Linear Programming bounds for codes in Grassmannian spaces

Christine Bachoc

***Abstract*— We develop the linear programming method to obtain bounds for the cardinality of Grassmannian codes endowed with the chordal distance. We obtain a bound and its asymptotic version that generalize the well-known bound for codes in the real projective space obtained by Kabatyanskiy and Levenshtein, and improve the Hamming bound for sufficiently large minimal distances.**

***Index Terms*— Grassmann manifold, chordal distance, codes, zonal functions, bounds, linear programming method**

I. Introduction

PHILIPPE Delsarte has introduced the so-called *linear programming method*, in order to find bounds for the size of codes with prescribed minimal distance, in the classical case of codes over finite fields. This method, also called *Delsarte method* or *polynomial method*, exploits a certain family of orthogonal polynomials attached to the situation, the Krawtchouk polynomials, and their positivity property. These polynomials and their properties are intimately related to the action of the symmetric group on the Hamming space. Delsarte method has proved to be very powerful, and was extended to many other situations, where the underlying space is symmetric of rank one, and is homogeneous under the action of a certain group of transformations. Examples of such spaces are: the Johnson space, the Grassmannian space over a finite field, the unit sphere of the Euclidean space, the projective spaces over the real, complex and quaternionic fields (for these last spaces see [1], [2]).

In recent years, codes over the real Grassmannian space have attracted attention, motivated by their application to information theory, more precisely to the so-called *space-time codes*, used for multi-antenna systems of communication. The distance usually considered is the chordal distance, introduced in [3], and defined in the following way (more details are given in the next subsection): The Grassmannian space of m-dimensional subspaces of $\mathbb{R}^n$, where $m \leq n/2$, is denoted by $\mathcal{G}_{m,n}$; to a pair (p,q) of elements of $\mathcal{G}_{m,n}$ is associated m principal angles $\theta_1, \ldots, \theta_m \in [0, \pi/2]$. Let $y_i := \cos^2 \theta_i$. Then

$$d_c(p,q) := \sqrt{\sum_{i=1}^{m} \sin^2 \theta_i} = \sqrt{m - \sum_{i=1}^{m} y_i}.$$

In [3], the authors give bounds for the size of Grassmannian codes, called the simplex and orthoplex bounds. The main drawback of these bounds is that they are only valid in a certain range of minimal distances. In [4], an asymptotic bound, derived from the Hamming bound, is given. Another approach is developed in [5], where bounds are given for codes which principal angles are subject to certain constraints (the so-called f-codes), which arise naturally from the notion of Grassmannian designs introduced in [6].

In this paper, we extend Delsarte method to the Grassmannian codes, exploiting the zonal polynomials attached to $\mathcal{G}_{m,n}$. These are symmetric polynomials in the m variables $y_1, \ldots, y_m$; they belong to the family of orthogonal *generalized Jacobi polynomials* (see the next subsection). In the second section, we recall, or settle the properties of these polynomials needed to perform linear programming bounds; these properties are easy to obtain by straightforward generalization of the arguments used in the classical cases. In fact, the principles underlying the LP method would remain true for the zonal polynomials attached to any symmetric space. The real difficulties start when one wants to actually perform explicit bounds, because the polynomials have (for $m \geq 2$) several variables. The low degree cases are still easy to manage; this is done in section III, where we recover the simplex bound as the bound arising from the case of degree one, and give new bounds from polynomials of degree 2 and 3. In the forth section, we propose a strategy based on the eigenvalues of certain symmetric endomorphisms, which extends the one variable method based on the zeros of the polynomials and on Christoffel-Darboux formula, but avoids to deal with zeros of polynomials in several variables. We obtain an upper bound for the size of a code C with minimal distance δ, which is expressed in terms of the largest eigenvalue (Theorem 4.4 and Corollary 9). Section V settles the asymptotic behavior of this largest eigenvalue (Theorem 5.3), and in section VI we derive the following asymptotic version of the bound:

Theorem 1.1: Let C be a code in $\mathcal{G}_{m,n}$ with minimal chordal distance δ, let $s := m - \delta^2 \in]0, m[$ and let

$$\rho := \frac{m}{2}(-1 + (1 - \frac{s}{m})^{-1/2}).$$

Then, when $n \to +\infty$,

$$\frac{1}{n} \log |C| \lesssim m\big((1+\rho)\log(1+\rho) - \rho \log(\rho)\big). \qquad (1)$$

Our bound coincides with the bound given by G. Kabatiansky and V. Levenshtein in [7] for the case of the real projective space, corresponding to $m = 1$. But it beats the Hamming bound of [4] only when the minimal distance is relatively big.

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A. Basic facts about Grassmannian spaces and their zonal polynomials.

We repeat here, without proofs, some well-known facts about Grassmannian spaces and their zonal polynomials. Some useful references for the mathematical background are: [8], [9] for the representations of the orthogonal group, [14], [15], for the Grassmannian spaces and harmonic analysis on it, [10], [11], [12], [13] for multivariate orthogonal polynomials.

The real Grassmannian space, denoted by $\mathcal{G}_{m,n}$ ($m \leq n/2$), is the set of m-dimensional $\mathbb{R}$-linear subspaces of $\mathbb{R}^n$. The orthogonal group $\mathrm{O}(n,\mathbb{R})$ acts transitively on $\mathcal{G}_{m,n}$; a transformation stabilizing a given element p_0 also stabilizes its orthogonal complement $p_0^{\perp}$ and therefore the stabilizer of p_0 is isomorphic to the direct product $\mathrm{O}(m,\mathbb{R}) \times \mathrm{O}(n-m,\mathbb{R})$. Hence we derive the identification of $\mathcal{G}_{m,n}$ with the set of classes:

$$\mathcal{G}_{m,n} \simeq \mathrm{O}(n,\mathbb{R})/\big(\mathrm{O}(m,\mathbb{R}) \times \mathrm{O}(n-m,\mathbb{R})\big)$$

from which $\mathcal{G}_{m,n}$ inherits the structure of a (compact) differential variety, and a $\mathrm{O}(n,\mathbb{R})$-invariant measure that will be normalized so that $\int_{\mathcal{G}_{m,n}} dp = 1$. It is worth noticing that the case $m = 1$ corresponds to the real projective space.

In order to understand the action of $\mathrm{O}(n,\mathbb{R})$ on pairs $(p,q) \in {\mathcal{G}_{m,n}}^2$, we need to introduce the *principal angles* between p and q. These are m angles $\theta_1, \ldots, \theta_m \in [0, \pi/2]$ defined in the following way:

Let $p_1 \subset p$, $q_1 \subset q$ be two lines such that the angle θ_1 between p_1 and q_1 is minimal. If $m = 1$ we have finished, otherwise let p' be the orthogonal complement of p_1 in p, q' be the orthogonal complement of q_1 in q; we define recursively $\theta_2, \ldots, \theta_m$ to be the principal angles associated to the pair (p', q') in $\mathcal{G}_{m-1,n}$. We introduce the notation $y_i := \cos^2\theta_i$; when needed, we may denote rather $y_i(p,q)$, $\theta_i(p,q)$. A classical result on the geometry of $\mathcal{G}_{m,n}$ is the following:

Proposition 1.2: Two pairs (p,q) and (p',q') are in the same orbit under the action of the group $\mathrm{O}(n,\mathbb{R})$, i.e. there exists $\sigma \in \mathrm{O}(n,\mathbb{R})$ such that $\sigma(p) = p'$ and $\sigma(q) = q'$, if and only if

$$y_i(p,q) = y_i(p',q') \text{ for all } 1 \leq i \leq m.$$

The previous proposition expresses the fact that the orbits under the action of $\mathrm{O}(n,\mathbb{R})$ of the pairs $(p,q) \in {\mathcal{G}_{m,n}}^2$ are characterized by the m-tuple of real numbers $(y_1(p,q), \ldots, y_m(p,q))$. It becomes clear that, for $m \geq 2$, $\mathcal{G}_{m,n}$ is not 2-point homogeneous, i.e. a single distance on $\mathcal{G}_{m,n}$ cannot characterize these orbits (while it is the case for other spaces of interest in coding theory, like the Hamming and binary Johnson spaces, or the unit sphere of the Euclidean space). It is the reason why we shall deal with mutivariate polynomials. Also, it shows that the choice of a distance on $\mathcal{G}_{m,n}$ is sort of arbitrary. We shall stick to the *chordal distance* in this paper, as introduced in [3]:

$$d_c(p,q) := \sqrt{\sum_{i=1}^{m} \sin^2\theta_i} = \sqrt{m - \sum_{i=1}^{m} y_i}.$$

Other possibilities are the Riemannian distance $\sqrt{\sum_{i=1}^{m} \theta_i^2}$ which behaves somewhat badly because it is not smooth; the max distance $\max_i \theta_i$, etc.. The "product distance" (which is not a distance in the metric sense) $\left(\prod_i \sin\theta_i\right)$ seems to be relevant in the context of space time codes.

Now we consider the space $L^2(\mathcal{G}_{m,n})$ of functions $f : \mathcal{G}_{m,n} \to \mathbb{C}$ such that $\int_{\mathcal{G}_{m,n}} |f|^2 dp < +\infty$. This is a $\mathbb{C}$-vector space, endowed with the hermitian product:

$$< f, g >= \int_{\mathcal{G}_{m,n}} f(p)\overline{g(p)}dp$$

and with the left action of the orthogonal group given by:

$$(\sigma \cdot f)(p) = f(\sigma^{-1}(p))$$

(for which the above hermitian product is of course invariant).

Its associated *algebra of zonal functions* (also called the Hecke algebra) is:

$$\mathcal{Z} := \{Z : {\mathcal{G}_{m,n}}^2 \to \mathbb{C} \mid \ Z(p,\cdot),\ Z(\cdot,q) \in L^2(\mathcal{G}_{m,n}) \text{ and } Z(\sigma(p),\sigma(q)) = Z(p,q) \text{ for all } \sigma \in \mathrm{O}(n,\mathbb{R})\}$$

Form Proposition 1.2, since $Z \in \mathcal{Z}$ is constant on the orbits of $\mathrm{O}(n,\mathbb{R})$ on ${\mathcal{G}_{m,n}}^2$, it can be given the form: $Z(p,q) = z(y_1(p,q), \ldots, y_m(p,q))$ for some function z.

The explicit decomposition into $\mathrm{O}(n,\mathbb{R})$-irreducible subspaces of $L^2(\mathcal{G}_{m,n})$, and the corresponding structure of $\mathcal{Z}$, where investigated for the first time by James and Constantine ([14]). It is now a standard result on the representation of the classical groups (see [9]).

Recall that the irreducible representations of $\mathrm{O}(n,\mathbb{R})$ are (up to a power of the determinant) naturally indexed by partitions $\kappa = (\kappa_1, \ldots, \kappa_n)$, where $\kappa_1 \geq \cdots \geq \kappa_n \geq 0$ (we may omit the last parts if they are equal to 0). Following [9], let them be denoted by V_n^{κ}. For example, $V_n^{()} = \mathbb{C}\mathbf{1}$, and $V_n^{(k)} = \mathrm{Harm}_k$ the space of homogeneous of degree k, harmonic polynomials in n variables.

The length $\ell(\kappa)$ of a partition κ is the number of its non zero parts, and its degree $\deg(\kappa)$ also denoted by $|\kappa|$ equals $\sum_{i=1}^{n} \kappa_i$.

Then, the decomposition of $L^2(\mathcal{G}_{m,n})$ is as follows:

$$L^2(\mathcal{G}_{m,n}) \simeq \oplus V_n^{2\kappa}$$

where κ runs over the partitions of length at most m and 2κ stands for $(2\kappa_1, \ldots, 2\kappa_m)$, meaning that only partitions with even parts enter the decomposition. We can see that the multiplicities in this decomposition are all equal to one, which translates the fact that the space $\mathcal{G}_{m,n}$ is a symmetric space. Consequently, to each irreducible component $V_n^{2\kappa}$ is associated a uniquely determined (up to a normalizing factor) zonal function $P_\kappa(y_1, \ldots, y_m)$, in the sense that

$$p \text{ (resp. } q) \mapsto P_\kappa(y_1(p,q), \ldots, y_m(p,q)) \in V_n^{2\kappa}$$

and

$$\mathcal{Z} = \oplus_{\kappa, \ell(\kappa) \leq m} \mathbb{C} P_\kappa.$$

It turns out that the P_κ are symmetric polynomials in the m variables $y_1, \ldots, y_m$, of degree $|\kappa|$, with rational coefficients once they are normalized by the condition $P_\kappa(1, \ldots, 1) = 1$.

