F_i^*) \leq \lambda_j + \lambda_i.$$

By Lemma 11, since we assume that  $\dim(F_i^{**}) = \dim F_i \geq \dim N_j$  we have  $Z \neq L$ , and by the induction hypothesis  $F_i \subsetneq F_{j-1}$  we have  $F_i^* \supsetneq F_{j-1}^*$ , so that  $Z \supsetneq F_i^*$  and  $Z^* \subsetneq F_i$ . Therefore  $Z^*$  and  $Z$  are not  $(j-1)$ -cells by the minimality of  $\dim F_{j-1}$  in  $\mathcal{S}_{j-1}$ , and hence not  $k$ -cells for  $k < j$ . Therefore  $\partial(Z) \geq \lambda_j$  and hence  $\partial(N_j \cap F_i^*) \leq \lambda_i$ . Now, by the hypothesis  $\dim F_i > \dim N_j$  we have that  $\dim(N_j \cap F_i^*) < \dim F_i$  and  $N_j \cap F_i^*$  can not be an  $i$ -cell. Therefore it is in  $\mathcal{S}_k$  for some  $k < i$ , which, by definition of  $i$ , implies that it is stabilized by  $F_k$  and hence by  $F_{i-1}$ . By applying this to  $N_j = xF_j$  for every  $x \in F_i^*$ , we get that the whole of  $F_i^*$  is stabilized by  $F_{i-1}$ : but this contradicts Lemma 15. Hence

$$\dim F_i \leq \dim F_j. \tag{3}$$

Next, consider  $x \in F_i$ . Suppose that for every  $x \in F_i$ ,  $x \neq 0$ ,  $xF_j \cap F_i$  is in  $\mathcal{S}_k$  for some  $k < i$ . Then every  $xF_j \cap F_i$  is stabilized by  $F_{i-1}$  and  $F_{i-1}F_i = F_i$  which contradicts Lemma 15. Therefore there exists  $x \in F_i$  such that  $xF_j \cap F_i$  is not in  $\mathcal{S}_k$  for every  $k < i$ . Let  $N_j = xF_j$  for such an  $x$ . This choice of  $x$  ensures that  $\partial(N_j \cap F_i) \geq \lambda_i$ . If we can show that  $\partial(N_j \cap F_i) = \lambda_i$  we will conclude that  $F_i \subset N_j$ , and since  $1 \in F_i$  we will have  $1 \in xF_j$  and  $xF_j = F_j$ , so that  $F_i \subset F_j$  and we will be done. Consider now

$$\partial(N_j + F_i) + \partial(N_j \cap F_i) \leq \lambda_j + \lambda_i.$$

This inequality will yield  $\partial(N_j \cap F_i) \leq \lambda_i$  and the desired result if we can show that

$$\partial(N_j + F_i) \geq \lambda_j. \tag{4}$$

Inequality (4) will in turn follow if we show that  $(N_j + F_i)^{**}$  is not a  $k$ -cell for  $k < j$ . We have  $N_j = xF_j \subset xF_{j-1}$  and by the induction hypothesis on  $j$ , we have  $F_i \subset F_{j-1}$  hence  $xF_{j-1} = F_{j-1}$  since  $x \in F_i$ . Therefore  $F_i + N_j \subset F_{j-1}$ . To show that  $(F_i + N_j)^{**}$  is

not a  $k$ -cell, it is enough to show that it is not stabilized by  $F_{j-1}$ , which will follow from  $(F_i + N_j)S \subsetneq F_{j-1}S$  which we now prove:

We have

$$\begin{aligned} \dim(N_j + F_i)S &\leq \dim(N_jS + F_iS) \\ &\leq \dim N_jS + \dim F_iS - \dim(N_jS \cap F_iS) \\ &\leq \dim N_jS + \dim F_iS - \dim S \\ &< \dim N_jS + \dim F_i \end{aligned}$$

since  $\dim F_iS \leq \dim F_i + \dim S - 1$ . But we know that  $N_jS \subsetneq F_{j-1}S$  otherwise  $N_j$  would not be saturated, and both  $N_jS$  and  $F_{j-1}$  are stabilised by the subfield  $F_j$ , therefore