Moreover, the set $(P_\kappa)_{|\kappa|\leq k}$ is a basis of the space of symmetric polynomials in the variables $y_1,\dots,y_m$ of degree at most equal to k, denoted by S_k.

Since the irreducible subspaces of $L^2(\mathcal{G}_{m,n})$ are pairwise non isomorphic, they are orthogonal for the $\mathrm{O}(n,\mathbb{R})$-invariant hermitian product defined above. This hermitian product induces an hermitian product on the space of symmetric polynomials, denoted by $[,]$, for which the polynomials P_κ are orthogonal. More precisely, it is given by the positive measure, calculated in [14],

$$d\mu = \lambda \prod_{\substack{i,j=1\\ i<j}}^{m} |y_i - y_j| \prod_{i=1}^{m} y_i^{-1/2}(1-y_i)^{n/2-m-1/2} dy_i$$

(where λ is chosen so that $\int_{[0,1]^m} d\mu(y) = 1$). and

$$[f,g] = \int_{[0,1]^m} f(y)\overline{g(y)} d\mu(y).$$

One recognizes a special case of the orthogonal measure associated to *generalized Jacobi polynomials* ([11]).

We let Π_k be the subspace of S_k generated by the polynomials P_κ, with $|\kappa| = k$, so that we have the orthogonal decomposition:

$$S_k = S_{k-1} \perp \Pi_k.$$

Let the dimensions of S_k, Π_k be denoted respectively by s_k, π_k. The number π_k is also equal to the number of partitions κ of k in at most m parts. These dimensions also depend on m, although it does not reflect on our notation, for the sake of simplicity.

In view of the explicit calculation of the polynomials P_κ, it is better to use the following characterization, which involves the polynomials C_κ, which are themselves the zonal polynomials associated to the symmetric space $\mathrm{GL}(m,\mathbb{R})/\mathrm{O}(m,\mathbb{R})$ (these are Jack polynomials, normalized by $C_\kappa(1,\dots,1)=1$, see [14], [12]), and the differential operator Δ induced on $\mathbb{C}[y_1,\dots,y_m]^{S_m}$ by the Laplace Beltrami operator of $\mathcal{G}_{m,n}$. The condition: for all $1 \leq i \leq m$, $\kappa_i \geq \mu_i$ is denoted by: $\kappa \geq \mu$.

(i) P_κ is an eigenvector for the operator

$$\begin{aligned}\Delta :=& \sum_{i=1}^{m} y_i^2 \frac{\partial^2}{\partial y_i^2} + \sum_{i\neq j=1}^{m} y_i^2 (y_i - y_j)^{-1} \frac{\partial}{\partial y_i} \\ &+ (\frac{n}{2} - m + 1) \sum_{i=1}^{m} y_i \frac{\partial}{\partial y_i} - \sum_{i=1}^{m} y_i \frac{\partial^2}{\partial y_i^2} \\ &- \sum_{i\neq j=1}^{m} y_i (y_i - y_j)^{-1} \frac{\partial}{\partial y_i} - \frac{1}{2} \sum_{i=1}^{m} \frac{\partial}{\partial y_i}\end{aligned}$$

(ii) $P_\kappa = \beta_\kappa C_\kappa + \sum_{\mu|\kappa>\mu} \beta_{\kappa,\mu} C_\mu$

(iii) $P_\kappa(1,\dots,1) = 1$.

Condition (ii) is needed to avoid the multiplicities of the operator Δ.

Examples: the effective computation of the polynomials P_κ following the method described above leads to, up to the normalization imposed by (iii):

$$\begin{aligned}P_0 &= 1\\ P_{(1)} &= s_1 - \frac{m^2}{n}\\ P_{(11)} &= \sigma_1 - \frac{(m-1)^2}{n-2} s_1 + \frac{m^2(m-1)^2}{2(n-1)(n-2)}\\ P_{(2)} &= s_2 + \frac{2}{3}\sigma_1 - \frac{2(m+2)^2}{3(n+4)} s_1 + \frac{m^2(m+2)^2}{3(n+2)(n+4)}\end{aligned}$$

where $s_1 = \sum_{1\leq i\leq m} y_i$, $s_2 = \sum_{1\leq i\leq m} y_i^2$, $\sigma_1 = \sum_{1\leq i<j\leq m} y_i y_j$.

Remark: The complex Grassmannian $\mathcal{G}_{m,n}(\mathbb{C})$ is more commonly used in the context of space-time coding. It affords the transitive action of the unitary group $U(\mathbb{C},n)$; similarly one defines principal angles $(\theta_1,\dots,\theta_m)$ between two elements of $\mathcal{G}_{m,n}(\mathbb{C})$. The $U(\mathbb{C},n)$ decomposition of $L^2(\mathcal{G}_{m,n}(\mathbb{C}))$ and the associated zonal polynomials are computed in [14] so one can play the same game concerning bounds of codes. We believe that an asymptotic bound obtained in this way would be the same as the bound obtained from the embedding $\mathcal{G}_{m,n}(\mathbb{C}) \subset \mathcal{G}_{2m,2n}(\mathbb{R})$ (if $(\theta_1,\dots,\theta_m)$ are the principal angles associated to a pair (p,q) of elements in $\mathcal{G}_{m,n}(\mathbb{C})$, the $2m$ principal angles associated to the pair (p,q), seen as elements of $\mathcal{G}_{2m,2n}(\mathbb{R})$, are simply $(\theta_1,\theta_1,\theta_2,\theta_2,\dots)$). This is what happens for $m=1$ (see [7]).

II. Zonal polynomials associated to $\mathcal{G}_{m,n}$ and the LP bound

In this section, we settle the properties of the polynomials P_κ relevant for the LP bound, settle this bound, and show how the Christoffel-Darboux formula can be exploited in that context.

The dimension of V_n^κ is denoted by d_κ. Explicit formulas for d_κ can be found in [8]; however we do not need them before Section 5.

Proposition 2.1: The polynomials P_κ, normalized by the condition $P_\kappa(1,\dots,1)=1$, satisfy:

(i) $[P_\kappa, P_\kappa] = d_{2\kappa}^{-1}$

(ii) (Positivity property): For all finite set $C \subset \mathcal{G}_{m,n}$,

$$\sum_{p,q\in C} P_\kappa(y_1(p,q),\dots,y_m(p,q)) \geq 0$$

(iii) Let $p_{\kappa,\mu}^\nu$ be defined by the property:

$$P_\kappa P_\mu = \sum_\nu p_{\kappa,\mu}^\nu P_\nu$$

The numbers $p_{\kappa,\mu}^\nu$ are non-negative numbers.

Proof: These properties where already pointed out in [5][Lemma 2.2] and step on very general arguments (see [16][Theorem 3.1]). For the sake of completeness, we briefly recall the arguments. Let $e_1,\dots,e_{d_{2\kappa}}$ be any orthonormal basis of the subspace $\mathrm{H}_{m,n}^{2\kappa}$ of $L^2(\mathcal{G}_{m,n})$ isomorphic to $V_n^{2\kappa}$.

Let $\tilde{P_\kappa}(p,q) := P_\kappa(y_1(p,q),\dots,y_m(p,q))$. It is well known that we have (this is called the *addition formula*)

$$\tilde{P_\kappa}(p,q) = \frac{1}{d_{2\kappa}} \sum_{i=1}^{d_{2\kappa}} e_i(p)\overline{e_i(q)}.$$

As a consequence, from the expression

$$[P_\kappa, P_\kappa] = \int_{\mathcal{G}_{m,n}} \tilde{P_\kappa}(p,q)\tilde{P_\kappa}(q,p)dq$$

(i) follows. Moreover,

$$\begin{aligned}\sum_{p,q\in C} \tilde{P_\kappa}(p,q) &= \frac{1}{d_{2\kappa}} \sum_{p,q\in C} \Big(\sum_{i=1}^{d_{2\kappa}} e_i(p)\overline{e_i(q)}\Big) \\ &= \frac{1}{d_{2\kappa}} \sum_{i=1}^{d_{2\kappa}} \Big(\sum_{p,q\in C} e_i(p)\overline{e_i(q)}\Big) \\ &= \frac{1}{d_{2\kappa}} \sum_{i=1}^{d_{2\kappa}} \Big| \sum_{p\in C} e_i(p)\Big|^2 \geq 0.\end{aligned}$$

hence (ii). More generally, for any function $\alpha : C \to \mathbb{C}$, we have:

$$\begin{aligned}\sum_{p,q\in C} \alpha(p)\overline{\alpha(q)}\tilde{P_\kappa}(p,q) =& \frac{1}{d_{2\kappa}} \sum_{i=1}^{d_{2\kappa}} \Big| \sum_{p\in C} \alpha(p)e_i(p)\Big|^2 \\ &\geq 0.\end{aligned}$$

Conversely, assume $F \in S_k$ is a polynomial with real coefficients, such that, for any finite set $C \subset \mathcal{G}_{m,n}$ and any function $\alpha : C \to \mathbb{C}$,

$$\sum_{p,q\in C} \alpha(p)\overline{\alpha(q)}\tilde{F}(p,q) \geq 0,$$

and let us prove that F expands on the P_κ with non-negative coefficients. Taking limits, we have, for any $\alpha \in L^2(\mathcal{G}_{m,n})$,

$$\int\int_{\mathcal{G}_{m,n}} \alpha(p)\overline{\alpha(q)}\tilde{F}(p,q)dpdq \geq 0$$

and hence, using the addition formula,

$$\int\int_{\mathcal{G}_{m,n}} \tilde{P_\kappa}(p,q)\tilde{F}(p,q)dpdq \geq 0.$$

If $F = \sum_{|\nu|\leq k} f_\nu P_\nu$, the left hand-side equals $f_\kappa/d_{2\kappa}$, which proves that the coefficients f_κ are non-negative numbers. Using once again the addition formula, it is easy to show that the product $P_\kappa P_\mu$ holds this general positivity property, and therefore expands on the P_ν with non-negative coefficients.

A. The principles of the LP bound

The positivity property of the polynomials P_κ is the basis of the linear programming method to upper bound the cardinality of δ-codes.

Definition 2.2: A Grassmannian code C satisfying the constraint:

$$\text{For all } p \neq q \in C^2, d_c(p,q) \geq \delta.$$

is called a δ-code.

Proposition 2.3: Assume $F_k \in S_k$ satisfy:

(i) $F_k = \sum_{|\kappa|\leq k} f_\kappa P_\kappa$ with $f_\kappa \geq 0$ for all κ, $f_0 > 0$

(ii) $F_k(y_1,\dots,y_m) \leq 0$ for all $(y_1,\dots,y_m) \in [0,1]^m$ such that $\sum_{i=1}^m y_i \leq m - \delta^2$

Then, the following bound holds for the cardinality $|C|$ of any δ-code:

$$|C| \leq \frac{F_k(1,\dots,1)}{f_0}.$$

Proof: This is a standard argument, that we recall here. Let C be a δ-code. As before, we let $\tilde{F_k}(p,q) = F_k(y_1(p,q),\dots,y_m(p,q))$. We calculate

$$\sum_{p,q\in C} \tilde{F_k}(p,q) = \sum_{|\kappa|\leq k} f_\kappa \Big(\sum_{p,q\in C} \tilde{P_\kappa}(p,q)\Big)$$

Assumption (ii) leads to $\tilde{F_k}(p,q) \leq 0$ when $p \neq q$. The remaining terms of the left hand-side, corresponding to $p = q$, give a contribution of $|C|F_k(1,\dots,1)$. Assumption (i), together with the positivity property of the polynomials P_κ (Proposition 2.1 (ii)), show that all the terms of the right hand-side are non-negative. When $\kappa = (0)$, $P_\kappa = 1$ and the contribution is $f_0|C|^2$. We obtain

$$|C|F_k(1,\dots,1) \geq f_0|C|^2$$

equivalently

$$|C| \leq \frac{F_k(1,\dots,1)}{f_0}.$$

It is worth noticing that equality in this inequality happens if and only if, for all $1 \leq |\kappa| \leq k$ such that $f_\kappa \neq 0$, $\sum_{p,q\in C} \tilde{P_\kappa}(p,q) = 0$ and, for all $p \neq q \in C$, $\tilde{F_k}(p,q) = 0$. The first condition says that C is a $2k$-design in the sense of [6] (when it holds for all $1 \leq |\kappa| \leq k$), and the second one that C is an F_k-code in the sense of [5].

B. The three-term relation and the Christoffel-Darboux formula

We join here more material on the sequence of polynomials P_κ, that will be of later use. The results presented here are essentially established in [13], except that we deal with symmetric polynomials. Following [13], the (column) vector of the polynomials P_κ with $|\kappa| = k$ is denoted by $\mathbb{P}_k$. If necessary, we order the partitions of the same degree in increasing lexicographic order.

We also set

$$\sigma := y_1 + y_2 + \cdots + y_m$$

and, when necessary, we make the involved variables explicit, by writing $\sigma(y)$ rather than σ. The $\pi_k \times \pi_k$ diagonal matrix, denoted by D_k, with entries

$$D_k[\kappa,\kappa] = d_{2\kappa} := \dim(V_n^{2\kappa})$$

is the inverse of the Gram matrix of $\mathbb{P}_k$.

Next result is an analogue of the so-called "three-term relation".

Theorem 2.4: For all $k \geq 1$, there exists matrices A_k, B_k, C_k, of size respectively $\pi_k \times \pi_{k+1}$, $\pi_k \times \pi_k$, $\pi_k \times \pi_{k-1}$, such that:

$$\sigma\mathbb{P}_k = A_k\mathbb{P}_{k+1} + B_k\mathbb{P}_k + C_k\mathbb{P}_{k-1}.$$

Moreover, $(D_kB_k)^t = D_kB_k$ and $D_kC_k = (D_{k-1}A_{k-1})^t$.

Proof: The polynomials σP_κ with $|\kappa| = k$ are symmetric of total degree $k+1$ so they afford a decomposition over the $(P_\mu)_{|\mu|\leq k+1}$. Moreover, $[\sigma P_\kappa, P_\mu] = [P_\kappa, \sigma P_\mu] = 0$ if $|\mu| \leq k-2$.

If $|\mu| = |\kappa| = k$, we have: $B_k[\kappa,\mu][P_\mu, P_\mu] = [\sigma P_\kappa, P_\mu] = [P_\kappa, \sigma P_\mu] = B_k[\mu,\kappa][P_\kappa, P_\kappa]$, which proves that the matrix D_kB_k is symmetric. The same argument shows that $D_kC_k = (D_{k-1}A_{k-1})^t$.

Notations: We want to define $\kappa^{(i)}$ (respectively $\kappa_{(i)}$) to be the partition obtained from κ by increasing (respectively decreasing) the i-th part κ_i by one. This is not possible for all i, since the result should be also a partition, i.e. the new parts should be in decreasing order. Hence we define (where $\kappa_{m+1} := 0$):

$$\begin{cases} u(\kappa) := \{1\} \cup \{i \in [2,m] \mid \kappa_{i-1} > \kappa_i\} \\ d(\kappa) := \{i \in [1,m] \mid \kappa_i > \kappa_{i+1}\} \end{cases}$$

The set $u(\kappa)$ is the set of indices i for which $\kappa^{(i)}$ makes sense (respectively $d(\kappa)$ for $\kappa_{(i)}$). Moreover, if $|\kappa| = k$, $|\kappa^{(i)}| = k+1$ and $|\kappa_{(i)}| = k-1$.

Otherwise explicitly mentioned, in the rest of this paper, κ, κ' are partitions of degree k, while μ, μ' are partitions of degree $k+1$ and ν, ν' are partitions of degree $k-1$.

Proposition 2.5: The following properties hold:

(i) For all κ, μ and κ', $A_k[\kappa,\mu] \geq 0$, and $B_k[\kappa,\kappa'] \geq 0$.

(ii) The coefficients of the matrix A_k are equal to zero, except the coefficients $A_k[\kappa,\kappa^{(i)}]$, which are positive.

Proof: The first assertions are equivalent to: $[\sigma P_\kappa, P_\mu] \geq 0$ for all κ, μ of any degree. But $[\sigma P_\kappa, P_\mu] = [1, \sigma P_\kappa P_\mu]$ and $\sigma = m(1-\frac{m}{n})P_1 + \frac{m^2}{n}$. Joint with Proposition 2.1(iii), we obtain $[1, \sigma P_\kappa P_\mu] \geq 0$.

The coefficients $A_k[\kappa,\mu]$ can be more precisely calculated, using ([17, Lemma 7.5.7]). Since we do not normalize the polynomials C_κ in the same way, we introduce coefficients $\begin{bmatrix}\mu\\ \kappa\end{bmatrix}$ such that

$$\sigma C_\kappa = \sum_{|\mu|=k+1} \begin{bmatrix}\mu\\ \kappa\end{bmatrix} C_\mu.$$

They differ by a positive multiplicative factor from the generalized binomial coefficients $\binom{\mu}{\kappa}$ defined in [17]; see also [10]. Then we have

$$A_k[\kappa,\mu] = \begin{bmatrix}\mu\\ \kappa\end{bmatrix} \left(\frac{\beta_\mu}{\beta_\kappa}\right)^{-1}.$$

It is known that the generalized binomial coefficients $\binom{\mu}{\kappa}$ are equal to zero when μ is not equal to one of the $\kappa^{(i)}$; consequently the same holds for $A_k[\kappa,\mu]$. Moreover, since $\binom{\kappa^{(i)}}{\kappa} > 0$, also $\begin{bmatrix}\kappa^{(i)}\\ \kappa\end{bmatrix} > 0$ and $A_k[\kappa,\kappa^{(i)}] \neq 0$.

Theorem 2.6 (Christoffel-Darboux Formula): Let

$$Q_\kappa := \sum_{|\mu|=k+1} A_k[\kappa,\mu] P_\mu \in \Pi_{k+1}.$$

With the previous notations, we have:

(i) For all $k \geq 0$,

$$\sum_{|\nu|\leq k} d_{2\nu} P_\nu(x) P_\nu(y) = \frac{\sum_{|\kappa|=k} d_{2\kappa}\big(Q_\kappa(x)P_\kappa(y) - P_\kappa(x)Q_\kappa(y)\big)}{\sigma(x)-\sigma(y)}$$

(ii) Moreover, if $\epsilon := \sum_{i=1}^m \frac{\partial}{\partial y_i}$,

$$\sum_{|\nu|\leq k} d_{2\nu}\big(P_\nu(y)\big)^2 = \sum_{|\kappa|=k} \frac{d_{2\kappa}}{m}\Big(\big(\epsilon Q_\kappa(y)\big)P_\kappa(y) - \big(\epsilon P_\kappa(y)\big)Q_\kappa(y)\Big).$$

Proof: The proof of (i) is the same as [13][Theorem 3.5.3]. Note that we cannot hope for a formula for each x_i like in [13], since we should stick to symmetric polynomials. If

$$\begin{aligned}\Sigma_s :&= (A_s\mathbb{P}_{s+1}(x))^t D_s \mathbb{P}_s(y) - \mathbb{P}_s(x)^t D_s A_s \mathbb{P}_{s+1}(y) \\ &= \sum_{|\kappa|=s} d_{2\kappa}\big(Q_\kappa(x)P_\kappa(y) - P_\kappa(x)Q_\kappa(y)\big),\end{aligned}$$

from the "three-term relation" of Theorem 2.4, we have:

$$\Sigma_s - \Sigma_{s-1} = (\sigma(x)-\sigma(y))\mathbb{P}_s(x)^t D_s \mathbb{P}_s(y).$$

The formula (i) follows from summing up these identities, for $1 \leq s \leq k$.

In the equation (i), we replace $Q_\kappa(x)P_\kappa(y) - P_\kappa(x)Q_\kappa(y)$ by

$$Q_\kappa(x)\big(P_\kappa(y) - P_\kappa(x)\big) - P_\kappa(x)\big(Q_\kappa(y) - Q_\kappa(x)\big).$$

Then, if we specialize $x_2 = y_2, \ldots, x_m = y_m$ and let x_1 tend to y_1, we obtain

$$\sum_{|\nu|\leq k} d_{2\nu}\big(P_\nu(y)\big)^2 = \sum_{\substack{|\kappa|=k\\ |\mu|=k+1}} d_{2\kappa} A_k[\kappa,\mu]\Big(\big(\frac{\partial Q_\kappa}{\partial y_1}(y)\big)P_\kappa(y) - \big(\frac{\partial P_\kappa}{\partial y_1}(y)\big)Q_\kappa(y)\Big).$$

The same identity holds when one replaces y_1 by any y_i; if we sum up all these identities, we obtain the more symmetric formula (ii).

Remark 2.7: The left hand side of the Christoffel-Darboux formula

$$K_k(x,y) := \sum_{|\nu|\leq k} d_{2\nu} P_\nu(x) P_\nu(y).$$

is the *reproducing kernel* of the space of symmetric polynomials of degree at most k. It satisfies the characteristic property: for all $Q \in S_k$, $[K_k(x,.), Q] = Q(x)$.

C. An LP bound from Christoffel-Darboux formula

In the classical cases, Christoffel-Darboux formula is involved in the setting up of bounds of the type $|C| \leq M(\delta)$ where $M(\delta)$ is an explicit function of δ. Usually the running interval of δ is divided into subintervals, related to the zeros of the zonal polynomials. This is the line followed in [18], and also in [7]; see [19] for a unified presentation. In this section, we follow this method, and analyze the difficulties arising from the several variables situation.

The numerator, of degree $k+1$, of the right hand side of Christoffel-Darboux formula (Theorem 2.6(i)) is denoted by $N_{k+1}(x,y)$. We consider the polynomial in the variables $y_1,\dots,y_m$, of degree $2k+1$,

$$\begin{aligned} F_{2k+1}(x,y) :&= -N_{k+1}(x,y)K_k(x,y) \\ &= \frac{N_{k+1}(x,y)^2}{\sigma(y)-\sigma(x)}. \end{aligned}$$

In order to make clear that only the $y_1,\dots,y_m$ are variables, while the $x_1,\dots,x_m$ will specialize to real values, we denote it by $F_{2k+1}(x,\cdot)$.

Proposition 2.8: Let $x \in [0,1]^m$ satisfy $\sigma(x) > s := m - \delta^2$. Assume the following conditions hold:

(i) For all κ, $|\kappa| \leq k$, $P_\kappa(x) \geq 0$
(ii) For all κ, $|\kappa| = k$, $Q_\kappa(x) \leq 0$

Then, $F_{2k+1}(x,\cdot)$ satisfies the conditions required in Proposition 2.3.

Proof: We have:

$$F_{2k+1}(x,y) = \frac{N_{k+1}(x,y)^2}{\sigma(y)-\sigma(x)}$$

hence condition (ii) is satisfied when $s < \sigma(x)$.

To prove condition (i), we point out that, if F and G are two polynomials with non-negative coefficients on the P_κ, then the product FG holds the same property. This is a direct consequence of Proposition 2.1(iii).

From the definition of $K_k(x,y)$, its coefficient on P_ν with $|\nu| \leq k$ equals $d_{2\nu}P_\nu(x)$ (and for higher degree partitions it is zero). On the other hand,

$$\begin{aligned} -N_{k+1}(x,y) &= \sum_{|\kappa|=k} d_{2\kappa}\big(P_\kappa(x)Q_\kappa(y) - Q_\kappa(x)P_\kappa(y)\big) \\ &= \sum_{|\mu|=k+1} \Big(\sum_{|\kappa|=k} d_{2\kappa}A_k[\kappa,\mu]P_\kappa(x)\Big)P_\mu(y) \\ &\quad - \sum_{|\kappa|=k} d_{2\kappa}Q_\kappa(x)P_\kappa(y) \end{aligned}$$

The coefficient $A_k[\kappa,\mu]$ is always non-negative. Clearly, under the conditions of the proposition, the coefficients of $-N_{k+1}(x,y)$ on the P_κ and P_μ are non-negative.