$$\dim N_jS \leq \dim F_{j-1}S - \dim F_j$$

hence

$$\dim(F_i + N_j)S < \dim F_{j-1}S - \dim F_j + \dim F_i \leq \dim F_{j-1}S,$$

by (3) and we are finished.  $\square$

**Proposition 18.**  *$F_i$  is a subfield.*

*Proof.* Let  $x \in F_i$ ; our aim is to show that  $xF_i \subset F_i$ . Since  $F_i$  is saturated it is enough to show that  $xF_iS \subset F_iS$ . By contradiction, if  $xF_iS \not\subset F_iS$ , then there exists a linear form  $\sigma$  such that  $\sigma(F_iS) = 0$  but  $\sigma(xF_iS) \neq 0$ . This last condition translates to  $x \notin F_i^*$  where duality is related to this very choice of non-zero linear form on  $L$ . We would then have  $1 \in F_i^*$  and  $F_i \not\subset F_i^*$ . Let us show now that this is not possible.

For  $Z := (F_i + F_i^*)^{**}$ , we have:

$$\partial(Z) + \partial(F_i \cap F_i^*) \leq \partial(F_i + F_i^*) + \partial(F_i \cap F_i^*) \leq 2\lambda_i.$$

Since we have proved that  $F_i \subset F_j$  for all  $j < i$ , and these inclusions are strict, we have  $F_i^* \supsetneq F_j^*$  hence  $Z \supsetneq F_j^*$ . Note that  $(F_i + F_i^*)^*$  is of finite dimension, so that  $Z \in \mathcal{S}$ . Since  $F_j^*$  is a  $j$ -cell whose dual has minimum dimension,  $Z$  can not be a  $j$ -cell for  $1 \leq j < i$ . By Lemma 11 we also have  $Z \neq L$  so we conclude that

$$\partial(Z) \geq \lambda_i.$$

Hence  $\partial(F_i \cap F_i^*) \leq \lambda_i$ , which implies  $F_i \subset F_i^*$  since  $F_i$  has smaller dimension than any  $j$ -saturated set for  $j < i$ .  $\square$

**Proposition 19.** *Every  $i$ -cell is stabilized by  $F_i$ .*

*Proof.* Let us suppose there exists an  $i$ -cell  $X$  that is not stabilized by  $F_i$  and work towards a contradiction. Without loss of generality we may assume that  $X$  is of finite dimension by Lemma 12.

The proof strategy consists in constructing smaller and smaller  $i$ -cells that are not stabilized by  $F_i$  until we eventually exhibit one that is included in  $xF_i$  for some  $x$ , which will yield a contradiction.

That  $X$  is not stabilized by  $F_i$  means there exists  $x \in X$ ,  $xF_i \not\subseteq X$ .

We first argue that there exists  $k$ ,  $1 \leq k \leq i-1$ , such that  $X \cap xF_k$  is an  $i$ -cell not stabilized by  $F_i$ .

We have

$$\partial(X + xF_i) + \partial(X \cap xF_i) \leq 2\lambda_i.$$

Since  $X \cap xF_i \subsetneq xF_i$  we have  $\partial(X \cap xF_i) > \lambda_i$ , therefore  $\partial(X + xF_i) < \lambda_i$ . Furthermore, by Lemma 11,  $(X + xF_i)^{**} \neq L$ , so that  $\partial((X + xF_i)^{**}) = \lambda_k$  for some  $1 \leq k \leq i-1$ . Now, since  $(X + xF_i)^{**} \in S_k$ , and  $k < i$ , we know that  $(X + xF_i)^{**}$  is stabilized by  $F_k$ , hence

$$(X + xF_i)^{**} = (X + xF_k)^{**}$$

and  $\partial(X + xF_k)^{**} = \lambda_k$ . We now write

$$\partial(X + xF_k)^{**} + \partial(X \cap xF_k) \leq \lambda_k + \lambda_i$$

from which we get  $\partial(X \cap xF_k) \leq \lambda_i$  which implies  $\partial(X \cap xF_k) = \lambda_i$ , since otherwise  $X \cap xF_k$  is an  $\ell$ -cell for some  $\ell < i$ , and therefore stabilized by  $F_\ell$ , and hence by  $F_i$ , which contradicts our assumption on  $x$ .