Corollary 2.9: Assume x satisfies the conditions of Proposition 2.8. Then, for all δ-code C,

$$|C| \leq \frac{\big(m-\sigma(x)\big)\big(\sum_{|\nu|\leq k} d_{2\nu}P_\nu(x)\big)^2}{-\sum_{|\kappa|=k} d_{2\kappa}P_\kappa(x)Q_\kappa(x)}$$

Proof: In order to apply Proposition 2.3, we are left with the computation of f_0 and of $F_{2k+1}(x,(1,\dots,1))$. Since $F_{2k+1}(x,y) = K_k(x,y)^2(\sigma(y)-\sigma(x))$, we have

$$F_{2k+1}(x,(1,\dots,1)) = \big(\sum_{|\nu|\leq k} d_{2\nu}P_\nu(x)\big)^2(m-\sigma(x)).$$

Using the orthogonality of the P_κ, we obtain

$$\begin{aligned} f_0 :&= [F_{2k+1}(x,\cdot),1] = -[K_k(x,\cdot),N_{k+1}(x,\cdot)] \\ &= -[\sum_{|\nu|\leq k} d_{2\nu}P_\nu(x)P_\nu, \sum_{|\kappa|=k} d_{2\kappa}Q_\kappa(x)P_\kappa] \\ &= -\sum_{|\kappa|=k} d_{2\kappa}P_\kappa(x)Q_\kappa(x). \end{aligned}$$

The main problem with this approach, is that, in general, we don't even know if the inequalities (i) and (ii) of Proposition 2.8 have a solution x. In case these inequalities define a non empty area of $\mathbb{R}^m$, a second problem would be to optimize the choice of x in this area. In the classical case $m=1$, $Q_k = P_{k+1}$ (up to a positive multiplicative factor). The interlacing property of the real zeros of the orthogonal polynomials P_k, ensures that one can take $x \in [z_k, z_{k+1}]$, where z_k is the largest zero of P_k, so that $P_{k+1}(x) \leq 0$ and $P_i(x) \geq 0$ for all $i \leq k$. Moreover, one uses asymptotic estimates of these zeros to derive an asymptotic bound for the size of codes.

In the general case $m \geq 2$, we don't have such tools to deal with the inequalities of Proposition 2.8, which seem to be intractable in general. The first case $k=1$, leading to a polynomial of degree 3, is however discussed in the next section. On the other hand, one can think of the zeros of orthogonal polynomials in one variable as being the eigenvalues of the so-called Jacobi matrices associated to the sequence of polynomials. We study in section IV the eigenvalues of the analogous matrices in the general case, and derive bounds for codes, which contain as a special case the bound obtained from a possible solution of these inequalities.

III. LP BOUNDS OF SMALL DEGREE

We take the following notations: let $s := m - \delta^2$, the maximal value of σ among pairs of points of a code C. We are looking for a function $M(s)$ such that $|C| \leq M(s)$. Obviously, $M(s)$ is an increasing function. In this section, we discuss the cases of small degree k, trying to optimize the choice of F_k in Proposition 2.3

A. Degree 1

Let $F_1 = 1 + f_1P_1$, with $f_1 \geq 0$ (condition (i)). We have $P_1 = \frac{n}{m(n-m)}(\sigma - \frac{m^2}{n})$.

When $\sigma \in [0,s]$, $1+f_1P_1$ should be non-positive (condition (ii)). Therefore, The zero of $1+f_1P_1$ should be greater than s. It leads to the condition:

$$s - \frac{m^2}{n} \leq -\frac{m(n-m)}{nf_1}.$$

Since $f_1 \geq 0$, we obtain the necessary condition $s \leq \frac{m^2}{n}$. The smallest value for f_1 is then

$$f_1 = \frac{-m(1-m/n)}{s - \frac{m^2}{n}}$$

corresponding to a polynomial proportional to $\sigma - s$. We obtain the bound

$$\text{if } s < \frac{m^2}{n},\ |C| \leq \frac{m-s}{\frac{m^2}{n} - s}$$

which is the so-called simplex bound proved in [3].

B. Degree 2

We restrict ourselves to polynomials which are divisible by $\sigma - s$. Then, such polynomials are polynomials in σ. We write:

$$F_2 = (\sigma - s)(\sigma - b) = f_2 P_2 + f_{11} P_{11} + f_1 P_1 + f_0.$$

with the condition that $b \leq 0$. With $t = s - m^2/n$, we find:

$$\begin{aligned}
f_2 :=& \frac{m(m+2)(n-m)(n-m+2)}{3(n+2)(n+4)} \\
f_{11} :=& \frac{2m(m-1)(n-m)(n-m-1)}{3(n-2)(n-1)} \\
f_1 :=& m\left(1 - \frac{m}{n}\right)\left(\frac{m^2}{n} + \frac{4(n-2m)^2}{n(n-2)(n+4)} - t - b\right) \\
f_0 :=& \frac{2m^2(n-m)^2}{n^2(n-1)(n+2)} - \frac{m^2}{n}t + bt
\end{aligned}$$

The condition $f_0 > 0$, when $t > 0$, is equivalent to

$$b > \frac{m^2}{n}\left(1 - \frac{2(n-m)^2}{tn(n-1)(n+2)}\right) \tag{2}$$

(and when $t <= 0$ is always fulfilled), which implies

$$t < \frac{2(n-m)^2}{n(n-1)(n+2)}. \tag{3}$$

The condition $f_1 \geq 0$ is equivalent to

$$b \leq \frac{m^2}{n} + \frac{4(n-2m)^2}{n(n-2)(n+4)} - t. \tag{4}$$

One can check that the right hand side of (4) is positive for $m \geq 2$, when t satisfies (3).

The bound $B = (f_2 + f_{11} + f_1 + f_0)/f_0$ equals

$$B = \delta^2 \frac{m-b}{f_0}. \tag{5}$$

Considered as a function of b, it is decreasing when $t \in [\frac{-2m(n-m)}{n(n-1)(n+2)}, \frac{2(n-m)^2}{n(n-1)(n+2)}[$, and hence the best choice of b is $b = 0$. We obtain the bound:

Theorem 3.1: If $s \in \left]0, \frac{m^2}{n} + \frac{2(n-m)^2}{n(n-1)(n+2)}\right[$,

$$|C| \leq \frac{n}{m}\left(\frac{m-s}{-s + \frac{m^2}{n} + \frac{2(n-m)^2}{n(n-1)(n+2)}}\right) \tag{6}$$

This bound, which is an increasing function of s, improves on the simplex bound when $s \geq \frac{m^2}{n} - \frac{2m(n-m)}{n(n-1)(n+2)}$. Their common value at $s = \frac{m^2}{n} - \frac{2m(n-m)}{n(n-1)(n+2)}$ is $\binom{n+1}{2}$. However, the orthoplex bound proved in [3, (5.6)] reads:

$$\begin{aligned}
s < m^2/n \Rightarrow |C| &\leq \binom{n+1}{2} \\
s = m^2/n \Rightarrow |C| &\leq (n-1)(n+2)
\end{aligned}$$

and is better than (6) in the range $]\frac{m^2}{n} - \frac{2m(n-m)}{n(n-1)(n+2)}, \frac{m^2}{n}[$. If we plug in (6) the value $s = m^2/n$, we find that $|C| \leq \frac{(n-1)(n+2)}{2(1-m/n)}$ which is better than the orthoplex bound when $m < n/2$. We recall that the orthoplex bound is attained for a family of codes with $n = 2^i$, $m = n/2$, constructed in [20, Theorem 1]. These codes are also optimal 6-designs (see [21]).

C. Degree 3

We do not study general polynomials of degree 3 but rather apply the approach described in subsection II-C

The polynomial F_3 has degree 3, and is again a polynomial in σ. In the following, we calculate the best choice for x (and discuss its existence). Let $u := \sigma(x) - m^2/n$. We should have:

(i) $u \geq s - m^2/n$
(ii) $u \geq 0$ (Condition (i))
(iii) $u^2 - \frac{4(n-2m)^2}{n(n-2)(n+4)}u - \frac{2m^2(n-m)^2}{n^2(n-1)(n+2)} \leq 0$ (Condition (ii))

The polynomial of degree 2 occurring in (iii) has a positive discriminant, and a unique positive root that we shall denote by u_2. Let b and c be the coefficients of this polynomial, so that it is equal to $u^2 - bu - c$, and let $d := \frac{2m(n-m)}{n(n-1)(n+2)}$. The bound is then equal to:

$$B(u) := -\frac{(n-1)(n+2)(u+d)^2(m-u-m^2/n)}{2u(u^2-bu-c)}.$$

The calculation of $B'(u)$ shows that it is increasing in the range $[u_1, u_2]$ (the numerator has the form: $u + d$ times a degree 3 polynomial with a unique real root u_1). Hence, for $s \in [u_1 + \frac{m^2}{n}, u_2 + \frac{m^2}{n}]$, the best choice for u is $u = s - m^2/n$.

We obtain:

Theorem 3.2: Let $d = \frac{2m(n-m)}{n(n-1)(n+2)}$, $b = \frac{4(n-2m)^2}{n(n-2)(n+4)}$, $c = \frac{2m^2(n-m)^2}{n^2(n-1)(n+2)}$.

$$\text{If } s \in \left]\frac{m^2}{n}, \frac{m^2}{n} + \frac{b}{2} + \sqrt{\frac{b^2}{4} + c}\right[,$$
$$|C| \leq \frac{(m-s)(s - \frac{m^2}{n} + d)^2(n-1)(n+2)}{2(s - \frac{m^2}{n})(-(s - \frac{m^2}{n})^2 + b(s - \frac{m^2}{n}) + c)}$$

IV. The endomorphisms T_k

We introduce an endomorphism $T_k : S_k \to S_k$ which eigenvalues will play the role of the zeros of the zonal polynomials in the rank one case.

Proposition 4.1: Let

$$\begin{aligned}
T_k : S_k &\to S_k \\
P &\mapsto \mathrm{pr}_{S_k}(\sigma P)
\end{aligned}$$

where the orthogonal projection on S_k is denoted by pr_{S_k} (note that, in general, σP does not belong to S_k but rather to S_{k+1}).

The endomorphism T_k is a symmetric endomorphism of S_k, and is an isomorphism.

Proof: We have, for all $P, Q \in S_k$, $[T_k(P), Q] = [\sigma P, Q] = [P, \sigma Q] = [P, T_k(Q)]$. Moreover, $[\sigma P, P] = [\sigma, P^2] > 0$ unless $P = 0$, because of the positivity of the measure on $\mathbb{R}[y_1, \ldots, y_m]^{S_m}$. Thus T_k is injective.

Let J_k be the matrix of this endomorphism in the basis $\{P_\kappa, |\kappa| \leq k\}$. From the three-term relation (Theorem 2.4), J_k is the block-tridiagonal matrix:

$$J_k = \begin{pmatrix} B_0 & A_0 & & & & \\ C_1 & B_1 & A_1 & & & \\ & C_2 & B_2 & A_2 & & \\ & & C_3 & \ddots & \ddots & \\ & & & \ddots & \ddots & A_{k-1} \\ & & & & C_k & B_k \end{pmatrix} \tag{7}$$

It is worth noticing that the matrix J_k itself is not symmetric, because the polynomials P_κ are not of norm 1. We shall later introduce and calculate the symmetric matrix J'_k obtained in the normalized basis.

In the end, we shall need some very precise information on the coefficients of J'_k. For the moment, the only, but crucial, property that we will exploit is the fact that it is *non-negative and irreducible*.

Lemma 4.2: The eigenvalues of T_k are real, and belong to $]0, m[$. The maximal eigenvalue of T_k, denoted by λ_k, is of multiplicity 1, and possesses an eigenvector with positive coordinates. Moreover, $\lambda_{k-1} < \lambda_k$.

Proof: The matrix J_k is non-negative and irreducible in the sense of [22], because of Proposition 2.5 (note that the coefficients $A_k[\kappa, \kappa^{(i)}]$ are positive). Moreover, it is the matrix of a symmetric endomorphism, so its eigenvalues are real. From [22, Perron-Frobenius Theorem], it follows that the maximal eigenvalue has multiplicity equal to 1, and that, if v is an eigenvector, either v or $-v$ has positive coordinates. Let us now prove that all its eigenvalues belong to $]0, m[$.

For any $v \in S_k$, $v \neq 0$, we have $[\sigma v, v] = \int \sigma v^2 d\mu(y)$, where $d\mu$ is a positive measure. We integrate on the domain $[0, 1]^m$, on which $0 \leq \sigma \leq m$, hence $0 < [\sigma v, v] < m[v, v]$. If v is an eigenvector of T_k associated with an eigenvalue λ, we have $[\sigma v, v] = [\lambda v, v] = \lambda[v, v]$, so we can conclude that $0 < \lambda < m$.

Now let v be an eigenvector of T_{k-1} for λ_{k-1}, assumed to be of norm 1. We have

$$\sigma v = \lambda_{k-1} v + u$$

with $u \in \Pi_k$. Obviously, since $\deg(\sigma v) = 1 + \deg(v)$, v must be of degree exactly $k - 1$ (and $u \neq 0$).

Since

$$\lambda_k = \max_{x \in S_k \setminus \{0\}} \frac{[T_k(x), x]}{[x, x]},$$

we have $[T_k(v), v] \leq \lambda_k$. But $[T_k(v), v] = [\sigma v, v] = \lambda_{k-1}$. The equality $\lambda_{k-1} = \lambda_k$ would mean that v is an eigenvector of T_k, which is not possible since it has degree $k - 1$.

In the case $m = 1$, the eigenvalues of T_k are exactly the zeros of the polynomial P_{k+1}. In the general case, we prove in next lemma that common zeros of the polynomials Q_κ give some of the eigenvalues. However, we do not know if such common zeros do exist, neither if all of the eigenvalues are obtained that way (and may be it is not so important):

Lemma 4.3: Let $\alpha \in [0, 1]^m$ be a common zero of the polynomials

$$Q_\kappa := \sum_{|\mu| = k+1} A_k[\kappa, \mu] P_\mu,$$

for all κ, $|\kappa| = k$. Then, $v := \sum_{|\nu| \leq k} d_{2\nu} P_\nu(\alpha) P_\nu$ is an eigenvector of T_k for the eigenvalue $\sigma(\alpha)$.

Proof: It is immediate from Christoffel-Darboux formula (Theorem 2.6(i)). If $Q_\kappa(\alpha) = 0$ for all κ, $|\kappa| = k$, we have

$$(\sigma(\alpha) - \sigma(y))v = - \sum_{\substack{|\kappa| = k \\ |\mu| = k+1}} d_{2\kappa} A_k[\kappa, \mu] P_\kappa(\alpha) P_\mu(y) \in \Pi_{k+1}$$

and, therefrom,

$$\sigma(\alpha) v = T_k(v).$$

We now show how to obtain a bound for the size of δ-codes, as a function of δ. Therefore, in order to cope with any possible δ, we must perturb the endomorphism T_k as explained next:

Theorem 4.4: Let $\epsilon \in \mathbb{R}^{\pi_k}$, with $\epsilon_\kappa \geq 0$. Let T_k^ϵ be the endomorphism defined on S_k by

$$T_k^\epsilon(v) = T_k(v) - \epsilon * v_k$$

where $\epsilon * v_k := \sum_{|\kappa| = k} \epsilon_\kappa v_\kappa P_\kappa$.

(i) T_k^ϵ has a unique maximal eigenvalue λ_k^ϵ, of multiplicity one, possessing an eigenvector v^ϵ with positive coefficients. Moreover, if $\epsilon \neq 0$,

$$\lambda_{k-1} < \lambda_k^\epsilon < \lambda_k$$

(ii) Let $\epsilon \neq 0$. Any δ-code C such that $s = m - \delta^2 < \lambda_k^\epsilon$ satisfies

$$|C| \leq \frac{\left(\sum_{|\kappa| = k} v_\kappa^\epsilon (\epsilon_\kappa + a_\kappa) \right)^2}{(m - \lambda_k^\epsilon) \left(\sum_{|\kappa| = k} d_{2\kappa}^{-1} \epsilon_\kappa {v_\kappa^\epsilon}^2 \right)}$$

where $a_\kappa := Q_\kappa(1, \ldots, 1) = \sum_{|\mu| = k+1} A_k[\kappa, \mu]$.

Proof:

(i) The matrix J_k^ϵ of T_k^ϵ is equal to J_k, except the diagonal elements lying in B_k. Replacing J_k^ϵ by $J_k^\epsilon + M \,\mathrm{Id}$ for some appropriate M, we obtain a non-negative matrix which is irreducible so its largest eigenvalue has multiplicity one and has an associated eigenvector with positive coordinates. It remains true for J_k^ϵ. Since, when $\epsilon \neq 0$, $J_k^\epsilon < J_k$, we have

$\lambda_k^\epsilon < \lambda_k$. The proof of the inequality $\lambda_{k-1} < \lambda_k^\epsilon$ is the same as the one of $\lambda_{k-1} < \lambda_k$.

(ii) We have $\sigma v^\epsilon = \lambda_k^\epsilon v^\epsilon + \epsilon * v_k^\epsilon + u$ where $u \in \Pi_{k+1}$. We need to compute u, and we set $u = \sum_{|\mu|=k+1} u_\mu P_\mu$. Let μ, $|\mu| = k+1$, we have:

$$\begin{aligned} u_\mu[P_\mu, P_\mu] = [u, P_\mu] &= [\sigma v^\epsilon, P_\mu] \\ &= \sum_{|\kappa|\le k} v_\kappa^\epsilon [\sigma P_\kappa, P_\mu] \\ &= \sum_{|\kappa|=k} v_\kappa^\epsilon A_k[\kappa,\mu][P_\mu, P_\mu] \end{aligned}$$

and we obtain $u_\mu = \sum_{|\kappa|=k} v_\kappa^\epsilon A_k[\kappa,\mu]$. We have found

$$u = \sum_{|\mu|=k+1} \big(\sum_{|\kappa|=k} v_\kappa^\epsilon A_k[\kappa,\mu]\big) P_\mu = \sum_{|\kappa|=k} v_\kappa^\epsilon Q_\kappa,$$

hence the *"generalized Christoffel-Darboux formula"*:

$$v^\epsilon = \sum_{s=0}^{k} v_s^\epsilon \mathbb{P}_s = \frac{\sum_{|\kappa|=k} v_\kappa^\epsilon (\epsilon_\kappa P_\kappa + Q_\kappa)}{\sigma - \lambda_k^\epsilon}. \tag{8}$$

Now we proceed like in Proposition 2.8. Let the numerator of the right hand side be denoted by $N_{k+1}(y)$, and let

$$F_{2k+1}(y) := \frac{N_{k+1}(y)^2}{\sigma(y) - \lambda_k^\epsilon} = N_{k+1}(y) v^\epsilon.$$

We have:

$$\begin{aligned} f_0 = [F_{2k+1}, 1] &= [N_{k+1}, v^\epsilon] \\ &= [\sum_{|\kappa|=k} v_\kappa^\epsilon \epsilon_\kappa P_\kappa, \sum_{|\kappa|=k} v_\kappa^\epsilon P_\kappa] \\ &= \sum_{|\kappa|=k} {v_\kappa^\epsilon}^2 \epsilon_\kappa d_{2\kappa}^{-1}. \end{aligned}$$

Since the coefficients of ϵ and of v^ϵ are non-negative numbers, and $f_0 \neq 0$ when $\epsilon \neq 0$, it follows that F_{2k+1} satisfies the condition (i) of Proposition 2.3. Condition (ii) is clearly fulfilled if $s < \lambda_k^\epsilon$. We calculate

$$F_{2k+1}(1, \dots, 1) = \frac{\big(\sum_{|\kappa|=k} v_\kappa^\epsilon (\epsilon_\kappa + a_\kappa)\big)^2}{m - \lambda_k^\epsilon}.$$

hence the announced bound.

Let us show that we have indeed generalized the situation described in subsection II-C and Proposition 2.8. Let $x \in \mathbb{R}^m$ such that $P_\kappa(x) > 0$, $Q_\kappa(x) \le 0$ for all $|\kappa| = k$, and $P_\kappa(x) \ge 0$ for all $|\kappa| \le k$. Let $\epsilon \in \mathbb{R}^{\pi_k}$ be defined by: $\epsilon_\kappa = -Q_\kappa(x)/P_\kappa(x)$. We can show that $\lambda_k^\epsilon = \sigma(x)$. Indeed, from (i) and (ii) of the proposition,

$$v^\epsilon(y) = \frac{\sum_{|\kappa|=k} v_\kappa^\epsilon \big(-(Q_\kappa(x)/P_\kappa(x)) P_\kappa(y) + Q_\kappa(y)\big)}{\sigma(y) - \lambda_k^\epsilon}.$$

When we let y tend to x, the numerator tends to 0. Since the coordinates of v^ϵ are positive and $P_\kappa(x) \ge 0$ for all $|\kappa| \le k$, the left hand side cannot be equal to zero when $y = x$ ($P_0 = 1$). So the denominator also tends to zero, and $\lambda_k^\epsilon = \sigma(x)$. The Christoffel-Darboux formula (Theorem 2.6(i)) shows that $v^\epsilon = \sum_{|\kappa|\le k} d_{2\kappa} P_\kappa(x) P_\kappa$.

When $m = 1$, $\pi_k = 1$ and any $\epsilon \ge 0$ is of this form. When $m \ge 2$, it is not clear.. It is not even clear that at least one x satisfying these inequalities exists.

Another natural question concerns the values that λ_k^ϵ takes. It is hoped of course that all values in the interval $]\lambda_{k-1}, \lambda_k]$ are attained. We have defined a mapping from $[0, +\infty[^{\pi_k}$ to $]\lambda_{k-1}, \lambda_k]$, sending ϵ to λ_k^ϵ, which is continuous, hence the image in an interval, containing λ_k, since clearly it is the image of $\epsilon = 0$. Let us prove that λ_k^ϵ tends to λ_{k-1} when ϵ tends to $+\infty$. To that end, we use the following inequality, valid for any non-negative matrix J with maximal eigenvalue λ ([22]):

$$\text{For all } x, x_i > 0,\ \lambda \le \sup_i \frac{(xJ)_i}{x_i}.$$

This inequality remains true for the matrix J_k^ϵ, although it is not non-negative, because we can apply it to some $J_k^\epsilon + M\,\mathrm{Id}$, an argument that we have already called for. We choose for $x \in \mathbb{R}^{s_k}$ a vector, which first s_{k-1} coefficients constitute a positive eigenvector of J_{k-1} for the eigenvalue λ_{k-1}. Its last π_k coordinates are denoted by $u = (u_\kappa)_{|\kappa|=k}$. We have:

$$\begin{cases} \text{If } |\nu| \le k-2, \frac{(xJ_k^\epsilon)_\nu}{x_\nu} = \lambda_{k-1} \\ \text{If } |\nu| = k-1, \frac{(xJ_k^\epsilon)_\nu}{x_\nu} = \lambda_{k-1} + \frac{\sum_{|\kappa|=k} u_\kappa C_k[\kappa,\nu]}{x_\nu} \\ \text{If } |\kappa| = k, \frac{(xJ_k^\epsilon)_\kappa}{x_\kappa} = \frac{\sum_{|\nu|=k-1} x_\nu A_{k-1}[\nu,\kappa]}{u_\kappa} \\ \qquad\qquad +(B_k[\kappa,\kappa] - \epsilon_\kappa). \end{cases}$$

The last equality relies on a result that is only proved in Section 5, Proposition 5.1(i), namely that $B_k[\kappa,\kappa'] = 0$ when $\kappa \neq \kappa'$.