The space  $X \cap xF_k$  is therefore an  $i$ -cell that is not stabilized by  $F_i$ , and we may therefore replace  $X$  by an  $i$ -cell which is included in some kernel. Specifically, let  $j \leq i$  be the largest integer such that there exists an  $i$ -cell  $X$  not stabilized by  $F_i$  and included in  $xF_j$  for some  $x$ . Clearly we can only have  $j \leq i-1$  since there is no  $i$ -cell included in but not equal to  $xF_i$ . We have just shown  $j \geq 1$ . Since  $X + xF_i \subset xF_j$  and  $xF_j$  is saturated we can not have  $(X + xF_i)^{**} \in S_k$  for  $k < j$ . Therefore the argument above implies that

$$(X + xF_i)^{**} \in S_j \tag{5}$$

otherwise  $X \cap xF_k$ ,  $k > j$ , is again an  $i$ -cell not stabilized by  $F_i$ , which contradicts our definition of  $j$ .

Our next objective is to construct an  $i$ -cell that is not stabilized by  $F_i$  and included in a  $(j+1)$ -kernel  $yF_{j+1}$ , which will contradict the definition of  $j$  and prove the proposition. For this we will need to apply Lemma 11 to the space  $X^*$  and to the  $(j+1)$ -kernel, for which we need the condition

$$\dim X \geq \dim F_{j+1} \tag{6}$$

which we now prove.

From  $F_{j+1}S \subsetneq F_jS$  (the  $F_j$  are saturated sets) we have, since the  $F_j$  are subfields,

$$\dim F_jS \geq \dim F_{j+1}S + \dim F_{j+1}. \tag{7}$$

For the same reason, since  $F_iS \subsetneq F_{j+1}S$ , we have

$$\dim F_{j+1}S \geq \dim F_iS + \dim F_i. \tag{8}$$

Now from

$$\partial(X + xF_i) + \partial(X \cap xF_i) \leq 2\lambda_i$$

we have  $\partial(X + xF_i) \leq \lambda_i$ , i.e.

$$\dim(X + xF_i)S - \dim(X + xF_i) \leq \lambda_i$$

meaning

$$\begin{aligned}\dim(X + xF_i)S &\leq \lambda_i + \dim(X + xF_i) \\ &< \lambda_i + \dim X + \dim F_i.\end{aligned}$$

From (5),  $X \subset xF_j$ , and the fact that  $j$ -cells are stabilized by  $F_j$ , we have  $(X + xF_i)S = xF_jS$ . Writing  $\lambda_i = \dim F_i S - \dim F_i$ , we get:

$$\begin{aligned}\dim F_j S &< \dim F_i S - \dim F_i + \dim X + \dim F_i \\ &< \dim F_i S + \dim X \\ &< \dim S + \dim F_i + \dim X.\end{aligned}$$

But on the other hand (7) and (8) imply

$$\dim F_j S \geq \dim F_i S + \dim F_i + \dim F_{j+1} \geq \dim S + \dim F_i + \dim F_{j+1}$$

hence  $\dim F_{j+1} < \dim X$ .

Now that we have proved (6) we are ready to construct an  $i$ -cell that is not stabilized by  $F_i$  and included in a  $(j+1)$ -kernel  $yF_{j+1}$ .

Since  $X$  is assumed not to be stabilized by  $F_i$ ,  $X^*$  is not stabilized by  $F_i$  either by Lemma 12. Therefore there exists  $y \in X^*$  such that  $yF_i \notin X^*$ . We write

$$\partial(X^* + yF_{j+1}) + \partial(X^* \cap yF_{j+1}) \leq \lambda_{j+1} + \lambda_i. \quad (9)$$

By the hypothesis  $X \subsetneq xF_j$ , we have  $X^* + yF_{j+1} \supset X^* \supsetneq (xF_j)^*$ , and  $(X^* + yF_{j+1})^* \subsetneq xF_j$ , therefore  $(X^* + yF_{j+1})^*$  and  $(X^* + yF_{j+1})^{**}$  do not belong to  $\mathbb{S}_k$  for any  $1 \leq k \leq j$ . Now (6) and Lemma 11 imply that  $(X^* + yF_{j+1})^{**} \neq L$ . Therefore,

$$\partial(X^* + yF_{j+1}) \geq \lambda_{j+1}.$$

From (9) we therefore get:

$$\partial(X^* \cap yF_{j+1}) \leq \lambda_i.$$

If  $X^* \cap yF_{j+1}$  were a  $k$ -cell for  $k < i$ , it would be stabilized by  $F_k$  and hence  $F_i$ : since this is assumed not to be the case, we have that  $X^* \cap yF_{j+1}$  must be an  $i$ -cell. This contradicts our assumption that  $j$  is the largest integer for which some  $i$ -saturated space not stabilized by  $F_i$  can be included in a multiplicative translate of  $F_j$ .  $\square$

Propositions 17, 18 and 19 imply that  $i = n$  and prove Theorem 13.

**Corollary 20.** *Let  $X$  be a finite-dimensional space such that  $\tilde{X}$  is a  $k$ -cell for  $1 \leq k \leq n$ . Then  $\tilde{X} = XF_k$ . Furthermore we have:*

$$\dim SX = \dim F_k S + \dim F_k X - \dim F_k.$$

*Proof.* We have  $\tilde{X}F_k \supset XF_k$ . Furthermore,

$$\begin{aligned}\dim X + \dim S - 1 &\geq \dim SX = \lambda_k + \dim \tilde{X} = \dim SF_k - \dim F_k + \dim \tilde{X} \\ &\geq \dim S - \dim F_k + \dim \tilde{X}\end{aligned}$$

hence

$$\dim \tilde{X} \leq \dim X + \dim F_k - 1. \quad (10)$$

Since we have  $\tilde{X}F_k = \tilde{X}$ , the strict inclusion  $\tilde{X} \supsetneq XF_k$  would imply  $\dim\tilde{X} \geq \dim X + \dim F_k$ , therefore (10) proves  $\tilde{X} = XF_k$ . Furthermore,

$$\dim SX = \lambda_k + \dim\tilde{X} = \dim SF_k - \dim F_k + \dim\tilde{X}$$

proves the last assertion.  $\square$

#### 4. PROOF OF THE MAIN THEOREM

*Proof of Theorem 2.* By replacing  $S$  if need be by  $s^{-1}S$  for some  $s \in S$ , we may always suppose  $1 \in S$ . Suppose first  $L = F(S)$ . If  $T = F(S)$  is the only saturated subspace of finite dimension such that

$$\dim ST < \dim S + \dim T - 1, \quad (11)$$

then the theorem holds with  $K = F(S)$ . Otherwise, if there exists a finite-dimensional subspace  $T$  satisfying (11) and such that  $ST \neq F(S)$ , then we are in the conditions of Theorem 13. In this case  $\tilde{T}$  is a  $k$ -cell for some  $1 \leq k \leq n-1$ , and  $\tilde{T}$ , hence  $ST = S\tilde{T}$ , is stabilised by  $F_k$ . Since  $F(S)$  and also every space  $F_k$  contain  $F_{n-1}$ , the conclusion of the theorem holds with  $K = F_{n-1}$ .

Consider now the case  $L \supsetneq F(S)$ . Let  $T$  be a subspace satisfying (11).

Let  $t \in T, t \neq 0$ . Let  $T_t = tF(S) \cap T$ : since  $T_t$  is an  $F$ -vector space, we may write  $T = T_t \oplus T'$  for some subspace  $T'$  of  $T$  with  $T' \cap T_t = \{0\}$ . Note that this implies  $T' \cap tF(S) = \{0\}$ . Since  $ST_t \subset tF(S)$ , we have  $ST_t \cap T' = \{0\}$ , and  $ST \supset ST_t \oplus T'$  implies

$$\dim ST \geq \dim ST_t + \dim T'.$$

From

$$\dim S + \dim T_t + \dim T' - 1 = \dim S + \dim T - 1 > \dim ST$$

we get

$$\dim S + \dim T_t - 1 > \dim ST_t.$$

Now we get from the case  $L = F(S)$  the existence of a subspace  $K$  such that  $STK = ST$  for any subspace  $T$  satisfying (11) and included in  $F(S)$ , or in a 1-dimensional  $F(S)$ -vector space. Therefore we have  $T_t K \subset ST$  for every  $t$  which proves the theorem.  $\square$

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