Let us now choose an arbitrary small $\alpha > 0$; we can choose the coefficients $u_\kappa > 0$ such that $\frac{\sum_{|\kappa|=k} u_\kappa C_k[\kappa,\nu]}{x_\nu} \le \alpha$ for all ν with $|\nu| = k-1$. Then we can choose $\epsilon_\kappa > 0$ such that $\frac{\sum_{|\nu|=k-1} x_\nu A_{k-1}[\nu,\kappa]}{u_\kappa} + (B_k[\kappa,\kappa] - \epsilon_\kappa) = 0$. We are left with:

$$\begin{cases} \text{If } |\nu| \le k-2, \frac{(xJ_k^\epsilon)_\nu}{x_\nu} = \lambda_{k-1} \\ \text{If } |\nu| = k-1, \frac{(xJ_k^\epsilon)_\nu}{x_\nu} \le \lambda_{k-1} + \alpha \\ \text{If } |\kappa| = k, \frac{(xJ_k^\epsilon)_\kappa}{x_\kappa} = 0. \end{cases}$$

Hence $\lambda_k^\epsilon \le \lambda_{k-1} + \alpha$ for that choice of ϵ.

Let us go back to the bound proved in Theorem 4.4. We can simplify further this bound, getting rid of the eigenvector. We obtain the following nicer, but weaker version:

Corollary 4.5: Let C be a δ-code such that $m - \delta^2 \le \lambda_{k-1}$. Then,

$$|C| \le \frac{4 \sum_{|\kappa|=k} d_{2\kappa} a_\kappa}{m - \lambda_k}. \tag{9}$$

Proof: If C satisfies $\delta^2 > m - \lambda_{k-1}$, since $\lambda_{k-1} < \lambda_k^\epsilon$ for all non-negative ϵ (from Theorem 4.4 (i)), the bound of Theorem 4.4 (iv) applies to C. We get, using Cauchy-Schwartz inequality, and $\lambda_k^\epsilon < \lambda_k$:

$$|C| \leq \frac{1}{m-\lambda_k^\epsilon} \frac{\left(\sum_{|\kappa|=k} v_\kappa^\epsilon(\epsilon_\kappa + a_\kappa)\right)^2}{\sum_{|\kappa|=k} d_{2\kappa}^{-1} \epsilon_\kappa {v_\kappa^\epsilon}^2}$$
$$\leq \frac{1}{m-\lambda_k} \sum_{|\kappa|=k} d_{2\kappa} \frac{(\epsilon_\kappa + a_\kappa)^2}{\epsilon_\kappa}.$$

The function $z \to \frac{(z+a)^2}{z}$ is minimized over $]0, +\infty[$ when $z = a$. We obtain, with $\epsilon_\kappa = a_\kappa$, the announced bound.

V. Asymptotic behavior of the largest eigenvalue λ_k of T_k

In this section, we compute the limit taken by λ_k when the quotient n/k tends to some fixed value (Theorem 5.3). This result is needed to pass to the asymptotic in the inequality (9) for the size of a Grassmannian code.

We first need some very explicit formulas for the coefficients of the symmetric matrix ${J'}_k$ associated to the endomorphism T_k, in the orthonormal basis $\{\sqrt{d_{2\kappa}} P_\kappa, |\kappa| \leq k\}$. From now on we change our usual convention: if not specified, κ is a partition of degree s. The diagonal coefficients of ${J'}_k$ are the same as the ones of J_k, while the other coefficients, denoted by ${A'}_s[\kappa, \mu]$, satisfy

$${A'}_s[\kappa, \mu] = A_s[\kappa, \mu] \sqrt{\frac{d_{2\kappa}}{d_{2\mu}}}.$$

To start with, we gather some known results on the polynomials C_κ.

A. Review of some properties of the polynomials C_κ

The coefficients $\begin{bmatrix} \mu \\ \kappa \end{bmatrix}$ and $\binom{\mu}{\kappa}$ are defined respectively by the following properties:

$$\sigma C_\kappa = \sum_{|\mu|=s+1} \begin{bmatrix} \mu \\ \kappa \end{bmatrix} C_\mu \tag{10}$$

$$\epsilon C_\kappa = \sum_{|\nu|=s-1} \binom{\kappa}{\nu} C_\nu \tag{11}$$

and have the following explicit expressions:

$$\begin{bmatrix} \kappa^{(i)} \\ \kappa \end{bmatrix} = \prod_{\substack{j=1 \\ j \neq i}}^{m} \frac{2\kappa_i - 2\kappa_j + j - i + 1}{2\kappa_i - 2\kappa_j + j - i} \tag{12}$$

$$\binom{\kappa^{(i)}}{\kappa} = (\kappa_i + 1 + \frac{m-i}{2}) \prod_{\substack{j=1 \\ j \neq i}}^{m} \frac{2\kappa_i - 2\kappa_j + j - i + 1}{2\kappa_i - 2\kappa_j + j - i + 2} \tag{13}$$

while any other values are equal to zero (see [17, Lemma 7.5.7], [17], [14, Th 14.1], [10]). The polynomials C_κ are intimately related to the decomposition of $\mathrm{GL}(m, \mathbb{R})$-modules ([9, Theorem 5.2.9]):

$$\mathcal{R}(\mathrm{GL}(m, \mathbb{R})/\mathrm{O}(m, \mathbb{R})) = \oplus_\kappa F_m^{2\kappa}.$$

For later use, we settle the notation: $\delta_\kappa := \dim(F_m^\kappa)$ and we recall the formula ([8]):

$$\delta_\kappa := \dim(F_m^\kappa) = \prod_{1 \leq i < j \leq m} \frac{\kappa_i - \kappa_j + j - i}{j - i}. \tag{14}$$

B. Formulas for the coefficients of the matrix ${J'}_k$

Proposition 5.1: The matrix B_s has the following properties:

(i) $B_s[\kappa, \kappa'] = 0$ for all $\kappa \neq \kappa'$.

(ii) If $m \leq n/2$,

$$2B_s[\kappa, \kappa] = \sum_{i \in u(\kappa)} \binom{\kappa^{(i)}}{\kappa} \begin{bmatrix} \kappa^{(i)} \\ \kappa \end{bmatrix} \frac{2\kappa_i + m + 1 - i}{2\kappa_i + n/2 + 1 - i}$$
$$- \sum_{i \in d(\kappa)} \binom{\kappa}{\kappa_{(i)}} \begin{bmatrix} \kappa \\ \kappa_{(i)} \end{bmatrix} \frac{2\kappa_i + m - 1 - i}{2\kappa_i + n/2 - 1 - i}.$$

(iii) If $m = n/2$, $B_s[\kappa, \kappa] = m/2$.

Proof: We recall that the coefficients $\beta_{\kappa,\nu}$ are defined by:

$$P_\kappa = \beta_\kappa C_\kappa + \sum_{\nu | \kappa > \nu} \beta_{\kappa,\nu} C_\nu.$$

Inverting these relations, we obtain coefficients $\alpha_{\kappa,\nu}$ such that

$$C_\kappa = \alpha_\kappa P_\kappa + \sum_{\nu | \kappa > \nu} \alpha_{\kappa,\nu} P_\nu.$$

Taking into account the formula (10), we obtain:

$$B_s[\kappa, \kappa'] = \beta_\kappa \sum_{|\mu|=s+1} \begin{bmatrix} \mu \\ \kappa \end{bmatrix} \alpha_{\mu,\kappa'} + \sum_{|\nu|=s-1} \beta_{\kappa,\nu} \begin{bmatrix} \kappa' \\ \nu \end{bmatrix} \alpha_{\kappa'}.$$

We use the following obvious relations: $\alpha_\kappa \beta_\kappa = 1$ and $\alpha_\kappa \beta_{\kappa,\kappa'} + \beta_{\kappa'} \alpha_{\kappa,\kappa'} = 0$ to rewrite

$$B_s[\kappa, \kappa'] = \frac{\beta_\kappa}{\beta_{\kappa'}} \Big(- \sum_{|\mu|=s+1} \begin{bmatrix} \mu \\ \kappa \end{bmatrix} \frac{\beta_{\mu,\kappa'}}{\beta_\mu} \tag{15}$$
$$+ \sum_{|\nu|=s-1} \begin{bmatrix} \kappa' \\ \nu \end{bmatrix} \alpha_{\kappa'} \frac{\beta_{\kappa,\nu}}{\beta_\kappa} \Big). \tag{16}$$

Let us assume first that $\kappa \neq \kappa'$. Since $\begin{bmatrix} \mu \\ \kappa \end{bmatrix}$ is non zero only if $\mu = \kappa^{(i)}$ for some index i, and also $\beta_{\mu,\kappa'}$ is non zero only if $\mu = {\kappa'}^{(j)}$ for some index j, at most one term in the first summation may be non zero, and the same argument holds for the second summation. We only have to consider the case when κ' satisfies: for some indexes $i \neq j$, $\kappa'_i = \kappa_i + 1$ and $\kappa'_j = \kappa_j - 1$. The remaining terms in the expression of $B_s[\kappa, \kappa']$ correspond to $\mu = \kappa^{(i)} = {\kappa'}^{(j)}$ and $\nu = \kappa_{(j)} = {\kappa'}_{(i)}$.

Moreover, the coefficients $\beta_{\mu,\kappa}$ are calculated in [14], and in particular we have:

$$\frac{\beta_{\kappa,\kappa_{(j)}}}{\beta_\kappa} = -\frac{1}{2} \binom{\kappa}{\kappa_{(j)}} \frac{2\kappa_j + m - 1 - j}{2\kappa_j + n/2 - 1 - j}. \tag{17}$$

Replacing in (15) we have

$$B_s[\kappa, \kappa'] = \frac{\beta_\kappa}{\beta_{\kappa'}} \frac{2\kappa_j + m - 1 - j}{2(2\kappa_j + n/2 - 1 - j)} \cdot$$
$$\left(\begin{bmatrix} \kappa^{(i)} \\ \kappa \end{bmatrix} \binom{\kappa^{(i)}}{\kappa'} - \binom{\kappa}{\kappa_{(j)}} \begin{bmatrix} \kappa' \\ \kappa_{(j)} \end{bmatrix} \right).$$

Combining (10) and (11) in the obvious relation:

$$(\epsilon\sigma - \sigma\epsilon) C_\kappa = m C_\kappa$$

leads to:

$$\sum_{|\mu|=s+1} \begin{bmatrix}\mu\\ \kappa\end{bmatrix}\binom{\mu}{\kappa'} - \sum_{|\nu|=s-1}\binom{\kappa}{\nu}\begin{bmatrix}\kappa'\\ \nu\end{bmatrix} = \begin{cases}0 & \text{if } \kappa \neq \kappa' \\ m & \text{if } \kappa = \kappa'\end{cases} \tag{18}$$

From (18) we can conclude that $B_s[\kappa,\kappa'] = 0$ when $\kappa \neq \kappa'$.

When $\kappa = \kappa'$, replacing (17) in (15) leads to the formula (ii). If moreover $n = m/2$, taking account of (18) we obtain $B_s[\kappa,\kappa] = m/2$.

We now give explicit formulas for the coefficients of J'_k:

Proposition 5.2: With the following notations:

$$q_i := 2\kappa_i - i + m$$

$$N := n - 2m$$

$$D(x) = \frac{x^2}{x^2-1} \quad \text{and}$$

$$\begin{cases} C(x) = \frac{(x+1)(x+N)}{(2x+N)(2x+N+2)} & x \neq 0 \\ C(0) = \frac{1}{N+2} \end{cases}$$

we have the expressions:

$$B_s[\kappa,\kappa] = \frac{m}{2} - \frac{N}{4}\sum_{i\in u(\kappa)}\Big(\prod_{\substack{j=1\\ j\neq i}}^{m} D(q_i-q_j+1)\Big)\frac{q_i+2}{2q_i+N+2} + \frac{N}{4}\sum_{i\in d(\kappa)}\Big(\prod_{\substack{j=1\\ j\neq i}}^{m} D(q_i-q_j-1)\Big)\frac{q_i}{2q_i+N-2}.$$

$$A'_s[\kappa,\kappa^{(i)}] = \Big(\Big(\prod_{j\neq i} D(q_i-q_j+1)D(q_i+q_j+N+1)\Big). C(q_i)C(q_i+1)\Big)^{1/2}.$$

Proof: For the calculation of B_s, we replace in Proposition 5.1 (ii) the formulas (12) and (13), and take account of Proposition 5.1 (iii).

In order to calculate $A'_s[\kappa,\kappa^{(i)}]$, we have already seen that:

$$A_s[\kappa,\kappa^{(i)}] = \begin{bmatrix}\kappa^{(i)}\\ \kappa\end{bmatrix}\left(\frac{\beta_{\kappa^{(i)}}}{\beta_\kappa}\right)^{-1}.$$

We need a formula for $\left(\frac{\beta_{\kappa^{(i)}}}{\beta_\kappa}\right)^{-1}$. Expressions for the leading coefficients of the polynomials C_κ and P_κ can be found in [12] and [23]. Putting them together we find:

$$\frac{\beta_{\kappa^{(i)}}}{\beta_\kappa} = \prod_{\substack{j=1\\ j\neq i}}^{m}\left(\frac{q_i+q_j+N}{q_i+q_j+N+1}\right)\frac{(2q_i+N)(2q_i+N+2)}{(q_i+N)(q_i+N+1)}$$

where the last fraction must be understood as $(N+2)/(N+1)$ when $q_i = 0$. Joined with (10), we obtain

$$A_s[\kappa,\kappa^{(i)}] = \prod_{\substack{j=1\\ j\neq i}}^{m}\left(\frac{q_i-q_j+1}{q_i-q_j}\right)\left(\frac{q_i+q_j+N+1}{q_i+q_j+N}\right). \frac{(q_i+N)(q_i+N+1)}{(2q_i+N)(2q_i+N+2)}.$$

Next we use ([8]):

$$\frac{d_{2\kappa^{(i)}}}{d_{2\kappa}} = \prod_{\substack{j=1\\ j\neq i}}^{m}\left(\frac{q_i-q_j+2}{q_i-q_j}\right)\left(\frac{q_i+q_j+N+2}{q_i+q_j+N}\right). \frac{(2q_i+N+4)(q_i+N)(q_i+N+1)}{(2q_i+N)(q_i+1)(q_i+2)}$$

where the last fraction must be understood as $(N+4)(N+2)/2$ when $q_i = 0$, and we obtain the announced formula for $A'[\kappa,\kappa^{(i)}]$.

C. The limit of λ_k

Now n varies with k so we rather denote by $T_k^{(n)}$ the endomorphism defined previously and $\lambda_k^{(n)}$ its largest eigenvalue.

Theorem 5.3: If $n/2k \to \ell$, while $n \to +\infty$ and $k \to +\infty$,

$$\lim \lambda_k^{(n)} = 4\frac{\ell + 1/m}{(\ell+2/m)^2}.$$

Proof: We give careful proofs in the cases $m = 1$ and $m = 2$, and will be more sketchy in the general case. As it was noticed previously, when $m = 1$ the eigenvalues are the zeros of the Jacobi polynomials; their asymptotic is calculated in [7], exploiting the differential equation for the Jacobi polynomials and Sturm's method. Another approach, using chain sequences, is used in [24]. However, none of these methods seem to generalize easily to the several variable case. Our argument will only use the fact that the matrix $J'^{(n)}_k$ is non-negative. More precisely, we use the following:

Lemma 5.4: [22] Let J be a non-negative symmetric matrix of size N, with largest eigenvalue λ.

(i) For all $x \in \mathbb{R}^N$ with $x_i > 0$, $\lambda \leq \max_i \frac{(xJ)_i}{x_i}$.

(ii) For all $x \in \mathbb{R}^N$ $x \neq 0$, $\lambda \geq \frac{(xJ)\cdot x}{x\cdot x}$.

The case $m = 1$**.** We recover from Proposition 5.2 the formulas:

$$2b_s = 1 - \frac{(n-2)(n-4)}{(4s+n)(4s+n-4)}$$

$$a'_s = \left(\frac{(2s+1)(2s+2)(2s+n-2)(2s+n-1)}{(4s+n-2)(4s+n)^2(4s+n+2)}\right)^{1/2}$$

From these expressions we see that both sequences are increasing with s. Moreover, we see easily that if $s \sim k$, $b_s \sim 2\frac{\ell+1}{(\ell+2)^2}$, and $a'_s \sim \frac{\ell+1}{(\ell+2)^2}$. Applying Lemma 5.4 (i) with $x_s = 1$ for all s leads to:

$$\lambda_k^{(n)} \leq a'_{k-1} + b_k + a'_k$$

and the right hand side tends to $4(\ell+1)/(\ell+2)^2$ when $n/2k$ tends to ℓ.

We lower bound $\lambda_k^{(n)}$ using Lemma 5.4 (ii) and a choice of x proposed in [24]: let x be defined by:

$$\begin{cases} x_s &= 0 \quad 1 \leq s \leq t := k - \lfloor\sqrt{k}\rfloor + 1 \\ x_s &= 1 \qquad t+1 \leq s \leq k+1 \end{cases}$$

so that $x_s = 1$ on the $\lfloor\sqrt{k}\rfloor$ last coordinates. Then,

$$\frac{(xJ'^{(n)}_k)\cdot x}{x\cdot x} \geq \frac{\sum_{s=t+1}^{k-1}(a'_{s-1}+b_s+a'_s)}{k-t+1} \geq \left(\frac{k-t-1}{k-t+1}\right)(a'_t+b_{t+1}+a'_{t+1})$$

Again, the right hand side tends to $4(\ell+1)/(\ell+2)^2$, hence the result.

The case $m=2$. From Proposition 5.2, we have, setting $s:=\kappa_1+\kappa_2$ and $v:=\kappa_1-\kappa_2$:

$$B_s[\kappa,\kappa]=1+\frac{(n-6)(n-4)}{8(4s+2n-6)}\left(\frac{(4s+2n-4)^2}{(4\kappa_1+n)(4\kappa_2+n-2)} -\frac{(4s+2n-8)^2}{(4\kappa_1+n-4)(4\kappa_2+n-6)}\right)$$

$$A'_s[\kappa,\kappa^{(1)}]=\left(\frac{(2v+2)^2}{(2v+2)^2-1}\right)^{1/2}\left(\frac{(2s+n-2)^2}{(2s+n-2)^2-1}\right)^{1/2}\cdot \left(\frac{(2\kappa_1+2)(2\kappa_1+3)(2\kappa_1+n-3)(2\kappa_1+n-2)}{(4\kappa_1+n-2)(4\kappa_1+n)^2(4\kappa_1+n+2)}\right)^{1/2}$$

$$A'_s[\kappa,\kappa^{(2)}]=\left(\frac{(2v)^2}{(2v)^2-1}\right)^{1/2}\left(\frac{(2s+n-2)^2}{(2s+n-2)^2-1}\right)^{1/2}\cdot \left(\frac{(2\kappa_2+1)(2\kappa_2+2)(2\kappa_2+n-4)(2\kappa_2+n-3)}{(4\kappa_2+n-4)(4\kappa_2+n-2)^2(4\kappa_2+n)}\right)^{1/2}$$

One can verify that these coefficients are increasing with s when v stays constant. This is easy to see for B_s, not so obvious for the two others because the second term is decreasing while the last big quotient is increasing.

In order to obtain a lower bound for $\lambda_k^{(n)}$ from Lemma 5.4 (ii), we choose $x=(x_\kappa)$ with: $x_\kappa=0,1$. We fix a number $V<k-\sqrt{k}$. Let $\mathcal{K}_{V,s}$ be the set of the V partitions of degree s with smallest $v=\kappa_1-\kappa_2$. Hence $\mathcal{K}_{V,s}=\{\kappa \mid |\kappa|=s, \kappa_2\geq \lfloor\frac{s}{2}\rfloor-V+1\}$. We set $x_\kappa=1$ iff $\deg(\kappa)\geq t:=k-\lfloor\sqrt{k}\rfloor+1$, and $\kappa\in\mathcal{K}_{V,|\kappa|}$. We need to avoid in $\mathcal{K}_{V,s}$ some partitions, namely the ones with $v=0$ and the ones with v maximal (for those partitions, some terms are either missing or are equal to zero in $(xJ'^{(n)}_k)_\kappa$). Let this new set be denoted by $\mathcal{K}'_{V,s}$. We have, when $x_\kappa=1$, $\kappa\in\mathcal{K}'_{V,s}$, $|\kappa|\neq t,k$,

$$(xJ'^{(n)}_k)_\kappa=B_s[\kappa,\kappa]+\sum_{i=1}^{2}A'_s[\kappa,\kappa^{(i)}]+\sum_{i=1}^{2}A'_{s-1}[\kappa_{(i)},\kappa]$$

In the expressions of $A'_s[\kappa,\kappa^{(i)}]$ we can minor the first term by 1 ($v\neq 0$), then minor each term by its minimal value in the sequence $v=cte$ to which it belongs. As was mentioned before, this minimal value is obtained when the degree is minimal, i.e. when $s=t$ or $s=t+1$. We do the same for $A'_{s-1}[\kappa_{(i)},\kappa]$ and for $B_s[\kappa,\kappa]$. Then we must consider the behavior when s is constant of $B_s[\kappa,\kappa]$, of:

$$A^1{}_s[\kappa]:=\frac{(2s+n-2)}{((2s+n-2)^2-1)^{1/2}}\cdot \left((C(q_1)C(q_1+1))^{1/2}+(C(q_2)C(q_2+1))^{1/2}\right)$$

and of the analogous expression $A^2{}_{s-1}[\kappa]$ corresponding to the last term. These expressions are increasing with κ_2. Let $B_{s,V}{}^{min}$, $A^i{}_{s,V}{}^{min}$ be their minimal values in $\mathcal{K}'_{V,s}$. For simplicity, we assume that $\min(B_{t,V}{}^{min},B_{t+1,V}{}^{min})=B_{t,V}{}^{min}$, and the same for A^1, A^2.

We obtain:

$$\frac{(xJ'^{(n)}_k)\cdot x}{x\cdot x}\geq \frac{(k-t-1)(V-2)}{(k-t+1)V}(B_{t,V}{}^{min}+A^1{}_{t,V}{}^{min}+A^2{}_{t,V}{}^{min}).$$

Now we let $n/2k$ tend to ℓ. Since $B_{t,V}{}^{min}$ is obtained at a partition essentially equal to $[t/2-V/2,t/2+V/2$, and since $t\sim k$, we see that $B_{t,V}{}^{min}$ tends to $2(\ell+1/2)/(\ell+1)^2$. For the same reason, $A^1{}_{t,V}{}^{min}$ and $A^2{}_{t,V}{}^{min}$ tend to $(\ell+1/2)/(\ell+1)^2$ (the parameter V is still fixed at this stage). So we obtain

$$\liminf \lambda_k^{(n)}\geq(1-\frac{2}{V})\cdot 4\frac{\ell+1/2}{(\ell+1)^2}.$$

Now we let V tend to $+\infty$ to obtain the appropriate lower bound.

The second and last step obtains an upper bound for $\lambda_k^{(n)}$ from Lemma 5.4 (i) with an appropriate choice of x. The choice $x_\kappa=1$ for all κ is not good enough here because $D(2v+2)^{1/2}+D(2v)^{1/2}\neq 2$. We need some x_κ that modify properly these factors. We choose $x_\kappa:=(2v+1)^{1/2}$ where $v=\kappa_1-\kappa_2$. We have

$$\frac{(xJ'^{(n)}_k)_\kappa}{x_\kappa}=B_s[\kappa,\kappa]+\sum_{i=1}^{2}A'_s[\kappa,\kappa^{(i)}]\frac{x_{\kappa^{(i)}}}{x_\kappa}+\sum_{i=1}^{2}A'_{s-1}[\kappa_{(i)},\kappa]\frac{x_{\kappa_{(i)}}}{x_\kappa}$$

Let:

$$A^\sharp[\kappa,\kappa^{(1)}] = \left(\frac{2v+2}{2v+1}\right)\left(\frac{(2s+n-2)^2}{(2s+n-2)^2-1}\right)^{1/2}\cdot \left(\frac{(2\kappa_1+2)(2\kappa_1+3)(2\kappa_1+n-3)(2\kappa_1+n-2)}{(4\kappa_1+n-2)(4\kappa_1+n)^2(4\kappa_1+n+2)}\right)^{1/2}$$

$$A^\sharp[\kappa,\kappa^{(2)}] = \left(\frac{2v}{2v+1}\right)\left(\frac{(2s+n-2)^2}{(2s+n-2)^2-1}\right)^{1/2}\cdot \left(\frac{(2\kappa_2+1)(2\kappa_2+2)(2\kappa_2+n-4)(2\kappa_2+n-3)}{(4\kappa_2+n-4)(4\kappa_2+n-2)^2(4\kappa_2+n)}\right)^{1/2}$$

$$A^\flat[\kappa_{(1)},\kappa] = \left(\frac{2v}{2v+1}\right)\left(\frac{(2s+n-4)^2}{(2s+n-4)^2-1}\right)^{1/2}\cdot \left(\frac{(2\kappa_1)(2\kappa_1+1)(2\kappa_1+n-5)(2\kappa_1+n-4)}{(4\kappa_1+n-6)(4\kappa_1+n-4)^2(4\kappa_1+n-2)}\right)^{1/2}$$

$$A^\flat[\kappa_{(2)},\kappa] = \left(\frac{2v+2}{2v+1}\right)\left(\frac{(2s+n-4)^2}{(2s+n-4)^2-1}\right)^{1/2}\cdot \left(\frac{(2\kappa_2-1)(2\kappa_2)(2\kappa_2+n-6)(2\kappa_2+n-5)}{(4\kappa_2+n-8)(4\kappa_2+n-6)^2(4\kappa_2+n-4)}\right)^{1/2}$$

Since

$$D(2v+2)^{1/2}\frac{(2v+3)^{1/2}}{(2v+1)^{1/2}} = \frac{2v+2}{2v+1},$$

and

$$D(2v)^{1/2}\frac{(2v-1)^{1/2}}{(2v+1)^{1/2}} = \frac{2v}{2v+1},$$

we have:

$$\frac{(xJ'^{(n)}_k)_\kappa}{x_\kappa} = B_s[\kappa,\kappa] + \sum_{i=1}^{2} A^\sharp[\kappa,\kappa^{(i)}] + \sum_{i=1}^{2} A^\flat[\kappa_{(i)},\kappa].$$

This expression is increasing with s when v is fixed. When s is fixed, $B_s[\kappa,\kappa]$, $\sum_{i=1}^2 A^\sharp[\kappa,\kappa^{(i)}]$ and $\sum_{i=1}^2 A^\flat[\kappa_{(i)},\kappa]$ are maximal at $\kappa = [s/2, s/2]$ (we extend the functions to partitions with real parts here). We obtain, with $\rho_k = [k/2, k/2]$,

$$\max_\kappa \frac{(xJ'^{(n)}_k)_\kappa}{x_\kappa} \leq B_s[\rho_k,\rho_k] + \sum_{i=1}^{2} A^\sharp[\rho_k,{\rho_k}^{(i)}] + \sum_{i=1}^{2} A^\flat[\rho_{k(i)},\rho_k].$$

The computation of these values shows that the right hand side tends to $4(\ell+1/2)/(\ell+1)^2$ when $n/2k \to \ell$.

The general case $\mathbf{m > 2}$ works the same. For the lower bound, we use $\mathcal{K}_{V,s} := \{\kappa \mid |\kappa| = s, \kappa_m \geq \lfloor \frac{s}{m} \rfloor - V + 1\}$. The cardinality of $\mathcal{K}_{V,s}$ only depends on $s \mod m$. We should avoid some partitions in $\mathcal{K}_{V,s}$, namely the ones with some parts equal and the ones with $\kappa_m = \lfloor \frac{s}{m} \rfloor - V + 1$. Their number is negligible compared to the cardinality of $\mathcal{K}_{V,s}$. Then, we proceed in the same way as for $m = 2$.

The upper bound is obtained with $x_\kappa = (\delta_{2\kappa})^{1/2}$. We have

$$\prod_{\substack{j=1\\ j\neq i}}^{m} D(q_i - q_j + 1)^{1/2}\left(\frac{\delta_{2\kappa^{(i)}}}{\delta_{2\kappa}}\right)^{1/2} = \prod_{\substack{j=1\\ j\neq i}}^{m}\frac{q_i - q_j + 1}{q_i - q_j}.$$

hence

$$\frac{(xJ'^{(n)}_k)_\kappa}{x_\kappa} = B_s[\kappa,\kappa] + \sum_{i=1}^{m} A^\sharp[\kappa,\kappa^{(i)}] + \sum_{i=1}^{m} A^\flat[\kappa_{(i)},\kappa],$$

where

$$A^\sharp[\kappa,\kappa^{(i)}] = \prod_{\substack{j=1\\ j\neq i}}^{m}\frac{q_i - q_j + 1}{q_i - q_j}\cdot \Big(\big(\prod_{\substack{j=1\\ j\neq i}}^{m} D(q_i + q_j + N + 1)\big)C(q_i)C(q_i+1)\Big)^{1/2}$$

and similarly

$$A^\flat[\kappa_{(i)},\kappa] = \prod_{\substack{j=1\\ j\neq i}}^{m}\frac{q_i - q_j - 1}{q_i - q_j}\cdot \Big(\big(\prod_{\substack{j=1\\ j\neq i}}^{m} D(q_i + q_j + N - 1)\big)C(q_i-2)C(q_i-1)\Big)^{1/2}.$$

We have the nice identity:

$$\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m}\frac{q_i - q_j + 1}{q_i - q_j} = m.$$

We do not have a reference for this last identity, so we give an argument here: from (14),

$$\frac{\dim F_m^{\kappa^{(i)}}}{\dim F_m^\kappa} = \prod_{\substack{j=1\\ j\neq i}}^{m}\frac{\kappa_i - \kappa_j + j - i + 1}{\kappa_i - \kappa_j + j - i}.$$

We obtain the demanded identity as the equality of the dimensions in the following decomposition of $\mathrm{GL}(m,\mathbb{R})$-modules (Pieri's rule, [9]):

$$F_m^{(1)} \otimes F_m^\kappa = \oplus_{i=1}^m F_m^{\kappa^{(i)}}.$$

It turns out that the coefficients $B[\kappa,\kappa]$, $A^\sharp[\kappa,\kappa^{(i)}]$ and $A^\flat[\kappa_{(i)},\kappa]$ are increasing when κ runs over a sequence of the type $(\nu + s[1,1,\ldots,1])_{s\geq 0}$ (when N is big enough), and that, on the space of partitions (with real parts) κ of fixed degree k, the maximum of the expressions $B[\kappa,\kappa]$, $\sum_{i=1}^m A^\sharp[\kappa,\kappa^{(i)}]$ and $\sum_{i=1}^m A^\flat[\kappa_{(i)},\kappa]$ is attained at $\kappa = \rho_k = [k/m, k/m, \ldots, k/m]$.

Moreover, it is easy to see that, when $n/2k \to \ell$,

$$\lim B_k[\rho_k,\rho_k] = 2\frac{\ell + 1/m}{(\ell + 2/m)^2}$$

$$\lim\left(\sum_{i=1}^{m} A^{\sharp}[\rho_k,\rho_k^{(i)}]\right)=\lim\left(\sum_{i=1}^{m} A^{\flat}[(\rho_k)_{(i)},\rho_k]\right)$$
$$=\frac{\ell+1/m}{(\ell+2/m)^2}.$$

Remark 5.5: One obvious consequence of Theorem 5.3 is that, for fixed n, the eigenvalue $\lambda_k^{(n)}$ runs over the whole interval $]0,m[$. Hence the bounds proved in section IV for the size of grassmannian codes potentially cover all possible minimal distance.

VI. An asymptotic bound for the size of Grassmannian codes

We are now ready to take the limit when n tends to $+\infty$ in the inequality (9), and prove Theorem 1.1.

We are left with the estimate of $\log(\sum_{|\kappa|=k} d_{2\kappa}a_\kappa)/n$.

Lemma 6.1: Let $\delta_\kappa := \dim(F_n^\kappa)$. If $n/2k \to \rho^{-1} \in \mathbb{R}$ while n and k tend to $+\infty$,

$$\limsup\frac{1}{n}\log(\sum_{\substack{|\kappa|=2k\\ \ell(\kappa)\le m}}\delta_\kappa)\le \tag{19}$$
$$m\big((1+\rho)\log(1+\rho)-\rho\log(\rho)\big). \tag{20}$$

Proof: In the case $m=1$, $\delta_{2k}=\dim S_{2k}=\binom{n+2k-1}{2k}$ and it is a classical result. The general case is probably well-known but since we lack a reference, we give a proof here. Let κ be a partition of length at most m and of degree $2k$, that we extend to a partition with n parts with an appropriate number of zeros. From (14),

$$\dim(F_n^\kappa)=\prod_{1\le i<j\le n}\frac{\kappa_i-\kappa_j+j-i}{j-i}.$$

Since $\kappa_j=0$ when $j>m$, we have

$$\dim(F_n^\kappa)=\prod_{1\le i<j\le m}\frac{\kappa_i-\kappa_j+j-i}{j-i}\prod_{1\le i\le m}\prod_{j>m}\frac{\kappa_i+j-i}{j-i}$$

We upper bound:

$$\prod_{j>m}\frac{\kappa_i+j-i}{j-i}\le\binom{n+\kappa_i-1}{\kappa_i}$$

and

$$\prod_{1\le i<j\le m}\frac{\kappa_i-\kappa_j+j-i}{j-i}\le(2k+1)^{m^2}$$

to obtain

$$\sum_{\substack{|\kappa|=2k\\ \ell(\kappa)\le m}}\delta_\kappa\le(2k+1)^{m^2}\sum_{\substack{|\kappa|=2k\\ \ell(\kappa)\le m}}\left(\prod_{i=1}^{m}\binom{n+\kappa_i-1}{\kappa_i}\right)$$
$$\le(2k+1)^{m^2}\left(\sum_{s=0}^{2k}\binom{n+s-1}{s}\right)^m$$
$$\le(2k+1)^{m^2}\binom{n+2k}{2k}^m$$

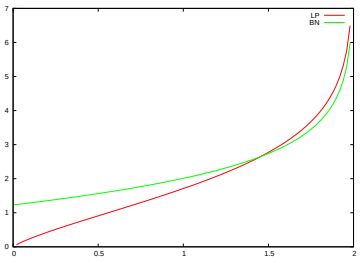

Fig. 1. LP and Hamming asymptotic bounds for $m=2$

and we obtain the announced limiting result using the classical

$$\lim_{n/2k\to\rho^{-1}}\frac{1}{n}\log\binom{n+2k}{2k}=(1+\rho)\log(1+\rho)-\rho\log(\rho).$$

From the three-term relation (2.4), specializing to $(1,\dots,1)$ we get trivially
$a_\kappa \le m$ and hence $\sum_{|\kappa|=k} d_{2\kappa}a_\kappa \le m\sum_{\substack{|\kappa|=2k\\ \ell(\kappa)\le m}}\delta_\kappa$ (obviously $d_{2\kappa}\le\delta_{2\kappa}$ since $V_n^{2\kappa}$ is contained in $F_n^{2\kappa}$).

Then we only have to solve the equation, involving the limiting result of Theorem 5.3,

$$s=\lim_{n/2k\to\rho^{-1}}\lambda_{k-1}^{(n)}=4\frac{\rho^{-1}+1/m}{(\rho^{-1}+2/m)^2}$$

which leads to

$$\rho=\frac{m}{2}(-1+(1-\frac{s}{m})^{-1/2}).$$

VII. LP versus Hamming

In [4], A. Barg and D. Nogin give an asymptotic bound for the size of Grassmannian codes, derived from the so-called Hamming bound. They prove, with the notations of Theorem 1.1:

Theorem 7.1: [4]

$$\frac{1}{n}\log|C|\lesssim -m\log\left(\sqrt{1-\sqrt{\frac{s+m}{2m}}}\right) \tag{21}$$

It turns out that our bound (1) is better than (21) only when s is small.[1] The crossing point s_0 for the two bounds has the approximate value:

m	2	3	4	5	6
s_0	1.4528	1.2714	1.1853	1.1372	1.1067

7	8	9	10
1.0856	1.0702	1.0584	1.0492

Figure 1 plots the two bounds for $m=2$.

[1] After this paper was submitted, the authors have further improved (21), see [25].

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